



The \mathcal{L}_B -cohomology on compact torsion-free G_2 manifolds and an application to ‘almost’ formality

Ki Fung Chan¹ · Spiro Karigiannis²  · Chi Cheuk Tsang¹

Received: 27 March 2018 / Accepted: 10 September 2018 / Published online: 25 September 2018
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Abstract

We study a cohomology theory H_φ^\bullet , which we call the \mathcal{L}_B -cohomology, on compact torsion-free G_2 manifolds. We show that $H_\varphi^k \cong H_{dR}^k$ for $k \neq 3, 4$, but that H_φ^k is infinite-dimensional for $k = 3, 4$. Nevertheless, there is a canonical injection $H_{dR}^k \rightarrow H_\varphi^k$. The \mathcal{L}_B -cohomology also satisfies a Poincaré duality induced by the Hodge star. The establishment of these results requires a delicate analysis of the interplay between the exterior derivative d and the derivation \mathcal{L}_B and uses both Hodge theory and the special properties of G_2 -structures in an essential way. As an application of our results, we prove that compact torsion-free G_2 manifolds are ‘almost formal’ in the sense that most of the Massey triple products necessarily must vanish.

Keywords G_2 manifolds · Cohomology · Formality

1 Introduction

Let (M, φ) be a manifold with G_2 -structure. Here φ is a smooth 3-form on M that is *nondegenerate* in a certain sense that determines a Riemannian metric g and a volume form vol , hence a dual 4-form ψ . We say that (M, φ) is a *torsion-free* G_2 manifold if $\nabla\varphi = 0$. Note that this implies that $\nabla\psi = d\varphi = d\psi = 0$ as well. In fact, it is now a classical result [7] that the pair of conditions $d\varphi = d\psi = 0$ is actually equivalent to $\nabla\varphi = 0$.

The forms φ and ψ can be used to construct a vector-valued 2-form B and a vector-valued 3-form K , respectively, by raising an index using the metric. These vector-valued forms were studied in detail by Kawai–Lê–Schwachhöfer in [16] in the context of the Frölicher–Nijenhuis bracket.

✉ Spiro Karigiannis
karigiannis@uwaterloo.ca

Ki Fung Chan
kifung@link.cuhk.edu.hk

Chi Cheuk Tsang
1155062770@link.cuhk.edu.hk

¹ Chinese University of Hong Kong, Shatin, Hong Kong

² Department of Pure Mathematics, University of Waterloo, Waterloo, Canada

These vector-valued forms B and K induce *derivations* \mathcal{L}_B and \mathcal{L}_K on the space Ω^\bullet of forms on M , of degree 2 and 3, respectively. From these derivations, we can define *cohomology theories*. We call these the \mathcal{L}_B -cohomology, denoted H_φ^\bullet , and the \mathcal{L}_K -cohomology, denoted H_ψ^\bullet . When M is compact, the \mathcal{L}_K -cohomology was studied extensively by Kawai–Lê–Schwachhöfer in [17]. In the present paper, we study in detail the \mathcal{L}_B -cohomology when M is compact. Specifically, we compute H_φ^k for all k . The results are summarized in Theorem 3.19, which we restate here:

Theorem 3.19. *The following relations hold.*

- $H_\varphi^k \cong H_{dR}^k$ for $k = 0, 1, 2, 5, 6, 7$.
- H_φ^k is infinite-dimensional for $k = 3, 4$.
- There is a canonical injection $\mathcal{H}^k \hookrightarrow H_\varphi^k$ for all k .
- The Hodge star induces isomorphisms $*$: $H_\varphi^k \cong H_\varphi^{7-k}$.

The proof involves a very delicate analysis of the interplay between the exterior derivative d and the derivation induced by B and uses Hodge theory in an essential way.

As an application of our results, we study the question of *formality* of compact torsion-free G_2 manifolds. This is a long-standing open problem. It has been studied by many authors, including Cavalcanti [3]. In particular, the paper [22] by Verbitsky has very close connections to the present paper. What is called d_c in [22] is \mathcal{L}_B in the present paper. Verbitsky’s paper contains many excellent ideas. Unfortunately, there are some gaps in several of the proofs in [22]. Most important for us, there is a gap in the proof of [22, Proposition 2.19], which is also used to prove [22, Proposition 2.20], among several other results in [22]. We give a different proof of this result, which is our Proposition 4.4. We then use this to prove our Theorem 4.6, which essentially says that a compact torsion-free G_2 manifold is ‘almost formal’ in the sense that its de Rham complex is equivalent to a differential graded algebra with all differentials trivial except one.

A consequence of our Theorem 4.6 is that almost all of the Massey triple products vanish on a compact torsion-free G_2 manifold. This gives a new topological obstruction to the existence of torsion-free G_2 -structures on compact manifolds. The precise statement is the following:

Corollary 4.9. *Let M be a compact torsion-free G_2 manifold. Consider cohomology classes $[\alpha]$, $[\beta]$, and $[\gamma] \in H_{dR}^\bullet$. If the Massey triple product $\langle [\alpha], [\beta], [\gamma] \rangle$ is defined and we have $|\alpha| + |\beta| \neq 4$ and $|\beta| + |\gamma| \neq 4$, then $\langle [\alpha], [\beta], [\gamma] \rangle = 0$.*

We also prove the following stronger result in the case of full holonomy G_2 (the ‘irreducible’ case):

Theorem 4.10. *Let M be a compact torsion-free G_2 manifold with full holonomy G_2 , and consider cohomology classes $[\alpha]$, $[\beta]$, and $[\gamma] \in H_{dR}^\bullet$. If the Massey triple product $\langle [\alpha], [\beta], [\gamma] \rangle$ is defined, then $\langle [\alpha], [\beta], [\gamma] \rangle = 0$ except possibly in the case when $|\alpha| = |\beta| = |\gamma| = 2$.*

The Massey triple products on a compact torsion-free G_2 manifold are not discussed in [22].

Organization of the paper In the rest of this section, we discuss the domains of validity of the various results in this paper in Remark 1.1; then, we consider notation and conventions and conclude with the statement of a trivial result from linear algebra that we use frequently.

Section 2 is the heart of the paper, where we establish the various relations between the derivations d , ι_B , ι_K , \mathcal{L}_B , and \mathcal{L}_K . We begin with a brief summary of known facts about G_2 -structures that we will need in Sect. 2.1. In Sect. 2.2, we study the operators d and Δ in detail. Some of the key results are Proposition 2.12, which establishes Fig. 1, and Corollary 2.13 and

Proposition 2.16 which establish second-order differential identities. These have appeared before (without proof) in a paper of Bryant [2, Section 5.2]. But see Remark 2.18. A new and crucial result in Sect. 2.2 is Theorem 2.19 which relates the kernels of various operators on Ω^1 . In Sect. 2.3, we introduce the derivations $\iota_B, \iota_K, \mathcal{L}_B$, and \mathcal{L}_K and study their basic properties. One of the highlights is Corollary 2.32, which establishes Figs. 4 and 5.

In Sect. 3, we study and compute the \mathcal{L}_B -cohomology H_φ^\bullet of a compact torsion-free G_2 manifold. We use heavily both the results of Sect. 2 and Hodge theory. This section culminates with the proof of Theorem 3.19. Then in Sect. 4, we apply the results of Sect. 3 to study the Massey triple products of compact torsion-free G_2 manifolds.

Remark 1.1 We summarize here the domains of validity of the various sections of the paper.

- All results of Sect. 2.1 except the last one (Proposition 2.8) are valid for any G_2 -structure.
- Proposition 2.8 and the entirety of Sect. 2.2 assume that (M, φ) is torsion-free.
- In Sect. 2.3, the results that only involve the algebraic derivations ι_B and ι_K , up to and including Proposition 2.31, are valid for any G_2 -structure.
- The rest of Sect. 2.3, beginning with Corollary 2.32, uses the results of Sect. 2.2 heavily and is only valid in the torsion-free setting.
- The cohomology theories introduced in Sect. 3.1 make sense on any torsion-free G_2 manifold. However, beginning in Sect. 3.2 and for the rest of the paper, we assume that (M, φ) is a compact torsion-free G_2 manifold, as we use Hodge theory throughout. \square

Notation and conventions We mostly follow the notation and conventions of [12], and we point out explicitly whenever our notation differs significantly. Let (M, g) be an oriented smooth Riemannian 7-manifold. Let $\{e_1, \dots, e_7\}$ be a local frame for TM with dual coframe $\{e^1, \dots, e^7\}$. It can be a local coordinate frame $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^7}\}$ with dual coframe $\{dx^1, \dots, dx^7\}$, but this is not necessary. Note that the metric dual 1-form of e_i is $(e_i)^\flat = g_{ij}e^j$.

We employ the Einstein summation convention throughout. We write Λ^k for the bundle $\Lambda^k(T^*M)$ and Ω^k for its space of smooth sections $\Gamma(\Lambda^k(T^*M))$. Then $\Lambda^\bullet = \bigoplus_{k=1}^n \Lambda^k$ is the exterior algebra of T^*M and $\Omega^\bullet = \bigoplus_{k=0}^n \Omega^k$ is the space of smooth differential forms on M . Similarly, we use $S^2(T^*M)$ to denote the second symmetric power of T^*M , and $S = \Gamma(S^2(T^*M))$ to denote the space of smooth symmetric 2-tensors on M .

The Levi–Civita covariant derivative of g is denoted by ∇ . Let $\nabla_p = \nabla_{e_p}$. The exterior derivative $d\alpha$ of a k -form α can be written in terms of ∇ as

$$\begin{aligned}
 d\alpha &= e^p \wedge \nabla_p \alpha, \\
 (d\alpha)_{i_1 i_1 \dots i_{k+1}} &= \sum_{j=1}^{k+1} \nabla_{i_j} \alpha_{i_1 \dots \hat{i}_j \dots i_k}.
 \end{aligned}
 \tag{1.1}$$

The adjoint d^* of d with respect to g satisfies $d^* = (-1)^k * d *$ on Ω^k . It can be written in terms of ∇ as

$$\begin{aligned}
 d^* \alpha &= -g^{pq} e_p \lrcorner \nabla_q \alpha, \\
 (d^* \alpha)_{i_1 \dots i_{k-1}} &= -g^{pq} \nabla_p \alpha_{q i_1 \dots i_{k-1}}.
 \end{aligned}
 \tag{1.2}$$

An element $h \in S$ can be decomposed as $h = \frac{\text{Tr}_g h}{7} g + h^0$, where $\text{Tr}_g h = g^{ij} h_{ij}$ is the trace and h^0 is the trace-free component of h , which is orthogonal to g . We use $S_0^2(T^*M)$ to denote the bundle whose sections $S_0 = \Gamma(S_0^2(T^*M))$ are the trace-free symmetric 2-tensors. Finally, if X is a vector field on M , we denote by X^\flat the 1-form metric dual to X with respect

to the metric g . Sometimes we abuse notation and write X^b as simply X when there is no danger of confusion.

We write H_{dR}^k for the k th de Rham cohomology over \mathbb{R} and \mathcal{H}^k for the space of harmonic k -forms. If $[\alpha]$ is a cohomology class, then $|\alpha|$ denotes the degree of any of its representative differential forms. That is, if $[\alpha] \in H_{dR}^k$, then $|\alpha| = k$.

We use C^\bullet to denote a \mathbb{Z} -graded complex of real vector spaces. A degree k map P of the complex C^\bullet maps C^i into C^{i+k} , and we write

$$\begin{aligned} (\ker P)^i &= \ker(P : C^i \rightarrow C^{i+k}), \\ (\text{im } P)^i &= \text{im}(P : C^{i-k} \rightarrow C^i). \end{aligned} \tag{1.3}$$

Lemma 1.2 *We state two trivial results from linear algebra that we use several times in Sect. 3.*

- (i) *Let $V \subseteq U \subseteq (V \oplus W)$ be nested subspaces. Then $U = V \oplus (W \cap U)$.*
- (ii) *Let $U = A \oplus B \oplus C$ be a direct sum decomposition of a vector space into complementary subspaces A, B, C . Let V, W be subspaces of U such that $V = A' \oplus B' \oplus C'$ and $W = A'' \oplus B'' \oplus C''$ where A', A'' are subspaces of A , and B', B'' are subspaces of B , and C', C'' are subspaces of C . Then $V \cap W = (A' \cap A'') \oplus (B' \cap B'') \oplus (C' \cap C'')$.*

2 Natural derivations on torsion-free G_2 manifolds

We first review some facts about torsion-free G_2 manifolds and the decomposition of the exterior derivative d . Then, we define two derivations on Ω^\bullet and discuss their properties.

2.1 G_2 -Structures and the decomposition of Ω^\bullet

Let (M^7, φ) be a manifold with a G_2 -structure. Here φ is the positive 3-form associated with the G_2 -structure, and we use ψ to denote the dual 4-form $\psi = *\varphi$ with respect to the metric g induced by φ . We will use the sign/orientation convention for G_2 -structures of [12]. In this section we collect some facts about G_2 -structures, taken from [12], that we will need. We recall the fundamental relation between φ and g , which allows one to extract the metric from the 3-form. This is:

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6g(X, Y)\text{vol}. \tag{2.1}$$

Lemma 2.1 *The tensors g, φ, ψ satisfy the following contraction identities in a local frame:*

$$\begin{aligned} \varphi_{ijk}\varphi_{abc}g^{kc} &= g_{ia}g_{jb} - g_{ib}g_{ja} - \psi_{ijab}, \\ \varphi_{ijk}\varphi_{abc}g^{jb}g^{kc} &= 6g_{ia}, \\ \varphi_{ijk}\varphi_{abc}g^{ia}g^{jb}g^{kc} &= 42, \\ \varphi_{ijk}\psi_{abcd}g^{kd} &= g_{ia}\varphi_{jbc} + g_{ib}\varphi_{ajc} + g_{ic}\varphi_{abj} - g_{aj}\varphi_{ibc} - g_{bj}\varphi_{aic} - g_{cj}\varphi_{abi}, \\ \varphi_{ijk}\psi_{abcd}g^{jc}g^{kd} &= -4\varphi_{iab}, \\ \varphi_{ijk}\psi_{abcd}g^{ib}g^{jc}g^{kd} &= 0, \\ \psi_{ijkl}\psi_{abcd}g^{ld} &= -\varphi_{ajk}\varphi_{ibc} - \varphi_{iak}\varphi_{jbc} - \varphi_{ija}\varphi_{kbc} \\ &\quad + g_{ia}g_{jb}g_{kc} + g_{ib}g_{jc}g_{ka} + g_{ic}g_{ja}g_{kb} - g_{ia}g_{jc}g_{kb} \end{aligned}$$

$$\begin{aligned}
 & -g_{ib}g_{ja}g_{kc} - g_{ic}g_{jb}g_{ka} \\
 & -g_{ia}\psi_{jkbc} \\
 & -g_{ja}\psi_{kibc} - g_{ka}\psi_{ijbc} + g_{ab}\psi_{ijkc} - g_{ac}\psi_{ijkb}, \\
 \psi_{ijkl}\psi_{abcd}g^{kc}g^{ld} &= 4g_{ia}g_{jb} - 4g_{ib}g_{ja} - 2\psi_{ijab}, \\
 \psi_{ijkl}\psi_{abcd}g^{jb}g^{kc}g^{ld} &= 24g_{ia}, \\
 \psi_{ijkl}\psi_{abcd}g^{ia}g^{jb}g^{kc}g^{ld} &= 168.
 \end{aligned}$$

Proof This is proved in Lemmas A.12, A.13, and A.14 of [12]. □

For $k = 0, \dots, 7$, the bundle $\Lambda^k := \Lambda^k(T^*M)$ decomposes as follows:

$$\begin{aligned}
 \Lambda^0 &= \Lambda_1^0, \\
 \Lambda^1 &= \Lambda_7^1, \\
 \Lambda^2 &= \Lambda_7^2 \oplus \Lambda_{14}^2, \\
 \Lambda^3 &= \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3, \\
 \Lambda^4 &= \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4, \\
 \Lambda^5 &= \Lambda_7^5 \oplus \Lambda_{14}^5, \\
 \Lambda^6 &= \Lambda_7^6, \\
 \Lambda^7 &= \Lambda_1^7.
 \end{aligned} \tag{2.2}$$

Here Λ_l^k is a rank l subbundle of Λ^k , and the decomposition is orthogonal with respect to g . Moreover, we have $\Lambda_l^{7-k} = *\Lambda_l^k$. In fact, there are isomorphisms $\Lambda_l^k \cong \Lambda_l^{k'}$, so the bundles in the same vertical column of (2.2) are all isomorphic. Moreover, the Hodge star $*$ and the operations of wedge product with φ or with ψ all commute with the projections π_l for $l = 1, 7, 14, 27$.

We will denote by Ω_l^k the space of smooth sections of Λ_l^k . The isomorphisms $\Lambda_l^k \cong \Lambda_l^{k'}$ induce isomorphisms $\Omega_l^k \cong \Omega_l^{k'}$. The descriptions of the Ω_l^k and the particular identifications that we choose to use in this paper are given explicitly as follows:

$$\begin{aligned}
 \Omega_1^0 &= C^\infty(M), \\
 \Omega_7^1 &= \Gamma(T^*M) \cong \Gamma(TM), \\
 \Omega_7^2 &= \{X \lrcorner \varphi : X \in \Gamma(TM)\} \cong \Omega_7^1 \\
 \Omega_{14}^2 &= \{\beta \in \Omega^2 : \beta \wedge \psi = 0\} = \{\beta \in \Omega^2 : \beta_{pq}g^{pi}q^{aj}\varphi_{ijk} = 0\}, \\
 \Omega_1^3 &= \{f\varphi : f \in C^\infty(M)\} \cong \Omega_1^0, \\
 \Omega_7^3 &= \{X \lrcorner \psi : X \in \Gamma(TM)\} \cong \Omega_7^1, \\
 \Omega_{27}^3 &= \{\beta \in \Omega^3 : \beta \wedge \varphi = 0 \text{ and } \beta \wedge \psi = 0\} = \{h_{ip}g^{pk}dx^i \wedge (\partial_k \lrcorner \varphi) : h \in \mathcal{S}_0\} \\
 \Omega_l^k &= \{*\beta : \beta \in \Omega_l^{7-k}\}, \quad \text{for } k = 4, 5, 6, 7.
 \end{aligned} \tag{2.3}$$

Remark 2.2 We emphasize that the particular identifications we have chosen in (2.3) are *not isometric*. Making them isometric identifications would require introducing irrational constant factors, but this will not be necessary. See also Remark 2.15. □

We will denote by π_l^k the orthogonal projection $\pi_l^k : \Omega^k \rightarrow \Omega_l^k$. We note for future reference that $\beta \in \Omega_1^3 \oplus \Omega_{27}^3$ if and only if $\beta \perp (X \lrcorner \psi)$ for all X , and $\beta \in \Omega_7^3 \oplus \Omega_{27}^3$ if and only if $\beta \perp \varphi$. In a local frame, these observations are

$$\begin{aligned} \beta \in \Omega_1^3 \oplus \Omega_{27}^3 &\iff \beta_{ijk} g^{ia} g^{jb} g^{kc} \psi_{abcd} = 0, \\ \beta \in \Omega_7^3 \oplus \Omega_{27}^3 &\iff \beta_{ijk} g^{ia} g^{jb} g^{kc} \varphi_{abc} = 0. \end{aligned} \tag{2.4}$$

Similarly, we have that $\gamma \in \Omega_1^4 \oplus \Omega_{27}^4$ if and only if $\gamma \perp (\varphi \wedge X)$ for all X , and $\gamma \in \Omega_7^4 \oplus \Omega_{27}^4$ if and only if $\gamma \perp \psi$. In a local frame, these observations are

$$\begin{aligned} \gamma \in \Omega_1^4 \oplus \Omega_{27}^4 &\iff \gamma_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{abc} = 0, \\ \gamma \in \Omega_7^4 \oplus \Omega_{27}^4 &\iff \gamma_{ijkl} g^{ia} g^{jb} g^{kc} g^{ld} \psi_{abcd} = 0. \end{aligned} \tag{2.5}$$

Lemma 2.3 *The following identities hold:*

$$\begin{aligned} *(\varphi \wedge X^b) &= X \lrcorner \psi, & *(\psi \wedge X^b) &= X \lrcorner \varphi, \\ \psi \wedge *(\varphi \wedge X^b) &= 0, & \varphi \wedge *(\psi \wedge X^b) &= -2\psi \wedge X^b, \\ \varphi \wedge (X \lrcorner \varphi) &= -2*(X \lrcorner \varphi), & \psi \wedge (X \lrcorner \varphi) &= 3*X^b, \\ \varphi \wedge (X \lrcorner \psi) &= -4*X^b, & \psi \wedge (X \lrcorner \psi) &= 0. \end{aligned}$$

Proof This is part of Proposition A.3 in [12]. □

Lemma 2.4 *Identify $\Omega^1 \cong \Gamma(TM)$ using the metric. The cross product $\times : \Omega^1 \times \Omega^1 \rightarrow \Omega^1$ is defined by $X \times Y = Y \lrcorner X \lrcorner \varphi = *(X \wedge Y \wedge \psi)$. It satisfies the identity*

$$X \times (X \times Y) = -g(X, X)Y + g(X, Y)X.$$

Proof This is part of Lemma A.1 in [12]. □

In terms of a local frame, we define a map $\ell_\varphi : \Gamma(T^*M \otimes T^*M) \rightarrow \Omega^3$ by

$$\ell_\varphi A = A_{ip} g^{pq} e^i \wedge (e_q \lrcorner \varphi). \tag{2.6}$$

In components, we have

$$(\ell_\varphi A)_{ijk} = A_{ip} g^{pq} \varphi_{qjk} + A_{jp} g^{pq} \varphi_{iqk} + A_{kp} g^{pq} \varphi_{ijq}.$$

Analogous to (2.6), we define $\ell_\psi : \Gamma(T^*M \otimes T^*M) \rightarrow \Omega^4$ by

$$\ell_\psi A = A_{ip} g^{pq} e^i \wedge (e_q \lrcorner \psi). \tag{2.7}$$

In components, we have

$$(\ell_\psi A)_{ijkl} = A_{ip} g^{pq} \psi_{qjkl} + A_{jp} g^{pq} \psi_{iqkl} + A_{kp} g^{pq} \psi_{ijql} + A_{lp} g^{pq} \psi_{ijkq}.$$

It is easy to see that when $A = g$ is the metric, then

$$\ell_\varphi g = 3\varphi, \quad \ell_\psi g = 4\psi. \tag{2.8}$$

In [12, Section 2.2], the map ℓ_φ is written as D , but we use ℓ_φ to avoid confusion with the many instances of ‘ D ’ throughout the present paper to denote various natural linear first-order differential operators. We can orthogonally decompose sections of $\Gamma(T^*M \otimes T^*M)$ into symmetric and skew-symmetric parts, which then further orthogonally decompose as

$$\Gamma(T^*M \otimes T^*M) = \Omega_1^0 \oplus S_0 \oplus \Omega_7^2 \oplus \Omega_{14}^2.$$

In [12, Section 2.2], it is shown that ℓ_φ has kernel Ω_{14}^2 and maps Ω_1^0 , \mathcal{S}_0 , and Ω_7^2 isomorphically onto Ω_1^3 , Ω_{27}^3 , and Ω_3^3 , respectively. One can similarly show that ℓ_ψ has kernel Ω_{14}^2 and maps Ω_1^0 , \mathcal{S}_0 , and Ω_7^2 isomorphically onto Ω_1^4 , Ω_{27}^4 , and Ω_7^4 , respectively. (See also [15] for a detailed proof.) In particular, we note for future references that

$$\begin{aligned} \beta \in \Omega_{14}^2 &\iff (\ell_\varphi\beta)_{ijk} = \beta_{ip}g^{pq}\varphi_{qjk} + \beta_{jp}g^{pq}\varphi_{iqk} + \beta_{kp}g^{pq}\varphi_{ijq} = 0, \\ &\iff (\ell_\psi\beta)_{ijkl} = \beta_{ip}g^{pq}\psi_{qjkl} + \beta_{jp}g^{pq}\psi_{iqkl} \\ &\quad + \beta_{kp}g^{pq}\psi_{ijql} + \beta_{lp}g^{pq}\psi_{ijkq} = 0. \end{aligned} \tag{2.9}$$

When restricted to \mathcal{S} , the map ℓ_φ is denoted by i in [12]. We use ℓ_φ rather than i , to avoid confusion with the algebraic derivations ι_B and ι_K that we introduce later in Sect. 2.3.

Lemma 2.5 *Let $h \in \mathcal{S}_0$. Then $\ast(\ell_\varphi h) = -\ell_\psi h$.*

Proof This is part of Proposition 2.14 in [12]. □

The next two propositions will be crucial to establish properties of the algebraic derivations ι_B and ι_K in Sect. 2.3.

Proposition 2.6 *Let $h = h_{ij}e^i e^j$ be a symmetric 2-tensor. The following identities hold:*

$$\begin{aligned} h^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner \varphi) &= -2(\text{Tr}_g h)\psi + 2\ell_\psi h, \\ h^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner \psi) &= 0, \\ h^{pq}(e_p \lrcorner \psi) \wedge (e_q \lrcorner \psi) &= 0. \end{aligned} \tag{2.10}$$

Proof Let $\alpha \in \Omega^k$ and $\beta \in \Omega^l$. Then, we have

$$(e_p \lrcorner \alpha) \wedge (e_q \lrcorner \beta) = e_p \lrcorner (\alpha \wedge (e_q \lrcorner \beta)) - (-1)^k \alpha \wedge (e_p \lrcorner e_q \lrcorner \beta).$$

Since the second term above is skew in p, q , when we contract with the symmetric tensor h^{pq} we obtain

$$h^{pq}(e_p \lrcorner \alpha) \wedge (e_q \lrcorner \beta) = h^{pq} e_p \lrcorner (\alpha \wedge (e_q \lrcorner \beta)). \tag{2.11}$$

We will repeatedly use the identities from Lemma 2.3. When $\alpha = \beta = \psi$ in (2.11), we have $\psi \wedge (e_q \lrcorner \psi) = 0$, establishing the third equation in (2.10). When $\alpha = \varphi$ and $\beta = \psi$ in (2.11), we have

$$\varphi \wedge (e_q \lrcorner \psi) = -4 \ast (g_{qm} e^m),$$

and hence using that $X \lrcorner (\ast\alpha) = -\ast (X^\flat \wedge \alpha)$ for $\alpha \in \Omega^1$, we find

$$\begin{aligned} h^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner \psi) &= h^{pq} e_p \lrcorner (-4 \ast g_{qm} e^m) = -4h^{pq} g_{qm} e_p \lrcorner (\ast e^m) \\ &= +4h^{pq} g_{qm} \ast ((e_p)^\flat \wedge e^m) = 4h^{pq} g_{qm} g_{pl} \ast (e^l \wedge e^m) \\ &= 4h_{lm} \ast (e^l \wedge e^m) = 0, \end{aligned}$$

establishing the second equation in (2.10). Finally, when $\alpha = \beta = \varphi$ in (2.11), we have

$$\varphi \wedge (e_q \lrcorner \varphi) = -2 \ast (e_q \lrcorner \varphi) = -2(\psi \wedge (e_q)^\flat) = -2g_{qm} e^m \wedge \psi,$$

and hence, using (2.7), we find

$$\begin{aligned} h^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner \varphi) &= h^{pq} e_p \lrcorner (-2g_{qm} e^m \wedge \psi) = -2h^{pq} g_{qm} e_p \lrcorner (e^m \wedge \psi) \\ &= -2h^{pq} g_{qm} \delta_p^m \psi + 2h^{pq} g_{qm} e^m \wedge (e_p \lrcorner \psi) \end{aligned}$$

$$= -2h^{pq}g_{pq}\psi + 2h_{ml}g^{lp}e^m \wedge (e_p \lrcorner \psi) = -2(\text{Tr}_g h)\psi + 2\ell_\psi h,$$

establishing the first equation in (2.10). □

Proposition 2.7 *For any fixed m , the following identities hold:*

$$\begin{aligned} g^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner e_m \lrcorner \varphi) &= 3(e_m \lrcorner \psi), \\ g^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner e_m \lrcorner \psi) &= -3 * (e_m \lrcorner \psi), \\ g^{pq}(e_p \lrcorner \psi) \wedge (e_q \lrcorner e_m \lrcorner \varphi) &= -3 * (e_m \lrcorner \psi), \\ g^{pq}(e_p \lrcorner \psi) \wedge (e_q \lrcorner e_m \lrcorner \psi) &= 4 * (e_m \lrcorner \varphi). \end{aligned} \tag{2.12}$$

Proof In this proof, we use e^{ijk} to denote $e^i \wedge e^j \wedge e^k$ and similarly for any number of indices. First, we compute

$$\begin{aligned} g^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner e_m \lrcorner \varphi) &= \frac{1}{2}(g^{pq}\varphi_{pij}\varphi_{mqk})e^{ijk} \\ &= \frac{1}{2}(g_{ik}g_{jm} - g_{im}g_{jk} - \psi_{ijkm})e^{ijk} \\ &= 0 - 0 - \frac{1}{2}\psi_{ijkm}e^{ijk} = 3\left(\frac{1}{6}\psi_{mijk}e^{ijk}\right) = 3(e_m \lrcorner \psi), \end{aligned}$$

establishing the first equation in (2.12).

Similarly, we compute

$$\begin{aligned} g^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner e_m \lrcorner \psi) &= \frac{1}{4}(g^{pq}\varphi_{pij}\psi_{mqkl})e^{ijkl} \\ &= \frac{1}{4}(g_{ik}\varphi_{jlm} + g_{il}\varphi_{kjm} + g_{im}\varphi_{klj} - g_{jk}\varphi_{ilm} - g_{jl}\varphi_{kim} - g_{jm}\varphi_{kli})e^{ijkl} \\ &= 0 + 0 + \frac{1}{4}g_{im}\varphi_{klj}e^{ijkl} + 0 + 0 - \frac{1}{4}g_{jm}\varphi_{kli}e^{ijkl} = \frac{1}{2}g_{im}\varphi_{jkl}e^{ijkl} \\ &= 3(g_{mi}e^i) \wedge \left(\frac{1}{6}\varphi_{jkl}e^{jkl}\right) = 3(e_m)^\flat \wedge \varphi = -3 * (e_m \lrcorner \psi), \end{aligned}$$

establishing the second equation in (2.12). Now let $h = g$ in the second equation of (2.10). Taking the interior product of $g^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner \psi) = 0$ with e_m , we obtain

$$g^{pq}(e_m \lrcorner e_p \lrcorner \varphi) \wedge (e_q \lrcorner \psi) + g^{pq}(e_p \lrcorner \varphi) \wedge (e_m \lrcorner e_q \lrcorner \psi) = 0,$$

which, after rearrangement and relabeling of indices, becomes

$$g^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner e_m \lrcorner \psi) = g^{pq}(e_p \lrcorner \psi) \wedge (e_q \lrcorner e_m \lrcorner \varphi),$$

establishing the third equation in (2.12).

Finally, we compute

$$\begin{aligned} g^{pq}(e_p \lrcorner \psi) \wedge (e_q \lrcorner e_c \lrcorner \psi) &= \frac{1}{12}(g^{pq}\psi_{pijk}\psi_{cqab})e^{ijkab} \\ &= -\frac{1}{12}(g^{pq}\psi_{ijkp}\psi_{abcq})e^{ijkab} \\ &= -\frac{1}{12}\left(-\varphi_{ajk}\varphi_{ibc} - \varphi_{iak}\varphi_{jbc} - \varphi_{ija}\varphi_{kbc} + g_{ia}g_{jb}g_{kc} + g_{ib}g_{jc}g_{ka} \right. \\ &\quad \left. + g_{ic}g_{ja}g_{kb} - g_{ia}g_{jc}g_{kb} - g_{ib}g_{ja}g_{kc} - g_{ic}g_{jb}g_{ka} - g_{ia}\psi_{jkb} \right. \\ &\quad \left. - g_{ja}\psi_{kibc} - g_{ka}\psi_{ijbc} + g_{ab}\psi_{ijkc} - g_{ac}\psi_{ijkb}\right)e^{ijkab}. \end{aligned}$$

The first three terms above combine, and all the remaining terms except the last one vanish. Thus using Lemma 2.3, we have

$$\begin{aligned}
 g^{pq}(e_p \lrcorner \psi) \wedge (e_q \lrcorner e_c \lrcorner \psi) &= -\frac{1}{12}(-3\varphi_{ajk}\varphi_{ibc} - g_{ac}\psi_{ijkb})e^{ijkab} \\
 &= -\frac{1}{4}(\varphi_{ajk}e^{ajk}) \wedge (\varphi_{cib}e^{ib}) - \frac{1}{12}(g_{ac}e^a) \wedge (\psi_{ijkb}e^{ijkb}) \\
 &= -3(\frac{1}{6}\varphi_{ajk}e^{ajk}) \wedge (\frac{1}{2}\varphi_{cib}e^{ib}) - 2(g_{ca}e^a) \wedge (\frac{1}{24}\psi_{ijkb}e^{ijkb}) \\
 &= -3\varphi \wedge (e_c \lrcorner \varphi) - 2(e_c)^\flat \wedge \psi = 6 * (e_c \lrcorner \varphi) \\
 &\quad - 2 * (e_c \lrcorner \varphi) = 4 * (e_c \lrcorner \varphi),
 \end{aligned}$$

establishing the fourth equation in (2.12). □

For the rest of this section and all of the next section, we assume (M, φ) is torsion-free. See also Remark 1.1.

Proposition 2.8 *Suppose (M, φ) is a torsion-free G_2 manifold. Then $*(\pi_{27}\mathcal{L}_X\varphi) = -\pi_{27}\mathcal{L}_X\psi$ for any vector field X .*

Proof Because φ and ψ are both parallel, from [12, equation (1.7)] we have

$$(\mathcal{L}_X\varphi) = (\nabla_i X_p)g^{pq}e^i \wedge (e_q \lrcorner \varphi), \quad (\mathcal{L}_X\psi) = (\nabla_i X_p)g^{pq}e^i \wedge (e_q \lrcorner \psi).$$

Applying π_{27} to both of the above expressions and using Lemma 2.5 yields the desired result. □

2.2 The exterior derivative d and the Hodge Laplacian Δ

In this section, we analyze the exterior derivative d and the Hodge Laplacian Δ on a manifold with torsion-free G_2 -structure. Much, but not all, of the results in this section have appeared before, without proof, in [2, Section 5.2]. See Remark 2.18 for details. Theorem 2.19, which relates kernels of various operators on Ω^1 , is fundamental to the rest of the paper and appears to be new.

We first define three first-order operators on torsion-free G_2 manifolds, which will be used to decompose $d : \Omega^k \rightarrow \Omega^{k+1}$ into components. More details can be found in [13, Section 4].

Definition 2.9 Let (M, φ) be a torsion-free G_2 manifold. We define the following first-order linear differential operators:

$$\begin{aligned}
 \text{grad} : \Omega_1^0 &\rightarrow \Omega_7^1, & f &\mapsto df, \\
 \text{div} : \Omega_7^1 &\rightarrow \Omega_1^0, & X &\mapsto -d^*X, \\
 \text{curl} : \Omega_7^1 &\rightarrow \Omega_7^1, & X &\mapsto *(\psi \wedge dX).
 \end{aligned}$$

In a local frame, these operators have the following form:

$$(\text{grad } f)_k = \nabla_k f, \quad \text{div } X = g^{ij}\nabla_i X_j, \quad (\text{curl } X)_k = (\nabla_i X_j)g^{ip}g^{jq}\varphi_{pqk}. \quad (2.13)$$

□

Definition 2.10 Denote by D_m^l the composition

$$D_m^l : \Omega_l^k \hookrightarrow \Omega^k \xrightarrow{d} \Omega^{k+1} \twoheadrightarrow \Omega_m^{k+1},$$

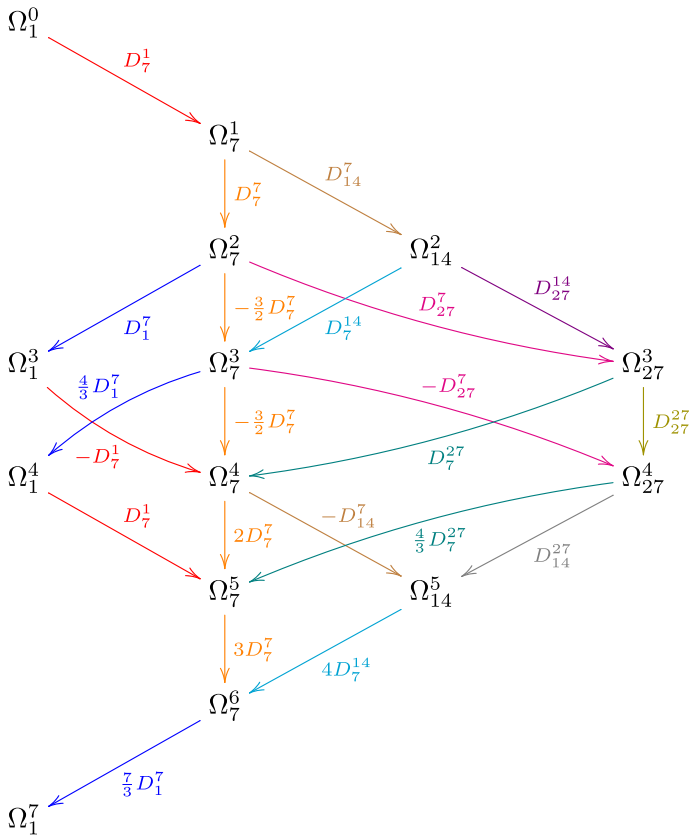


Fig. 1 Decomposition of the exterior derivative d into components

where k is the smallest integer such that this composition makes sense. Here the surjection is the projection π_m^{k+1} . That is, $D_m^k = \pi_m^{k+1} \circ d|_{\Omega_1^k}$. □

Proposition 2.11 *The operators $D_1^1, D_{14}^1, D_1^{14}, D_{27}^1, D_1^{27}$, and D_{14}^{14} are all zero.*

Proof It is clear from (2.2) that $D_{14}^{14} = 0$. The operators $D_1^1 : \Omega_1^3 \rightarrow \Omega_1^4$ and $D_{27}^1 : \Omega_1^3 \rightarrow \Omega_{27}^4$ are both zero because $d(f\varphi) = (df) \wedge \psi \in \Omega_7^4$. Similarly, since $d(f\psi) = (df) \wedge \psi \in \Omega_7^5$, we also have $D_{14}^2 = 0$. If $\beta \in \Omega_{14}^2$, then $\beta \wedge \psi = 0$, so $(d\beta) \wedge \psi = 0$, and thus $\pi_1(d\beta) = 0$, hence $D_{14}^2 = 0$. Similarly, if $\beta \in \Omega_{27}^3$, then $\beta \wedge \varphi = 0$, so $(d\beta) \wedge \varphi = 0$, and thus $\pi_1(d\beta) = 0$, hence $D_1^{27} = 0$. □

Proposition 2.12 *With respect to the identifications described in (2.3), the components of the exterior derivative d satisfy the relations given in Fig. 1.*

Proof We will use repeatedly the contraction identities of Lemma 2.1 and descriptions (2.3) of the Ω_1^k spaces.

- (i) We establish the relations for $\pi_7 d \pi_1 : \Omega_1^k \rightarrow \Omega_7^{k+1}$ for $k = 0, 3, 4$.
 $k = 0$: Let $f \in \Omega_1^0$. By Definition 2.10, we have $D_7^1 f = df$.

- $k = 3$: Let $\beta = f\varphi \in \Omega_1^3$. Since $d\beta = (df) \wedge \varphi \in \Omega_7^4$, we have $\pi_7(d\beta) = -\varphi \wedge (df) = - * ((df) \lrcorner \psi)$, so $\pi_7 d\pi_1 : \Omega_1^3 \rightarrow \Omega_7^4$ is identified with $-D_7^1$.
- $k = 4$: Let $\gamma = f\psi \in \Omega_1^4$. Since $d\gamma = (df) \wedge \psi \in \Omega_7^5$, we have $\pi_7(d\gamma) = \psi \wedge (df) = *((df) \lrcorner \varphi)$, so $\pi_7 d\pi_1 : \Omega_1^4 \rightarrow \Omega_7^5$ is identified with D_7^1 .
- (ii) We establish the relations for $\pi_1 d\pi_7 : \Omega_7^k \rightarrow \Omega_1^{k+1}$ for $k = 2, 3, 6$.
- $k = 2$: Let $\alpha = X \lrcorner \varphi \in \Omega_7^2$. Then $(\pi_1 d\alpha)_{ijk} = f\varphi_{ijk}$ for some function f . Using (2.4), we compute

$$\begin{aligned} (\alpha)_{ijk} g^{ia} g^{jb} g^{kc} \varphi_{abc} &= (\pi_1 d\alpha)_{ijk} g^{ia} g^{jb} g^{kc} \varphi_{abc} = f\varphi_{ijk} g^{ia} g^{jb} g^{kc} \varphi_{abc} = 42f, \\ &= (\nabla_i \alpha_{jk} + \nabla_j \alpha_{ki} + \nabla_k \alpha_{ij}) g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= 3(\nabla_i \alpha_{jk}) g^{ia} g^{jb} g^{kc} \varphi_{abc}, \end{aligned}$$

and thus $f = \frac{3}{42}(\nabla_i \alpha_{jk}) g^{ia} g^{jb} g^{kc} \varphi_{abc}$. Substituting $\alpha_{jk} = X^m \varphi_{mjk}$, we obtain

$$f = \frac{3}{42}(\nabla_i X^m) \varphi_{mjk} \varphi_{abc} g^{ia} g^{jb} g^{kc} = \frac{18}{42}(\nabla_i X^m) g^{ia} g_{ma} = \frac{3}{7} \nabla_i X^i,$$

and comparing with Definition 2.9, we find that

$$D_1^7 X = \pi_1 d(X \lrcorner \varphi) = f\varphi = (\frac{3}{7} \operatorname{div} X)\varphi. \tag{2.14}$$

$k = 3$: Let $\beta = X \lrcorner \psi \in \Omega_7^3$. Then $(\pi_1 d\beta)_{ijkl} = f\psi_{ijkl}$ for some function f . Using (2.5), we compute

$$\begin{aligned} (d\beta)_{ijkl} g^{ia} g^{jb} g^{kc} g^{ld} \psi_{abcd} &= (\pi_1 d\beta)_{ijkl} g^{ia} g^{jb} g^{kc} g^{ld} \psi_{abcd} \\ &= f\psi_{ijkl} g^{ia} g^{jb} g^{kc} g^{ld} \psi_{abcd} = 168f, \\ &= (\nabla_i \beta_{jkl} - \nabla_j \beta_{ikl} + \nabla_k \beta_{ijl} - \nabla_l \beta_{ijk}) g^{ia} g^{jb} g^{kc} g^{ld} \psi_{abcd} \\ &= 4(\nabla_i \beta_{jkl}) g^{ia} g^{jb} g^{kc} g^{ld} \psi_{abcd}, \end{aligned}$$

and thus $f = \frac{4}{168}(\nabla_i \beta_{jkl}) g^{ia} g^{jb} g^{kc} g^{ld} \psi_{abcd}$. Substituting $\beta_{jkl} = X^m \psi_{mjkl}$, we obtain

$$f = \frac{4}{168}(\nabla_i X^m) \psi_{mjkl} \psi_{abcd} g^{ia} g^{jb} g^{kc} g^{ld} = \frac{4 \cdot 24}{168}(\nabla_i X^m) g^{ia} g_{ma} = \frac{4}{7} \nabla_i X^i,$$

and comparing with (2.14), we find that $\pi_1 d\pi_7 : \Omega_7^3 \rightarrow \Omega_1^4$ is identified with $\frac{4}{3} D_7^1$.

$k = 6$: Let $*X \in \Omega_7^6$. Then $\pi_1 d(*X) = d*X = *^2 d*X = -* (d*X) = -(d*X) \operatorname{vol}$, where we have used $d^* = -*d*$ on odd forms. Comparing with Definition 2.9 and (2.14), we find that $\pi_1 d\pi_7 : \Omega_7^6 \rightarrow \Omega_1^7$ is identified with $\frac{7}{3} D_7^1$.

- (iii) We establish the relations for $\pi_7 d\pi_7 : \Omega_7^k \rightarrow \Omega_7^{k+1}$ for $k = 1, 2, 3, 4, 5$.

$k = 1$: Let $X \in \Omega_7^1$. Then $(\pi_7 dX)_{ij} = Y^m \varphi_{mij}$ for some vector field Y . We compute

$$\begin{aligned} (dX)_{ij} g^{ia} g^{jb} \varphi_{kab} &= (\pi_7 dX)_{ij} g^{ia} g^{jb} \varphi_{kab} = Y^m \varphi_{mij} g^{ia} g^{jb} \varphi_{kab} = 6Y_k, \\ &= (\nabla_i X_j - \nabla_j X_i) g^{ia} g^{jb} \varphi_{kab} = 2(\nabla_i X_j) g^{ia} g^{jb} \varphi_{abk}, \end{aligned}$$

from which it follows from Definition 2.10 that

$$D_7^1 X = \pi_7 dX = Y = \frac{1}{3} \operatorname{curl} X. \tag{2.15}$$

$k = 2$: Let $\alpha = X \lrcorner \varphi \in \Omega_7^2$. Then $(\pi_7 d\alpha)_{ijk} = Y^m \psi_{mijk}$ for some vector field Y . Using (2.4), we compute

$$(d\alpha)_{ijk} g^{ia} g^{jb} g^{kc} \psi_{labc}$$

$$\begin{aligned} &= (\pi_7 d\alpha)_{ijk} g^{ia} g^{jb} g^{kc} \psi_{labc} = Y^m \psi_{mijk} g^{ia} g^{jb} g^{kc} \psi_{labc} = 24Y_l, \\ &= (\nabla_i \alpha_{jk} + \nabla_j \alpha_{ki} + \nabla_k \alpha_{ij}) g^{ia} g^{jb} g^{kc} \psi_{labc} \\ &= 3(\nabla_i \alpha_{jk}) g^{ia} g^{jb} g^{kc} \psi_{labc}, \end{aligned}$$

and thus $Y_l = \frac{1}{8}(\nabla_i \alpha_{jk}) g^{ia} g^{jb} g^{kc} \psi_{labc}$. Substituting $\alpha_{jk} = X^m \varphi_{mjk}$, we obtain

$$Y_l = \frac{1}{8}(\nabla_i X^m) \varphi_{mjk} \psi_{labc} g^{ia} g^{jb} g^{kc} = -\frac{4}{8}(\nabla_i X^m) g^{ia} \varphi_{mla} = -\frac{1}{2} \text{curl } X,$$

and comparing with (2.15), we find that $\pi_7 d\pi_7 : \Omega_7^2 \rightarrow \Omega_7^3$ is identified with $-\frac{3}{2}D_7^7$. $k = 3$: Let $\beta = X \lrcorner \psi \in \Omega_7^3$. Then $\pi_7(d\beta) = *(Y \lrcorner \psi) = \varphi \wedge Y$ for some vector field Y . We have $(\pi_7 d\beta)_{ijkl} = \varphi_{ijk} Y_l - \varphi_{ijl} Y_k + \varphi_{ikl} Y_j - \varphi_{jkl} Y_i$. Using (2.4), we compute

$$\begin{aligned} &(\text{d}\beta)_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= (\pi_7 \text{d}\beta)_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= (\varphi_{ijk} Y_l - \varphi_{ijl} Y_k + \varphi_{ikl} Y_j - \varphi_{jkl} Y_i) g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= 42Y_l - 3\varphi_{ijl} Y_k g^{ia} g^{jb} g^{kc} \varphi_{abc} = 42Y_l - 3(6Y_k g^{kc} g_{lc}) = 24Y_l. \end{aligned}$$

But we also have

$$\begin{aligned} (\text{d}\beta)_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{abc} &= (\nabla_i \beta_{jkl} - \nabla_j \beta_{ikl} + \nabla_k \beta_{ijl} - \nabla_l \beta_{ijk}) g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= 3(\nabla_i \beta_{jkl}) g^{ia} g^{jb} g^{kc} \varphi_{abc} - (\nabla_l \beta_{ijk}) g^{ia} g^{jb} g^{kc} \varphi_{abc}. \end{aligned}$$

Substituting $\beta_{ijk} = X^m \psi_{mijk}$, we obtain

$$\begin{aligned} (\text{d}\beta)_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{cab} &= 3(\nabla_i X^m) \psi_{mljk} g^{ia} g^{jb} g^{kc} \varphi_{abc} - (\nabla_l X^m) \psi_{mijk} g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= -12(\nabla_i X^m) g^{ia} \varphi_{aml} - 0, \end{aligned}$$

and thus $Y_l = -\frac{12}{24}(\nabla_i X^m) g^{ia} \varphi_{aml} = -\frac{1}{2} \text{curl } X$. Comparing with (2.15), we find that $\pi_7 d\pi_7 : \Omega_7^3 \rightarrow \Omega_7^4$ is identified with $-\frac{3}{2}D_7^7$.

$k = 4$: Let $\gamma = *(X \lrcorner \psi) = \varphi \wedge X \in \Omega_7^4$. Then $\pi_7(d\gamma) = \pi_7 d(\varphi \wedge X) = -\pi_7(\varphi \wedge dX) = *(Y \lrcorner \varphi)$ for some vector field Y . We compute

$$*(Y \lrcorner \varphi) = -\pi_7(\varphi \wedge dX) = -\varphi \wedge (\pi_7 dX) = 2 * (\pi_7 dX).$$

Comparing with (2.15), we find that $\pi_7 d\pi_7 : \Omega_7^4 \rightarrow \Omega_7^5$ is identified with $2D_7^7$.

$k = 5$: Let $\eta = *(X \lrcorner \varphi) = \psi \wedge X \in \Omega_7^5$. Then $\pi_7(d\eta) = d\eta = d(\psi \wedge X) = \psi \wedge dX = *Y$ for some vector field Y . Using Definition 2.9, we compute

$$Y = *(\psi \wedge dX) = \text{curl } X.$$

Comparing with (2.15) we find that $\pi_7 d\pi_7 : \Omega_7^5 \rightarrow \Omega_7^6$ is identified with $3D_7^7$.

(iv) We establish the relations for $\pi_{14} d\pi_7 : \Omega_7^k \rightarrow \Omega_{14}^{k+1}$ for $k = 1, 4$.

$k = 1$: Let $X \in \Omega_7^1$. By definition, we have $\pi_{14} dX = D_{14}^7 X$.

$k = 4$: Let $\gamma = *(X \lrcorner \psi) = \varphi \wedge X \in \Omega_7^4$. Then $d\gamma = -\varphi \wedge (dX)$, so $\pi_{14} d\gamma = -\pi_{14}(\varphi \wedge dX) = -\varphi \wedge (\pi_{14} dX) = -*(\pi_{14} dX)$. Thus, we find that $\pi_{14} d\pi_7 : \Omega_7^4 \rightarrow \Omega_{14}^5$ is identified with $-D_{14}^7$.

(v) We establish the relations for $\pi_7 d\pi_{14} : \Omega_{14}^k \rightarrow \Omega_7^{k+1}$ for $k = 2, 5$.

$k = 1$: Let $\alpha \in \Omega_{14}^2$. By definition, we have $\pi_7 d\alpha = D_7^{14} X$.

$k = 4$: Let $\eta = *\beta \in \Omega_{14}^5$ where $\beta \in \Omega_{14}^2$. We have $*\beta = \varphi \wedge \beta$, so $\pi_7 d(*\beta) = d(*\beta) = -\varphi \wedge d\beta \in \Omega_7^6$. We can write $\pi_7 d\beta = Y \lrcorner \psi \in \Omega_7^3$ for some vector field Y . Then using

Lemma 2.3, we find $\pi_7 d(*\beta) = -\varphi \wedge d\beta = -\varphi \wedge (\pi_7 d\beta) = -\varphi \wedge (Y \lrcorner \psi) = 4 * Y$. Thus, we find that $\pi_7 d\pi_{14} : \Omega_{14}^5 \rightarrow \Omega_7^6$ is identified with $4D_7^{14}$.

(vi) We establish the relations for $\pi_{27} d\pi_7 : \Omega_7^k \rightarrow \Omega_{27}^{k+1}$ for $k = 2, 3$.

$k = 2$: Let $\alpha \in \Omega_7^2$. By definition, we have $\pi_{27} d\alpha = D_{27}^7 \alpha$.

$k = 3$: Let $\beta = X \lrcorner \psi \in \Omega_7^3$. Then $\pi_{27} d\beta = \pi_{27} d(X \lrcorner \psi) = \pi_{27} \mathcal{L}_X \psi$. Consider $\alpha = X \lrcorner \varphi$. Then similarly, we have $\pi_{27} d\alpha = \pi_{27} \mathcal{L}_X \varphi$. By Proposition 2.8, we have $\pi_{27} d(X \lrcorner \psi) = -*(\pi_{27} d(X \lrcorner \varphi))$. Thus, we find that $\pi_{27} d\pi_7 : \Omega_{27}^3 \rightarrow \Omega_7^4$ is identified with $-D_{27}^7$.

(vii) We establish the relations for $\pi_7 d\pi_{27} : \Omega_{27}^k \rightarrow \Omega_7^{k+1}$ for $k = 3, 4$.

$k = 3$: Let $\beta = \ell_\varphi h \in \Omega_{27}^3$, where $h \in \mathcal{S}_0$. Then $\pi_7(d\beta) = *(Y \lrcorner \psi) = \varphi \wedge Y$ for some vector field Y . We have $(\pi_7 d\beta)_{ijkl} = \varphi_{ijk} Y_l - \varphi_{ijl} Y_k + \varphi_{ikl} Y_j - \varphi_{jkl} Y_i$. Using (2.5), we compute

$$\begin{aligned} (d\beta)_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{abc} &= (\pi_7 d\beta)_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= (\varphi_{ijk} Y_l - \varphi_{ijl} Y_k + \varphi_{ikl} Y_j - \varphi_{jkl} Y_i) g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= 42Y_l - 3\varphi_{ijl} Y_k g^{ia} g^{jb} g^{kc} \varphi_{abc} = 42Y_l - 3(6Y_k g^{kc} g_{lc}) = 24Y_l. \end{aligned}$$

But we also have

$$\begin{aligned} (d\beta)_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{abc} &= (\nabla_i \beta_{jkl} - \nabla_j \beta_{ikl} + \nabla_k \beta_{ijl} - \nabla_l \beta_{ijk}) g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= 3(\nabla_i \beta_{jkl}) g^{ia} g^{jb} g^{kc} \varphi_{abc} - (\nabla_l \beta_{ijk}) g^{ia} g^{jb} g^{kc} \varphi_{abc}. \end{aligned}$$

Substituting $\beta_{ijk} = h_{ip} g^{pq} \varphi_{qjk} + h_{jp} g^{pq} \varphi_{qki} + h_{kp} g^{pq} \varphi_{qjk}$, we obtain

$$\begin{aligned} 24Y_l &= (d\beta)_{ijkl} g^{ia} g^{jb} g^{kc} \varphi_{cab} \\ &= 3(\nabla_i (h_{jp} g^{pq} \varphi_{qkl} + h_{kp} g^{pq} \varphi_{qlj} + h_{lp} g^{pq} \varphi_{qjk})) g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &\quad - (\nabla_l (h_{ip} g^{pq} \varphi_{qjk} + h_{jp} g^{pq} \varphi_{qki} + h_{kp} g^{pq} \varphi_{qjk})) g^{ia} g^{jb} g^{kc} \varphi_{abc} \\ &= 6(\nabla_i h_{jp}) g^{pq} g^{ia} g^{jb} (g^{kc} \varphi_{lqk} \varphi_{abc}) + 3(\nabla_i h_{lp}) g^{pq} g^{ia} (g^{jb} g^{kc} \varphi_{qjk} \varphi_{abc}) \\ &\quad - 3(\nabla_l h_{ip}) g^{pq} g^{ia} (g^{jb} g^{kc} \varphi_{qjk} \varphi_{abc}). \end{aligned}$$

We further simplify this as

$$\begin{aligned} 24Y_l &= 6(\nabla_i h_{jp}) g^{pq} g^{ia} g^{jb} (g_{la} g_{qb} - g_{lb} g_{qa} - \psi_{lqab}) \\ &\quad + 3(\nabla_i h_{lp}) g^{pq} g^{ia} (6g_{qa}) - 3(\nabla_l h_{ip}) g^{pq} g^{ia} (6g_{qa}) \\ &= 6(\nabla_l h_{jp}) g^{jp} - 6(\nabla_i h_{lp}) g^{ip} - 0 + 18(\nabla_i h_{lp}) g^{ip} - 18(\nabla_l h_{ip}) g^{ip} \\ &= 6\nabla_l(\text{Tr } h) - 6(\nabla_i h_{jl}) g^{ij} + 18(\nabla_i h_{jl}) g^{ij} - 18\nabla_l(\text{Tr } h) = 12g^{ij} (\nabla_i h_{jl}), \end{aligned}$$

and thus $Y_l = \frac{1}{2} g^{ij} (\nabla_i h_{jl})$. It follows from Definition 2.10 that

$$D_7^{27} h = \pi_7 d(\ell_\varphi h) = *(Y \lrcorner \psi), \quad \text{where } Y_l = \frac{1}{2} g^{ij} (\nabla_i h_{jl}). \tag{2.16}$$

$k = 4$: Let $\gamma = *(\ell_\varphi h) \in \Omega_{27}^4$, where $h \in \mathcal{S}_0$. Then $\pi_7(d\gamma) = *(Y \lrcorner \varphi)$ for some vector field Y . Taking Hodge star of both sides, we have $Y \lrcorner \varphi = *\pi_7(d * \ell_\varphi h) = \pi_7 * d * (\ell_\varphi h) = -\pi_7 d^*(\ell_\varphi h)$. Thus, we have

$$\begin{aligned} -(d^*(\ell_\varphi h))_{ij} g^{ia} g^{jb} \varphi_{kab} &= -(\pi_7 d^*(\ell_\varphi h))_{ij} g^{ia} g^{jb} \varphi_{kab} = (Y \lrcorner \varphi)_{ij} g^{ia} g^{jb} \varphi_{kab} \\ &= Y^m \varphi_{mij} g^{ia} g^{jb} \varphi_{kab} = 6Y_k. \end{aligned}$$

But we also have

$$\begin{aligned}
 & - (d^*(\ell_\varphi h))_{ij} g^{ia} g^{jb} \varphi_{kab} \\
 & = g^{pq} (\nabla_p (\ell_\varphi h)_{qij}) g^{ia} g^{jb} \varphi_{kab} \\
 & = g^{pq} (\nabla_p (h_{ql} g^{lm} \varphi_{mij} + h_{il} g^{lm} \varphi_{mj} + h_{jl} g^{lm} \varphi_{mqi})) g^{ia} g^{jb} \varphi_{kab} \\
 & = g^{pq} (\nabla_p h_{ql}) g^{lm} (g^{ia} g^{jb} \varphi_{mij} \varphi_{kab}) + 2g^{pq} (\nabla_p h_{il}) g^{lm} g^{ia} (g^{jb} \varphi_{qmj} \varphi_{kab}) \\
 & = g^{pq} (\nabla_p h_{ql}) g^{lm} (6g_{mk}) + 2g^{pq} (\nabla_p h_{il}) g^{lm} g^{ia} (g_{qk} g_{ma} - g_{qa} g_{mk} - \psi_{qmka}) \\
 & = 6g^{pq} (\nabla_p h_{qk}) + 2\nabla_k (\text{Tr } h) - 2g^{ip} (\nabla_p h_{ik}) - 0 = 4g^{ij} (\nabla_i h_{jk})
 \end{aligned}$$

Thus, we have $Y_k = \frac{2}{3} g^{ij} (\nabla_i h_{jk}) = \frac{4}{3} (\frac{1}{2} g^{ij} (\nabla_i h_{jk}))$. Comparing with (2.16), we find that $\pi_7 d\pi_{27} : \Omega_{27}^4 \rightarrow \Omega_7^5$ is identified with $\frac{4}{3} D_7^{27}$.

□

Corollary 2.13 *The operators of Definition 2.10 satisfy the following fourteen relations:*

$$\begin{aligned}
 D_7^7 D_7^1 &= 0, & D_{14}^7 D_7^1 &= 0, \\
 D_1^7 D_7^7 &= 0, & \frac{3}{2} D_7^7 D_7^7 - D_7^{14} D_{14}^7 &= 0, \\
 -D_7^1 D_7^1 + \frac{9}{4} D_7^7 D_7^7 + D_7^{27} D_{27}^7 &= 0, & \frac{3}{2} D_{14}^7 D_7^7 - D_{14}^{27} D_{27}^7 &= 0, \\
 \frac{3}{2} D_{27}^7 D_7^7 + D_{27}^{27} D_{27}^7 &= 0, & D_{27}^7 D_7^7 + D_{27}^{14} D_{14}^7 &= 0, \\
 D_1^7 D_7^{14} &= 0, & \frac{3}{2} D_7^7 D_7^{14} - D_7^{27} D_{27}^{14} &= 0, \\
 D_{27}^7 D_7^{14} - D_{27}^{27} D_{27}^{14} &= 0, & D_7^7 D_7^{27} + D_7^{14} D_{14}^{27} &= 0, \\
 \frac{3}{2} D_7^7 D_7^{27} + D_7^{27} D_{27}^{27} &= 0, & D_{14}^7 D_7^{27} - D_{14}^{27} D_{27}^{27} &= 0.
 \end{aligned} \tag{2.17}$$

Proof These relations all follow from Fig. 1 and the fact that $d^2 = 0$, by computing $\pi_{l'} d^2 \pi_l : \Omega_l^k \rightarrow \Omega_{l'}^{k+2}$ for all $l, l' \in \{1, 7, 14, 27\}$ and all $k = 0, \dots, 5$. Some of the relations arise multiple times this way. Moreover, there are two distinct relations for $(l, l') = (7, 7), (7, 27)$, and $(27, 7)$. □

Corollary 2.14 *Consider the maps $D_m^l : \Omega_l^k \rightarrow \Omega_m^{k+1}$ introduced in Definition 2.10. Recall these were only defined for the smallest integer k where the composition makes sense. The formal adjoint is a map $(D_m^l)^* : \Omega_m^{k+1} \rightarrow \Omega_l^k$. With respect to the identifications described in (2.3), these adjoint maps are given by*

$$\begin{aligned}
 (D_7^1)^* &= -\frac{7}{3} D_7^1, & (D_7^7)^* &= 3D_7^7, & (D_{14}^7)^* &= 4D_7^{14}, \\
 (D_1^7)^* &= -D_1^7, & (D_{27}^7)^* &= -\frac{4}{3} D_7^{27}, & (D_{14}^{14})^* &= D_{14}^7, \\
 (D_{27}^{14})^* &= -D_{14}^{27}, & (D_7^{27})^* &= -D_{27}^7, & (D_{27}^{27})^* &= D_{27}^{27}, \\
 (D_{14}^{27})^* &= -D_{27}^{14}.
 \end{aligned} \tag{2.18}$$

Proof These follow from Fig. 1 and the facts that $d^* = (-1)^k * d^*$ on Ω^k and that $*$ is compatible with the identifications given in (2.3). □

Remark 2.15 One has to be very careful with the ‘equations’ in (2.18). In particular, taking the adjoint of both sides of an equation in (2.18) in general violates $P^{**} = P$. This is because these are not really equalities, but identifications, and recall that unfortunately the identifications in (2.2) are not isometries, as explained in Remark 2.2. However, this will

not cause us any problems, because the notation D_m^l will always only refer to the maps introduced in Definition 2.10, and we will never have need to consider the adjoints of any other components of d . □

We can now describe the Hodge Laplacian $\Delta = dd^* + d^*d$ on each summand Ω_l^k in terms of the operators of Definition 2.10.

Proposition 2.16 *On Ω_l^k , the Hodge Laplacian Δ can be written as follows:*

$$\begin{aligned}
 \Delta|_{\Omega_1^k} &= -\frac{7}{3}D_1^7D_1^7 && \text{for } k = 0, 3, 4, 7, \\
 \Delta|_{\Omega_7^k} &= 9D_7^7D_7^7 - \frac{7}{3}D_7^1D_7^1 && \text{for } k = 1, 2, 3, 4, 5, 6, \\
 \Delta|_{\Omega_{14}^k} &= 5D_{14}^7D_{14}^{14} - D_{14}^{27}D_{14}^{14} && \text{for } k = 2, 5, \\
 \Delta|_{\Omega_{27}^k} &= -\frac{7}{3}D_{27}^7D_{27}^{27} - D_{27}^{14}D_{27}^{14} + (D_{27}^{27})^2 && \text{for } k = 3, 4.
 \end{aligned}
 \tag{2.19}$$

Proof Recall that $d^* = (-1)^k * d*$ on Ω^k and that $*$ is compatible with the identifications given in (2.3). The expressions in (2.19) can be checked on a case-by-case basis using these facts, Fig. 1, and the relations in Corollary 2.13. Note that one can show from general principles that Δ_d preserves splittings (2.2) when φ is parallel, which we always assume. (See [11] for details.) However, the proof of the present proposition gives an explicit verification of this fact, viewing it as a consequence of fundamental relations (2.17). □

Remark 2.17 We emphasize that for Proposition 2.12, Corollary 2.13, and Proposition 2.16, the torsion-free assumption is essential, as the proofs frequently made use of $\nabla\varphi = \nabla\psi = d\varphi = d\psi = 0$. For G_2 -structures with torsion, there would be many additional terms involving torsion, and in particular the Laplacian Δ would *not* preserve splittings (2.2). See also Remark 1.1. □

Remark 2.18 As mentioned in Introduction, the results of Proposition 2.12, Corollary 2.13, and Proposition 2.16 have appeared before in [2, Section 5.2, Tables 1–3], where Bryant says the results follow by routine computation. We have presented all the details for completeness and for readers to be able to use the computational techniques for possible future applications. Note that one has to be careful to compare our results with [2]. First, we use a different orientation convention, which effectively replaces $*$ by $-*$ and ψ by $-\psi$, although Bryant denotes the 3-form by σ . Secondly, we use slightly different *identifications* between the spaces Ω_l^k for different values of k . Finally, Bryant defines the ‘fundamental’ operators differently. For example, Bryant’s d_7^7 is our $3D_7^7$, and Bryant’s $-\frac{3}{7}d_7^1$ is our D_7^1 . We did notice at least one typographical error in [2]. The equation $d(\alpha \wedge *_\sigma\sigma) = -*_\sigma d_7^7\alpha$ in Table 1 is inconsistent with the definition $d_7^7\alpha = *_\sigma(d(\alpha \wedge *_\sigma\sigma))$ on the previous page, since $(*_\sigma)^2 = +1$, not -1 . □

From now on, we assume M is *compact*, as we will be using Hodge theory throughout. Moreover, we can integrate by parts, so if P is a linear operator on forms, then $P\alpha = 0 \iff P^*P\alpha = 0$, which we will use often. The next result relates the kernel of the operators in Definition 2.10 with harmonic 1-forms. This result is *fundamental* and is used often in the rest of the paper.

Theorem 2.19 We have $\ker D_7^7 = \ker D_{14}^7$. Furthermore, let $\mathcal{H}^1 = \ker \Delta|_{\Omega^1}$ denote the space of harmonic 1-forms. Then, we have

$$\begin{aligned} \mathcal{H}^1 &= \ker D_1^7 \cap \ker D_7^7 \cap \ker D_{14}^7 \\ &= \ker D_1^7 \cap \ker D_7^7 \\ &= \ker D_1^7 \cap \ker D_{27}^7 \\ &= \ker D_7^7 \cap \ker D_{27}^7. \end{aligned} \tag{2.20}$$

In particular, the intersection of any two of the three spaces $\ker D_1^7$, $\ker D_7^7$, $\ker D_{27}^7$ is \mathcal{H}^1 .

Proof From Corollary 2.14, on Ω_7^1 we have that $d^* = (D_7^1)^* : \Omega_7^1 \rightarrow \Omega_1^0$ equals $-\frac{7}{3}D_1^7$, and thus

$$\begin{aligned} \mathcal{H}^1 &= (\ker d)^1 \cap (\ker d^*)^1 = \ker(D_7^7 + D_{14}^7) \cap \ker(-\frac{7}{3}D_1^7) \\ &= \ker D_7^7 \cap \ker D_{14}^7 \cap \ker D_1^7, \end{aligned}$$

establishing the first equality in (2.20).

Similarly, from Corollary 2.14, we have $(D_7^7)^* = 3D_7^7$ and $(D_{14}^7)^* = 4D_7^{14}$. Hence, using $D_7^{14}D_{14}^7 = \frac{3}{2}D_7^7D_7^7$ from (2.17), we have

$$\begin{aligned} D_7^7\alpha = 0 &\iff (D_7^7)^*D_7^7\alpha = 3D_7^7D_7^7\alpha = 0 \\ &\iff D_7^{14}D_{14}^7\alpha = \frac{1}{4}(D_{14}^7)^*D_{14}^7\alpha = 0 \\ &\iff D_{14}^7\alpha = 0. \end{aligned}$$

Thus, we deduce that $\ker D_7^7 = \ker D_{14}^7$ as claimed, and hence, the second equality in (2.20) follows.

Finally, from Corollary 2.14 we have $(D_1^7)^* = -D_7^1$ and $(D_{27}^7)^* = -\frac{4}{3}D_7^{27}$ and $(D_7^7)^* = 3D_7^7$. Thus, the relation $-D_7^1D_1^7 + \frac{9}{4}D_7^7D_7^7 + D_7^{27}D_{27}^7 = 0$ from (2.17) can be written as

$$(D_1^7)^*D_1^7 + \frac{3}{4}(D_7^7)^*D_7^7 - \frac{3}{4}(D_{27}^7)^*D_{27}^7 = 0.$$

From the above relation, we easily deduce again by integration by parts that any two of $D_1^7\alpha = 0$, $D_7^7\alpha = 0$, $D_{27}^7\alpha = 0$ implies the third, establishing the remaining equalities in (2.20). \square

2.3 The derivations \mathcal{L}_B and \mathcal{L}_K and their properties

We begin with a brief discussion of derivations on Ω^\bullet arising from vector-valued forms on a general n -manifold M . A good reference for this material is [18]. We use notation similar to [4,6].

Let $\Omega_{TM}^r = \Gamma(\Lambda^r(T^*M) \otimes TM)$ be the space of vector-valued r -forms on M . Given an element $K \in \Omega_{TM}^r$, it induces two derivations on Ω^\bullet . They are the algebraic derivation ι_K , of degree $r - 1$, and the Nijenhuis–Lie derivation \mathcal{L}_K , of degree r . They are defined as follows. Let $\{e_1, \dots, e_n\}$ be a (local) tangent frame with dual coframe $\{e^1, \dots, e^n\}$. Then locally $K = K^j e_j$ where each K^j is an r -form. The operation $\iota_K : \Omega^k \rightarrow \Omega^{k+r-1}$ is defined to be

$$\iota_K\alpha = K^j \wedge (e_j \lrcorner \alpha), \tag{2.21}$$

where $e_j \lrcorner \cdot$ is the interior product with e_j . The operation ι_K is well defined and is a derivation on Ω^\bullet . Moreover, ι_K vanishes on functions, so $\iota_K(h\alpha) = h(\iota_K\alpha)$ for any $h \in \Omega^0$ and $\alpha \in \Omega^k$, which justifies why ι_K is called algebraic. If $Y \in \Omega^1$, then

$$(\iota_K Y)(X_1, \dots, X_r) = Y(K(X_1, \dots, X_r)). \tag{2.22}$$

The operation $\mathcal{L}_K : \Omega^k \rightarrow \Omega^{k+r}$ is defined to be

$$\mathcal{L}_K \alpha = \iota_K(d\alpha) - (-1)^{r-1} d(\iota_K \alpha) = [\iota_K, d]\alpha. \tag{2.23}$$

That is, \mathcal{L}_K is the graded commutator of ι_K and d . The graded Jacobi identity on the space of graded linear operators on Ω^\bullet and $d^2 = 0$ together implies that

$$[d, \mathcal{L}_K] = d\mathcal{L}_K - (-1)^r \mathcal{L}_K d = 0. \tag{2.24}$$

From now on, let g be a Riemannian metric on M .

Lemma 2.20 *Let $K \in \Omega^r_{TM}$ be obtained from an $(r + 1)$ -form η by raising the last index. That is, $g(K(X_1, \dots, X_r), X_{r+1}) = \eta(X_1, \dots, X_{r+1})$. In a local frame, we have $K^q_{i_1 \dots i_r} = \eta_{i_1 \dots i_r p} g^{pq}$. The operator ι_K is of degree $r - 1$. For any $\alpha \in \Omega^k$, the $(k + r - 1)$ -form $\iota_K \alpha$ is given by*

$$\iota_K \alpha = (-1)^r g^{pq} (e_p \lrcorner \eta) \wedge (e_q \lrcorner \alpha). \tag{2.25}$$

Proof In a local frame, we have $K = \frac{1}{k!} K^q_{i_1 \dots i_r} e^{i_1} \wedge \dots \wedge e^{i_r} \otimes e_q$, and thus, from (2.21), we have

$$\begin{aligned} \iota_K \alpha &= \frac{1}{k!} K^q_{i_1 \dots i_r} e^{i_1} \wedge \dots \wedge e^{i_r} \wedge (e_q \lrcorner \alpha) \\ &= \frac{1}{k!} \eta_{i_1 \dots i_r p} g^{pq} e^{i_1} \wedge \dots \wedge e^{i_r} \wedge (e_q \lrcorner \alpha) \\ &= (-1)^r g^{pq} \left(\frac{1}{k!} \eta_{p i_1 \dots i_r} e^{i_1} \wedge \dots \wedge e^{i_r} \right) \wedge (e_q \lrcorner \alpha) \\ &= (-1)^r g^{pq} (e_p \lrcorner \eta) \wedge (e_q \lrcorner \alpha) \end{aligned}$$

as claimed. □

Corollary 2.21 *Let K be as in Lemma 2.20. If $\alpha \in \Omega^{n-(r-1)}$, then $\iota_K \alpha = 0$ in Ω^n .*

Proof Let $\alpha \in \Omega^{n-(r-1)}$. Since $e_p \lrcorner \eta \in \Omega^r$, the form $(e_p \lrcorner \eta) \wedge \alpha$ is of degree $(n + 1)$ and hence zero. Taking the interior product with e_q , we have

$$0 = e_q \lrcorner ((e_p \lrcorner \eta) \wedge \alpha) = (e_q \lrcorner e_p \lrcorner \eta) \wedge \alpha + (-1)^r (e_p \lrcorner \eta) \wedge (e_q \lrcorner \alpha).$$

Thus, by the skew symmetry of $e_q \lrcorner e_p \lrcorner \eta$ in p, q , we find from (2.25) that

$$\iota_K \alpha = (-1)^r g^{pq} (e_p \lrcorner \eta) \wedge (e_q \lrcorner \alpha) = -g^{pq} (e_q \lrcorner e_p \lrcorner \eta) \wedge \alpha = 0$$

as claimed. □

Corollary 2.22 *Let K be as in Lemma 2.20. Then the adjoint ι_K^* is a degree $-(r - 1)$ operator on Ω^\bullet and satisfies*

$$\iota_K^* \beta = (-1)^{nk+r+k+nr+n+1} * \iota_K * \beta \quad \text{for } \beta \in \Omega^k. \tag{2.26}$$

Proof Let $\alpha \in \Omega^{k-(r-1)}$ and $\beta \in \Omega^k$. Then $\alpha \wedge * \beta \in \Omega^{n-(r-1)}$, so by Lemma 2.20 we have $\iota_K(\alpha \wedge * \beta) = 0$. Since ι_K is a derivation of degree $r - 1$, and $\iota_K * \beta$ is an $(n - k + r - 1)$ -form, this can be written as

$$\begin{aligned} 0 &= (\iota_K \alpha) \wedge * \beta + (-1)^{(r-1)(k-(r-1))} \alpha \wedge (\iota_K * \beta) \\ &= g(\iota_K \alpha, \beta) \text{vol} + (-1)^{rk+k+r+1} \alpha \wedge (-1)^{(n-k+r-1)(k-r+1)} * (* \iota_K * \beta) \\ &= g(\iota_K \alpha, \beta) \text{vol} + (-1)^{rk+k+r+1} (-1)^{k+r+1+nk+nr+n} g(\alpha, * \iota_K * \beta) \text{vol} \\ &= g(\iota_K \alpha, \beta) \text{vol} + (-1)^{nk+rk+nr+n} g(\alpha, * \iota_K * \beta) \text{vol}, \end{aligned}$$

and hence $\iota_K^* \beta = (-1)^{nk+rk+nr+n+1} * \iota_K * \beta$ as claimed. □

Now let (M, φ) be a manifold with G_2 -structure. In particular, $n = 7$ from now on.

Definition 2.23 From the G_2 -structure φ on M , we obtain two particular vector-valued forms $B \in \Omega^2_{TM}$ and $K \in \Omega^3_{TM}$ by raising the last index on the forms φ and ψ , respectively. That is,

$$g(B(X, Y), Z) = \varphi(X, Y, Z), \quad g(K(X, Y, Z), W) = \psi(X, Y, Z, W).$$

In local coordinates, we have

$$B^q_{ij} = \varphi_{ijp} g^{pq}, \quad K^q_{ijk} = \psi_{ijkp} g^{pq}.$$

The vector-valued 2-form B is also called the *cross product* induced by φ , and, up to a factor of $-\frac{1}{2}$, the vector-valued 3-form K is called the *associator*. (See [9, p.116] for details.) Thus, ι_B and ι_K are algebraic derivations on Ω^\bullet of degrees 1 and 2, respectively. We also have the associated Nijenhuis–Lie derivations \mathcal{L}_B and \mathcal{L}_K . From (2.23), we have

$$\mathcal{L}_B = \iota_B d + d \iota_B, \quad \mathcal{L}_K = \iota_K d - d \iota_K. \tag{2.27}$$

The operators \mathcal{L}_B and \mathcal{L}_K are of degree 2 and 3, respectively. □

Remark 2.24 In much of the literature, the associator K is denoted by χ , but we are following the convention of [4,6] of denoting vector-valued forms by capital Roman letters. □

Proposition 2.25 Let $\iota_B, \iota_K, \mathcal{L}_B$, and \mathcal{L}_K be as in Definition 2.23. Then on Ω^k , we have

$$\begin{aligned} \iota_B^* &= (-1)^k * \iota_B *, & \iota_K^* &= - * \iota_K *, \\ \mathcal{L}_B^* &= - * \mathcal{L}_B *, & \mathcal{L}_K^* &= (-1)^k * \mathcal{L}_K *. \end{aligned} \tag{2.28}$$

Proof The first pairs of equations follow from (2.26) with $n = 7$ and $r = 2, 3$, respectively. In odd dimensions, $d^* = (-1)^k * d *$ on k -forms, and $*^2 = 1$. The second pair of equations follows from these facts and taking adjoints of (2.27). □

The operations ι_B and ι_K are morphisms of G_2 -representations, and in fact, they are constants on $\Omega^l_{\mathcal{V}}$ after our identifications (2.3). We will prove this in Propositions 2.30 and 2.31, but first we need to collect several preliminary results.

Lemma 2.26 Let $f \in \Omega^0$ and $X \in \Omega^1$. The following identities hold:

$$\begin{aligned} \iota_B f &= 0, & \iota_K f &= 0, \\ \iota_B X &= X \lrcorner \varphi, & \iota_K X &= -X \lrcorner \psi. \end{aligned} \tag{2.29}$$

Proof The first pair of equations is immediate since any algebraic derivation vanishes on functions. Letting $\alpha = X$ in (2.25) gives $\iota_K X = (-1)^r g^{pq} (e_p \lrcorner \eta) \wedge X_q = (-1)^r X^p e_p \lrcorner \eta = (-1)^r X \lrcorner \eta$. The second pair of equations now follows using $r = 2$ for $\eta = \varphi$ and $r = 3$ for $\eta = \psi$. \square

Lemma 2.27 *The following identities hold:*

$$\begin{aligned} \iota_B \varphi &= -6\psi, & \iota_K \varphi &= 0, \\ \iota_B \psi &= 0, & \iota_K \psi &= 0. \end{aligned} \tag{2.30}$$

Proof To establish each of these, we use (2.25) and Proposition 2.6 with $h = g$. First, using (2.8) and $\text{Tr}_g g = 7$, we have

$$\iota_B \varphi = g^{pq} (e_p \lrcorner \varphi) \wedge (e_q \lrcorner \varphi) = -2(\text{Tr}_g g)\psi + 2\ell_\psi g = -14\psi + 8\psi = -6\psi.$$

Similarly, from Proposition 2.6, we find that

$$\iota_B \psi = g^{pq} (e_p \lrcorner \varphi) \wedge (e_q \lrcorner \psi) = 0,$$

and hence also $\iota_K \varphi = -g^{pq} (e_p \lrcorner \psi) \wedge (e_q \lrcorner \varphi) = -\iota_B \psi = 0$. Finally, again from Proposition 2.6 we deduce that

$$\iota_K \psi = -g^{pq} (e_p \lrcorner \psi) \wedge (e_q \lrcorner \psi) = 0$$

as well. \square

Lemma 2.28 *Let $X \in \Omega^1$. The following identities hold:*

$$\begin{aligned} \iota_B (X \lrcorner \varphi) &= 3(X \lrcorner \psi), & \iota_K (X \lrcorner \varphi) &= 3 * (X \lrcorner \psi), \\ \iota_B (X \lrcorner \psi) &= -3 * (X \lrcorner \psi), & \iota_K (X \lrcorner \psi) &= -4 * (X \lrcorner \varphi). \end{aligned} \tag{2.31}$$

Proof Let $X = X^m e_m$. By linearity of derivations and (2.25), we have

$$\begin{aligned} \iota_B (X \lrcorner \beta) &= X^m g^{pq} (e_p \lrcorner \varphi) \wedge (e_q \lrcorner e_m \lrcorner \beta), \\ \iota_K (X \lrcorner \beta) &= -X^m g^{pq} (e_p \lrcorner \psi) \wedge (e_q \lrcorner e_m \lrcorner \beta). \end{aligned}$$

The equations in (2.31) now follow immediately from Proposition 2.7. \square

Lemma 2.29 *Let $\beta \in \Omega^2_{14}$. The following identities hold:*

$$\iota_B \beta = 0, \quad \iota_K \beta = 0. \tag{2.32}$$

Proof We use the notation of Proposition 2.7. Let $\beta \in \Omega^2_{14}$. Using (2.25) and (2.9), we compute

$$\begin{aligned} \iota_B \beta &= g^{pq} (e_p \lrcorner \varphi) \wedge (e_q \lrcorner \beta) \\ &= \frac{1}{2} g^{pq} \varphi_{pij} \beta_{qk} e^{ijk} \\ &= -\frac{1}{6} (\beta_{kq} g^{qp} \varphi_{pij} + \beta_{iq} g^{qp} \varphi_{pj k} + \beta_{jq} g^{qp} \varphi_{pki}) e^{ijk} = 0. \end{aligned}$$

Similarly, again using (2.25) and (2.9), we compute

$$\begin{aligned} \iota_K \beta &= -g^{pq} (e_p \lrcorner \psi) \wedge (e_q \lrcorner \beta) \\ &= -\frac{1}{6} g^{pq} \psi_{pijk} \beta_{ql} e^{ijkl} \\ &= +\frac{1}{24} (\beta_{lq} g^{qp} \psi_{pijk} - \beta_{iq} g^{qp} \psi_{pljk} - \beta_{jq} g^{qp} \psi_{pilk} - \beta_{kq} g^{qp} \psi_{pijl}) e^{ijkl} = 0 \end{aligned}$$

as claimed. \square

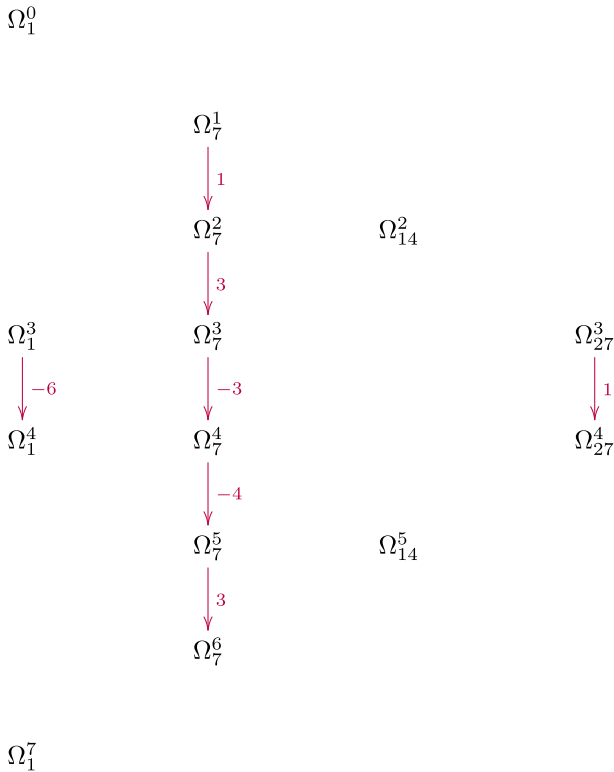


Fig. 2 Decomposition of the algebraic derivation ι_B into components

We are now ready to establish the actions of ι_B and ι_K on the summands of Ω^\bullet with respect to identifications (2.3).

Proposition 2.30 *With respect to the identifications described in (2.3), the components of the operator ι_B satisfy the relations given in Fig. 2.*

Proof The derivation ι_B is of degree 1, so it vanishes on Ω^7 . Moreover, by Corollary 2.21 it also vanishes on Ω^6 . We establish the rest of Fig. 2 by each vertical column.

Ω_1^k column: This follows from (2.29) and (2.30). In particular, the map $\iota_B : \Omega_1^3 \rightarrow \Omega_1^4$ is identified with multiplication by -6 .

Ω_7^k column: The map $\iota_B : \Omega_7^1 \rightarrow \Omega_7^2$ is identified with multiplication by 1 by (2.29). The maps $\iota_B : \Omega_7^2 \rightarrow \Omega_7^3$ and $\iota_B : \Omega_7^3 \rightarrow \Omega_7^4$ are identified with multiplication by 3 and -3 , respectively, by (2.31). Let $*(X \lrcorner \psi) = \varphi \wedge X \in \Omega_7^4$. Then

$$\begin{aligned} \iota_B(* (X \lrcorner \psi)) &= \iota_B(\varphi \wedge X) = (\iota_B \varphi) \wedge X - \varphi \wedge (\iota_B X) \\ &= (-6\psi) \wedge X - \varphi \wedge (X \lrcorner \varphi) = -6 * (X \lrcorner \varphi) + 2 * (X \lrcorner \varphi) \\ &= -4 * (X \lrcorner \varphi), \end{aligned}$$

and hence, the map $\iota_B : \Omega_7^4 \rightarrow \Omega_7^5$ is identified with multiplication by -4 . Finally, let $*(X \lrcorner \varphi) = \psi \wedge X \in \Omega_7^5$. Then

$$\iota_B(* (X \lrcorner \varphi)) = \iota_B(\psi \wedge X) = (\iota_B \psi) \wedge X + \psi \wedge (\iota_B X)$$

$$= 0 + \psi \wedge (X \lrcorner \varphi) = 3 * X,$$

and hence, the map $\iota_B : \Omega_7^5 \rightarrow \Omega_7^6$ is identified with multiplication by 3.

Ω_{14}^k column: The map ι_B on Ω_{14}^2 is zero by Lemma 2.29. Let $\mu = *\beta \in \Omega_{14}^5$ where $\beta \in \Omega_{14}^2$. Then $\mu = *\beta = \varphi \wedge \beta$, so $\iota_B \mu = (\iota_B \varphi) \wedge \beta - \varphi \wedge (\iota_B \beta) = -6\psi \wedge \beta - 0 = 0$, by the description of Ω_{14}^2 in (2.3).

Ω_{27}^k column: Let $\gamma = \ell_\varphi h \in \Omega_{27}^3$, where $h \in S_0^2(T^*M)$. By (2.6) we have $\gamma = h_{kl} g^{lm} e^k \wedge (e_m \lrcorner \varphi)$. Since ι_B is algebraic, we can pull out functions, and using (2.29) and (2.31) we compute

$$\begin{aligned} \iota_B \gamma &= \iota_B (h_{kl} g^{lm} e^k \wedge (e_m \lrcorner \varphi)) \\ &= h_{kl} g^{lm} ((\iota_B e^k) \wedge (e_m \lrcorner \varphi) - e^k \wedge \iota_B (e_m \lrcorner \varphi)) \\ &= h_{kl} g^{lm} (g^{kp} (e_p \lrcorner \varphi) \wedge (e_m \lrcorner \varphi) - e^k \wedge (3e_m \lrcorner \psi)) \\ &= h^{pm} (e_p \lrcorner \varphi) \wedge (e_m \lrcorner \varphi) - 3h_{kl} g^{lm} e^k \wedge (e_m \lrcorner \psi). \end{aligned}$$

By (2.10) and (2.7), since $\text{Tr}_g h = 0$, the above expression is

$$\iota_B \gamma = 2\ell_\psi h - 3\ell_\psi h = -\ell_\psi h.$$

Using Lemma 2.5, we conclude that $\iota_B(\ell_\varphi h) = *(\ell_\varphi h)$, and thus, the map $\iota_B : \Omega_{27}^3 \rightarrow \Omega_{27}^4$ is identified with multiplication by 1. Finally, let $\eta = \ell_\psi h \in \Omega_{27}^4$, where $h \in S_0^2(T^*M)$. By (2.7), we have $\eta = h_{kl} g^{lm} e^k \wedge (e_m \lrcorner \psi)$. Computing as before, we find

$$\begin{aligned} \iota_B \eta &= \iota_B (h_{kl} g^{lm} e^k \wedge (e_m \lrcorner \psi)) \\ &= h_{kl} g^{lm} ((\iota_B e^k) \wedge (e_m \lrcorner \psi) - e^k \wedge \iota_B (e_m \lrcorner \psi)) \\ &= h_{kl} g^{lm} (g^{kp} (e_p \lrcorner \varphi) \wedge (e_m \lrcorner \psi) - e^k \wedge (-3 * (e_m \lrcorner \psi))) \\ &= h^{pm} (e_p \lrcorner \varphi) \wedge (e_m \lrcorner \psi) + 3h_{kl} g^{lm} e^k \wedge (\varphi \wedge (e_m)^b). \end{aligned}$$

Using (2.10), the above expression becomes

$$\iota_B \eta = 0 + 3h_{kl} g^{lm} e^k \wedge \varphi \wedge (g_{mp} e^p) = -3h_{kp} e^k \wedge e^p \wedge \varphi = 0,$$

so the map ι_B on Ω_{27}^4 is zero. □

Proposition 2.31 *With respect to the identifications described in (2.3), the components of the operator ι_K satisfy the relations given in Fig. 3.*

Proof The derivation ι_K is of degree 2, so it vanishes on Ω^6 and Ω^7 . Moreover, by Corollary 2.21 it also vanishes on Ω^5 . We establish the rest of Fig. 2 by each vertical column. Note that ι_K preserves the parity (even/odd) of forms.

Ω_1^k column: This follows from (2.29) and (2.30).

Ω_7^k column: The map $\iota_K : \Omega_7^1 \rightarrow \Omega_7^3$ is identified with multiplication by -1 by (2.29). The maps $\iota_K : \Omega_7^2 \rightarrow \Omega_7^4$ and $\iota_K : \Omega_7^3 \rightarrow \Omega_7^5$ are identified with multiplication by 3 and -4 , respectively, by (2.31). Let $*(X \lrcorner \psi) = \varphi \wedge X \in \Omega_7^4$. Then, since ι_K is an even derivation,

$$\begin{aligned} \iota_K (* (X \lrcorner \psi)) &= \iota_K(\varphi \wedge X) = (\iota_K \varphi) \wedge X + \varphi \wedge (\iota_K X) \\ &= 0 + \varphi \wedge (-X \lrcorner \psi) = -\varphi \wedge (X \lrcorner \psi) = 4 * X \end{aligned}$$

and hence, the map $\iota_K : \Omega_7^4 \rightarrow \Omega_7^6$ is identified with multiplication by 4.

Ω_{14}^k column: The map ι_K on Ω_{14}^2 is zero by Lemma 2.29.

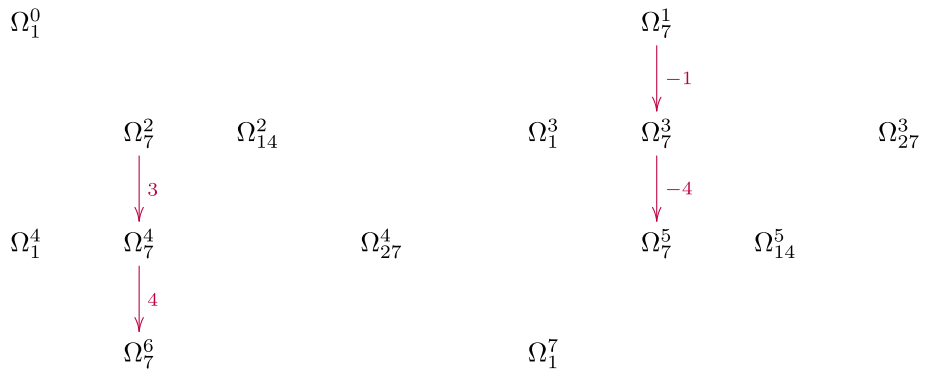


Fig. 3 Decomposition of the algebraic derivation ι_K into components

Ω_{27}^k column: Let $\gamma = \ell_\varphi h \in \Omega_{27}^3$, where $h \in S_0^2(T^*M)$. By (2.6), we have $\gamma = h_{kl}g^{lm}e^k \wedge (e_m \lrcorner \varphi)$. Computing as in the proof of Proposition 2.31, we find that

$$\begin{aligned} \iota_K \gamma &= \iota_K(h_{kl}g^{lm}e^k \wedge (e_m \lrcorner \varphi)) \\ &= h_{kl}g^{lm}((\iota_K e^k) \wedge (e_m \lrcorner \varphi) - e^k \wedge \iota_K(e_m \lrcorner \varphi)) \\ &= h_{kl}g^{lm}(-g^{kp}(e_p \lrcorner \psi) \wedge (e_m \lrcorner \varphi) - e^k \wedge (3 * (e_m \lrcorner \psi))) \\ &= -h^{pm}(e_p \lrcorner \psi) \wedge (e_m \lrcorner \varphi) - 3h_{kl}g^{lm}e^k \wedge \varphi \wedge (e_m)^\flat. \end{aligned}$$

The first term vanishes by (2.10) and the second term vanishes as it is $-3h_{kl}g^{lm}g_{mp}e^k \wedge \varphi \wedge e^p = 3h_{kp}e^k \wedge e^p \wedge \varphi = 0$. Thus, the map ι_K vanishes on Ω_{27}^3 . Finally, let $\eta = \ell_\psi h \in \Omega_{27}^4$, where $h \in S_0^2(T^*M)$. By (2.7), we have $\eta = h_{kl}g^{lm}e^k \wedge (e_m \lrcorner \psi)$. Computing as before, we find

$$\begin{aligned} \iota_K \eta &= \iota_K(h_{kl}g^{lm}e^k \wedge (e_m \lrcorner \psi)) \\ &= h_{kl}g^{lm}((\iota_K e^k) \wedge (e_m \lrcorner \psi) - e^k \wedge \iota_K(e_m \lrcorner \psi)) \\ &= h_{kl}g^{lm}(-g^{kp}(e_p \lrcorner \psi) \wedge (e_m \lrcorner \psi) - e^k \wedge (-4 * (e_m \lrcorner \varphi))) \\ &= -h^{pm}(e_p \lrcorner \psi) \wedge (e_m \lrcorner \psi) + 4h_{kl}g^{lm}e^k \wedge (\psi \wedge (e_m)^\flat). \end{aligned}$$

Again, the first term vanishes by (2.10) and the second term vanishes as it is $4h_{kl}g^{lm}g_{mp}e^k \wedge \psi \wedge e^p = 4h_{kp}e^k \wedge e^p \wedge \psi = 0$. Thus, the map ι_K vanishes on Ω_{27}^4 . □

From now on in the paper, we always assume that (M, φ) is torsion-free. See also Remark 1.1.

Corollary 2.32 *With respect to the identifications described in (2.3), the components of the operators \mathcal{L}_B and \mathcal{L}_K satisfy the relations given in Figs. 4 and 5.*

Proof This is straightforward to verify from Figs. 1, 2, and 3 using the equations in (2.27). □

Next we discuss some properties of \mathcal{L}_B and \mathcal{L}_K .

Lemma 2.33 *Let α be a form. In a local frame, the actions of \mathcal{L}_B and \mathcal{L}_K are given by*

$$\begin{aligned} \mathcal{L}_B \alpha &= g^{pq}(e_p \lrcorner \varphi) \wedge (\nabla_q \alpha), \\ \mathcal{L}_K \alpha &= -g^{pq}(e_p \lrcorner \psi) \wedge (\nabla_q \alpha). \end{aligned} \tag{2.33}$$

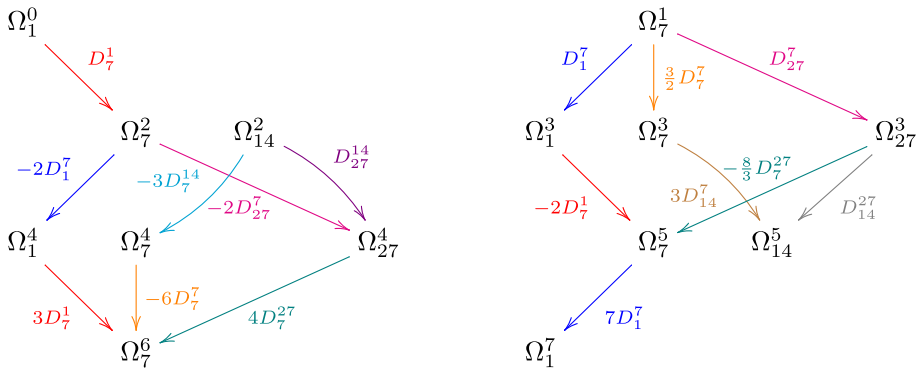


Fig. 4 Decomposition of the Nijenhuis–Lie derivation \mathcal{L}_B into components

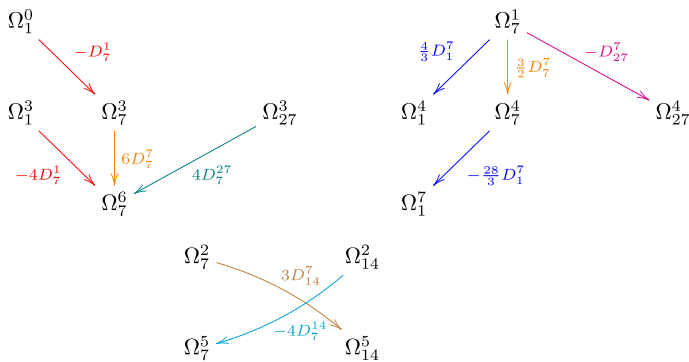


Fig. 5 Decomposition of the Nijenhuis–Lie derivation \mathcal{L}_K into components

Proof It is clear that both expressions in (2.33) are independent of the choice of frame. To establish these expressions at $x \in M$, we choose a local frame determined by Riemannian normal coordinates centered at x . In particular, at the point x we have $\nabla_p e_j =$ and $\nabla_p e^j = 0$. Recalling that M is torsion-free, so $\nabla\varphi = 0$, using (2.27), (2.25), and (1.1) at the point x we compute

$$\begin{aligned} \mathcal{L}_B\alpha &= (\iota_B d + d\iota_B)\alpha \\ &= \iota_B(e^m \wedge \nabla_m \alpha) + e^m \wedge \nabla_m(\iota_B \alpha) \\ &= g^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner (e^m \wedge \nabla_m \alpha)) + e^m \wedge \nabla_m(g^{pq}(e_p \lrcorner \varphi) \wedge (e_q \lrcorner \alpha)) \\ &= g^{pq}(e_p \lrcorner \varphi) \wedge (\delta_q^m \nabla_m \alpha - e^m \wedge (e_q \lrcorner \nabla_m \alpha)) + g^{pq} e^m \wedge (e_p \lrcorner \varphi) \wedge (e_q \lrcorner \nabla_m \alpha) \\ &= g^{pq}(e_p \lrcorner \varphi) \wedge \nabla_q \alpha, \end{aligned}$$

establishing the first equation in (2.33). The other equation is proved similarly using $\nabla\psi = 0$. \square

Corollary 2.34 For any for α , we have

$$\begin{aligned} \mathcal{L}_B\alpha &= -d^*(\varphi \wedge \alpha) - \varphi \wedge d^*\alpha, \\ \mathcal{L}_K\alpha &= d^*(\psi \wedge \alpha) - \psi \wedge d^*\alpha. \end{aligned} \tag{2.34}$$

Proof Consider a local frame determined by Riemannian normal coordinates centered at $x \in M$ as in the proof of Lemma 2.33. Using (2.33) and (1.2), we compute

$$\begin{aligned} \mathcal{L}_B \alpha &= g^{pq} (e_p \lrcorner \varphi) \wedge (\nabla_q \alpha) \\ &= g^{pq} (e_p \lrcorner (\varphi \wedge \nabla_q \alpha) + \varphi \wedge (e_p \lrcorner \nabla_q \alpha)) \\ &= g^{pq} e_p \lrcorner \nabla_q (\varphi \wedge \alpha) + \varphi \wedge (g^{pq} e_p \lrcorner \nabla_q \alpha) \\ &= -d^*(\varphi \wedge \alpha) - \varphi \wedge (d^* \alpha), \end{aligned}$$

establishing the first equation in (2.34). The other equation is proved similarly. □

Proposition 2.35 *The derivations \mathcal{L}_B and \mathcal{L}_K satisfy the following identities:*

$$\mathcal{L}_B d^* = d^* \mathcal{L}_B, \quad \mathcal{L}_K d^* = -d^* \mathcal{L}_K, \quad (2.35)$$

$$\mathcal{L}_B \Delta = \Delta \mathcal{L}_B, \quad \mathcal{L}_K \Delta = \Delta \mathcal{L}_K, \quad (2.36)$$

$$\mathcal{L}_B \mathcal{L}_K = \mathcal{L}_K \mathcal{L}_B = 0, \quad (\mathcal{L}_K)^2 = 0, \quad (2.37)$$

and

$$\mathcal{L}_B = \mathcal{L}_K = 0 \text{ on } \mathcal{H}^k \text{ if } M \text{ is compact.} \quad (2.38)$$

Proof The identities in (2.35)–(2.37) can be verified directly from the Figs. 1, 4, and 5 using $d^* = (-1)^k * d * \text{on } \Omega^k$ and $\Delta = dd^* + d^*d$, the identities in Corollary 2.13, and recalling that our identifications were chosen compatible with $*$.

However, we now give an alternative proof of the first equation in (2.35) that is less tedious and more illuminating. A similar proof establishes the second equation in (2.35). (In fact, this proof can be found in [17]). Using (2.34) and $(d^*)^2 = 0$, we compute

$$\begin{aligned} \mathcal{L}_B d^* \alpha &= -d^*(\varphi \wedge d^* \alpha) - \varphi \wedge (d^*(d^* \alpha)) \\ &= d^*(-\varphi \wedge (d^* \alpha) - d^*(\varphi \wedge \alpha)) \\ &= d^* \mathcal{L}_B \alpha. \end{aligned}$$

The equations in (2.36) can also be established from (2.35), (2.24), and $\Delta = dd^* + d^*d$.

Equation (2.38) can be similarly verified using Figs. 1, 4, and 5, noting that in the compact case, the space \mathcal{H}^k of harmonic k -forms coincides with the space of d -closed and d^* -closed k -forms. □

Remark 2.36 For a k -form γ , let L_γ be the linear operator of degree k on Ω^\bullet given by $L_\gamma \alpha = \gamma \wedge \alpha$. In terms of graded commutators, in the torsion-free case Corollary 2.34 says that $[d^*, L_\varphi] = -\mathcal{L}_B$ and $[d^*, L_\psi] = \mathcal{L}_K$, and Proposition 2.35 says that $[d^*, \mathcal{L}_B] = [d^*, \mathcal{L}_K] = 0$, $[\Delta, \mathcal{L}_B] = [\Delta, \mathcal{L}_K] = 0$, and $[\mathcal{L}_B, \mathcal{L}_K] = [\mathcal{L}_K, \mathcal{L}_K] = 0$. (In fact, the first equation in (2.37) is actually stronger than $[\mathcal{L}_B, \mathcal{L}_K] = 0$.) These graded commutators and others are considered more generally for G_2 manifolds with torsion in [14] using the general framework developed in [6] in the case of Um -structures. □

3 The \mathcal{L}_B -cohomology H^\bullet_φ of M and its computation

In this section, we define two cohomologies on a torsion-free G_2 manifold using the derivations \mathcal{L}_B and \mathcal{L}_K . The cohomology determined by \mathcal{L}_K was studied extensively by

Kawai–Lê–Schwachhöfer in [17]. We recall one of the main results of [17] on the \mathcal{L}_K -cohomology, stated here as Theorem 3.2. We then proceed to compute the cohomology determined by \mathcal{L}_B . This section culminates with the proof of Theorem 3.19, which is our analog of Theorem 3.2 for the \mathcal{L}_B -cohomology. An application to formality of compact torsion-free G_2 manifolds is given in Sect. 4.

3.1 Cohomologies determined by \mathcal{L}_B and \mathcal{L}_K

Recall from (2.37) that $(\mathcal{L}_K)^2 = 0$. This observation motivates the following definition.

Definition 3.1 For any $0 \leq k \leq 7$, we define

$$H_\psi^k := \frac{\ker(\mathcal{L}_K : \Omega^k \rightarrow \Omega^{k+3})}{\text{im}(\mathcal{L}_K : \Omega^{k-3} \rightarrow \Omega^k)}.$$

We call these groups the \mathcal{L}_K -cohomology groups. □

The \mathcal{L}_K -cohomology is studied extensively in [17]. Here is one of the main results of [17].

Theorem 3.2 (Kawai–Lê–Schwachhöfer [17]) *The following relations hold.*

- $H_\psi^k \cong H_{\text{dR}}^k$ for $k = 0, 1, 6, 7$.
- H_ψ^k is infinite-dimensional for $k = 2, 3, 4, 5$.
- There is a canonical injection $\mathcal{H}^k \hookrightarrow H_\psi^k$ for all k .
- The Hodge star induces isomorphisms $*$: $H_\psi^k \cong H_\psi^{7-k}$.

Proof This is part of [17, Theorem 1.1]. □

From Fig. 4 and (2.17), we see that in general $(\mathcal{L}_B)^2 \neq 0$. Because of this, we *cannot* directly copy the definition of H_ψ^k to define \mathcal{L}_B -cohomology groups. However, we can make the following definition.

Definition 3.3 For any $0 \leq k \leq 7$, we define

$$H_\varphi^k := \frac{\ker(\mathcal{L}_B : \Omega^k \rightarrow \Omega^{k+2})}{\text{im}(\mathcal{L}_B : \Omega^{k-2} \rightarrow \Omega^k) \cap \ker(\mathcal{L}_B : \Omega^k \rightarrow \Omega^{k+2})}. \tag{3.1}$$

We call these groups the \mathcal{L}_B -cohomology groups. □

In Sects. 3.2 and 3.3, we compute these \mathcal{L}_B -cohomology groups, and then in Sect. 3.4, we prove Theorem 3.19, which is the analog to Theorem 3.2.

3.2 Computation of the groups $H_\varphi^0, H_\varphi^1, H_\varphi^2$, and H_φ^3

From now on, we always assume that (M, φ) is a compact torsion-free G_2 manifold as we use Hodge theory frequently. See also Remark 1.1.

Remark 3.4 In particular, we will often use the following observations. (There is no summation over l, l', l'' in this remark. The symbols $l, l', l'' \in \{1, 7, 14, 27\}$ are not indices.) By Corollary 2.14, we have $D_l^{l'} = c(D_l^{l'})^*$ for some $c \neq 0$. Thus, by integration by parts,

$$\text{whenever } D_l^{l'} D_l^{l'} \omega = 0 \text{ for some } \omega, \text{ then } D_l^{l'} \omega = 0.$$

More generally, by Corollary 2.14 an equation of the form $aD_l^l D_l^l \omega + bD_l^l D_l^l \omega = 0$ can be rewritten as $\tilde{a}(D_l^l)^* D_l^l \omega + \tilde{b}(D_l^l)^* D_l^l \omega = 0$ for some \tilde{a}, \tilde{b} . If \tilde{a}, \tilde{b} have the same sign, then again by integration by parts, we conclude that both $D_l^l \omega = 0$ and $D_l^l \omega = 0$. \square

In the first two propositions, we establish that $H_\varphi^k \cong H_{dR}^k$ for $k = 0, 1, 2$.

Proposition 3.5 *We have $H_\varphi^0 = \mathcal{H}^0$ and $H_\varphi^1 = \mathcal{H}^1$.*

Proof From Figs. 4 and 1, we observe that

$$\begin{aligned} \text{im}(\mathcal{L}_B : \Omega^{-2} \rightarrow \Omega^0) &= 0, \\ \ker(\mathcal{L}_B : \Omega^0 \rightarrow \Omega^2) &= \ker(D_7^1) = \mathcal{H}^0, \end{aligned}$$

and thus that $H_\varphi^0 = \mathcal{H}^0$.

Similarly, using Fig. 4 and Theorem 2.19, we observe that

$$\begin{aligned} \text{im}(\mathcal{L}_B : \Omega^{-1} \rightarrow \Omega^1) &= 0, \\ \ker(\mathcal{L}_B : \Omega^1 \rightarrow \Omega^3) &= \ker(D_1^7) \cap \ker(D_7^7) \cap \ker(D_{27}^7) = \mathcal{H}^1 \end{aligned}$$

and hence $H_\varphi^1 = \mathcal{H}^1$. \square

In the remainder of this section and the next, we will often use the notation introduced in (1.3).

Proposition 3.6 *We have $H_\varphi^2 \cong \mathcal{H}^2$.*

Proof We first show that the denominator in (3.1) is trivial. Let $\omega \in (\ker \mathcal{L}_B)^2 \cap (\text{im } \mathcal{L}_B)^2$. Then by Fig. 4, we have

$$\omega = \mathcal{L}_B f = D_7^1 f \quad \text{for some } f \in \Omega_1^0$$

and also that

$$0 = \mathcal{L}_B \omega = -2D_1^7(D_7^1 f) - 2D_{27}^7(D_7^1 f).$$

Projecting onto the Ω_1^4 component, we find that $D_1^7 D_7^1 f = 0$. By Remark 3.4, we deduce that $\omega = D_7^1 f = 0$. Thus, we have shown that $(\ker \mathcal{L}_B)^2 \cap (\text{im } \mathcal{L}_B)^2 = 0$. Hence, $H_\varphi^2 = (\ker \mathcal{L}_B)^2$.

Write $\omega = \omega_7 + \omega_{14} \in \Omega_7^2 \oplus \Omega_{14}^2$. By Fig. 4 we have

$$\omega \in (\ker \mathcal{L}_B)^2 \iff \left\{ \begin{aligned} -2D_1^7 \omega_7 &= 0, \\ -3D_7^{14} \omega_{14} &= 0, \\ -2D_{27}^7 \omega_7 + D_{27}^{14} \omega_{14} &= 0. \end{aligned} \right. \tag{3.2}$$

Taking D_7^{27} of the third equation in (3.2), using Corollary 2.13 to write $D_7^{27} D_{27}^{14} = \frac{3}{2} D_7^7 D_7^{14}$, and using the second equation in (3.2), we find that

$$\begin{aligned} 0 &= D_7^{27} (-2D_{27}^7 \omega_7 + D_{27}^{14} \omega_{14}) \\ &= -2D_7^{27} D_{27}^7 \omega_7 + \frac{3}{2} D_7^7 D_7^{14} \omega_{14} = -2D_7^{27} D_{27}^7 \omega_7, \end{aligned}$$

implying by Remark 3.4 that $D_{27}^7\omega_7 = 0$. Therefore, we have established that

$$\omega \in (\ker \mathcal{L}_B)^2 \iff \left\{ \begin{array}{l} D_1^7\omega_7 = 0, \\ D_7^{14}\omega_{14} = 0, \\ D_{27}^7\omega_7 = 0, \\ D_{27}^{14}\omega_{14} = 0, \end{array} \right\} \iff \left\{ \begin{array}{l} \omega_7 \in \mathcal{H}_7^2 \cong \mathcal{H}_7^1 \text{ by Theorem 2.19,} \\ \omega_{14} \in \mathcal{H}_{14}^2 \text{ by Fig. 1 and Corollary 2.14.} \end{array} \right\}$$

We conclude that $H_\varphi^2 = (\ker \mathcal{L}_B)^2 = \mathcal{H}^2$. □

Proposition 3.7 *We have $H_\varphi^3 = \mathcal{H}^3 \oplus ((\text{im } d^*)^3 \cap (\ker \mathcal{L}_B)^3)$.*

Proof We first show that the denominator in (3.1) is trivial. Let $\omega \in (\ker \mathcal{L}_B)^3 \cap (\text{im } \mathcal{L}_B)^3$. Then by Fig. 4 we have

$$\omega = \mathcal{L}_B\alpha = D_1^7\alpha + \frac{3}{2}D_7^7\alpha + D_{27}^7\alpha \text{ for some } \alpha \in \Omega_7^1.$$

Also, using Corollary 2.13 to write $D_{14}^{27}D_{27}^7 = \frac{3}{2}D_{14}^7D_7^7$ and $D_7^{27}D_{27}^7 = D_7^1D_1^7 - \frac{9}{4}D_7^7D_7^7$, we have that

$$\begin{aligned} 0 = \mathcal{L}_B\omega &= -2D_7^1(D_1^7\alpha) + 3D_{14}^7\left(\frac{3}{2}D_7^7\alpha\right) + \left(-\frac{8}{3}D_7^{27} + D_{14}^{27}\right)(D_{27}^7\alpha) \\ &= -2D_7^1D_1^7\alpha - \frac{8}{3}D_7^{27}D_{27}^7\alpha + \frac{9}{4}D_{14}^7D_7^7\alpha + D_{14}^{27}D_{27}^7\alpha \\ &= -2D_7^1D_1^7\alpha - \frac{8}{3}\left(D_7^1D_1^7\alpha - \frac{9}{4}D_7^7D_7^7\alpha\right) + \frac{9}{4}D_{14}^7D_7^7\alpha + \frac{3}{2}D_{14}^{27}D_{27}^7\alpha \\ &= \left(-\frac{14}{3}D_7^1D_1^7\alpha + 6D_7^7D_7^7\alpha\right) + \frac{15}{4}D_{14}^7D_7^7\alpha. \end{aligned}$$

Projecting onto the Ω_7^5 component, we find that

$$-\frac{14}{3}D_7^1D_1^7\alpha + 6D_7^7D_7^7\alpha = 0.$$

Using Corollary 2.14, the above expression becomes

$$\frac{14}{3}(D_1^7)^*D_1^7\alpha + 2(D_7^7)^*D_7^7\alpha = 0,$$

and hence, by Remark 3.4 we deduce that $D_1^7\alpha = 0$ and $D_7^7\alpha = 0$. By Theorem 2.19, we then have $D_{27}^7\alpha = 0$ automatically. Therefore, we have shown that $(\ker \mathcal{L}_B)^3 \cap (\text{im } \mathcal{L}_B)^2 = 0$, and so $H_\varphi^3 = (\ker \mathcal{L}_B)^3$.

Write $\omega = \omega_1 + \omega_7 + \omega_{27} \in \Omega_1^3 \oplus \Omega_7^3 \oplus \Omega_{27}^3$. By Fig. 4, we have

$$\omega \in (\ker \mathcal{L}_B)^3 \iff \left\{ \begin{array}{l} -2D_7^1\omega_1 - \frac{8}{3}D_7^{27}\omega_{27} = 0, \\ 3D_{14}^7\omega_7 + D_{14}^{27}\omega_{27} = 0. \end{array} \right\} \tag{3.3}$$

Taking D_7^{14} of the second equation in (3.3), using Corollary 2.13 to write $D_7^{14}D_{14}^{27} = -D_7^7D_7^{27}$ and $D_7^7D_7^1 = 0$, and using $D_7^{27}\omega_{27} = -\frac{3}{4}D_7^1\omega_1$ from the first equation in (3.3), we find that

$$\begin{aligned} 0 &= D_7^{14}(3D_{14}^7\omega_7 + D_{14}^{27}\omega_{27}) \\ &= 3D_7^{14}D_{14}^7\omega_7 - D_7^7D_7^{27}\omega_{27} \\ &= 3D_7^{14}D_{14}^7\omega_7 + \frac{3}{4}D_7^7D_7^1\omega_1 = 3D_7^{14}D_{14}^7\omega_7, \end{aligned}$$

implying by Remark 3.4 that $D_{14}^7 \omega_7 = 0$. Therefore, we have established that

$$\omega \in (\ker \mathcal{L}_B)^3 \iff \left\{ \begin{array}{l} 2D_7^1 \omega_1 + \frac{8}{3} D_7^{27} \omega_{27} = 0, \\ D_{14}^7 \omega_7 = 0, \\ D_{14}^{27} \omega_{27} = 0, \end{array} \right\} \xrightarrow{\text{Theorem 2.19}} \left\{ \begin{array}{l} 2D_7^1 \omega_1 + \frac{8}{3} D_7^{27} \omega_{27} = 0, \\ D_7^7 \omega_7 = 0, \\ D_{14}^{27} \omega_{27} = 0. \end{array} \right\} \tag{3.4}$$

From $d^* = - * d *$ on Ω^3 and Fig. 1 we find that

$$d^* \omega = 0 \iff \left\{ \begin{array}{l} D_7^1 \omega_1 + 2D_7^7 \omega_7 + \frac{4}{3} D_7^{27} \omega_{27} = 0, \\ -D_{14}^7 \omega_7 + D_{14}^{27} \omega_{27} = 0. \end{array} \right\} \tag{3.5}$$

Now Eqs. (3.4) and (3.5) together imply that $(\ker \mathcal{L}_B)^3 \subseteq (\ker d^*)^3$. By the Hodge theorem, we have $(\ker d^*)^3 = \mathcal{H}^3 \oplus (\text{im } d^*)^3$, and by (2.38) we have $\mathcal{H}^3 \subset (\ker \mathcal{L}_B)^3$. Thus,

$$\mathcal{H}^3 \subseteq (\ker \mathcal{L}_B)^3 \subseteq \mathcal{H}^3 \oplus (\text{im } d^*)^3.$$

Applying Lemma 1.2(i), we conclude that $H_\varphi^3 = (\ker \mathcal{L}_B)^3 = \mathcal{H}^3 \oplus ((\text{im } d^*)^3 \cap (\ker \mathcal{L}_B)^3)$. □

We have thus far computed half of the \mathcal{L}_B -cohomology groups H_φ^k , for $k = 0, 1, 2, 3$. The other half, for $k = 4, 5, 6, 7$, will be computed rigorously in Sect. 3.3. However, we can predict the duality result that $H_\varphi^k \cong H_\varphi^{7-k}$ by the following formal manipulation:

$$\begin{aligned} H_\varphi^k &= \frac{(\ker \mathcal{L}_B)^k}{(\text{im } \mathcal{L}_B)^k \cap (\ker \mathcal{L}_B)^k} \cong \frac{(\ker \mathcal{L}_B)^k + (\text{im } \mathcal{L}_B)^k}{(\text{im } \mathcal{L}_B)^k} \text{ by the second isomorphism theorem} \\ &\cong \frac{(\ker \mathcal{L}_B)^{7-k} + (\text{im } \mathcal{L}_B)^{7-k}}{(\text{im } \mathcal{L}_B)^{7-k}} \text{ by applying } * \text{ and using equation (2.28)} \\ &\stackrel{(!)}{=} \frac{((\text{im } \mathcal{L}_B)^{7-k})^\perp + ((\ker \mathcal{L}_B)^{7-k})^\perp}{((\ker \mathcal{L}_B)^{7-k})^\perp} \\ &= \frac{((\text{im } \mathcal{L}_B)^{7-k} \cap (\ker \mathcal{L}_B)^{7-k})^\perp}{((\ker \mathcal{L}_B)^{7-k})^\perp} \text{ by properties of orthogonal complement} \\ &\stackrel{(!!)}{\cong} \frac{(\ker \mathcal{L}_B)^{7-k}}{(\text{im } \mathcal{L}_B)^{7-k} \cap (\ker \mathcal{L}_B)^{7-k}} = H_\varphi^{7-k}. \end{aligned}$$

Note that the above formal manipulation is not a rigorous proof of duality because at step (!), we do not have $\text{im } P^* = (\ker P)^\perp$ in general for an arbitrary operator P , and step (!!) is also not justified. Because Ω^k is not complete with respect to the \mathcal{L}^2 -norm, the usual Hilbert space techniques do not apply. We will use elliptic operator theory to give a rigorous computation of H_φ^k for $k = 4, 5, 6, 7$, in the next section.

3.3 Computation of the groups $H_\varphi^4, H_\varphi^5, H_\varphi^6$, and H_φ^7

The material on regular operators in this section is largely based on Kawai–Lê–Schwachhöfer [17].

Definition 3.8 Let P be a linear differential operator of degree r on Ω^* . Then $P : \Omega^{k-r} \rightarrow \Omega^k$ is said to be *regular* if $\Omega^k = \text{im } P \oplus \ker P^*$, where by $\ker P^*$ we mean the kernel of

the formal adjoint $P^* : \Omega^k \rightarrow \Omega^{k-r}$ with respect to the L^2 inner product. The operator P is said to be elliptic, overdetermined elliptic, underdetermined elliptic, if the principal symbol $\sigma_\xi(P)$ of P is bijective, injective, surjective, respectively, for all $\xi \neq 0$. \square

Remark 3.9 It is a standard result in elliptic operator theory (see [1, p.464; 32 Corollary]) that elliptic, overdetermined elliptic, and underdetermined elliptic operators are all regular. \square

Proposition 3.10 *The operator $\mathcal{L}_B : \Omega^{k-2} \rightarrow \Omega^k$ is regular for all $k = 0, \dots, 9$.*

Proof Consider the symbol $P = \sigma_\xi(\mathcal{L}_B)$. By (2.33), this operator is $P(\omega) = (\xi \lrcorner \varphi) \wedge \omega$. Note that this is an algebraic (pointwise) map, and thus, at each point it is a linear map between finite-dimensional vector spaces. We will show that $P : \Omega^{k-2} \rightarrow \Omega^k$ is injective for $k = 0, 1, 2, 3, 4$ and surjective for $k = 5, 6, 7, 8, 9$. The claim will then follow by Remark 3.9.

First we claim that injectivity of $P : \Omega^{k-2} \rightarrow \Omega^k$ for $k = 0, 1, 2, 3, 4$ implies surjectivity of $P : \Omega^{k-2} \rightarrow \Omega^k$ for $k = 5, 6, 7, 8, 9$. Suppose $P : \Omega^{k-2} \rightarrow \Omega^k$ is injective. Then the dual map $P^* : \Omega^k \rightarrow \Omega^{k-2}$ is surjective. But we have

$$P^* = (\sigma_\xi(\mathcal{L}_B))^* = \sigma_\xi(\mathcal{L}_B^*),$$

and by (2.28), this equals $\sigma_\xi(-* \mathcal{L}_B^*) = -* \sigma_\xi(\mathcal{L}_B)^* = -* P^*$. Since $* : \Omega^l \rightarrow \Omega^{7-l}$ is bijective, and we have that $*P^* : \Omega^k \rightarrow \Omega^{k-2}$ is surjective, we deduce that $P : \Omega^{(9-k)-2} \rightarrow \Omega^{9-k}$ is surjective. But $9-k \in \{5, 6, 7, 8, 9\}$ if $k = \{0, 1, 2, 3, 4\}$. Thus, the claim is proved.

It remains to establish injectivity of $P : \Omega^{k-2} \rightarrow \Omega^k$ for $k = 0, 1, 2, 3, 4$. This is automatic for $k = 0, 1$ since $\Omega^{k-2} = 0$ in these cases.

If $k = 2$, then $P : \Omega^0 \rightarrow \Omega^2$ is given by $Pf = (\xi \lrcorner \varphi) \wedge f = f(\xi \lrcorner \varphi)$. Suppose $Pf = 0$. Since $\xi \neq 0$, we have $\xi \lrcorner \varphi \neq 0$, and thus, $f = 0$. So P is injective for $k = 2$.

If $k = 3$, then $P : \Omega^1 \rightarrow \Omega^3$ is given by $P\alpha = (\xi \lrcorner \varphi) \wedge \alpha$. Suppose $P\alpha = 0$. Taking the wedge product of $P\alpha = 0$ with ψ and using Lemma 2.3 gives

$$\begin{aligned} 0 &= \psi \wedge (\xi \lrcorner \varphi) \wedge \alpha = 3(*\xi) \wedge \alpha \\ &= 3g(\xi, \alpha)\text{vol}. \end{aligned}$$

Thus, $g(\xi, \alpha) = 0$. Similarly, taking the wedge product of $P\alpha = 0$ with φ and using Lemmas 2.3 and 2.4 gives

$$\begin{aligned} 0 &= \varphi \wedge (\xi \lrcorner \varphi) \wedge \alpha = -2(*(\xi \lrcorner \varphi)) \wedge \alpha \\ &= -2\psi \wedge \xi \wedge \alpha = -2* (\xi \times \alpha). \end{aligned}$$

Thus, $\xi \times \alpha = 0$. Taking the cross product of this with ξ and using Lemma 2.4 gives

$$-g(\xi, \xi)\alpha + g(\xi, \alpha)\xi = 0.$$

Since $g(\xi, \alpha) = 0$ and $\xi \neq 0$, we conclude that $\alpha = 0$. So P is injective for $k = 3$.

If $k = 4$, then $P : \Omega^2 \rightarrow \Omega^4$ is given by $P\beta = (\xi \lrcorner \varphi) \wedge \beta$. Suppose $P\beta = 0$. This means

$$(\xi \lrcorner \varphi) \wedge \beta = 0. \tag{3.6}$$

Write $\beta = \beta_7 + \beta_{14} \in \Omega_7^2 \oplus \Omega_{14}^2$, where by (2.3) we can write $\beta_7 = Y \lrcorner \varphi$ for some unique Y . Taking the wedge product of (3.6) with φ and using (2.3) and (2.1), we have

$$\begin{aligned} 0 &= (\xi \lrcorner \varphi) \wedge \varphi \wedge \beta = (\xi \lrcorner \varphi) \wedge (-2 * \beta_7 + * \beta_{14}) \\ &= -2(\xi \lrcorner \varphi) \wedge * \beta_7 + 0 = (\xi \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6g(\xi, Y)\text{vol}. \end{aligned}$$

Thus, we have

$$g(\xi, Y) = 0. \tag{3.7}$$

Now we take the interior product of (3.6) with ξ . This gives $(\xi \lrcorner \varphi) \wedge (\xi \lrcorner \beta) = 0$. By the injectivity of P for $k = 3$, we deduce that

$$\xi \lrcorner \beta = 0. \tag{3.8}$$

Using (3.8) and (2.3), we can rewrite (3.6) as

$$0 = \xi \lrcorner (\varphi \wedge \beta) = \xi \lrcorner (-2 * \beta_7 + * \beta_{14}).$$

Taking $*$ of the above equation and using $*(\xi \lrcorner * \gamma) = \pm \xi \wedge \gamma$, where in general the sign depends on the dimension of the manifold and the degree of γ , we find that

$$-2\xi \wedge \beta_7 + \xi \wedge \beta_{14} = 0. \tag{3.9}$$

Equation (3.9) implies that

$$\xi \wedge \beta = \xi \wedge \beta_7 + \xi \wedge \beta_{14} = 3\xi \wedge \beta_7. \tag{3.10}$$

Taking the interior product of (3.10) with ξ and using (3.8) yields

$$g(\xi, \xi)\beta = 3g(\xi, \xi)\beta_7 - 3\xi \wedge (\xi \lrcorner \beta_7). \tag{3.11}$$

By Lemma 2.4, we have $\xi \lrcorner \beta_7 = \xi \lrcorner Y \lrcorner \varphi = Y \times \xi$. Thus, (3.11) becomes

$$g(\xi, \xi)\beta = 3g(\xi, \xi)\beta_7 - 3\xi \wedge (Y \times \xi). \tag{3.12}$$

Now we take the wedge product of (3.12) with ψ , use Lemma 2.4 again, and the fact that $\beta_{14} \wedge \psi = 0$ from (2.3). We obtain

$$\begin{aligned} g(\xi, \xi)\beta_7 \wedge \psi &= 3g(\xi, \xi)\beta_7 \wedge \psi - 3\xi \wedge (Y \times \xi) \wedge \psi \\ &= 3g(\xi, \xi)\beta_7 \wedge \psi - 3 * (\xi \times (Y \times \xi)), \end{aligned}$$

which can be rearranged to give, using Lemma 2.4 and (3.7), that

$$-2g(\xi, \xi)\beta_7 \wedge \psi = 3 * (\xi \times (\xi \times Y)) = -3 * (g(\xi, \xi)Y). \tag{3.13}$$

But from Lemma 2.3 we find $\beta_7 \wedge \psi = (Y \lrcorner \varphi) \wedge \psi = 3 * Y$. Substituting this into (3.13) and taking $*$, we find that

$$-3g(\xi, \xi)Y = -2g(\xi, \xi) * (3 * Y) = -6g(\xi, \xi)Y.$$

Since $\xi \neq 0$, we deduce that $Y = 0$, and thus, $\beta_7 = 0$. Substituting back into (3.11) then gives $g(\xi, \xi)\beta_{14} = 0$, and thus, $\beta_{14} = 0$ as well. So P is injective for $k = 4$. \square

Corollary 3.11 *For any $k = 0, \dots, 7$, we have*

$$(\text{im } \mathcal{L}_B)^k = *((\ker \mathcal{L}_B)^{7-k})^\perp. \tag{3.14}$$

Proof By (2.28), we have $(\text{im } \mathcal{L}_B)^k = *(\text{im } \mathcal{L}_B^*)^{7-k}$, and because \mathcal{L}_B is regular by Proposition 3.10, we have $(\text{im } \mathcal{L}_B^*)^{7-k} = ((\ker \mathcal{L}_B)^{7-k})^\perp$. The result follows. \square

Proposition 3.12 *We have $H_\varphi^7 \cong \mathcal{H}^7$ and $H_\varphi^6 \cong \mathcal{H}^6$.*

Proof In the proof of Proposition 3.5, we showed that $(\ker \mathcal{L}_B)^0 = \mathcal{H}^0$ and $(\ker \mathcal{L}_B)^1 = \mathcal{H}^1$. Thus, using (3.14) we have

$$\begin{aligned} (\operatorname{im} \mathcal{L}_B)^7 &= *((\ker \mathcal{L}_B)^0)^\perp = *(\mathcal{H}^0)^\perp \\ &= (\operatorname{im} d)^7 \oplus (\operatorname{im} d^*)^7 \quad \text{by the Hodge decomposition.} \end{aligned}$$

In exactly the same way we get $(\operatorname{im} \mathcal{L}_B)^6 = (\operatorname{im} d)^6 \oplus (\operatorname{im} d^*)^6$.

Moreover, since \mathcal{L}_B has degree two, we have $(\ker \mathcal{L}_B)^6 = \Omega^6$ and $(\ker \mathcal{L}_B)^7 = \Omega^7$. Thus, we conclude that

$$H_\varphi^k = \frac{\Omega^k}{(\operatorname{im} d)^k \oplus (\operatorname{im} d^*)^k} \cong \mathcal{H}^k \quad \text{for } k = 6, 7.$$

by the Hodge decomposition. □

Proposition 3.13 *We have $H_\varphi^5 \cong \mathcal{H}^5$.*

Proof In the proof of Proposition 3.6, we showed that $(\ker \mathcal{L}_B)^2 = \mathcal{H}^2$, so using (3.14) just as in the proof of Proposition 3.12 we deduce that

$$(\operatorname{im} \mathcal{L}_B)^5 = (\operatorname{im} d)^5 \oplus (\operatorname{im} d^*)^5. \tag{3.15}$$

Let $\alpha \in \Omega^6$. Then, since $d^* = *d*$ on Ω^6 , we find from Fig. 1 that up to our usual identifications, $d^*\alpha = D_7^2\alpha + D_{14}^7\alpha \in \Omega_7^5 \oplus \Omega_{14}^5$. Then, Fig. 4 and (2.17) gives

$$\mathcal{L}_B d^*\alpha = \mathcal{L}_B(D_7^2\alpha + D_{14}^7\alpha) = 7D_7^1 D_7^2\alpha + 0 = 0,$$

so $(\operatorname{im} d^*)^5 \subset (\ker \mathcal{L}_B)^5$. We also have $\mathcal{H}^5 \subset (\ker \mathcal{L}_B)^5$ by (2.37). Using the Hodge decomposition of Ω^5 , we therefore have

$$\mathcal{H}^5 \oplus (\operatorname{im} d^*)^5 \subseteq (\ker \mathcal{L}_B)^5 \subseteq \Omega^5 = \mathcal{H}^5 \oplus (\operatorname{im} d^*)^5 \oplus (\operatorname{im} d)^5.$$

Applying Lemma 1.2(i), we deduce that

$$(\ker \mathcal{L}_B)^5 = \mathcal{H}^5 \oplus (\operatorname{im} d^*)^5 \oplus ((\operatorname{im} d)^5 \cap (\ker \mathcal{L}_B)^5). \tag{3.16}$$

Applying Lemma 1.2(ii) to (3.15), (3.16), as subspaces of $\Omega^5 = \mathcal{H}^5 \oplus (\operatorname{im} d)^5 \oplus (\operatorname{im} d^*)^5$, we obtain

$$(\operatorname{im} \mathcal{L}_B)^5 \cap (\ker \mathcal{L}_B)^5 = (\operatorname{im} d^*)^5 \oplus ((\operatorname{im} d)^5 \cap (\ker \mathcal{L}_B)^5). \tag{3.17}$$

Therefore, we find that

$$\begin{aligned} H_\varphi^5 &= \frac{(\ker \mathcal{L}_B)^5}{(\ker \mathcal{L}_B)^5 \cap (\operatorname{im} \mathcal{L}_B)^5} \\ &= \frac{\mathcal{H}^5 \oplus (\operatorname{im} d^*)^5 \oplus ((\operatorname{im} d)^5 \cap (\ker \mathcal{L}_B)^5)}{(\operatorname{im} d^*)^5 \oplus (\operatorname{im} d)^5 \cap (\ker \mathcal{L}_B)^5} \quad \text{by (3.16) and (3.17)} \\ &\cong \mathcal{H}^5 \end{aligned}$$

as claimed. □

Before we can compute H_φ^4 , we need two preliminary results.

Lemma 3.14 *We have*

$$(\ker \mathcal{L}_B)^4 \cap ((\operatorname{im} d^*)^4 \oplus (\operatorname{im} d)^4) = ((\ker \mathcal{L}_B)^4 \cap (\operatorname{im} d^*)^4) \oplus ((\ker \mathcal{L}_B)^4 \cap (\operatorname{im} d)^4). \tag{3.18}$$

Proof Let $\beta = \beta_7 + \beta_{14} \in \Omega_7^5 \oplus \Omega_{14}^5$, and $\gamma = \gamma_1 + \gamma_7 + \gamma_{27} \in \Omega_1^3 \oplus \Omega_7^3 \oplus \Omega_{27}^3$. We need to prove that

$$\begin{aligned} \mathcal{L}_B d^*(\beta_7 + \beta_{14}) + \mathcal{L}_B d(\gamma_1 + \gamma_7 + \gamma_{27}) &= 0 \\ \iff \mathcal{L}_B d^*(\beta_7 + \beta_{14}) &= \mathcal{L}_B d(\gamma_1 + \gamma_7 + \gamma_{27}) = 0. \end{aligned} \tag{3.19}$$

From $d^* = - * d *$ on Ω^5 and Figs. 1 and 4, we have

$$\begin{aligned} \mathcal{L}_B d^*(\beta_7 + \beta_{14}) &= \mathcal{L}_B (-D_1^7 \beta_7 + \frac{3}{2} D_7^7 \beta_7 - D_{27}^7 \beta_7 - D_7^{14} \beta_{14} - D_{27}^{14} \beta_{14}) \\ &= 3 D_1^7 (-D_1^7 \beta_7) - 6 D_7^7 (\frac{3}{2} D_7^7 \beta_7 - D_7^{14} \beta_{14}) + 4 D_{27}^{27} (-D_{27}^7 \beta_7 - D_{27}^{14} \beta_{14}) \\ &= -3 D_1^1 D_1^7 \beta_7 - 9 D_7^7 D_7^7 \beta_7 + 6 D_7^7 D_7^{14} \beta_{14} - 4 D_{27}^{27} D_{27}^7 \beta_7 - 4 D_{27}^{27} D_{27}^{14} \beta_{14}. \end{aligned}$$

Using the relations in (2.17), the above expression simplifies to

$$\mathcal{L}_B d^*(\beta_7 + \beta_{14}) = -7 D_7^1 D_1^7 \beta_7. \tag{3.20}$$

Similarly, from Figs. 1 and 4 and $D_7^7 D_7^1 = 0$, we have

$$\begin{aligned} \mathcal{L}_B d(\gamma_1 + \gamma_7 + \gamma_{27}) &= \mathcal{L}_B (\frac{4}{3} D_1^7 \gamma_1 + (-D_1^7 \gamma_1 - \frac{3}{2} D_7^7 \gamma_1 + D_{27}^{27} \gamma_{27}) + (-D_{27}^7 \gamma_7 + D_{27}^{27} \gamma_{27})) \\ &= 3 D_1^7 (\frac{4}{3} D_1^7 \gamma_1) - 6 D_7^7 (-D_1^7 \gamma_1 - \frac{3}{2} D_7^7 \gamma_1 + D_{27}^{27} \gamma_{27}) + 4 D_{27}^{27} (-D_{27}^7 \gamma_7 + D_{27}^{27} \gamma_{27}) \\ &= 4 D_7^1 D_1^7 \gamma_1 + 9 D_7^7 D_7^7 \gamma_1 - 6 D_7^7 D_{27}^{27} \gamma_{27} - 4 D_{27}^{27} D_{27}^7 \gamma_7 + 4 D_{27}^{27} D_{27}^{27} \gamma_{27}. \end{aligned}$$

Using the relations in (2.17), the above expression simplifies to

$$\mathcal{L}_B d(\gamma_1 + \gamma_7 + \gamma_{27}) = 18 D_7^7 D_7^7 \gamma_1 - 12 D_7^7 D_{27}^{27} \gamma_{27}. \tag{3.21}$$

Combining Eqs. (3.20) and (3.21), if $\mathcal{L}_B d^*(\beta_7 + \beta_{14}) + \mathcal{L}_B d(\gamma_1 + \gamma_7 + \gamma_{27}) = 0$, then we have

$$-7 D_7^1 D_1^7 \beta_7 + 18 D_7^7 D_7^7 \gamma_1 - 12 D_7^7 D_{27}^{27} \gamma_{27} = 0,$$

and thus, applying D_1^7 and using $D_1^7 D_7^7 = 0$, we deduce that

$$7 D_1^7 D_7^1 D_1^7 \beta_7 = D_1^7 D_7^7 (18 D_7^7 \gamma_1 - 12 D_{27}^{27} \gamma_{27}) = 0.$$

Thus, we have $D_1^7 D_7^1 D_1^7 \beta_7 = 0$. Applying Remark 3.4, we deduce that $D_7^1 D_1^7 \beta_7 = 0$, and thus, by (3.20) that $\mathcal{L}_B d^*(\beta_7 + \beta_{14}) = 0$. Thus, we have established (3.19) and consequently

$$(\ker \mathcal{L}_B)^4 \cap ((\text{im } d^*)^4 \oplus (\text{im } d)^4) = ((\ker \mathcal{L}_B)^4 \cap (\text{im } d^*)^4) \oplus ((\ker \mathcal{L}_B)^4 \cap (\text{im } d)^4)$$

as claimed. □

Lemma 3.15 *We have*

$$(\text{im } d)^4 \cap (\ker \mathcal{L}_B)^4 \cap (\text{im } \mathcal{L}_B)^4 = 0. \tag{3.22}$$

Proof Let $\omega \in (\text{im } d)^4 \cap (\ker \mathcal{L}_B)^4 \cap (\text{im } \mathcal{L}_B)^4$. We write $\omega = \mathcal{L}_B(\alpha_7 + \alpha_{14})$ for some $\alpha_7 + \alpha_{14} \in \Omega_7^2 \oplus \Omega_{14}^2$. Using Fig. 4, we find

$$\begin{aligned} \omega = \mathcal{L}_B(\alpha_7 + \alpha_{14}) &= (-2 D_1^7 \alpha_7) + (-3 D_7^{14} \alpha_{14}) + (-2 D_{27}^7 \alpha_7 + D_{27}^{14} \alpha_{14}) \\ &= \omega_1 + \omega_7 + \omega_{27} \in \Omega_1^4 \oplus \Omega_7^4 \oplus \Omega_{27}^4. \end{aligned}$$

That is, we have

$$\begin{aligned}\omega_1 &= -2D_1^7\alpha_7, \\ \omega_7 &= -3D_7^{14}\alpha_{14}, \\ \omega_{27} &= -2D_{27}^7\alpha_7 + D_{27}^{14}\alpha_{14}.\end{aligned}\tag{3.23}$$

Using Fig. 4 again, the equation $\mathcal{L}_B\mathcal{L}_B(\alpha_7 + \alpha_{14}) = \mathcal{L}_B\omega = 0$ gives

$$\begin{aligned}0 &= \mathcal{L}_B(-2D_1^7\alpha_7 - 3D_7^{14}\alpha_{14} + (-2D_{27}^7\alpha_7 + D_{27}^{14}\alpha_{14})) \\ &= 3D_7^1(-2D_1^7\alpha_7) - 6D_7^7(-3D_7^{14}\alpha_{14}) + 4D_7^{27}(-2D_{27}^7\alpha_7 + D_{27}^{14}\alpha_{14}) \\ &= -6D_7^1D_1^7\alpha_7 + 18D_7^7D_7^{14}\alpha_{14} - 8D_7^{27}D_{27}^7\alpha_7 + 4D_7^{27}D_{27}^{14}\alpha_{14}.\end{aligned}$$

Using relations (2.17), we can rewrite the above expression in two different ways, both of which will be useful. These are

$$-6D_7^1D_1^7\alpha_7 - 8D_7^{27}D_{27}^7\alpha_7 + 24D_7^7D_7^{14}\alpha_{14} = 0,\tag{3.24}$$

$$-14D_7^1D_1^7\alpha_7 + 18D_7^7D_7^{14}\alpha_{14} + 24D_7^7D_7^{14}\alpha_{14} = 0.\tag{3.25}$$

Applying D_1^7 to (3.25) and using $D_1^7D_7^7 = 0$, we deduce that

$$14D_1^7D_7^1D_1^7\alpha_7 = (D_1^7D_7^7)(18D_7^7\alpha_7 + 24D_7^{14}\alpha_{14}) = 0.$$

Thus, we have $D_1^7D_7^1D_1^7\alpha_7 = 0$. Applying Remark 3.4 twice, we deduce first that $D_7^1D_1^7\alpha_7 = 0$ and then that

$$D_1^7\alpha_7 = 0.\tag{3.26}$$

Comparing (3.26) and (3.23), we find that $\omega_1 = 0$. Since $\omega \in (\text{im } d)^4$, it is d-closed. Using Figs. 1 and 4, the conditions $\pi_{7d}\omega = 0$ and $\mathcal{L}_B\omega = 0$ give, respectively,

$$\begin{aligned}2D_7^7\omega_7 + \frac{4}{3}D_7^{27}\omega_{27} &= 0, \\ -6D_7^7\omega_7 + 4D_7^{27}\omega_{27} &= 0.\end{aligned}$$

These two equations together force

$$D_7^7\omega_7 = 0 \quad \text{and} \quad D_7^{27}\omega_{27} = 0.\tag{3.27}$$

Also, from (3.23) we have $\omega_7 = -3D_7^{14}\alpha_{14}$, and thus, since $D_1^7D_7^{14} = 0$, we deduce that

$$D_1^7\omega_7 = 0.\tag{3.28}$$

Combining the first equation in (3.27) with (3.28), we find by Theorem 2.19 that, considered as a 1-form, $\omega_7 \in \mathcal{H}^1$ and in particular

$$D_{14}^7\omega_7 = 0 \quad \text{and} \quad D_{27}^7\omega_7 = 0.\tag{3.29}$$

From Fig. 1, the condition $\pi_{14d}\omega = 0$ gives $-D_{14}^7\omega_7 + D_{14}^{27}\omega_{27} = 0$, which, by the first equation in (3.29), implies that

$$D_{14}^{27}\omega_{27} = 0.\tag{3.30}$$

Recalling from (3.23) that $\omega_7 = -3D_7^{14}\alpha_{14}$, substituting (3.26) into (3.24) and using the first equation in (3.27) now gives

$$0 = -8D_7^{27}D_{27}^7\alpha_7 - 8D_7^7\omega_7 = -8D_7^{27}D_{27}^7\alpha_7,$$

which by Remark 3.4 implies that

$$D_{27}^7 \alpha_7 = 0. \tag{3.31}$$

Combining (3.31) with (3.26) and using Theorem 2.19, we find that α_7 is harmonic.

Recalling from (3.23) that $\omega_{27} = -2D_{27}^7 \alpha_7 + D_{27}^{14} \alpha_{14}$, substituting (3.31) and taking D_{27}^{27} , we obtain by the relations in (2.17) that

$$D_{27}^{27} \omega_{27} = D_{27}^{27} D_{27}^{14} \alpha_{14} = D_{27}^7 D_7^{14} \alpha_{14}.$$

Substituting $D_7^{14} \alpha_{14} = -\frac{1}{3} \omega_7$ from (3.23) into the above expression and using the second equation in (3.29), we find that

$$D_{27}^{27} \omega_{27} = -\frac{1}{3} D_{27}^7 \omega_7 = 0. \tag{3.32}$$

Combining the second equation in (3.27), Eq. (3.30), and (3.32), with Eq. (2.19), we deduce that ω_{27} is a harmonic Ω_{27}^4 form. We already showed that ω_7 is a harmonic Ω_7^4 form and that $\omega_1 = 0$. Thus, we have $\omega \in \mathcal{H}^4$, and moreover, we assumed that $\omega \in (\text{im } d)^4$. By Hodge theory, we conclude that $\omega = 0$ as claimed. \square

Proposition 3.16 *We have $H_\varphi^4 \cong \mathcal{H}^4 \oplus ((\text{im } d)^4 \cap (\ker \mathcal{L}_B)^4)$.*

Proof In the proof of Proposition 3.7, we showed that

$$H_\varphi^3 = (\ker \mathcal{L}_B)^3 = \mathcal{H}^3 \oplus ((\text{im } d^*)^3 \cap (\ker \mathcal{L}_B)^3).$$

We also have $\mathcal{H}^3 \subset (\ker \mathcal{L}_B)^3$ by (2.38). Thus,

$$\mathcal{H}^3 \subseteq (\ker \mathcal{L}_B)^3 \subseteq \mathcal{H}^3 \oplus (\text{im } d^*)^3.$$

Taking orthogonal complements of the above chain of nested subspaces and using the Hodge decomposition $\Omega^3 = \mathcal{H}^3 \oplus (\text{im } d)^3 \oplus (\text{im } d^*)^3$, we find

$$(\text{im } d)^3 \oplus (\text{im } d^*)^3 \supseteq ((\ker \mathcal{L}_B)^3)^\perp \supseteq (\text{im } d)^3.$$

Taking the Hodge star of the above chain of nested subspaces and using $(\text{im } \mathcal{L}_B)^4 = *((\ker \mathcal{L}_B)^3)^\perp$ from (3.14), we obtain

$$(\text{im } d^*)^4 \subseteq (\text{im } \mathcal{L}_B)^4 \subseteq (\text{im } d^*)^4 \oplus (\text{im } d)^4.$$

Applying Lemma 1.2(i) to the above yields

$$(\text{im } \mathcal{L}_B)^4 = (\text{im } d^*)^4 \oplus ((\text{im } d)^4 \cap (\text{im } \mathcal{L}_B)^4). \tag{3.33}$$

Now recall that $\mathcal{H}^4 \subseteq (\ker \mathcal{L}_B)^4$ by (2.38). Thus, we have

$$\mathcal{H}^4 \subseteq (\ker \mathcal{L}_B)^4 \subseteq \Omega^4 = \mathcal{H}^4 \oplus (\text{im } d)^4 \oplus (\text{im } d^*)^4.$$

Applying Lemma 1.2(i) to the above and using Lemma 3.14 gives

$$(\ker \mathcal{L}_B)^4 = \mathcal{H}^4 \oplus ((\text{im } d^*)^4 \cap (\ker \mathcal{L}_B)^4) \oplus ((\text{im } d)^4 \cap (\ker \mathcal{L}_B)^4). \tag{3.34}$$

Thus, applying Lemma 1.2(ii) to (3.33), (3.34), as subspaces of $\Omega^4 = \mathcal{H}^4 \oplus (\text{im } d)^4 \oplus (\text{im } d^*)^4$, we obtain

$$(\ker \mathcal{L}_B)^4 \cap (\text{im } \mathcal{L}_B)^4 = ((\text{im } d^*)^4 \cap (\ker \mathcal{L}_B)^4) \oplus ((\text{im } d)^4 \cap (\ker \mathcal{L}_B)^4 \cap (\text{im } \mathcal{L}_B)^4). \tag{3.35}$$

By Lemma 3.15, Eq. (3.35) simplifies to

$$(\ker \mathcal{L}_B)^4 \cap (\text{im } \mathcal{L}_B)^4 = (\text{im } d^*)^4 \cap (\ker \mathcal{L}_B)^4. \tag{3.36}$$

Hence, by (3.34) and (3.36), we have

$$\begin{aligned} \mathcal{H}_\varphi^4 &= \frac{(\ker \mathcal{L}_B)^4}{(\ker \mathcal{L}_B)^4 \cap (\operatorname{im} \mathcal{L}_B)^4} \\ &= \frac{\mathcal{H}^4 \oplus ((\operatorname{im} d^*)^4 \cap (\ker \mathcal{L}_B)^4) \oplus ((\operatorname{im} d)^4 \cap (\ker \mathcal{L}_B)^4)}{(\operatorname{im} d^*)^4 \cap (\ker \mathcal{L}_B)^4} \\ &\cong \mathcal{H}^4 \oplus (\operatorname{im} d)^4 \cap (\ker \mathcal{L}_B)^4 \end{aligned}$$

as claimed. □

Lemma 3.17 *We have $(\operatorname{im} d^*)^3 \cap (\ker \mathcal{L}_B)^3 = (\operatorname{im} d^*)^3 \cap (\ker \mathcal{L}_B^*)^3$.*

Proof Let $\omega = d^*(\gamma_1 + \gamma_7 + \gamma_{27}) \in (\operatorname{im} d^*)^3$ where $\gamma_1 + \gamma_7 + \gamma_{27} \in \Omega_1^4 \oplus \Omega_7^4 \oplus \Omega_{27}^4$. From $d^* = *d*$ on Ω^5 and Fig. 1, we find that

$$d^*(\gamma_1 + \gamma_7 + \gamma_{27}) = \frac{4}{3}D_1^7\gamma_7 + (-D_7^1\gamma_1 - \frac{3}{2}D_7^7\gamma_7 + D_7^{27}\gamma_{27}) + (-D_{27}^7\gamma_7 + D_{27}^{27}\gamma_{27}). \tag{3.37}$$

Using (3.37) and Fig. 4, we have

$$\begin{aligned} \mathcal{L}_B d^*(\gamma_1 + \gamma_7 + \gamma_{27}) &= -2D_7^1(\frac{4}{3}D_1^7\gamma_7) + 3D_{14}^7(-D_7^1\gamma_1 - \frac{3}{2}D_7^7\gamma_7 + D_7^{27}\gamma_{27}) \\ &\quad + (-\frac{8}{3}D_7^{27} + D_{14}^{27})(-D_{27}^7\gamma_7 + D_{27}^{27}\gamma_{27}) \\ &= (-\frac{8}{3}D_7^1D_1^7\gamma_7 + \frac{8}{3}D_7^{27}D_{27}^7\gamma_7 - \frac{8}{3}D_7^{27}D_{27}^{27}\gamma_{27}) \\ &\quad + (-3D_{14}^7D_7^1\gamma_1 - \frac{9}{2}D_{14}^7D_7^7\gamma_7 + 3D_{14}^7D_7^{27}\gamma_{27} \\ &\quad - D_{14}^{27}D_{27}^7\gamma_7 + D_{14}^{27}D_{27}^{27}\gamma_{27}). \end{aligned}$$

Using the various relations in (2.17), the above expression simplifies to

$$\begin{aligned} \mathcal{L}_B d^*(\gamma_1 + \gamma_7 + \gamma_{27}) &= -6D_7^1D_7^7\gamma_7 + 4D_7^7D_7^{27}\gamma_{27} - 6D_{14}^7D_7^7\gamma_7 + 4D_{14}^7D_7^{27}\gamma_{27} \\ &= 2D_7^7(-3D_7^7\gamma_7 + 2D_7^{27}\gamma_{27}) + 2D_{14}^7(-3D_7^7\gamma_7 + 2D_7^{27}\gamma_{27}). \end{aligned} \tag{3.38}$$

Using $\mathcal{L}_B^* = -*\mathcal{L}_B*$ from (2.28), Eq. (3.37), and Fig. 4 again, we also have that

$$\begin{aligned} \mathcal{L}_B^* d^*(\gamma_1 + \gamma_7 + \gamma_{27}) &= -3D_7^1(\frac{4}{3}D_1^7\gamma_7) + 6D_7^7(-D_7^1\gamma_1 - \frac{3}{2}D_7^7\gamma_7 + D_7^{27}\gamma_{27}) \\ &\quad - 4D_7^{27}(-D_{27}^7\gamma_7 + D_{27}^{27}\gamma_{27}) \\ &= -4D_7^1D_1^7\gamma_7 - 6D_7^7D_7^1\gamma_1 - 9D_7^7D_7^7\gamma_7 + 6D_7^7D_7^{27}\gamma_{27} \\ &\quad + 4D_7^{27}D_{27}^7\gamma_7 - 4D_7^{27}D_{27}^{27}\gamma_{27}. \end{aligned}$$

Using the various relations in (2.17), the above expression simplifies to

$$\begin{aligned} \mathcal{L}_B^* d^*(\gamma_1 + \gamma_7 + \gamma_{27}) &= -18D_7^7D_7^7\gamma_7 + 12D_7^7D_7^{27}\gamma_{27} \\ &= 6D_7^7(-3D_7^7\gamma_7 + 2D_7^{27}\gamma_{27}). \end{aligned} \tag{3.39}$$

Thus, for $\omega \in (\operatorname{im} d^*)^3$ we conclude that

$$\begin{aligned} \omega \in (\ker \mathcal{L}_B)^3 &\iff \left\{ \begin{array}{l} D_7^7(-3D_7^7\gamma_7 + 2D_7^{27}\gamma_{27}) = 0, \\ D_{14}^7(-3D_7^7\gamma_7 + 2D_7^{27}\gamma_{27}) = 0, \end{array} \right\} && \text{by (3.38)} \\ &\iff D_7^7(-3D_7^7\gamma_7 + 2D_7^{27}\gamma_{27}) = 0 && \text{by Theorem 2.19} \\ &\iff \omega \in (\ker \mathcal{L}_B^*)^3 && \text{by (3.39)} \end{aligned}$$

which is what we wanted to show. □

Corollary 3.18 *We have*

$$H_\varphi^3 = \mathcal{H}^3 \oplus ((\text{im } d^*)^3 \cap (\ker \mathcal{L}_B)^3 \cap (\ker \mathcal{L}_B^*)^3),$$

$$H_\varphi^4 = \mathcal{H}^4 \oplus ((\text{im } d)^4 \cap (\ker \mathcal{L}_B)^4 \cap (\ker \mathcal{L}_B^*)^4).$$

Proof Lemma 3.17 says that

$$(\text{im } d^*)^3 \cap (\ker \mathcal{L}_B)^3 = (\text{im } d^*)^3 \cap (\ker \mathcal{L}_B)^3 \cap (\ker \mathcal{L}_B^*)^3 = (\text{im } d^*)^3 \cap (\ker \mathcal{L}_B^*)^3.$$

Applying $*$ to the above equation and using (2.28) gives

$$(\text{im } d)^4 \cap (\ker \mathcal{L}_B)^4 = (\text{im } d)^4 \cap (\ker \mathcal{L}_B)^4 \cap (\ker \mathcal{L}_B^*)^4 = (\text{im } d)^4 \cap (\ker \mathcal{L}_B^*)^4.$$

The claim now follows from Propositions 3.7 and 3.16. □

3.4 The main theorem on \mathcal{L}_B -cohomology

We summarize the results of Sect. 3 in the following theorem, which is intentionally stated in a way to mirror Theorem 3.2.

Theorem 3.19 *The following relations hold.*

- $H_\varphi^k \cong H_{dR}^k$ for $k = 0, 1, 2, 5, 6, 7$.
- H_φ^k is infinite-dimensional for $k = 3, 4$.
- There is a canonical injection $\mathcal{H}^k \hookrightarrow H_\varphi^k$ for all k .
- The Hodge star induces isomorphisms $*$: $H_\varphi^k \cong H_\varphi^{7-k}$.

Proof All that remains to show is that H_φ^3 is indeed infinite-dimensional. But observe by (3.38) that for all $\alpha \in \Omega_1^4$, we have $\mathcal{L}_B d^* \alpha = 0$. Therefore, $\{d^* \alpha : \alpha \in \Omega_1^4\} \cong \text{im } D_7^1 \cong (\text{im } d)^1$ is an infinite-dimensional subspace of $(\text{im } d^*)^3 \cap (\ker \mathcal{L}_B)^3 \subseteq H_\varphi^3$. □

4 An application to ‘almost’ formality

In this section, we consider an application of our results to the question of formality of compact torsion-free G_2 manifolds. We discover a new topological obstruction to the existence of torsion-free G_2 -structures on compact manifolds and discuss an explicit example in detail.

4.1 Formality and Massey triple products

Recall from (2.24) that d commutes with \mathcal{L}_B . Hence, d induces a natural map

$$d : H_\varphi^k \rightarrow H_\varphi^{k+1}.$$

Also, because \mathcal{L}_B is a derivation, it is easy to check that the wedge product on Ω^\bullet descends to H_φ^\bullet , with the Leibniz rule $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge (d\eta)$ still holding on H_φ^\bullet . These two facts say that the complex (H_φ^\bullet, d) is a *differential graded algebra*, henceforth abbreviated *dga*.

Additionally, because $[d, \mathcal{L}_B] = 0$, we also have that $((\ker \mathcal{L}_B)^\bullet, d)$ is a subcomplex of the de Rham complex (Ω^\bullet, d) . The natural injection and projection give homomorphisms of *dga*’s

$$(\Omega^\bullet, d) \hookrightarrow ((\ker \mathcal{L}_B)^\bullet, d) \twoheadrightarrow (H_\varphi^\bullet, d).$$

One goal of this section is to show that these two homomorphisms of dga’s are both *quasi-isomorphisms*. This means that they induce *isomorphisms* on the cohomologies of the complexes. As mentioned in Introduction, some of the results in this section appeared earlier in work of Verbitsky [22]. For example, our Proposition 4.1 is exactly [22, Proposition 2.21], with the same proof. However, the proof of [22, Proposition 2.19] has several errors. The critical error is the following: first Verbitsky correctly shows that $\alpha - \Pi\alpha$ is an element of both $(\text{im } d_c + \text{im } d_c^*)$ and $(\text{im } d_c^*)^\perp$. But then he incorrectly concludes that $\alpha - \Pi\alpha$ must be an element of $\text{im } d_c$. This conclusion is only valid if $(d_c)^2 = 0$, which is not true in general. We give a correct proof of this result, which is our Proposition 4.4. One consequence is the result about the Massey triple product in our Corollary 4.9, which appears to be new.

Proposition 4.1 (Verbitsky [22]) *The inclusion $((\ker \mathcal{L}_B)^\bullet, d) \hookrightarrow (\Omega^\bullet, d)$ is a quasi-isomorphism.*

Proof This is proved in [22, Proposition 2.11]. We reproduce the short proof here for completeness and convenience of the reader. Since the differential for both complexes Ω^\bullet and $(\ker \mathcal{L}_B)^\bullet$ is the same exterior derivative d , we will omit it from the notation for simplicity.

It is well known that the Hodge Laplacian Δ determines an eigenspace decomposition $\Omega^k = \bigoplus_\lambda \Omega_\lambda^k$ where the sum is over all eigenvalues λ of Δ , which form a discrete set of nonnegative real numbers, and $\Omega_\lambda^k = \{\alpha \in \Omega^k : \Delta\alpha = \lambda\alpha\}$ are the associated eigenspaces. Note that $\Omega_0^k = \mathcal{H}^k$ is the space of harmonic k -forms. It is well known that the cohomology of Ω_λ^k is trivial for $\lambda > 0$. This is because, if $\alpha \in \Omega_\lambda^k$ with $\lambda > 0$ and $d\alpha = 0$, then

$$\alpha = \frac{1}{\lambda} \Delta\alpha = \frac{1}{\lambda} (dd^*\alpha + d^*d\alpha) = d(\frac{1}{\lambda} d^*\alpha) \tag{4.1}$$

is exact.

By (2.36), the operator \mathcal{L}_B commutes with Δ , and thus, we obtain a decomposition

$$(\ker \mathcal{L}_B)^k = \bigoplus_\lambda (\Omega_\lambda^k \cap (\ker \mathcal{L}_B)^k).$$

Note by (2.38) that $\Omega_0^k \cap (\ker \mathcal{L}_B)^k = \mathcal{H}^k \cap (\ker \mathcal{L}_B)^k = \mathcal{H}^k = \Omega_0^k$. Thus, it remains to show that the cohomology of $\Omega_\lambda^k \cap (\ker \mathcal{L}_B)^k$ is also trivial for all $\lambda > 0$. But if $\alpha \in \Omega_\lambda^k \cap (\ker \mathcal{L}_B)^k$, we have $\mathcal{L}_B\alpha = 0$ and $\alpha = d(\frac{1}{\lambda} d^*\alpha)$ by (4.1). Since \mathcal{L}_B commutes with d^* by (2.35), we have $\mathcal{L}_B(\frac{1}{\lambda} d^*\alpha) = \frac{1}{\lambda} d^*\mathcal{L}_B\alpha = 0$, so the class of α in the cohomology of $(\ker \mathcal{L}_B)^k$ is indeed trivial. □

In Sect. 3, while computing H_φ^\bullet , we explicitly computed the complex $((\ker \mathcal{L}_B)^\bullet, d)$. The results are collected in Fig. 6. The isomorphisms displayed in Fig. 6 are explained in Corollary 4.2.

Corollary 4.2 *For all $0 \leq k \leq 7$, we have $(\text{im } d)^k \cap (\ker \mathcal{L}_B)^k = d(\ker \mathcal{L}_B)^{k-1}$.*

Proof Let $\Omega^k = \mathcal{H}^k \oplus (\text{im } d)^k \oplus (\text{im } d^*)^k$ denote the Hodge decomposition of Ω^k . For simplicity in this proof, we will write $A^k = \mathcal{H}^k$, $B^k = (\text{im } d)^k$, and $C^k = (\text{im } d^*)^k$. Thus, $\Omega^k = A^k \oplus B^k \oplus C^k$. We can see from Fig. 6 that for all $0 \leq k \leq 7$, we have $(\ker \mathcal{L}_B)^k = A^k \oplus \tilde{B}^k \oplus \tilde{C}^k$, where \tilde{B}^k and \tilde{C}^k are subspaces of B^k and C^k , respectively. Depending on k , we can have $\tilde{B}^k = 0$, $\tilde{B}^k = B^k$, or $0 \subsetneq \tilde{B}^k \subsetneq B^k$ and similarly for \tilde{C}^k . By Hodge theory, $(\ker d)^k = A^k \oplus B^k$, so applying Lemma 1.2(ii) we find that

$$(\ker \mathcal{L}_B)^k \cap (\ker d)^k = A^k \oplus \tilde{B}^k. \tag{4.2}$$

Applying d to $(\ker \mathcal{L}_B)^{k-1}$, we have

$$d(\ker \mathcal{L}_B)^{k-1} = d(\tilde{C}^{k-1}) \subseteq \tilde{B}^k. \tag{4.3}$$

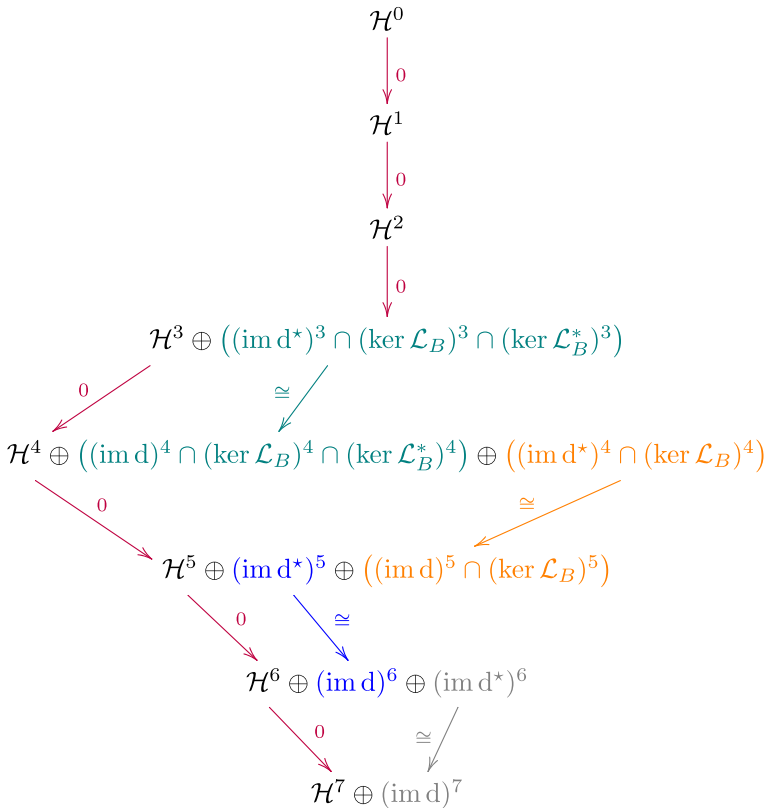


Fig. 6 The complex $((\ker \mathcal{L}_B)^\bullet, d)$

By Proposition 4.1, the cohomology of (Ω^k, d) equals the cohomology of $((\ker \mathcal{L}_B)^k, d)$. But by Hodge theory, the cohomology of (Ω^k, d) is $\mathcal{H}^k = A^k$, and Eqs. (4.2) and (4.3) say that the cohomology of $((\ker \mathcal{L}_B)^k, d)$ is $A^k \oplus (\tilde{B}^k/d\tilde{C}^{k-1})$. Thus, in fact, we have $d\tilde{C}^{k-1} = \tilde{B}^k$, and since d is injective on C^k , we deduce that

$$d \text{ maps } \tilde{C}^{k-1} \text{ isomorphically onto } \tilde{B}^k \text{ for all } 0 \leq k \leq 7. \tag{4.4}$$

From $(\text{im } d)^k \cap (\ker \mathcal{L}_B)^k = \tilde{B}^k$, and $d(\ker \mathcal{L}_B)^{k-1} = d(A^{k-1} \oplus \tilde{B}^{k-1} \oplus \tilde{C}^{k-1}) = d\tilde{C}^{k-1}$, we conclude that $(\text{im } d)^k \cap (\ker \mathcal{L}_B)^k = d(\ker \mathcal{L}_B)^{k-1}$ as claimed. \square

Remark 4.3 Corollary 4.2 may be related to a G_2 -analog of the *generalized* $\partial\bar{\partial}$ -lemma, called the $d\mathcal{L}_J$ -lemma, introduced by the authors in [4] in the context of Um -structures. See [4, Equation (3.27)]. \square

Proposition 4.4 *The quotient map $((\ker \mathcal{L}_B)^\bullet, d) \twoheadrightarrow (H_\varphi^\bullet, d)$ is a quasi-isomorphism.*

Proof We have a short exact sequence of chain complexes

$$0 \rightarrow ((\ker \mathcal{L}_B)^\bullet \cap (\text{im } \mathcal{L}_B)^\bullet, d) \rightarrow ((\ker \mathcal{L}_B)^\bullet, d) \rightarrow (H_\varphi^\bullet, d) \rightarrow 0,$$

so it suffices to show that the cohomology of $((\ker \mathcal{L}_B)^\bullet \cap (\text{im } \mathcal{L}_B)^\bullet, d)$ is trivial. In Sect. 3, while computing H_φ^\bullet , we explicitly computed the complex $((\ker \mathcal{L}_B)^\bullet \cap (\text{im } \mathcal{L}_B)^\bullet, d)$. The

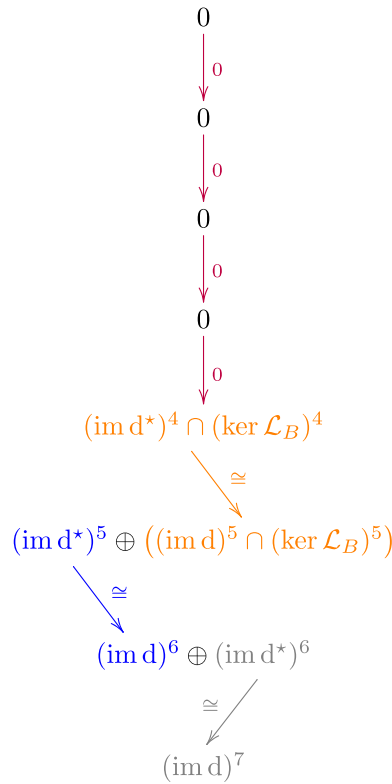
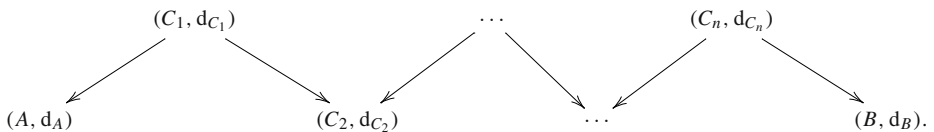


Fig. 7 The complex $((\ker \mathcal{L}_B \cap \text{im } \mathcal{L}_B)^\bullet, d)$

results are collected in Fig. 7. The isomorphisms in Fig. 7 are a subset of the isomorphisms from Fig. 6 and are colored in the same way. It is clear from Fig. 7 that the cohomology of $((\ker \mathcal{L}_B)^\bullet \cap (\text{im } \mathcal{L}_B)^\bullet, d)$ is trivial. \square

The next two definitions are taken from [10, Section 3.A].

Definition 4.5 Let (A, d_A) and (B, d_B) be two differential graded algebras (dga’s). We say that A and B are *equivalent* if there exists a finite sequence of *dga quasi-isomorphisms*



A dga (A, d_A) is called *formal* if it is equivalent to a dga (B, d_B) with $d_B = 0$. \square

It is well known [10, Section 3.A] that a *compact Kähler manifold* is formal. That is, the de Rham complex of a compact Kähler manifold is equivalent to a dga with zero differential. It is still an open question whether or not compact torsion-free G_2 manifolds are formal. We show in Theorem 4.6 that compact torsion-free G_2 manifolds are ‘almost formal’ in the sense that the de Rham complex is equivalent to a dga which has *only one nonzero differential*.

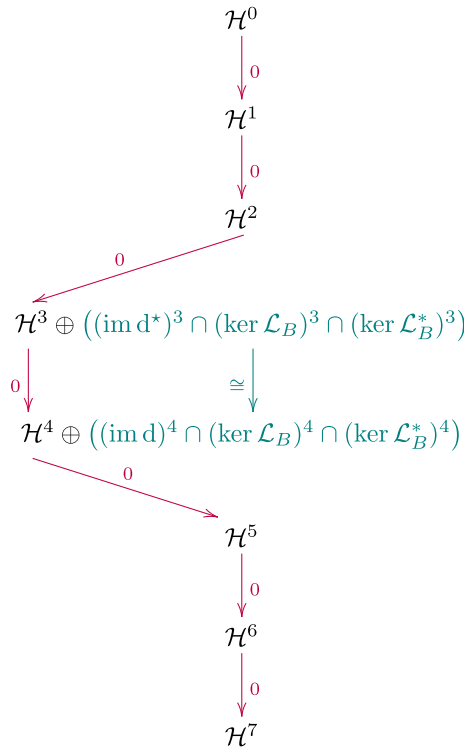


Fig. 8 The complex (H_φ^\bullet, d)

Theorem 4.6 *The de Rham complex of a compact torsion-free G_2 manifold (Ω^\bullet, d) is equivalent to (H_φ^\bullet, d) , which is a dga with all differentials trivial except for $d : H_\varphi^3 \rightarrow H_\varphi^4$.*

Proof In Sect. 3, we explicitly computed the complex (H_φ^\bullet, d) . The results are collected in Fig. 8. The isomorphism in Fig. 8 appeared already in Fig. 6 and is colored in the same way. The zero maps in Fig. 8 are a consequence of $\mathcal{H}^k \subseteq (\ker d)^k$. \square

One consequence of almost-formality is that *most of the Massey triple products of the de Rham complex will vanish*. This is established in Corollary 4.9 below.

Definition 4.7 Let (A, d_A) be a dga, and denote by $H^k(A)$ the degree k cohomology of A with respect to d_A . Let $[\alpha] \in H^p(A)$, $[\beta] \in H^q(A)$, $[\gamma] \in H^r(A)$ be cohomology classes satisfying

$$[\alpha][\beta] = 0 \in H^{p+q}(A) \quad \text{and} \quad [\beta][\gamma] = 0 \in H^{q+r}(A).$$

Then $\alpha\beta = df$ and $\beta\gamma = dg$ for some $f \in A^{p+q-1}$ and $g \in A^{q+r-1}$. Consider the class

$$[f\gamma - (-1)^p\alpha g] \in H^{p+q+r-1}(A).$$

It can be checked that this class is well defined up to an element of $H^{p+q-1} \cdot H^r + H^p \cdot H^{q+r-1}$. That is, it is well defined as an element of the quotient

$$\frac{H^{p+q+r-1}(A)}{H^{p+q-1} \cdot H^r + H^p \cdot H^{q+r-1}}.$$

We call this element the *Massey triple product* and write it as $\langle [\alpha], [\beta], [\gamma] \rangle$. It is easy to see that the Massey triple product is linear in each of its three arguments. \square

In the following, we only consider the case when $(A, d_A) = (\Omega^*, d)$ is the dga of smooth differential forms. If the dga (A, d_A) is formal, then *all the Massey triple products vanish* due to the naturality of the triple product (see [10, Proposition 3.A.33] for details). In fact, the proof of [10, Proposition 3.A.33] actually yields the following more general result.

Corollary 4.8 *Let (A, d_A) be a dga such that the differentials d_A are all zero except for $d : A^{k-1} \rightarrow A^k$. Then if the Massey triple product $\langle [\alpha], [\beta], [\gamma] \rangle$ is defined and we have $|\alpha| + |\beta| \neq k$ and $|\beta| + |\gamma| \neq k$, then $\langle [\alpha], [\beta], [\gamma] \rangle = 0$.*

Combining Corollary 4.8 and Theorem 4.6 yields the following.

Corollary 4.9 *Let M be a compact torsion-free G_2 manifold. Consider cohomology classes $[\alpha], [\beta]$, and $[\gamma] \in H_{dR}^\bullet$. If the Massey triple product $\langle [\alpha], [\beta], [\gamma] \rangle$ is defined and we have $|\alpha| + |\beta| \neq 4$ and $|\beta| + |\gamma| \neq 4$, then $\langle [\alpha], [\beta], [\gamma] \rangle = 0$.*

In Theorem 4.10 in the next section, we establish a stronger version of Corollary 4.9 when the holonomy of the metric on M is exactly G_2 .

4.2 A new topological obstruction to existence of torsion-free G_2 -structures

A key feature of the criterion in Corollary 4.9 is that it is *topological*. That is, *it does not depend on the differentiable structure on M* . Therefore, it gives a *new topological obstruction* to the existence of torsion-free G_2 -structures on compact 7-manifolds. There are several previously known topological obstructions to the existence of a torsion-free G_2 -structure on a compact 7-manifold. These obstructions are discussed in detail in [11, Chapter 10]. We summarize them here. Let φ be a torsion-free G_2 -structure on a compact manifold M with induced metric g_φ . Let $b_M^k = \dim H_{dR}^k(M)$. Then

$$\left. \begin{aligned}
 &b_M^3 \geq b_M^1 + b_M^0, \\
 &b_M^2 \geq b_M^1, \\
 &b_M^1 \in \{0, 1, 3, 7\}, \\
 &\text{if } g_\varphi \text{ is not flat, then } p_1(M) \neq 0, \text{ where } p_1(M) \text{ is the first Pontryagin class of } TM, \\
 &\text{if } g_\varphi \text{ has full holonomy } G_2, \text{ then the fundamental group } \pi_1(M) \text{ is finite.}
 \end{aligned} \right\} \tag{4.5}$$

Note that the first three conditions are simply obstructions to the existence of torsion-free G_2 -structures. The fourth condition can be used to rule out *non-flat* torsion-free G_2 -structures, and the fifth condition can be used to rule out *non-irreducible* torsion-free G_2 -structures. In fact, the third condition determines the reduced holonomy of g_φ , which is $\{1\}, \text{SU}(2), \text{SU}(3)$, or G_2 , if $b_M^1 = 7, 3, 1$, or 0 , respectively.

Theorem 4.10 *Let M be a compact torsion-free G_2 manifold with full holonomy G_2 , and consider cohomology classes $[\alpha], [\beta]$, and $[\gamma] \in H_{dR}^\bullet$. If the Massey triple product $\langle [\alpha], [\beta], [\gamma] \rangle$ is defined, then $\langle [\alpha], [\beta], [\gamma] \rangle = 0$ except possibly in the case when $|\alpha| = |\beta| = |\gamma| = 2$.*

Proof Recall that the hypothesis of full holonomy G_2 implies that $b_M^1 = 0$, so $H_{dR}^1 = \{0\}$. Suppose $|\alpha| = 1$. Then $[\alpha] \in H_{dR}^1$, so $[\alpha] = 0$, and by linearity it follows that

$\langle [\alpha], [\beta], [\gamma] \rangle = 0$. The same argument holds if $|\beta| = 1$ or $|\gamma| = 1$. Suppose $|\alpha| = 0$. Then α is a constant function. The condition $[\alpha\beta] = [\alpha][\beta] = 0$ forces the form $\alpha\beta$ to be exact, so either $\alpha = 0$ (in which case the Massey product vanishes), or β is exact, so $[\beta] = 0$ and again the Massey product vanishes. A similar argument holds if $|\beta| = 0$ or $|\gamma| = 0$.

Thus, we must have $|\alpha|, |\beta|, |\gamma| \geq 2$ if the Massey product has any chance of being nontrivial. Moreover, since $\langle [\alpha], [\beta], [\gamma] \rangle$ lies in a quotient of $H_{\text{dR}}^{|\alpha|+|\beta|+|\gamma|-1}$, we also need $|\alpha| + |\beta| + |\gamma| \leq 8$. Finally, Corollary 4.9 tells us that we must have either $|\alpha| + |\beta| = 4$ or $|\beta| + |\gamma| = 4$. Hence, the only possibilities for the triple $(|\alpha|, |\beta|, |\gamma|)$ to obtain a nontrivial Massey product are $(2, 2, 2), (2, 2, 3), (2, 2, 4), (3, 2, 2)$, and $(4, 2, 2)$. For $(2, 2, 3)$ or $(3, 2, 2)$, the Massey product lies in a quotient of H_{dR}^6 , which is zero since $b_M^6 = b_M^1 = 0$. For $(2, 2, 4)$ or $(4, 2, 2)$, the Massey product lies inside $H_{\text{dR}}^7 / (H_{\text{dR}}^2 \cdot H_{\text{dR}}^6 + H_{\text{dR}}^3 \cdot H_{\text{dR}}^4)$, but $H_{\text{dR}}^3 \cdot H_{\text{dR}}^4 = H_{\text{dR}}^7$ since $\varphi \wedge \psi = 7\text{vol}$ is a generator of H_{dR}^7 . Thus, in this case, the quotient space is zero. We conclude that the only possibly nontrivial Massey product corresponds to the case $(|\alpha|, |\beta|, |\gamma|) = (2, 2, 2)$. \square

In the remainder of this section, we will apply our new criterion to a particular nontrivial example. Consider a smooth compact connected oriented 7-manifold M of the form $M = W \times L$, where W and L are smooth compact connected oriented manifolds of dimensions 3 and 4, respectively. In order for M to admit G_2 -structures, we must have $w_2(M) = 0$, where $w_2(M)$ is the second Stiefel–Whitney class of TM , by [19, pp. 348–349].

Take W to be the real Iwasawa manifold, which is defined to be the quotient of the set

$$\left\{ \begin{pmatrix} 1 & t_1 & t_2 \\ 0 & 1 & t_3 \\ 0 & 0 & 1 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\} \cong \mathbb{R}^3$$

by the left multiplication of the group

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

The manifold W is a compact orientable 3-manifold, so it is parallelizable and hence $w_2(W) = 0$. Moreover, it is shown in [10, Example 3.A.34] that $b_W^1 = 2$ and that

$$\text{there exist } \alpha, \beta \in H_{\text{dR}}^1(W) \text{ such that } \langle \alpha, \beta, \beta \rangle \neq 0. \tag{4.6}$$

By the Whitney product formula, we have $w_2(M) = w_2(W) + w_2(L)$. Thus, if we choose L to have vanishing w_2 , then $w_2(M)$ will vanish as required, and $M = W \times L$ will admit G_2 -structures.

Theorem 4.11 *Let L be a smooth compact connected oriented 4-manifold with $w_2(L) = 0$, and let W be the real Iwasawa manifold described above. Then $M = W \times L$ admits G_2 -structures but cannot admit any torsion-free G_2 -structures.*

Proof Let $\pi : M \rightarrow W$ be the projection map. Consider the classes $\pi^*\alpha, \pi^*\beta \in H_{\text{dR}}^1(M)$. By naturality of the Massey triple product, and since $p = q = r = 1$, we have

$$\langle \pi^*\alpha, \pi^*\beta, \pi^*\beta \rangle = \pi^*\langle \alpha, \beta, \beta \rangle \in \frac{H^2(M)}{H^1(M) \cdot H^1(M)}.$$

Let $s : W \rightarrow W \times L$ be any section of π . Since $s^*\pi^* = (\pi \circ s)^* = \text{Id}$, we deduce that

$$\pi^* : \frac{H^2(W)}{H^1(W) \cdot H^1(W)} \rightarrow \frac{H^2(M)}{H^1(M) \cdot H^1(M)} \text{ is injective.}$$

Thus, since $\langle \alpha, \beta, \beta \rangle \neq 0$ we have

$$\langle \pi^* \alpha, \pi^* \beta, \pi^* \beta \rangle = \pi^* \langle \alpha, \beta, \beta \rangle \neq 0.$$

Since $|\pi^* \alpha| = |\pi^* \beta| = 1$ and $1 + 1 \neq 4$, we finally conclude by Corollary 4.9 that M does not admit a torsion-free G_2 -structure. \square

It remains to find an L with $w_2(L) = 0$ such that no previously known topological obstructions (4.5) are violated, so that we have indeed established something new. We first collect several preliminary results that we will require.

By Poincaré duality $b_L^3 = b_L^1$ and $b_W^2 = b_W^1 = 2$. The Künneth formula therefore yields

$$\left. \begin{aligned} b_M^1 &= b_W^1 + b_L^1 = 2 + b_L^1, \\ b_M^2 &= b_W^2 + b_W^1 b_L^1 + b_L^2 = 2 + 2b_L^1 + b_L^2, \\ b_M^3 &= b_W^3 + b_W^2 b_L^1 + b_W^1 b_L^2 + b_L^3 = 1 + 2b_L^1 + 2b_L^2 + b_L^1 = 1 + 3b_L^1 + 2b_L^2. \end{aligned} \right\} (4.7)$$

Remark 4.12 Let M, N be smooth compact oriented n -manifolds. There is a canonical way to make the connected sum $M\#N$ smooth, by smoothing around the S^{n-1} with which we paste them together. With coefficients in either $R = \mathbb{Z}$ or $R = \mathbb{Z}/2\mathbb{Z}$, we have $H^k(M\#N, R) \cong H^k(M, R) \oplus H^k(N, R)$ for $k = 1, \dots, n - 1$. This can be seen using the Mayer–Vietoris sequence. The isomorphism is induced by the map $p : M\#N \rightarrow M$ collapsing N , and the map $q : M\#N \rightarrow N$ collapsing M . For $k = n$, we have $H^k(M\#N) \cong H^n(M, R) \cong H^n(N, R)$ with isomorphisms induced by p and q as before. \square

Lemma 4.13 Let L be a simply connected smooth compact oriented 4-manifold, with intersection form

$$Q : H_2(L, \mathbb{Z}) \times H_2(L, \mathbb{Z}) \rightarrow \mathbb{Z}.$$

If the signature of Q is (p, q) , let $\sigma(L) = p - q$. Then we have

- $w_2(L) = 0$ if and only if $Q(a, a) \in 2\mathbb{Z}$ for all $a \in H^2(L, \mathbb{Z})$;
- $p_1(L) = 0$ if and only if $\sigma(L)$ is zero.

Proof The first statement can be found in [19, Corollary 2.12]. The Hirzebruch signature theorem for 4-manifolds, which can be found in [8, Theorem 1.4.12], says that $p_1(L) = 3\sigma(L)$. This immediately implies the second statement. \square

Recall that K3 is the unique connected simply connected smooth manifold underlying any compact complex surface with vanishing first Chern class. One way to define the K3 surface is by

$$K3 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}.$$

It is well known (see [8, Page 75] or [21, Pages 127–133]) that K3 has intersection form $Q_{K3} = -2E_8 \oplus 3H$, where E_8 is a certain even positive definite bilinear form, and $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is also even and has signature 0. It follows that Q_{K3} has signature $(3, 19)$, and thus, $\sigma(K3) = -16$. We also have that the Betti numbers of K3 are $b_{K3}^1 = b_{K3}^3 = 0$ and $b_{K3}^2 = 22$.

Proposition 4.14 Let $L = K3\#(S^1 \times S^3)$. Then $w_2(L) = 0$, and for $M = W \times L$ where W is the real Iwasawa manifold, none of the first four topological obstructions (4.5) are violated. Thus, M cannot admit any torsion-free G_2 -structure.

Proof Since $b_{S^1 \times S^3}^1 = b_{S^1 \times S^3}^3 = 1$ and $b_{S^1 \times S^3}^2 = 0$, Remark 4.12 tells us that the Betti numbers of L are $b_L^1 = b_L^3 = 1$ and $b_L^2 = 22$. In [8, Pages 20, 456], it is shown that $Q_{M\#N} = Q_M \oplus Q_N$, and consequently $\sigma(M\#N) = \sigma(M) + \sigma(N)$. Since $Q_{S^1 \times S^3} = 0$, we find that Q_L is even and has nonzero signature. Thus, by Lemma 4.13, we deduce that $p_1(L) \neq 0$ and $w_2(L) = 0$. Now Eq. (4.7) tell us that the Betti numbers of M are $b_M^1 = 3$, $b_M^2 = 26$, and $b_M^3 = 48$. In particular, the first three conditions in (4.5) are satisfied.

We now claim that $p_1(M) \neq 0$. To see this, consider the inclusion $\iota : L \rightarrow M = W \times L$ into some vertical fiber $\{*\} \times L$ of M over W . Then $\iota^*(TM) = TL \oplus E$ where W is the trivial rank 3 real vector bundle over L . If $p_1(TM) = 0$, then by naturality we have $p_1(TL) = \iota^*(p_1(TM)) = 0$, which we showed was not the case. Thus, the fourth condition in (4.5) is satisfied. \square

Remark 4.15 Because $b_M^1 = 3$, if M had any compact torsion-free G_2 -structure, it would have reduced holonomy $SU(2)$. We have shown in Proposition 4.14 that such a Riemannian metric cannot exist on M . It is not clear whether there is any simpler way to rule out such a Riemannian metric on M . \square

Other examples of compact orientable spin 7-manifolds that cannot be given a torsion-free G_2 -structure can likely be constructed similarly.

Remark 4.16 The formality of compact 7-manifolds with additional structure has been studied by several authors, in particular recently by Crowley–Nordström [5] and Muñoz–Tralle [20]. Two of the results in [5] are: There exist non-formal compact 7-manifolds that have only trivial Massey triple products, and a non-formal compact manifold M with G_2 holonomy must have $b^2(M) \geq 4$. One of the results in [20] is that a compact simply connected 7-dimensional Sasakian manifold is formal if and only if all its triple Massey products vanish. \square

Remark 4.17 A natural question is: Can we actually establish *formality* by extending our chain of quasi-isomorphisms? One idea is to quotient out the unwanted summands, but such a quotient map is not a dga morphism. One can also try to involve \mathcal{L}_K or other operators that can descend to H_φ^\bullet , but the authors have so far had no success in this direction. \square

Acknowledgements These results were obtained in 2017 as part of the collaboration between the COSINE program organized by the Chinese University of Hong Kong and the URA program organized by the University of Waterloo. The authors thank both universities for this opportunity. Part of the writing was done, while the second author held a Fields Research Fellowship at the Fields Institute. The second author thanks the Fields Institute for their hospitality. The authors also thank the anonymous referee for pointing out that we had actually also established Theorem 4.10, which is the stronger version of Corollary 4.9 in the case of full G_2 holonomy.

References

1. Besse, A.L.: Einstein Manifolds. Springer, New York (1987)
2. Bryant, R.L.: Some remarks on G_2 -structures. In: Proceedings of Gökova Geometry-Topology Conference 2005, pp. 75–109. [arXiv:math/0305124](https://arxiv.org/abs/math/0305124)
3. Cavalcanti, G.: New aspects of the dd^c lemma. Ph.D. Thesis. http://www.staff.science.uu.nl/~caval101/homepage/Research_files/thesis.pdf. Accessed 15 Nov 2017
4. Chan, K.F., Karigiannis, S., Tsang, C.C.: Cohomologies on almost complex manifolds and the $\partial\bar{\partial}$ -lemma. Preprint [arXiv:1710.04695](https://arxiv.org/abs/1710.04695)
5. Crowley, D., Nordström, J.: The rational homotopy type of $(n - 1)$ -connected manifolds of dimension up to $5n - 3$. Preprint [arXiv:1505.04184v2](https://arxiv.org/abs/1505.04184v2)

6. de la Ossa, X., Karigiannis, S., Svanes, E.: Geometry of Um -structures: Kähler identities, the dd^c lemma, and Hodge theory (in preparation)
7. Fernández, M., Gray, A.: Riemannian manifolds with structure group G_2 . *Ann. Mat. Pura Appl.* **32**, 19–45 (1982)
8. Gompf, R.E., Stipsicz, A.I.: *4-Manifolds and Kirby Calculus*. American Mathematical Society, Providence (1999)
9. Harvey, R., Lawson, H.B.: Calibrated geometries. *Acta Math.* **148**, 47–157 (1982)
10. Huybrechts, D.: *Complex Geometry*, Universitext. Springer, Berlin (2005)
11. Joyce, D.D.: *Compact Manifolds with Special Holonomy*. Oxford University Press, Oxford (2000)
12. Karigiannis, S.: Flows of G_2 -structures, I. *Q. J. Math.* **60**, 487–522 (2009). [arXiv:math.DG/0702077](https://arxiv.org/abs/math/0702077)
13. Karigiannis, S.: Some notes on G_2 and $Spin7$ geometry. In: *Recent Advances in Geometric Analysis; Advanced Lectures in Mathematics*, vol. 11, pp. 129–146. International Press. [arxiv:math/0608618](https://arxiv.org/abs/math/0608618) (2010)
14. Karigiannis, S.: Geometry of G_2 -structures (in preparation)
15. Karigiannis, S., Lin, C., Loftin, J.: Octonionic–algebraic structure and curvature of the Teichmüller space of G_2 manifolds (in preparation)
16. Kawai, K., Lê, H.V., Schwachhöfer, L.: The Frölicher–Nijenhuis bracket and the geometry of G_2 - and $Spin7$ -manifolds. *L. Ann. Mat.* (2017). <https://doi.org/10.1007/s10231-017-0685-9>
17. Kawai, K., Lê, H.V., Schwachhöfer, L.: Frölicher–Nijenhuis cohomology on G_2 - and $Spin7$ -manifolds. [arXiv:1703.05133](https://arxiv.org/abs/1703.05133)
18. Kolář, I., Michor, P.W., Slovák, J.: *Natural Operations in Differential Geometry*. Springer, Berlin (1993)
19. Lawson, H.B., Michelsohn, M.L.: *Spin Geometry*. Princeton University Press, Princeton (1989)
20. Muñoz, V., Tralle, A.: Simply-connected K -contact and Sasakian manifolds of dimension 7. Preprint [arXiv:1408.2496](https://arxiv.org/abs/1408.2496)
21. Scorpan, A.L.: *The Wild World of 4-manifolds*. American Mathematical Society, Providence (2005)
22. Verbitsky, M.: Manifolds with parallel differential forms and Kähler identities for G_2 manifolds. *J. Geom. Phys.* **61**, 1001–1016 (2011). [arXiv:math/0502540v8](https://arxiv.org/abs/math/0502540v8)