Annals of Global Analysis and Geometry (2019) 55:703–717
https://doi.org/10.1007/s10455-019-09648-x

Toric nearly Kähler manifolds

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Received: 8 November 2018 / Accepted: 2 January 2019 / Published online: 12 January 2019

Abstract
We show that 6-dimensional strict nearly Kähler manifolds admitting effective $\mathbb{T}^3$ actions by automorphisms are completely characterized in the neigborhood of each point by a function on $\mathbb{R}^3$ satisfying a certain Monge–Ampère-type equation.

Keywords Killing vector field · Nearly Kähler manifold · Toric structure

Mathematics Subject Classification 53C12 · 53C24 · 53C55

1 Introduction

Nearly Kähler manifolds were originally introduced as the class $\mathcal{W}_1$ in the Gray–Hervella classification of almost Hermitian manifolds [7]. More precisely, an almost Hermitian manifold (M, g, J) is called nearly Kähle (NK in short) if $(\nabla_X J)(X) = 0$ for every vector field X on M, where ∇ denotes the Levi-Civita covariant derivative of g. A NK manifold is called *strict* if $\nabla J \neq 0$.

In [12], it was shown that every NK manifold is locally a product of one of the following types of factors:

- Kähle manifolds;
- 3-symmetric spaces;
- twistor spaces of positive quaternion-Kähle manifolds;
- 6-dimensional strict NK manifolds.

It is thus crucial to understand the 6-dimensional case, to which we will restrict in the sequel. In dimension 6, strict NK are important for several further reasons: They admit real Killing spinors [5]; in particular, they are Einstein with positive scalar curvature, and they can

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be characterized in terms of exterior differential systems as manifolds with special generic 3-forms in the sense of Hitchin [8].

Until 2015, the only known examples of compact 6-dimensional strict NK manifolds were the 3-symmetric spaces $S^6 = \mathrm{G}_2/\mathrm{SU}(3)$, $F(1,2) = \mathrm{SU}_3/S^1 \times S^1$, $CP^3 = \mathrm{Sp}_2/S^1 \times \mathrm{Sp}_1$ and $S^3 \times S^3 = \mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_1/\mathrm{Sp}_1$. Moreover, J.-B. Butruille has shown in [1] that these are the only homogeneous examples.

A breakthrough was achieved very recently by L. Foscolo and M. Haskins, who studied cohomogeneity one NK metrics and obtained the first examples of non-homogeneous NK structures on S^6 and $S^3 \times S^3$, cf. [3,4]. The corresponding metrics are shown to exist, but cannot be constructed explicitly. However, their isometry group is known and is equal to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in both cases.

It is easy to show that a torus acting by automorphisms of a NK structure (M^6, g, J) has dimension at most 3 (Corollary 3.2), and if equality holds, then the corresponding commuting vector fields span a totally real distribution on a dense open set of M (cf. Lemma 3.4). In the present paper, we study 6-dimensional nearly Kähle manifolds whose automorphism group has maximal possible rank. We call them *toric NK structures* by analogy with the Kähle case.

Our main result is to give a local characterization of toric NK structures in terms of a single function of 3 real variables satisfying a certain Monge–Ampère-type equation. We conjecture that the only compact toric NK manifold is $S^3 \times S^3$ with its 3-symmetric NK structure.

2 Structure equations

Let M^6 be an oriented manifold. An SU(3)-structure on M is a triple (g, J, ψ), where g is a Riemannian metric, J is a compatible almost complex structure (i.e., $\omega := g(J\cdot,\cdot)$ is a 2-form), and $\psi = \psi^+ + i\psi^-$ is a (3, 0) complex volume form satisfying

$$\psi \wedge \bar{\psi} = -8i\,\mathrm{vol}_g. \tag{2.1}$$

Following Hitchin [8], it is possible to characterize SU(3)-structures in terms of exterior forms only. If ψ^+ is a 3-form on M, one can define $K \in \mathrm{End(TM)} \otimes 3^6\mathrm{M}$ by

$$X \mapsto K(X) := (X \lrcorner \psi^+) \wedge \psi^+ \in \Lambda^5 M \simeq \mathrm{TM} \otimes \Lambda^6\mathrm{M}.$$

Lemma 2.1 ([8]) *A non-degenerate 2-form ω on M, and a 3-form $\psi^+ \in \Lambda^3 M$ satisfying*

(i) $\omega \wedge \psi^+ = 0$.
(ii) $\mathrm{tr}K^2 = -\frac{1}{6}(\omega^3)^2 \in (\Lambda^6 M)^{\otimes 2}$.
(iii) $\omega(X, K(X))/\omega^3 > 0$ *for every* $X \neq 0$.

define an SU(3)*-structure on* M.

Proof It is easy to check that

$$K^2 = \frac{1}{6}\mathrm{Id} \otimes \mathrm{tr}(K^2) \in \mathrm{End(TM)} \otimes (3^6\mathrm{M})^{\otimes 2}. \tag{2.2}$$

From (ii), we see that $J := 6K/\omega^3$ is an almost complex structure on M. The tensor g defined by $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is symmetric by (i) and positive definite by (iii). Finally, it is straightforward to check that $\psi^+ + i\psi^-$ is a (3, 0) complex volume form satisfying (2.1), where $\psi^- := -\psi^+(J\cdot,\cdot,\cdot)$. ◻

Since $\mathrm{vol}_g = \frac{1}{6}\omega^3$, (2.1) is equivalent to

$$\psi^+ \wedge \psi^- = \frac{2}{3}\omega^3. \tag{2.3}$$

Definition 2.1 A strict NK structure on M^6 is an SU(3)-structure $(\psi^\pm, \omega)$ satisfying

$$\mathrm{d}\omega = 3\psi^+ \tag{2.4}$$

and

$$\mathrm{d}\psi^- = -2\omega \wedge \omega. \tag{2.5}$$

For an alternative definition and more details on NK manifolds, we refer to [6] or [10].

Let g denote the Riemannian metric induced by $(\psi^\pm, \omega)$, with Levi-Civita covariant derivative ∇, and let J denote the induced almost complex structure. From now on, we identify vectors and 1-forms, as well as skew-symmetric endomorphisms and 2-forms using g.

We then have the relations (cf. [10]):

$$JX \lrcorner \psi^+ = (X \lrcorner \psi^+) \circ J = -J \circ (X \lrcorner \psi^+), \qquad \forall X \in \mathrm{TM}, \tag{2.6}$$
$$\nabla_X J = X \lrcorner \psi^+, \qquad \forall X \in \mathrm{TM}. \tag{2.7}$$

3 Torus actions by automorphisms

Suppose that $(M^6, \psi^\pm, \omega, g, J)$ is a strict NK structure carrying a toric action by automorphisms. More precisely, we assume that there exists some positive integer $d \geq 1$ and k linearly independent Killing vector fields ζ_i, $1 \leq i \leq d$ such that $[\zeta_i, \zeta_j] = 0$ for $1 \leq i, j \leq d$, which are pseudo-holomorphic in the sense that $L_{\zeta_i} J = 0$ for $1 \leq i \leq d$. This last condition is equivalent with the requirement that

$$L_{\zeta_i}\psi^\pm = 0,\ L_{\zeta_i}\omega = 0, \qquad 1 \leq i \leq d. \tag{3.1}$$

Notice that if M is compact and not isometric with the standard sphere, (3.1) follows directly from the Killing condition (cf. [10], Proposition 3.1).

We define the smooth functions μ_{ij} on M by setting $\mu_{ij} := \omega(\zeta_i, \zeta_j)$.

Lemma 3.1 *The following relations hold for every* $i, j, k \in \{1, \ldots, d\}$:

(i) $\mathrm{d}\mu_{ij} = -3\zeta_i \lrcorner \zeta_j \lrcorner \psi^+$.
(ii) $\psi^+(\zeta_i, \zeta_j, \zeta_k) = 0$.
(iii) $[\zeta_i, J\zeta_j] = 0$.
(iv) $[J\zeta_i, J\zeta_j] = 4(J\zeta_j \lrcorner \zeta_i \lrcorner \psi^+)^\sharp$.

Proof (i) From (2.4) together with the Cartan formula, we get

$$0 = L_{\zeta_j}\omega = \zeta_j \lrcorner \mathrm{d}\omega + \mathrm{d}(\zeta_j \lrcorner \omega) = 3\zeta_j \lrcorner \psi^+ + \mathrm{d}(\zeta_j \lrcorner \omega).$$

Taking now the interior product with ζ_i yields

$$0 = 3\zeta_i \lrcorner \zeta_j \lrcorner \psi^+ + \zeta_i \lrcorner \mathrm{d}(\zeta_j \lrcorner \omega)$$

and the claim follows by taking into account that

$$\zeta_i \lrcorner \mathrm{d}(\zeta_j \lrcorner \omega) = L_{\zeta_i}(\zeta_j \lrcorner \omega) - \mathrm{d}(\zeta_i \lrcorner \zeta_j \lrcorner \omega) = \mathrm{d}\mu_{ij}.$$

(ii) Using (i), we can write

$$\psi^+(\zeta_i, \zeta_j, \zeta_k) = -\frac{1}{3}\mathrm{d}\mu_{jk}(\zeta_i) = -\frac{1}{3}L_{\zeta_i}(\omega(\zeta_j, \zeta_k)) = 0.$$

(iii) Follows directly from $L_{\zeta_i} J = 0$ and the fact that the ζ_i's mutually commute.
(iv) On every almost Hermitian manifold, the Nijenhuis tensor

$$N(X, Y) := [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY]$$

can be expressed as

$$N(X, Y) = J(L_X J)Y - (L_{JX} J)Y \tag{3.2}$$

for all vector fields X, Y. On the other hand, (2.7) shows that on every NK manifold, the Nijenhuis tensor satisfies

$$N(X, Y) = J(\nabla_X J)Y - J(\nabla_Y J)X - (\nabla_{JX} J)Y + (\nabla_{JY} J)X = -4Y \lrcorner JX \lrcorner \psi^+. \tag{3.3}$$

Applying (3.2) and (3.3) to $X = \zeta_i$, and using the fact that $L_{\zeta_i} J = 0$ yields

$$(L_{J\zeta_i} J) = 4J\zeta_i \lrcorner \psi^+. \tag{3.4}$$

This, together with (iii), finishes the proof. □

Lemma 3.2 *If ξ is a Killing vector field, $J\xi$ cannot be Killing on any open set U.*

Proof From Corollary 3.3 and Lemma 3.4 in [10], we have

$$(\mathrm{d}J\xi)^{(2,0)} = \mathrm{d}J\xi = -\xi \lrcorner \mathrm{d}\omega = -3\xi \lrcorner \psi^+$$

and

$$(\mathrm{d}\xi)^{(2,0)} = -J\xi \lrcorner \psi^+$$

for every Killing vector field ξ. If $J\xi$ were Killing on some open set, the same relations applied to $J\xi$ would read

$$(\mathrm{d}\xi)^{(2,0)} = 3J\xi \lrcorner \psi^+$$

and

$$(\mathrm{d}J\xi)^{(2,0)} = \xi \lrcorner \psi^+,$$

a contradiction. □

Assume from now on that the dimension of the torus acting by automorphisms satisfies $d \geq 2$.

Lemma 3.3 *For every $i \neq j$ in $\{1, \ldots, d\}$, the vector fields $\{\zeta_i, \zeta_j, J\zeta_i, J\zeta_j\}$ are linearly independent on a dense open subset of M.*

Proof One can of course assume $i = 1$, $j = 2$. If the contrary holds, there exists some open set U on which ζ_1 does not vanish and functions $a, b : U \to \mathbb{R}$ such that

$$\zeta_2 = a\zeta_1 + bJ\zeta_1. \tag{3.5}$$

We differentiate this relation on U with respect to the Levi-Civita covariant derivative ∇ and obtain the following relation between endomorphisms of TM:

$$\begin{aligned}\nabla\zeta_2 &= \mathrm{d}a \otimes \zeta_1 + a\nabla\zeta_1 + \mathrm{d}b \otimes J\zeta_1 + b\nabla J\zeta_1 \\ &= \mathrm{d}a \otimes \zeta_1 + a\nabla\zeta_1 + \mathrm{d}b \otimes J\zeta_1 - b\zeta_1 \lrcorner \psi^+ + bJ \circ (\nabla\zeta_1).\end{aligned}$$

Taking the symmetric parts in this equation yields

$$0 = \mathrm{d}a \odot \zeta_1 + \mathrm{d}b \odot J\zeta_1 + b(J \circ (\nabla\zeta_1))^{\mathrm{sym}}.$$

Since $\nabla\zeta_1$ is skew-symmetric, $(J \circ (\nabla\zeta_1))^{\mathrm{sym}}$ commutes with J, whence J commutes with $\mathrm{d}a \odot \zeta_1 + \mathrm{d}b \odot J\zeta_1$. On the other hand, J commutes with $\mathrm{d}a \odot \zeta_1 + J\mathrm{d}a \odot J\zeta_1$; thus, it commutes with $(\mathrm{d}b - J\mathrm{d}a) \odot J\zeta_1$. This implies $\mathrm{d}b = J\mathrm{d}a$. Differentiating this again with respect to ∇ yields

$$\nabla\mathrm{d}b = \nabla(J\mathrm{d}a) = -\mathrm{d}a \lrcorner \psi^+ + J \circ \nabla\mathrm{d}a.$$

Taking the skew-symmetric part in this equality shows that

$$\mathrm{d}a \lrcorner \psi^+ = (J \circ \nabla\mathrm{d}a)^{\mathrm{skew}}.$$

But the left-hand side anti-commutes with J, whereas the right- hand side commutes with J (since $\nabla\mathrm{d}a$ is symmetric). Thus $\mathrm{d}a = 0$, so a and b are constants. From (3.5), we obtain that $J\zeta_1$ is a Killing vector field on U, which is impossible by Lemma 3.2. This contradiction concludes the proof. □

Corollary 3.1 *The vector fields $\{\zeta_1, \zeta_2, J\zeta_1, J\zeta_2, \zeta_1 \lrcorner \zeta_2 \lrcorner \psi^+, J\zeta_1 \lrcorner \zeta_2 \lrcorner \psi^+\}$ are linearly independent on a dense open subset of M.*

Proof This follows from Lemma 3.3 using the fact that the vectors $\zeta_1 \lrcorner \zeta_2 \lrcorner \psi^+$ and $J\zeta_1 \lrcorner \zeta_2 \lrcorner \psi^+$ are orthogonal to $\zeta_1, \zeta_2, J\zeta_1$ and $J\zeta_2$, and they both are non-vanishing at each point where $\{\zeta_1, \zeta_2, J\zeta_1, J\zeta_2\}$ are linearly independent. □

From now on, we assume that $d \geq 3$.

Lemma 3.4 *For every mutually distinct $1 \leq i, j, k \leq d$, the 6 vector fields ζ_i, ζ_j, ζ_k, $J\zeta_i$, $J\zeta_j$, $J\zeta_k$ are linearly independent on a dense open subset M_0 of M.*

Proof We may assume that $i = 1$, $j = 2$ and $k = 3$. Like before, if the statement does not hold, there exists some open set U on which ζ_1 does not vanish and functions $a_1, b_1, a_2, b_2 : U \to \mathbb{R}$ such that

$$\zeta_3 = a_1\zeta_1 + b_1 J\zeta_1 + a_2\zeta_2 + b_2 J\zeta_2. \tag{3.6}$$

By Lemma 3.3, one may assume that $\{\zeta_1, \zeta_2, J\zeta_1, J\zeta_2\}$ are linearly independent on U. Taking the Lie derivative with respect to $J\zeta_1$ in (3.6) and using Lemma 3.1 (iii) and (iv) yields

$$0 = J\zeta_1(a_1)\zeta_1 + J\zeta_1(b_1)J\zeta_1 + J\zeta_1(a_2)\zeta_2 + J\zeta_1(b_2)J\zeta_2 + 4b_2 J\zeta_2 \lrcorner \zeta_2 \lrcorner \psi^+.$$

From Corollary 3.1, we get $b_2 = 0$. Similarly, taking the Lie derivative with respect to $J\zeta_2$ in (3.6), we get $b_1 = 0$. Therefore, (3.6) becomes

$$\zeta_3 = a_1\zeta_1 + a_2\zeta_2. \tag{3.7}$$

Differentiating this equation with respect to ∇ and taking the symmetric part yields

$$0 = \mathrm{d}a_1 \odot \zeta_1 + \mathrm{d}a_2 \odot \zeta_2.$$

Since ζ_1 and ζ_2 are linearly independent on U, this implies $\mathrm{d}a_1 = c\zeta_2$ and $\mathrm{d}a_2 = -c\zeta_1$ for some function $c : U \to \mathbb{R}$. On the other hand, taking the Lie derivative with respect to ζ_2 in (3.7) yields $0 = \zeta_2(a_1)\zeta_1 + \zeta_2(a_2)\zeta_2$; thus, $\zeta_2(a_1) = 0$, so finally $c|\zeta_2|^2 = g(\mathrm{d}a_1, \zeta_2) = \zeta_2(a_1) = 0$, whence $c = 0$. This shows that a_1 and a_2 are constant, contradicting the hypothesis that ζ_1, ζ_2 and ζ_3 are linearly independent Killing vector fields. This proves the lemma. □

Corollary 3.2 *The rank d of the automorphism group of M is at most* 3.

Proof Assume for a contradiction that $d \geq 4$, there exist 4 linearly independent mutually commuting Killing vector fields $\zeta_1, \dots, \zeta_4$ on M preserving the almost complex structure J. From Lemma 3.4, there exist functions a_i and b_i $(i = 1, 2, 3)$ on M_0 such that

$$\zeta_4 = \sum_{j=1}^{3} a_j \zeta_j + b_j J\zeta_j. \tag{3.8}$$

From Lemma 3.1 (ii), we get $\psi^+(\zeta_1, \zeta_2, \zeta_3) = \psi^+(\zeta_1, \zeta_2, \zeta_4) = 0$. Using (3.8) together with the fact that $\psi^+(X, JX, \cdot) = 0$ for every X, we get $b_3\psi^+(\zeta_1, \zeta_2, J\zeta_3) = 0$.

Assume that b_3 is not identically zero on M. Then $\psi^+(\zeta_1, \zeta_2, J\zeta_3) = 0$ on some non-empty open set U. On the other hand, the 1-form $\psi^+(\zeta_1, \zeta_2, \cdot)$ vanishes when applied to ζ_1, $J\zeta_1$, ζ_2, $J\zeta_2$ and ζ_3; so, by Lemma 3.4, $\psi^+(\zeta_1, \zeta_2, \cdot)$ vanishes on the non-empty open set $U \cap M_0$. This contradicts Corollary 3.1. Consequently $b_3 \equiv 0$, and similarly $b_2 = b_1 \equiv 0$. We thus get

$$\zeta_4 = \sum_{j=1}^{3} a_j \zeta_j. \tag{3.9}$$

Taking the Lie derivative in (3.9) with respect to ζ_i and $J\zeta_i$ for $i = 1, 2, 3$ and using Lemma 3.1 (iii) we obtain $\zeta_i(a_j) = J\zeta_i(a_j) = 0$ for every $i, j \in \{1, 2, 3\}$, so a_j are constant on M_0, thus showing that ζ_4 is a linear combination of $\zeta_1, \zeta_2, \zeta_3$, a contradiction. □

4 Toric NK structures

In view of Corollary 3.2 we can now introduce the following:

Definition 4.1 A 6-dimensional strict NK manifold is called toric if its automorphism group has rank 3, or equivalently, if it carries 3 linearly independent mutually commuting pseudo-holomorphic Killing vector fields $\zeta_1, \zeta_2, \zeta_3$.

Assume from now on that $(M^6, g, J, \zeta_1, \zeta_2, \zeta_3)$ is a toric NK manifold and consider on the dense open subset M_0 given by Lemma 3.4 the basis $\{\theta^1, \theta^2, \theta^3, \gamma^1, \gamma^2, \gamma^3\}$ of $\Lambda^1 M_0$ dual to $\{\zeta_1, \zeta_2, \zeta_3, J\zeta_1, J\zeta_2, J\zeta_3\}$, together with the function

$$\varepsilon := \psi^-(\zeta_1, \zeta_2, \zeta_3). \tag{4.1}$$

For further use, let us also introduce the symmetric 3×3 matrix

$$C := (C_{ij}) = (g(\zeta_i, \zeta_j)). \tag{4.2}$$

As a direct consequence of Lemma 3.4, we have that $\zeta + J\zeta = \mathrm{TM}_0$, where ζ is the 3-dimensional distribution spanned by ζ_k, $1 \leq k \leq 3$. This enables us to express ψ^+, and ψ^- in terms of the basis $\{\theta^i, \gamma^j\}$ and of the function ε, simply by checking that the two terms are equal when applied to elements of the basis $\{\zeta_i, J\zeta_j\}$ of TM_0:

$$\begin{aligned} \psi^+ &= \varepsilon\Big(\gamma^{123} - \theta^{12} \wedge \gamma^3 - \theta^{31} \wedge \gamma^2 - \theta^{23} \wedge \gamma^1\Big), \\ \psi^- &= \varepsilon\Big(\theta^{123} - \gamma^{12} \wedge \theta^3 - \gamma^{31} \wedge \theta^2 - \gamma^{23} \wedge \theta^1\Big), \end{aligned} \tag{4.3}$$

where here and henceforth the notation γ^{123} stands for $\gamma^1 \wedge \gamma^2 \wedge \gamma^3$, etc. Recalling the definition of $\mu_{ij} := \omega(\zeta_i, \zeta_j)$, the fundamental 2-form $\omega := g(J\cdot, \cdot)$ can be expressed by the formula:

$$\omega = \sum_{1 \le i < j \le 3} \mu_{ij}(\theta^{ij} + \gamma^{ij}) + \sum_{i=1}^{3} \theta^i \wedge c^i \tag{4.4}$$

where the 1-forms c^i in $\Lambda^1(J\zeta^*)$ are given by $c^i = \sum_{j=1}^{3} C_{ij}\gamma^j$. A short computation yields

$$\omega^3 = -6\theta^{123} \wedge c^{123} + 6\theta^{123} \wedge c \wedge \eta, \tag{4.5}$$

where η in $\Lambda^2(J\zeta^*)$ is given by

$$\eta := \sum_{1 \le i < j \le 3} \mu_{ij}\gamma^{ij}$$

and c in $\Lambda^1(J\zeta^*)$ is given by

$$c := \mu_{23}c^1 + \mu_{31}c^2 + \mu_{12}c^3.$$

Therefore, from the compatibility relations (2.3), it follows that

$$c^{123} = \varepsilon^2\gamma^{123} + c \wedge \eta, \tag{4.6}$$

which is equivalent to

$$\det C = \varepsilon^2 +^t VCV, \tag{4.7}$$

where we denote by

$$V := \begin{pmatrix} \mu_{23} \\ \mu_{31} \\ \mu_{12} \end{pmatrix}. \tag{4.8}$$

Lemma 4.1 *The following relations hold*:

(i) $\mathrm{d}\mu_{12} = -3\varepsilon\gamma^3,\ \mathrm{d}\mu_{31} = -3\varepsilon\gamma^2,\ \mathrm{d}\mu_{23} = -3\varepsilon\gamma^1$;
(ii) $\mathrm{d}\varepsilon = 4c$.

Proof (i) Using (2.4), (4.3) and the Cartan formula, we can write

$$\mathrm{d}\mu_{12} = \mathrm{d}(\zeta_2 \lrcorner \zeta_1 \lrcorner \omega) = \zeta_2 \lrcorner \zeta_1 \lrcorner \mathrm{d}\omega = 3\zeta_2 \lrcorner \zeta_1 \lrcorner \psi^+ = -3\varepsilon\gamma^3.$$

The other formulas are similar.

(ii) Using (2.5), (4.4) and the Cartan formula again, we get

$$\begin{aligned} \mathrm{d}\varepsilon &= \mathrm{d}(\zeta_3 \lrcorner \zeta_2 \lrcorner \zeta_1 \lrcorner \psi^-) = -\zeta_3 \lrcorner \zeta_2 \lrcorner \zeta_1 \lrcorner \mathrm{d}\psi^- \\ &= 2\zeta_3 \lrcorner \zeta_2 \lrcorner \zeta_1 \lrcorner \omega^2 = 4(\mu_{23}c^1 + \mu_{31}c^2 + \mu_{12}c^3). \end{aligned}$$

□

We will now show that Eq. (2.5) is equivalent to some exterior system involving the 1-forms θ^i.

Lemma 4.2 *Equation* (2.5) *holds if and only if the forms* θ_i, $1 \le i \le 3$ *satisfy the differential system*:

$$\begin{aligned} \frac{1}{4}\varepsilon \mathrm{d}\theta^1 &= c^2 \wedge c^3 - \mu_{23}\eta \\ \frac{1}{4}\varepsilon \mathrm{d}\theta^2 &= c^3 \wedge c^1 - \mu_{31}\eta \\ \frac{1}{4}\varepsilon \mathrm{d}\theta^3 &= c^1 \wedge c^2 - \mu_{12}\eta \end{aligned} \tag{4.9}$$

Proof Assume that (2.5) holds. By (4.3)

$$\zeta_2 \lrcorner \zeta_1 \lrcorner \psi^- = \varepsilon\theta^3. \tag{4.10}$$

Since ζ_k, $1 \le k \le 3$ are commuting Killing vector fields preserving the whole SU(3)-structure, (4.4) yields

$$\mathrm{d}(\zeta_2 \lrcorner \zeta_1 \lrcorner \psi^-) = \zeta_2 \lrcorner \zeta_1 \lrcorner \mathrm{d}\psi^- = -2\zeta_2 \lrcorner \zeta_1 \lrcorner (\omega \wedge \omega) = -4\theta^3 \wedge c - 4\mu_{12}\eta + 4c^1 \wedge c^2.$$

hence by (4.10) and Lemma 4.1 (ii), we get

$$\frac{1}{4}\varepsilon \mathrm{d}\theta^3 = \frac{1}{4}\mathrm{d}(\varepsilon\theta^3) - \frac{1}{4}\mathrm{d}\varepsilon \wedge \theta^3 = -\theta^3 \wedge c - \mu_{12}\eta + c^1 \wedge c^2 - c \wedge \theta^3 = c^1 \wedge c^2 - \mu_{12}\eta.$$

The proof of the two other relations is similar.

Conversely, we notice that (2.5) holds if and only if

$$\begin{cases} \zeta_i \lrcorner \zeta_j \lrcorner \mathrm{d}\psi^- = -2\zeta_i \lrcorner \zeta_j \lrcorner \omega^2, & \forall\, 1 \le i, j \le 3, \\ J\zeta_1 \lrcorner J\zeta_2 \lrcorner J\zeta_3 \lrcorner \mathrm{d}\psi^- = -2J\zeta_1 \lrcorner J\zeta_2 \lrcorner J\zeta_3 \lrcorner \omega^2. \end{cases}$$

The first relation was just shown to be equivalent to (4.9). It remains to check, by a straightforward calculation, that the second relation is automatically fulfilled. □

We finally interpret Eq. (2.4) in terms of the frame $\{c^i\}$.

Lemma 4.3 *Equation* (2.4) *holds if and only if* (4.6) *holds and the forms* εc^k *are closed for* $1 \le k \le 3$.

Proof Taking the interior product with ζ_1 in (2.4) and using (4.3), (4.4) and Lemma 4.1 (i) yields

$$\begin{aligned} 3\varepsilon(-\theta^2 \wedge \gamma^3 + \theta^3 \wedge \gamma^2) &= 3\zeta_1 \lrcorner \psi^+ = \zeta_1 \lrcorner \mathrm{d}\omega = -\mathrm{d}(\zeta_1 \lrcorner \omega) = -\mathrm{d}(\mu_{12}\theta^2 - \mu_{31}\theta^3 + c^1) \\ &= 3\varepsilon\gamma^3 \wedge \theta^2 - \mu_{12}\mathrm{d}\theta^2 - 3\varepsilon\gamma^2 \wedge \theta^3 + \mu_{31}\mathrm{d}\theta^3 - \mathrm{d}c^1, \end{aligned}$$

whence

$$\mathrm{d}c^1 = \mu_{31}\mathrm{d}\theta^3 - \mu_{12}\mathrm{d}\theta^2.$$

From Lemma 4.2 and 4.1 (ii), we thus obtain

$$\begin{aligned} \mathrm{d}(\varepsilon c^1) &= 4c \wedge c^1 + 4\Big[\mu_{31}(c^1 \wedge c^2 - \mu_{12}\eta) - \mu_{12}(c^3 \wedge c^1 - \mu_{31}\eta)\Big] \\ &= 4(\mu_{23}c^1 + \mu_{31}c^2 + \mu_{12}c^3) \wedge c^1 + 4(\mu_{31}c^1 \wedge c^2 - \mu_{12}c^3 \wedge c^1) = 0. \end{aligned}$$

Conversely, we notice that (2.4) holds if and only if

$$\begin{cases} \zeta_i \lrcorner \mathrm{d}\omega = 3\zeta_i \lrcorner \psi^+, & \forall\, 1 \le i \le 3, \\ J\zeta_1 \lrcorner J\zeta_2 \lrcorner J\zeta_3 \lrcorner \mathrm{d}\omega = 3J\zeta_1 \lrcorner J\zeta_2 \lrcorner J\zeta_3 \lrcorner \psi^+. \end{cases}$$

We have just shown that the first equation is equivalent to εc^k being closed. The component of $\mathrm{d}\omega = 3\psi^+$ on $\Lambda^3 J\zeta$ is given by

$$\mathrm{d}\eta + \sum_{k=1}^{3} \mathrm{d}\theta^k \wedge c^k = 3\varepsilon\gamma^{123},$$

so using (4.9), the second equation is equivalent to (4.7). □

Let us now consider the 3-dimensional quotient $U := M_0/\zeta$ of the open set M_0 by the action of the 3-dimensional torus generated by the Killing vector fields ζ_i. Clearly, the natural projection $\pi : M \to U$ is a submersion. We shall now interpret the geometry of the situation down on U. Since $\zeta_i(\mu_{jk}) = 0$, there exist functions y_i on U such that $\pi^* y_1 = \mu_{23}, \pi^* y_2 = \mu_{31}, \pi^* y_3 = \mu_{12}$. Moreover, since ε does not vanish on M_0, Lemma 4.1 (i) shows that $\{y_i\}$ define a global coordinate system on U. From now on, we will identify the projectable functions or exterior forms on M with their projection on U. Since everything is local, we may suppose that U is contractible.

Remark 4.1 By Lemma 3.1 (i), it follows that the map $\mu : M \to \Lambda^2\mathbb{R}^3 \cong \mathfrak{so}(3)$ defined by

$$\mu := \begin{pmatrix} 0 & \mu_{12} & \mu_{13} \\ \mu_{21} & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{pmatrix} = \pi^* \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}$$

is the multi-moment map of the strong geometry (M, ψ^+) defined by Madsen and Swann in [9] and studied further by Dixon [2] in the particular case where $M = S^3 \times S^3$. Similarly, the function ε can be seen as the multi-moment map associated with the closed 4-form $\mathrm{d}\psi^-$. These maps will play an important role in Sects. 5 and 6 below.

Proposition 4.1 *There exists a function φ on U (defined up to an affine function) such that* $\mathrm{Hess}(\varphi) = C$ *in the coordinates* $\{y_i\}$.

Proof From Lemma 4.3, there exist functions f_i on U such that $\mathrm{d}f_i = \varepsilon c^i$ for $1 \le i \le 3$. Notice that by Lemma 4.1 (i), this is equivalent to

$$\frac{\partial f_i}{\partial y_j} = -3C_{ij}. \tag{4.11}$$

From Lemma 4.1 (i), we get

$$\mathrm{d}\left(\sum_{i=1}^{3} f_i \mathrm{d}y_i\right) = \sum_{i=1}^{3} \mathrm{d}f_i \wedge \mathrm{d}y_i = -3\sum_{i=1}^{3} \varepsilon c^i \wedge \varepsilon\gamma^i = \sum_{i,j=1}^{3} \varepsilon^2 C_{ij}\gamma^j \wedge \gamma^i = 0,$$

so there exists some function φ such that

$$\mathrm{d}\varphi = -\frac{1}{3}\sum_{i=1}^{3} f_i \mathrm{d}y_i.$$

This means that $\frac{\partial\varphi}{\partial y_i} = -\frac{1}{3} f_i$, which together with (4.11) finishes the proof. □

Let us introduce the operator ∂_r of radial differentiation, acting on functions on U by

$$\partial_r f := \sum_{i=1}^{3} y_i \frac{\partial f}{\partial y_i}.$$

Proposition 4.2 *The function φ can be chosen in such a way that*

$$\varepsilon^2 = \frac{8}{3}(\varphi - \partial_r\varphi). \tag{4.12}$$

Proof It is clearly enough to show that the exterior derivatives of the two terms coincide. Since

$$\frac{\partial(\partial_r\varphi)}{\partial y_j} = \sum_{i=1}^{3} \frac{\partial^2\varphi}{\partial y_i \partial y_j} y_i + \frac{\partial\varphi}{\partial y_j},$$

Lemma 4.1 yields

$$-\frac{8}{3}\mathrm{d}(\partial_r\varphi - \varphi) = -\frac{8}{3}\sum_{i,j=1}^{3} C_{ij} y_i \mathrm{d}y_j = 8\sum_{i,j=1}^{3} C_{ij} y_i \varepsilon\gamma^j = 8\varepsilon c = \mathrm{d}(\varepsilon^2).$$

□

Summing up, we get the following result:

Corollary 4.1 *The function φ given in the previous proposition satisfies the equation*

$$\det(\mathrm{Hess}(\varphi)) = \frac{8}{3}\varphi - \frac{11}{3}\partial_r\varphi + \partial_r^2\varphi. \tag{4.13}$$

Proof We have

$$\partial_r^2\varphi = \partial_r\left(\sum_{i=1}^{3} y_i \frac{\partial\varphi}{\partial y_i}\right) = \sum_{i=1}^{3} y_i \frac{\partial\varphi}{\partial y_i} + \sum_{i,j=1}^{3} y_i y_j \frac{\partial^2\varphi}{\partial y_i \partial y_j} = \partial_r\varphi +{}^t VCV, \tag{4.14}$$

so (4.13) is a consequence of (4.7) and (4.12). □

5 The inverse construction

In this section, we will show that conversely, every solution φ of Eq. (4.13) on some open set $U \subset \mathbb{R}^3$ defines a NK structure with 3 linearly independent commuting Killing vector fields on $U_0 \times \mathbb{T}^3$, where U_0 is some open subset of U. More precisely, let y_1, y_2, y_3 be the standard coordinates on U and let μ be the 3×3 skew-symmetric matrix

$$\mu := \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}. \tag{5.1}$$

Define the 6×6 symmetric matrix

$$D := \begin{pmatrix} \mathrm{Hess}(\varphi) & -\mu \\ \mu & \mathrm{Hess}(\varphi) \end{pmatrix}.$$

Let U_0 denote the open set

$$U_0 := \Big\{x \in U \mid \varphi(x) - \partial_r\varphi(x) > 0 \text{ and } D \text{ is positive definite}\Big\}. \tag{5.2}$$

The next result is straightforward:

Lemma 5.1 *The matrix D is positive definite if and only if*

(i) $C = \mathrm{Hess}(\varphi)$ *is positive definite and*
(ii) $\langle \mu a, b\rangle^2 < \langle Ca, a\rangle\langle Cb, b\rangle$ *for all* $(a, b) \in (\mathbb{R}^3 \times \mathbb{R}^3) \setminus (0, 0)$.

On U_0 we define a positive function ε by (4.12), 1-forms γ^i by $\mathrm{d}y_i = -3\varepsilon\gamma^i$ and a 2-form $\eta := y_1\gamma^2 \wedge \gamma^3 + y_2\gamma^3 \wedge \gamma^1 + y_3\gamma^1 \wedge \gamma^2$. We denote as before by C the Hessian of φ and define $c^i := \sum_{j=1}^3 C_{ij}\gamma^j$.

Lemma 5.2 *The following hold*:

(i) *The* 1*-forms* εc^i *are exact.*
(ii) *The* 2*-forms* $\tau_1 := (c^2\wedge c^3 - y_1\eta)/\varepsilon$, $\tau_2 := (c^3\wedge c^1 - y_2\eta)/\varepsilon$ *and* $\tau_3 := (c^1\wedge c^2 - y_3\eta)/\varepsilon$ *are closed.*

Proof (i) We have:

$$\mathrm{d}\left(-\frac{1}{3}\frac{\partial\varphi}{\partial y_i}\right) = -\frac{1}{3}\sum_{j=1}^{3}\frac{\partial^2\varphi}{\partial y_i\partial y_j}\mathrm{d}y_j = -\frac{1}{3}\sum_{j=1}^{3}C_{ij}\mathrm{d}y_j = \varepsilon c^i.$$

(ii) We first compute using (i):

$$\begin{aligned}\mathrm{d}(\varepsilon^3\tau_1) &= \mathrm{d}(\varepsilon^2(c^2\wedge c^3 - y_1\eta)) = -\mathrm{d}(y_1\varepsilon^2\eta)\\ &= -\mathrm{d}(y_1^2\varepsilon^2\gamma^{23} + y_1y_2\varepsilon^2\gamma^{31} + y_1y_3\varepsilon^2\gamma^{12}) = 12y_1\varepsilon^3\gamma^{123}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{d}(\varepsilon^3)\wedge\tau_1 &= 3\varepsilon^2\mathrm{d}\varepsilon\wedge\tau_1 = 12\varepsilon\left(\sum_{j=1}^{3}y_jc^j\right)\wedge(c^2\wedge c^3 - y_1\eta)\\ &= 12\varepsilon y_1\left(\det C - \sum_{i,j=1}^{3}C_{ij}y_iy_j\right)\gamma^{123} = 12y_1\varepsilon^3\gamma^{123},\end{aligned}$$

the last equality (which is the converse to (4.7)) following from (4.12), (4.13) and (4.14). These two relations show that τ_1 is closed. The proof that $\mathrm{d}\tau_2 = \mathrm{d}\tau_3 = 0$ is similar. □

By replacing U_0 with a smaller open subset if necessary, one can find 1-forms σ_i such that $\mathrm{d}\sigma_i = 4\tau_i$. Consider now the 6-dimensional manifold $M := U_0 \times \mathbb{T}^3$ with coordinates y_1, y_2, y_3 and x_1, x_2, x_3 (locally defined). The 1-forms $\theta^i := \mathrm{d}x_i + \sigma_i$ satisfy the differential system (4.9). We define $\psi^\pm$ and ω by (4.3) and (4.4) and we claim that they determine a strict NK structure on M whose automorphism group contains a 3-torus.

Let us first check that $(\psi^\pm, \omega)$ satisfy the conditions of Lemma 2.1. The relation (i) is straightforward, (ii) is equivalent to (4.7), and (iii) holds from the definition (5.2) of U_0.

In order to prove that $(\psi^\pm, \omega)$ defines a NK structure, we need to check (2.4) and (2.5). By Lemma 4.3, (2.4) is equivalent to εc^i being closed (Lemma 5.2 (i)) together with (4.7). Similarly, Lemma 4.2 shows that (2.5) is equivalent to the system (4.9) together with (4.7) again.

It remains to check that the automorphism group contains a 3-torus. This is actually clear: The action of $\mathbb{T}^3$ on $M = U_0 \times \mathbb{T}^3$ by multiplication on the first factor preserves the SU(3) structure. We have proved the following result:

Theorem 5.1 *Every solution of the Monge–Ampère-type equation* (4.13) *on some open set* U *in* $\mathbb{R}^3$ *defines in a canonical way a NK structure with* 3 *linearly independent commuting infinitesimal automorphisms on* $U_0 \times \mathbb{T}^3$, *where* U_0 *is defined by* (5.2).

6 Examples

We will illustrate the above computations on a specific example of toric nearly Kähler manifold, namely the 3-symmetric space $S^3 \times S^3$.

Let $K := \mathrm{SU}_2$ with Lie algebra $\mathfrak{k} = \mathfrak{su}_2$ and $G := K \times K \times K$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k}$. We consider the 6-dimensional manifold $M = G/K$, where K is diagonally embedded in G. The tangent space of M at $o = eK$ can be identified with

$$\mathfrak{p} = \{(X, Y, Z) \in \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k} \mid X + Y + Z = 0\}.$$

Consider the invariant scalar product B on $\mathfrak{su}_2$ such that the scalar product

$$\langle (X, Y, Z), (X, Y, Z) \rangle := B(X, X) + B(Y, Y) + B(Z, Z)$$

defines the homogeneous nearly Kähler metric g of scalar curvature 30 on $M = S^3 \times S^3$ (cf. [11], Lemma 5.4).

The G-automorphism σ of order 3 defined by $\sigma(a_1, a_2, a_3) = (a_2, a_3, a_1)$ induces a canonical almost complex structure on the 3-symmetric space M by the relation

$$\sigma = \frac{-\mathrm{Id} + \sqrt{3}J}{2}, \qquad \text{on } \mathfrak{p},$$

whence

$$J(X, Y, Z) = \tfrac{2}{\sqrt{3}}(Y, Z, X) + \tfrac{1}{\sqrt{3}}(X, Y, Z), \qquad \forall (X, Y, Z) \in \mathfrak{p}. \tag{6.1}$$

Let ξ be a unit vector in $\mathfrak{su}_2$ with respect to B. The right-invariant vector fields on G generated by the elements

$$\tilde{\zeta}_1 = (\xi, 0, 0), \qquad \tilde{\zeta}_2 = (0, \xi, 0), \qquad \tilde{\zeta}_3 = (0, 0, \xi)$$

of $\mathfrak{g}$, define three commuting Killing vector fields $\zeta_1, \zeta_2, \zeta_3$ on M.

Let us compute $g(\zeta_1, J\zeta_2)$ at some point $aK \in M$, where $a = (a_1, a_2, a_3)$ is some element of G. By the definition of J, we have

$$\begin{aligned}
g(\zeta_1, J\zeta_2)_{aK} &= \left\langle \left(a^{-1}\tilde{\zeta}_1 a\right)_{\mathfrak{p}}, J\left(a^{-1}\tilde{\zeta}_2 a\right)_{\mathfrak{p}} \right\rangle = \left\langle \left(a_1^{-1}\xi a_1, 0, 0\right)_{\mathfrak{p}}, J\left(0, a_2^{-1}\xi a_2, 0\right)_{\mathfrak{p}} \right\rangle \\
&= \frac{1}{9}\left\langle \left(2a_1^{-1}\xi a_1, -a_1^{-1}\xi a_1, -a_1^{-1}\xi a_1\right), J\left(-a_2^{-1}\xi a_2, 2a_2^{-1}\xi a_2, -a_2^{-1}\xi a_2\right) \right\rangle \\
&= \frac{1}{9}\left\langle \left(2a_1^{-1}\xi a_1, -a_1^{-1}\xi a_1, -a_1^{-1}\xi a_1\right), \sqrt{3}\left(a_2^{-1}\xi a_2, 0, -a_2^{-1}\xi a_2\right) \right\rangle \\
&= \frac{1}{\sqrt{3}} B\left(a_1^{-1}\xi a_1, a_2^{-1}\xi a_2\right).
\end{aligned}$$

We introduce the functions $y_1, y_2, y_3 : G \to \mathbb{R}$ defined by

$$y_i(a_1, a_2, a_3) = \frac{1}{\sqrt{3}} B\left(a_j^{-1}\xi a_j, a_k^{-1}\xi a_k\right),$$

for every permutation (i, j, k) of $(1, 2, 3)$. The previous computation yields

$$g(\zeta_2, J\zeta_3)_{aK} = y_1(a), \qquad g(\zeta_3, J\zeta_1)_{aK} = y_2(a), \qquad g(\zeta_1, J\zeta_2)_{aK} = y_3(a), \qquad \forall a \in G.$$

A similar computation yields

$$g(\zeta_i, \zeta_j)_{aK} = 2\delta_{ij} + \frac{1}{\sqrt{3}} y_k(a)$$

for every even permutation (i, j, k) of $(1, 2, 3)$. In other words, the matrix C defined in (4.2) satisfies

$$C_{ij} = 2\delta_{ij} + \frac{1}{\sqrt{3}} y_k,$$

where by a slight abuse of notation we keep the same notations y_i for the functions defined on M by the K-invariant functions y_i on G.

The function φ in the coordinates y_i such that $\mathrm{Hess}(\varphi) = C$ is determined by

$$\varphi(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 + \frac{1}{\sqrt{3}} y_1 y_2 y_3 + h, \tag{6.2}$$

up to some affine function h in the coordinates y_i. On the other hand, since

$$\det(C) = -\frac{2}{3}\left(y_1^2 + y_2^2 + y_3^2\right) + \frac{2}{3\sqrt{3}} y_1 y_2 y_3 + 8,$$

an easy computation shows that the function φ given by (6.2) satisfies indeed the Monge–Ampère-type equation (4.13) for $h = 3$. For the sake of completeness, we list the other functions involved in the previous section, in the particular case of the present situation:

$$\varepsilon^2 = -\frac{8}{3}\left(y_1^2 + y_2^2 + y_3^2\right) - \frac{16}{3\sqrt{3}} y_1 y_2 y_3 + 8,$$
$${}^tVCV = 2\left(y_1^2 + y_2^2 + y_3^2\right) + 2\sqrt{3} y_1 y_2 y_3,$$

where ε is defined in (4.1) and V in (4.8).

6.1 Radial solutions

We search here particular solutions to Eq. (4.13), namely when φ is a radial function on (some open subset of) $\mathbb{R}^3$ with coordinates y_k, $1 \le k \le 3$. Let therefore $\varphi(y_1, y_2, y_3) := x(\frac{r^2}{2})$ where x is a function of one real variable and $r^2 = y_1^2 + y_2^2 + y_3^2$. A direct computation yields

$$\begin{aligned}\mathrm{Hess}(\varphi) &= \begin{pmatrix} y_1^2 x'' + x' & y_1 y_2 x'' & y_1 y_3 x'' \\ y_1 y_2 x'' & y_2^2 x'' + x' & y_2 y_3 x'' \\ y_1 y_3 x'' & y_2 y_3 x'' & y_3^2 x'' + x' \end{pmatrix} \\ &= x' \mathrm{Id} + x''(\frac{r^2}{2}) V \cdot {}^t V\end{aligned}$$

where $V := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. In particular,

$$\det \mathrm{Hess}(\varphi) = (x')^2 x'' r^2 + (x')^3$$
$$\partial_r \varphi = r^2 x', \ \partial_r^2 \varphi = r^4 x'' + 2r^2 x',$$

whence after making the substitution $t := \frac{r^2}{2}$ we get:

Proposition 6.1 *Radial solutions to the Monge–Ampère-type equation* (4.13) *are given by solutions of the second-order ODE*

$$x'' = F(t, x, x') \tag{6.3}$$

where $F(t, p, q) := \frac{8p-(10tq+3q^3)}{6(q^2t-2t^2)}$.

To decide which solutions to (6.3) yield genuine Riemannian metrics in dimension six, we observe that

Proposition 6.2 *For a radial solution* $\varphi = x(\frac{r^2}{2})$ *to* (4.13)*, the set* U_0 *defined in* (5.2) *is*

$$U_0 = \left\{ t > 0 \mid x(t) > 2tx'(t) > 2t\sqrt{2t} \right\}.$$

Proof Having $\varphi - \partial_r \varphi > 0$ is equivalent with

$$2tx'(t) - x(t) < 0.$$

The matrix $\mathrm{Hess}(\varphi)$ has the eigenvalues $x'(\frac{r^2}{2})$ with eigenspace $E := \{a \in \mathbb{R}^3 \mid \langle a, y\rangle = 0\}$ and $x'(\frac{r^2}{2}) + r^2 x''(\frac{r^2}{2})$ with eigenvector y. Therefore, $\mathrm{Hess}(\varphi) > 0$ if and only if

$$x'(t) > 0, \; x'(t) + 2tx''(t) > 0. \tag{6.4}$$

However, $x'(t) + 2tx''(t) = \frac{8(x-2tx')}{3((x')^2-2t)}$ from (6.3), thus showing that the system (6.4) is equivalent to $x'(t) > \sqrt{2t}$. By Lemma 5.1, it remains to interpret the condition

$$\langle \mu a, b\rangle^2 < \langle Ca, a\rangle\langle Cb, b\rangle \tag{6.5}$$

for all $(a, b) \in (\mathbb{R}^3 \times \mathbb{R}^3) \setminus (0, 0)$.

We split $a = \lambda_1 y + v_1$, $b = \lambda_2 y + v_2$ with $v_1, v_2 \in E$ and take into account that C preserves the orthogonal decomposition $\mathbb{R}^3 = \mathbb{R}y \oplus E$ and also that y belongs to $\ker \mu$. Then,

$$\langle Ca, a\rangle\langle Cb, b\rangle = \left(\lambda_1^2 \langle Cy, y\rangle + \langle Cv_1, v_1\rangle\right)\left(\lambda_2^2 \langle Cy, y\rangle + \langle Cv_2, v_2\rangle\right)$$

and since μ is skew-symmetric,

$$\langle \mu a, b\rangle^2 = \langle \mu v_1, v_2\rangle^2.$$

Thus, (6.5) holds if and only if $\langle Cv_1, v_1\rangle\langle Cv_2, v_2\rangle > \langle \mu v_1, v_2\rangle^2$ for all nonzero $v_1, v_2 \in E$. This is equivalent to

$$\langle \mu v_1, v_2\rangle^2 < (x'(t))^2 |v_1|^2 |v_2|^2 \tag{6.6}$$

for all v_1, v_2 in $E \setminus \{0\}$. By the Cauchy–Schwartz inequality, this is equivalent to $-\frac{1}{2}\mathrm{tr}(\mu^2) < (x')^2(t)$ and since $\mathrm{tr}(\mu^2) = -2r^2 = -4t$, (6.6) is equivalent to $x'(t) > \sqrt{2t}$. However, this was already known and the proof is finished. □

Remark 6.1 The solutions of the ODE (6.3) of the form $x = kt^l$ with $k, l \in \mathbb{R}$ are $x_{1,2} = \pm\frac{2\sqrt{2}}{9}t^{\frac{3}{2}}$ and $x_3 = kt^{\frac{1}{2}}$, corresponding to

$$\varphi_{1,2} = \pm\frac{r^3}{9}, \qquad \varphi_3 = \frac{k}{\sqrt{2}}r.$$

However, they do not satisfy the positivity requirements from Proposition 6.2.

Solutions to the Cauchy problem (6.3), admissible in the sense of Proposition 6.2, are obtained by requiring the initial data $(t_0, x(t_0), x'(t_0))$ belong to

$$\mathcal{S} := \left\{ (t, p, q) \in \mathbb{R}^3 : t > 0, \; p > 2tq > 2t\sqrt{2t} \right\}.$$

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