Annals of Global Analysis and Geometry (2019) 55:197–213
https://doi.org/10.1007/s10455-018-9623-3

Dual quadratic differentials and entire minimal graphs in Heisenberg space

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Received: 20 April 2018 / Accepted: 12 July 2018 / Published online: 21 July 2018

Abstract
We define holomorphic quadratic differentials for spacelike surfaces with constant mean curvature in the Lorentzian homogeneous spaces $\mathbb{L}(\kappa, \tau)$ with isometry group of dimension 4, which are dual to the Abresch–Rosenberg differentials in the Riemannian counterparts $\mathbb{E}(\kappa, \tau)$, and obtain some consequences. On the one hand, we classify explicitly those surfaces in $\mathbb{L}(\kappa, \tau)$ with zero differential. On the other hand, we prove that entire minimal graphs in Heisenberg space have negative Gauss curvature.

Keywords Minimal surfaces · Constant mean curvature · Homogeneous 3-manifolds · Heisenberg group · Lorentz–Minkowski space · Quadratic differentials · Gauss curvature

Mathematics Subject Classification Primary 53A10; Secondary 53C30

1 Introduction

The theory of constant mean curvature surfaces in homogeneous 3-manifolds has been a rather active research field during the last decades, and specially since Abresch and Rosenberg [1,2] defined holomorphic quadratic differentials on constant mean curvature H surfaces (H-surfaces in the sequel) immersed in simply connected homogeneous Riemannian 3-manifolds $\mathbb{E}(\kappa, \tau)$ with isometry group of dimension 4. These differentials extended the classical Hopf differentials in space forms and revealed that H-spheres in $\mathbb{E}(\kappa, \tau)$, which exist if and only if $4H^2 + \kappa > 0$, are rotationally invariant. The value of H, if any, such that $4H^2 + \kappa = 0$ is usually called the *critical mean curvature*. The classification of H-spheres

This research has been partially supported by Spanish MINECO-Feder research grants MTM2014-52368-P and MTM2017-89677-P (Universidad de Granada) and SEV-2015-0554 (ICMAT Severo Ochoa), and by the EPSRC Grant No. EP/M024512/1. The author is grateful to Hojoo Lee who, when heard about this manuscript, provided the author with an unpublished preprint from 2009, in which he also proves Lemma 3.1, and other results closely related to Proposition A.1.

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has been generalized recently to the rest of homogeneous 3-manifolds by Meeks, Mira, Pérez, and Ros [24].

Given $\kappa, \tau \in \mathbb{R}$, the space $\mathbb{E}(\kappa, \tau)$ is characterized by admitting a Killing submersion with unitary Killing vector field and constant bundle curvature τ over the complete simply connected surface $\mathbb{M}^2(\kappa)$ with constant curvature κ, see [7,19]. They include the product spaces $\mathbb{H}^2(\kappa) \times \mathbb{R}$ and $\mathbb{S}^2(\kappa) \times \mathbb{R}$, the Heisenberg space $\mathrm{Nil}_3(\tau)$, and the Lie groups SU(2) and $\widetilde{\mathrm{Sl}}_2(\mathbb{R})$ endowed with special left-invariant metrics. There are homogeneous Lorentzian counterparts $\mathbb{L}(\kappa, \tau)$ enjoying the same description, but such that the Killing direction is timelike [15,16]. Some space forms also live among these spaces, namely the Euclidean space $\mathbb{R}^3 = \mathbb{E}(0, 0)$, the round spheres $\mathbb{S}^3(\kappa) = \mathbb{E}(\kappa, \frac{1}{2}\sqrt{\kappa})$ with $\kappa > 0$, the Lorentz–Minkowski space $\mathbb{L}^3 = \mathbb{L}(0, 0)$, and the universal covers of the anti-de Sitter spaces $\mathbb{H}^3_1(\kappa) = \mathbb{L}(\kappa, \frac{1}{2}\sqrt{-\kappa})$ with $\kappa < 0$.

A celebrated result for H-surfaces in $\mathbb{E}(\kappa, \tau)$-spaces was the solution to the Bernstein problem in Heisenberg space $\mathrm{Nil}_3(\frac{1}{2}) = \mathbb{E}(0, \frac{1}{2})$ by Fernández and Mira [12]: For each holomorphic quadratic differential Q on a non-compact simply connected Riemann surface Σ (with $Q \neq 0$ if Σ is parabolic), there is a 2-parameter family of entire minimal graphs (i.e., global minimal sections of the Killing submersion) in $\mathrm{Nil}_3(\frac{1}{2})$ with Abresch–Rosenberg differential Q. They use Wan and Au's classification of entire spacelike $\frac{1}{2}$-graphs in $\mathbb{L}^3$ in terms of their Hopf differential [29,30].

More recently, Lee [15] found an interesting duality between H-graphs in $\mathbb{E}(\kappa, \tau)$ and spacelike τ-graphs in $\mathbb{L}(\kappa, H)$ over simply connected subdomains of $\mathbb{M}^2(\kappa)$. This duality swaps the mean curvature and the bundle curvature, and generalizes the classical Calabi duality [4] between minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{L}^3$. It easily extends to spacelike conformal immersions with constant mean curvature which are not necessarily graphs [20]. The duality gives a reason why surfaces with critical mean curvature enjoy special properties: they are those whose dual surfaces lie in constant sectional curvature spaces, where the geometry is richer. Furthermore, when the duality is restricted to minimal surfaces in $\mathrm{Nil}_3(\frac{1}{2})$, we get a shortcut in Fernández and Mira's arguments, avoiding the Daniel correspondence [7] and the Gauss maps of critical mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathrm{Nil}_3(\tau)$, see also [8]. This shortcut will be discussed in "Appendix", where we will also provide a 2-parameter geometric deformation of entire minimal graphs in $\mathrm{Nil}_3(\tau)$ which preserves the Abresch–Rosenberg differential.

The main goal of this paper is to obtain new applications of the duality connecting the conformal theories of H-surfaces in the Riemannian and Lorentzian settings. On the one hand, we define holomorphic quadratic differentials for spacelike τ-surfaces in $\mathbb{L}(\kappa, H)$, which are dual to the Abresch–Rosenberg differentials (see Theorem 2.3). In particular, the Abresch–Rosenberg differential of minimal graphs in $\mathrm{Nil}_3(\tau)$ and the Hopf differential of τ-graphs in $\mathbb{L}^3$ are shown to be dual. We will elude the use of conformal parameters to obtain explicit expressions of the differentials in terms of the functions defining the graphs. We will also give an explicit classification of τ-surfaces with zero differential in $\mathbb{L}(\kappa, H)$, dual to the classification in $\mathbb{E}(\kappa, \tau)$ given in [10] by Domínguez-Vázquez and the author. In particular, it follows that a τ-surface Σ immersed in $\mathbb{L}(\kappa, H)$ with zero differential must be embedded and equivariant, but possibly Σ cannot be extended to a complete surface.

On the other hand, we employ the duality to prove that entire graphs with critical mean curvature in $\mathbb{E}(\kappa, \tau)$, $\kappa - 4\tau^2 \neq 0$, have negative Gauss curvature. To this effect, we shall use a curvature estimate for entire τ-graphs in $\mathbb{L}^3$ due to Cheng and Yau [5], see also [6]. The proof of our curvature estimate will also rely on a fancy relation between the Hessian determinants of dual graphs, which follows from comparing the moduli of the dual differentials (see

Lemma 3.1 and Remark 3.2). It is worth mentioning that the Gauss equation for surfaces in $\mathbb{E}(\kappa,\tau)$ reads

$$K = \det(A) + \tau^2 + (\kappa - 4\tau^2)\nu^2, \tag{1.1}$$

where K, A and ν denote the Gauss curvature, the shape operator, and the angle function of the surface, respectively [7,11]. If $\kappa + 4H^2 = 0$, then Equation (1.1) gives the estimate $K \le \tau^2 + H^2$, which is not sharp for entire graphs. Nonetheless, there do exist complete properly embedded minimal surfaces in $\mathrm{Nil}_3(\tau)$ whose Gauss curvature changes sign (e.g., some helicoids, see Remark 3.6).

By a simple comparison with the Euclidean plane, we deduce that entire minimal graphs in $\mathrm{Nil}_3(\tau)$ have at least quadratic area growth, and it is quadratic if and only if the graph has finite total curvature by the results of Hartman [14] and Li [17]. This allows us to rephrase (in terms of curvature) the conjecture posed by Pérez, Rodríguez and the author in [21] that there are no entire minimal graphs in $\mathrm{Nil}_3(\tau)$ with quadratic area growth. We also get the necessary condition that such an entire graph with intrinsic quadratic area growth must converge to a vertical plane along any diverging sequence, so the Gauss curvature and the angle function of the graph must vanish at infinity, see Proposition 3.7. This condition is not sufficient, and minimal umbrellas are counterexamples, see Remark 3.5. No monotonicity formula for minimal surfaces in $\mathrm{Nil}_3(\tau)$ has been found so far, which is one of the key techniques leading to the same result in $\mathbb{R}^3$, and Theorem 3.4 is the first lower bound for the area growth of entire minimal graphs in $\mathrm{Nil}_3(\tau)$. We remark that Nelli and the author [22] proved the intrinsic area growth of an entire minimal graph in $\mathrm{Nil}_3(\tau)$ is at most cubic, which is a sharp upper bound.

2 The conformal duality

Throughout the paper $\mathbb{E}(\kappa,\tau)$ (resp. $\mathbb{L}(\kappa,\tau)$), with $\kappa,\tau\in\mathbb{R}$, will denote the unique simply connected Riemannian (resp. Lorentzian) 3-manifold admitting a Killing submersion π (resp. $\widetilde{\pi}$) with constant bundle curvature τ over the complete simply connected surface $\mathbb{M}^2(\kappa)$ with constant curvature κ, see [7,19]. The duality is a natural correspondence between constant mean curvature surfaces.

Theorem 2.1 (Conformal duality [15,20]) *Let Σ be a simply connected Riemann surface, and let $\tau,\kappa,H\in\mathbb{R}$. There is a duality between*

(a) *nowhere vertical conformal H-immersions $X:\Sigma\to\mathbb{E}(\kappa,\tau)$,*
(b) *spacelike conformal τ-immersions $\widetilde{X}:\Sigma\to\mathbb{L}(\kappa,H)$.*

The dual immersions X and $\widetilde{X}$ are determined up to a vertical translation, and satisfy $\pi\circ X = \widetilde{\pi}\circ\widetilde{X}$.

We will introduce coordinates in order to build up an appropriate framework to develop the duality. (A coordinate free approach was given in [20].) Let us define

$$\lambda(x,y) = \left(1 + \tfrac{\kappa}{4}(x^2+y^2)\right)^{-1}$$

over the simply connected domain $\Omega_\kappa = \left\{(x,y)\in\mathbb{R}^2 : 1 + \tfrac{\kappa}{4}(x^2+y^2) > 0\right\}$. Then $(\Omega_\kappa, \lambda^2(\mathrm{d}x^2+\mathrm{d}y^2))$ has constant curvature κ, and the spaces $\mathbb{E}(\kappa,\tau)$ and $\mathbb{L}(\kappa,H)$ can be locally modeled as the manifold $\Omega_\kappa\times\mathbb{R}$ endowed with the metrics

$$\begin{aligned} \mathrm{d}s^2_{\mathbb{E}(\kappa,\tau)} &= \lambda^2(\mathrm{d}x^2+\mathrm{d}y^2) + (\mathrm{d}z^2 + \tau\lambda(y\mathrm{d}x - x\mathrm{d}y))^2,\\ \mathrm{d}s^2_{\mathbb{L}(\kappa,H)} &= \lambda^2(\mathrm{d}x^2+\mathrm{d}y^2) - (\mathrm{d}z^2 - H\lambda(y\mathrm{d}x - x\mathrm{d}y))^2. \end{aligned} \tag{2.1}$$

If $\kappa \leq 0$, then (2.1) provides global models. If $\kappa > 0$, then $\mathbb{E}(\kappa, 0)$ and $\mathbb{L}(\kappa, 0)$ are diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ and the model omits a whole fiber; if $\kappa > 0$, and $\tau, H \neq 0$, then $\mathbb{E}(\kappa, \tau)$ and $\mathbb{L}(\kappa, H)$ are diffeomorphic to $\mathbb{S}^3$ and the model represents the universal cover of $\mathbb{S}^3$ minus a fiber of the Hopf fibration. In this last case, $\mathbb{E}(\kappa, \tau)$ and $\mathbb{L}(\kappa, H)$ are the Lie group SU(2) equipped with Riemannian or Lorentzian left-invariant metrics with four-dimensional isometry group. However, since these coordinates will be used in local computations, the global topology of the spaces will not become actually an issue.

We can define global orthonormal frames $\{E_1, E_2, E_3\}$ and $\{\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3\}$ for $\mathrm{d}s^2_{\mathbb{E}(\kappa,\tau)}$ and $\mathrm{d}s^2_{\mathbb{L}(\kappa,H)}$, respectively, where $\widetilde{E}_3$ is timelike, as

$$\begin{aligned} &E_1 = \tfrac{1}{\lambda}\partial_x - \tau y \partial_z, \quad &E_2 = \tfrac{1}{\lambda}\partial_y + \tau x \partial_z, \quad &E_3 = \partial_z,\\ &\widetilde{E}_1 = \tfrac{1}{\lambda}\partial_x + H y \partial_z, \quad &\widetilde{E}_2 = \tfrac{1}{\lambda}\partial_y - H x \partial_z \quad &\widetilde{E}_3 = \partial_z. \end{aligned} \tag{2.2}$$

The Killing submersion in this model is nothing but $(x, y, z) \mapsto (x, y)$, and the vertical Killing vector field is ∂_z. The Levi–Civita connection $\overline{\nabla}$ of $\mathbb{E}(\kappa, \tau)$ can be evaluated at the frame $\{E_1, E_2, E_3\}$ by means of Koszul formula, giving rise to

$$\begin{aligned} &\overline{\nabla}_{E_1} E_1 = \tfrac{\kappa}{2} y E_2, \quad &\overline{\nabla}_{E_1} E_2 = -\tfrac{\kappa}{2} y E_1 + \tau E_3, \quad &\overline{\nabla}_{E_1} E_3 = -\tau E_2,\\ &\overline{\nabla}_{E_2} E_1 = -\tfrac{\kappa}{2} x E_2 - \tau E_3, \quad &\overline{\nabla}_{E_2} E_2 = \tfrac{\kappa}{2} x E_1, \quad &\overline{\nabla}_{E_2} E_3 = \tau E_1,\\ &\overline{\nabla}_{E_3} E_1 = -\tau E_2, \quad &\overline{\nabla}_{E_3} E_2 = \tau E_1, \quad &\overline{\nabla}_{E_3} E_3 = 0, \end{aligned} \tag{2.3}$$

The Levi–Civita connection $\widetilde{\nabla}$ of $\mathbb{L}(\kappa, H)$ admits a representation very similar to (2.3), since it also satisfies the same Koszul formula (see [25]):

$$\begin{aligned} &\widetilde{\nabla}_{\widetilde{E}_1} \widetilde{E}_1 = \tfrac{\kappa}{2} y \widetilde{E}_2, \quad &\widetilde{\nabla}_{\widetilde{E}_1} \widetilde{E}_2 = -\tfrac{\kappa}{2} y \widetilde{E}_1 - H \widetilde{E}_3, \quad &\widetilde{\nabla}_{\widetilde{E}_1} \widetilde{E}_3 = -H \widetilde{E}_2,\\ &\widetilde{\nabla}_{\widetilde{E}_2} \widetilde{E}_1 = -\tfrac{\kappa}{2} x \widetilde{E}_2 + H \widetilde{E}_3, \quad &\widetilde{\nabla}_{\widetilde{E}_2} \widetilde{E}_2 = \tfrac{\kappa}{2} x \widetilde{E}_1, \quad &\widetilde{\nabla}_{\widetilde{E}_2} \widetilde{E}_3 = H \widetilde{E}_1,\\ &\widetilde{\nabla}_{\widetilde{E}_3} \widetilde{E}_1 = -H \widetilde{E}_2, \quad &\widetilde{\nabla}_{\widetilde{E}_3} \widetilde{E}_2 = H \widetilde{E}_1, \quad &\widetilde{\nabla}_{\widetilde{E}_3} \widetilde{E}_3 = 0. \end{aligned} \tag{2.4}$$

2.1 The duality in coordinates

For any $u \in C^\infty(\Omega)$ over some open domain $\Omega \subset \Omega_\kappa$, we can define the vertical graph of u over the section $(x, y) \mapsto (x, y, 0)$, as the surface $\Sigma_u \subset \Omega_\kappa \times \mathbb{R}$ given by

$$\Sigma_u = \{(x, y, u(x, y)) : (x, y) \in \Omega\}.$$

Observe that the duality in Theorem 2.1 restricts to a duality between graphs over simply connected domains. If $\Sigma_u \subset \mathbb{E}(\kappa, \tau)$ and $\Sigma_v \subset \mathbb{L}(\kappa, H)$ are dual graphs over $\Omega \subseteq \mathbb{M}^2(\kappa)$ for some $u, v \in C^\infty(\Omega)$, let us define the quantities

$$\begin{aligned} &\alpha = \tfrac{1}{\lambda} u_x + \tau y, \quad &\beta = \tfrac{1}{\lambda} u_y - \tau x, \quad &\omega = (1 + \alpha^2 + \beta^2)^{1/2},\\ &\widetilde{\alpha} = \tfrac{1}{\lambda} v_x - H y, \quad &\widetilde{\beta} = \tfrac{1}{\lambda} v_y + H x, \quad &\widetilde{\omega} = (1 - \widetilde{\alpha}^2 - \widetilde{\beta}^2)^{1/2}. \end{aligned} \tag{2.5}$$

Then Σ_u and Σ_v are dual graphs if and only if they satisfy the so-called *twin relations* [15], namely the equivalent identities

$$(\widetilde{\alpha}, \widetilde{\beta}) = \left(\frac{\beta}{\omega}, \frac{-\alpha}{\omega}\right) \iff (\alpha, \beta) = \left(\frac{-\widetilde{\beta}}{\widetilde{\omega}}, \frac{\widetilde{\alpha}}{\widetilde{\omega}}\right), \tag{2.6}$$

in which case we also have $\widetilde{\omega} = \frac{1}{\omega}$.

Consider the global frames $\{e_1, e_2\}$ in Σ_u and $\{\widetilde{e}_1, \widetilde{e}_2\}$ in Σ_v defined as

$$\begin{aligned} e_1 &= E_1 + \alpha E_3, \quad \widetilde{e}_1 = \widetilde{E}_1 + \widetilde{\alpha}\widetilde{E}_3, \\ e_2 &= E_2 + \beta E_3, \quad \widetilde{e}_2 = \widetilde{E}_2 + \widetilde{\beta}\widetilde{E}_3. \end{aligned} \tag{2.7}$$

The first fundamental forms of Σ_u and Σ_v in these frames are given by

$$\mathrm{I} \equiv \begin{pmatrix} 1+\alpha^2 & \alpha\beta \\ \alpha\beta & 1+\beta^2 \end{pmatrix}, \quad \widetilde{\mathrm{I}} \equiv \begin{pmatrix} 1-\widetilde{\alpha}^2 & -\widetilde{\alpha}\widetilde{\beta} \\ -\widetilde{\alpha}\widetilde{\beta} & 1-\widetilde{\beta}^2 \end{pmatrix}, \tag{2.8}$$

respectively, so (2.6) implies that the two matrices in (2.8) are proportional with $\widetilde{\mathrm{I}} = \omega^{-2}\mathrm{I} = \widetilde{\omega}^2\mathrm{I}$. This means that the global diffeomorphism

$$\Phi : \Sigma_u \to \Sigma_v, \qquad \Phi(x, y, u(x, y)) = (x, y, v(x, y)). \tag{2.9}$$

is conformal. In terms of conformal immersions $X : \Sigma \to \Sigma_u$ and $\widetilde{X} : \Sigma \to \Sigma_v$, the condition $\pi \circ X = \widetilde{\pi} \circ \widetilde{X}$ yields $\Phi = \widetilde{X} \circ X^{-1}$, so the metrics induced by dual immersions are related by the following identity:

$$X^*\mathrm{d}s^2_{\mathbb{E}(\kappa,\tau)} = \omega^2 \widetilde{X}^*\mathrm{d}s^2_{\mathbb{L}(\kappa,H)}. \tag{2.10}$$

From (2.7), we deduce that

$$N = -\frac{\alpha}{\omega}E_1 - \frac{\beta}{\omega}E_2 + \frac{1}{\omega}E_3, \quad \widetilde{N} = \frac{\widetilde{\alpha}}{\widetilde{\omega}}\widetilde{E}_1 + \frac{\widetilde{\beta}}{\widetilde{\omega}}\widetilde{E}_2 + \frac{1}{\widetilde{\omega}}\widetilde{E}_3. \tag{2.11}$$

are the upward-pointing unit normal vector fields to Σ_u and Σ_v, respectively. Hence (2.6) implies that the *angle functions* $\nu = \langle N, E_3\rangle = \omega^{-1}$ and $\widetilde{\nu} = \langle \widetilde{N}, \widetilde{E}_3\rangle = \widetilde{\omega}^{-1}$ are multiplicative inverse. Moreover, N (resp. $\widetilde{N}$) allows us to define a $\frac{\pi}{2}$-rotation in the tangent bundle as $JX = N \times X$ (resp. $JX = \widetilde{N} \times X$). Here the orientation in $\mathbb{E}(\kappa, \tau)$ and $\mathbb{L}(\kappa, H)$ is essential so that one can define the cross product $\times$, namely $\langle N \times X, Y\rangle = \det(N, X, Y)$ for all vector fields X, Y in both Riemannian and Lorentzian cases, see [18].

In particular, from (2.7) and (2.11), we deduce the following identities, which will be helpful in Section 2.2:

$$\begin{aligned} Je_1 &= -\frac{\alpha\beta}{\omega}e_1 + \frac{1+\alpha^2}{\omega}e_2, \quad J\widetilde{e}_1 = \frac{\widetilde{\alpha}\widetilde{\beta}}{\widetilde{\omega}}\widetilde{e}_1 + \frac{1-\widetilde{\alpha}^2}{\widetilde{\omega}}\widetilde{e}_2, \\ Je_2 &= -\frac{1+\beta^2}{\omega}e_1 + \frac{\alpha\beta}{\omega}e_2, \quad J\widetilde{e}_2 = -\frac{1-\widetilde{\beta}^2}{\widetilde{\omega}}\widetilde{e}_1 - \frac{\widetilde{\alpha}\widetilde{\beta}}{\widetilde{\omega}}\widetilde{e}_2. \end{aligned} \tag{2.12}$$

The second fundamental forms of Σ_u and Σ_v are defined as $\sigma(X, Y) = \langle \overline{\nabla}_X Y, N\rangle$ and $\widetilde{\sigma}(X, Y) = \langle \widetilde{\nabla}_X Y, \widetilde{N}\rangle$, respectively. Using (2.7), (2.8), (2.11), and the Levi–Civita connection (2.3), we reach the following identities for σ:

$$\begin{aligned} \sigma(e_1, e_1) &= \frac{1}{\omega}\left(\frac{u_{xx}}{\lambda^2} + 2\tau\alpha\beta + \frac{\kappa}{2}(x\alpha - y\beta) - \frac{\kappa\tau}{2}xy\right), \\ \sigma(e_1, e_2) &= \frac{1}{\omega}\left(\frac{u_{xy}}{\lambda^2} + \tau(\beta^2 - \alpha^2) + \frac{\kappa}{2}(x\beta + y\alpha) + \frac{\kappa\tau}{4}(x^2 - y^2)\right), \\ \sigma(e_2, e_2) &= \frac{1}{\omega}\left(\frac{u_{yy}}{\lambda^2} - 2\tau\alpha\beta - \frac{\kappa}{2}(x\alpha - y\beta) + \frac{\kappa\tau}{2}xy\right). \end{aligned} \tag{2.13}$$

Analogously, we get the corresponding expressions for $\widetilde{\sigma}$ by means of (2.4):

$$\begin{aligned}\widetilde{\sigma}(\widetilde{e}_1,\widetilde{e}_1)&=\frac{-1}{\widetilde{\omega}}\left(\frac{v_{xx}}{\lambda^2}+2H\widetilde{\alpha}\widetilde{\beta}+\frac{\kappa}{2}(x\widetilde{\alpha}-y\widetilde{\beta})+\frac{\kappa H}{2}xy\right),\\ \widetilde{\sigma}(\widetilde{e}_1,\widetilde{e}_2)&=\frac{-1}{\widetilde{\omega}}\left(\frac{v_{xy}}{\lambda^2}+H(\widetilde{\beta}^2-\widetilde{\alpha}^2)+\frac{\kappa}{2}(x\widetilde{\beta}+y\widetilde{\alpha})-\frac{\kappa H}{4}(x^2-y^2)\right),\\ \widetilde{\sigma}(\widetilde{e}_2,\widetilde{e}_2)&=\frac{-1}{\widetilde{\omega}}\left(\frac{v_{yy}}{\lambda^2}-2H\widetilde{\alpha}\widetilde{\beta}-\frac{\kappa}{2}(x\widetilde{\alpha}-y\widetilde{\beta})-\frac{\kappa H}{2}xy\right).\end{aligned} \tag{2.14}$$

2.2 Holomorphic quadratic differentials

Given arbitrary $\kappa,\tau,H\in\mathbb{R}$, let us consider constants $a,\widetilde{a},b,\widetilde{b}\in\mathbb{C}$ satisfying the following linear identities

$$\begin{aligned}(\kappa-4\tau^2)a+2(H+i\tau)b=0,\\ (\kappa+4H^2)\widetilde{a}+2i(H+i\tau)\widetilde{b}=0,\end{aligned} \tag{2.15}$$

We define a quadratic differential $Q_{a,b}$ for immersed H-surfaces in $\mathbb{E}(\kappa,\tau)$, and a quadratic differential $\widetilde{Q}_{\widetilde{a},\widetilde{b}}$ for immersed spacelike τ-surfaces in $\mathbb{L}(\kappa,H)$ as

$$\begin{aligned}Q_{a,b}(X,Y)&=a\,\sigma(X-iJX,Y-iJY)+b\langle X-iJX,E_3\rangle\langle Y-iJY,E_3\rangle,\\ \widetilde{Q}_{\widetilde{a},\widetilde{b}}(X,Y)&=\widetilde{a}\,\widetilde{\sigma}(X-iJX,Y-iJY)+\widetilde{b}\langle X-iJX,\widetilde{E}_3\rangle\langle Y-iJY,\widetilde{E}_3\rangle,\end{aligned} \tag{2.16}$$

i.e., $Q_{a,b}$ and $\widetilde{Q}_{\widetilde{a},\widetilde{b}}$ are the $(2,0)$-components of linear combinations of the complexified second fundamental forms (denoted by σ and $\widetilde{\sigma}$, respectively), and the height differentials. Recall that J is the $\frac{\pi}{2}$-rotation defined in the previous section.

The differential $Q_{a,b}$ is nothing but the Abresch–Rosenberg differential of an immersed H-surface in $\mathbb{E}(\kappa,\tau)$, see [9, Theorem 2.2.1], which is holomorphic. The definition of $\widetilde{Q}_{\widetilde{a},\widetilde{b}}$ is motivated by the duality, as shown in Theorem 2.3.

Remark 2.2 Changing the sign of N or $\widetilde{N}$ in (2.16) implies a change of the signs of J and of the second fundamental form, which gives other (non-holomorphic) quadratic differentials. In other words, some compatibility in the orientation is needed, in the sense that $JX=N\times X$ must hold true when we choose the orientation in the ambient space that defines the bundle curvature (note that the cross product $\times$ and the sign of τ depend upon the choice of orientation, see also [19]).

Theorem 2.3 *Let $\kappa,\tau,H\in\mathbb{R}$, and let $X:\Sigma\to\mathbb{E}(\kappa,\tau)$ and $\widetilde{X}:\Sigma\to\mathbb{L}(\kappa,H)$ be dual conformal immersions. Given $a,\widetilde{a},b,\widetilde{b}\in\mathbb{C}$ satisfying (2.15), it follows that*

$$X^*Q_{a,b}=\widetilde{X}^*\widetilde{Q}_{\widetilde{a},\widetilde{b}},$$

whenever $\widetilde{a}=-ia$ and $(\kappa-4\tau^2)\widetilde{b}+(\kappa+4H^2)b=0$.

Remark 2.4 (special cases) Note that $\kappa-4\tau^2$ and $\kappa+4H^2$ cannot be zero simultaneously unless $\kappa=\tau=H=0$, so the relation between (a,b) and $(\widetilde{a},\widetilde{b})$ is well defined. In the *degenerate* case $\kappa=\tau=H=0$, the conditions in (2.15) become trivial, but we get the original Calabi duality between minimal graphs in $\mathbb{R}^3$ and maximal graphs in $\mathbb{L}^3$. It is well known [3] that the classical Hopf differentials in $\mathbb{R}^3$ and $\mathbb{L}^3$ agree via the duality. This will be also discussed in Remark 2.6.

If $\kappa + 4H^2 = 0$, then X has critical constant mean curvature, and $\mathbb{L}(\kappa, \tau)$ becomes a Lorentzian space form, and the Abresch–Rosenberg differential $Q_{a,b}$ corresponds to the classical Hopf differential $\widetilde{Q}_{-ia,0}$ in $\mathbb{L}^3$ or $\mathbb{H}^3_1(\kappa)$. Likewise, if $\kappa - 4\tau^2 = 0$, then $\mathbb{E}(\kappa, \tau)$ becomes a Riemannian space form and $b = 0$, i.e., $Q_{a,b}$ is the Hopf differential in $\mathbb{R}^3$ or $\mathbb{S}^3(\kappa)$.

Proof of Theorem 2.3 Since the result is local, we will consider dual graphs $\Sigma_u \subset \mathbb{E}(\kappa, \tau)$ and $\Sigma_v \subset \mathbb{L}(\kappa, H)$ over some common domain of $\mathbb{M}^2(\kappa)$. Following the notation in Section 2.1, it suffices to check that $Q_{a,b}(e_i, e_j) = \widetilde{Q}_{\widetilde{a},\widetilde{b}}(\widetilde{e}_i, \widetilde{e}_j)$ for all $i, j \in \{1, 2\}$. Since (2.15) are linear relations, we will choose $a = 2(H + i\tau)$, $b = -(\kappa - 4\tau^2)$, $\widetilde{a} = 2(\tau - iH)$, and $\widetilde{b} = \kappa + 4H^2$, see Remark 2.4, and the subindexes (a, b) and $(\widetilde{a}, \widetilde{b})$ will be omitted in the sequel.

The computations are somewhat cumbersome, so we will just sketch the proof of $Q(e_1, e_1) = \widetilde{Q}(\widetilde{e}_1, \widetilde{e}_1)$. Since $\sigma(JX, JY) = 2H\langle X, Y\rangle - \sigma(X, Y)$ for all vector fields X and Y on Σ_u, we obtain from Equation (2.16) that

$$\begin{aligned} Q(e_1, e_1) &= 4(H + i\tau)\big(\sigma(e_1, e_1) - i\sigma(e_1, Je_1) - H\langle e_1, e_1\rangle\big) \\ &\quad - (\kappa - 4\tau^2)\big(\langle e_1, E_3\rangle - i\langle Je_1, E_3\rangle\big)^2. \end{aligned} \tag{2.17}$$

We can make use of (2.7), (2.8), (2.12), and (2.13) to evaluate the right-hand side of (2.17), getting to an expression of $Q(e_1, e_1)$ in terms of u_{xx}, u_{xy}, and the first-order symbols α and β. On the one hand, α and β can be written directly in terms of $\widetilde{\alpha}$ and $\widetilde{\beta}$ by means of the twin relations (2.6), which give the following explicit first-order PDE system relating u and v (see also [15,20]):

$$\alpha = \frac{\widetilde{\beta}}{\widetilde{\omega}} \quad \Leftrightarrow \quad u_x = \frac{v_y + Hx\lambda}{\sqrt{1 - (\frac{v_x}{\lambda} - Hy)^2 - (\frac{v_y}{\lambda} + Hx)^2}} - \tau y\lambda, \tag{2.18}$$

$$\beta = \frac{-\widetilde{\alpha}}{\widetilde{\omega}} \quad \Leftrightarrow \quad u_y = \frac{-v_x + Hy\lambda}{\sqrt{1 - (\frac{v_x}{\lambda} - Hy)^2 - (\frac{v_y}{\lambda} + Hx)^2}} + \tau x\lambda. \tag{2.19}$$

Therefore, u_{xx} (resp. u_{xy}) can be worked out by differentiating (2.18) (resp. (2.19)) with respect to x, which gives an expression in terms of v_{xx}, v_{xy}, $\widetilde{\alpha}$ and $\widetilde{\beta}$. Plugging such expressions for u_{xx} and u_{xy} in the result of evaluating (2.17), we get

$$Q(e_1, e_1) = \frac{4(H + i\tau)\lambda^2(\widetilde{\alpha}\widetilde{\beta} + i\widetilde{\omega})}{\widetilde{\omega}^2} v_{xx} + \frac{4(H + i\tau)\lambda^2(1 - \widetilde{\alpha}^2)}{\widetilde{\omega}^2} v_{xy} + L, \tag{2.20}$$

where L represents the lower-order terms.

The proof will finish if we check that (2.20) coincides with $\widetilde{Q}(\widetilde{e}_1, \widetilde{e}_1)$. Using (2.16) and the fact that $\widetilde{\sigma}(JX, JY) = -2\tau\langle X, Y\rangle - \widetilde{\sigma}(X, Y)$ for all vector fields X and Y on Σ_v (note the change of sign with respect to the Riemannian case due to the definition of mean curvature in the Lorentzian setting, see, e.g., [18]), we reach

$$\begin{aligned} \widetilde{Q}(\widetilde{e}_1, \widetilde{e}_1) &= 4(\tau - iH)\big(\widetilde{\sigma}(\widetilde{e}_1, \widetilde{e}_1) - i\widetilde{\sigma}(\widetilde{e}_1, J\widetilde{e}_1) + \tau\langle\widetilde{e}_1, \widetilde{e}_1\rangle\big) \\ &\quad + (\kappa + 4H^2)\big(\langle\widetilde{e}_1, \widetilde{E}_3\rangle - i\langle J\widetilde{e}_1, \widetilde{E}_3\rangle\big)^2. \end{aligned} \tag{2.21}$$

Transforming the right-hand side of (2.21) by means of (2.7), (2.8), (2.12), and (2.13), it is long but straightforward to verify that it is equal to (2.20). □

Corollary 2.5 *Given $\widetilde{a}, \widetilde{b} \in \mathbb{C}$ satisfying (2.15), the differential $\widetilde{Q}_{\widetilde{a},\widetilde{b}}$ is holomorphic on spacelike τ-surfaces immersed in $\mathbb{L}(\kappa, H)$.*

Remark 2.6 In the case of a minimal graph $\Sigma_u \subset \mathbb{E}(0,\tau)$ (i.e., in the Heisenberg space $\mathrm{Nil}_3(\tau)$ or in $\mathbb{R}^3$), let $Q = Q_{1,-2i\tau}$. Evaluating Q at $\{e_1 = \partial_x + u_x\partial_z, e_2 = \partial_y + u_x\partial_z\}$ as in the proof of Theorem 2.3, we get the simple expressions

$$\begin{aligned} Q(e_1,e_1) &= \frac{2u_{xx}}{\omega} + \frac{2i}{\omega^2}\left(\alpha\beta u_{xx} - (1+\alpha^2)u_{xy}\right), \\ Q(e_1,e_2) &= \frac{2u_{xy}}{\omega} + \frac{2i}{\omega^2}\left((1+\beta^2)u_{xx} - \alpha\beta u_{xy}\right), \\ Q(e_2,e_2) &= \frac{2u_{yy}}{\omega} + \frac{2i}{\omega^2}\left((1+\beta^2)u_{xy} - \alpha\beta u_{yy}\right). \end{aligned} \tag{2.22}$$

The Hopf differential $\widetilde{Q} = \widetilde{Q}_{-i,0}$ of the dual τ-graph $\Sigma_v \subset \mathbb{L}^3$ at the dual frame $\{\widetilde{e}_1 = \partial_x + v_x\partial_z, \widetilde{e}_2 = \partial_y + v_y\partial_z\}$ satisfies the formulas

$$\begin{aligned} \widetilde{Q}(\widetilde{e}_1,\widetilde{e}_1) &= \frac{2}{\widetilde{\omega}^2}\left(\widetilde{\alpha}\widetilde{\beta}v_{xx} + (1-\widetilde{\alpha}^2)v_{xy}\right) + \frac{2i}{\widetilde{\omega}}v_{xx} - 2i\tau(1-\widetilde{\alpha}^2), \\ \widetilde{Q}(\widetilde{e}_1,\widetilde{e}_2) &= \frac{-2}{\widetilde{\omega}^2}\left((1-\widetilde{\beta}^2)v_{xx} + \widetilde{\alpha}\widetilde{\beta}v_{xy}\right) + \tau\widetilde{\omega} + \frac{2i}{\widetilde{\omega}}v_{xy} + 2i\tau\widetilde{\alpha}\widetilde{\beta}, \\ \widetilde{Q}(\widetilde{e}_2,\widetilde{e}_2) &= \frac{-2}{\widetilde{\omega}^2}\left((1-\widetilde{\beta}^2)v_{xx} + \widetilde{\alpha}\widetilde{\beta}v_{xy}\right) + \frac{2i}{\widetilde{\omega}}v_{yy} - 2i\tau(1-\widetilde{\beta}^2). \end{aligned} \tag{2.23}$$

It follows that $Q_{1,-2\tau i} = \Phi^*\widetilde{Q}_{-i,0}$, where $\Phi : \Sigma_u \to \Sigma_v$ is the conformal diffeomorphism defined in (2.9). If $\tau = 0$, then the classical Hopf differentials $Q_{1,0}$ and $\widetilde{Q}_{1,0}$ of dual graphs in $\mathbb{R}^3$ and $\mathbb{L}^3$ are related by $Q_{1,0} = \Phi^*\widetilde{Q}_{-i,0} = -i\Phi^*\widetilde{Q}_{1,0}$.

The minimal surface equation for $\Sigma_u \subset \mathrm{Nil}_3(\tau)$ and the mean curvature τ equation for $\Sigma_v \subset \mathbb{L}^3$, which are the integrability conditions for the twin relations and will be used below, can be written as

$$\begin{aligned} (1+\beta^2)u_{xx} - 2\alpha\beta u_{xy} + (1+\alpha^2)u_{yy} &= 0, \\ (1-\widetilde{\beta}^2)v_{xx} + 2\widetilde{\alpha}\widetilde{\beta}v_{xy} + (1-\widetilde{\alpha}^2)v_{yy} &= 2\tau\widetilde{\omega}^3. \end{aligned} \tag{2.24}$$

2.3 Behavior under isometric deformations

Consider the standard Killing submersion structures $\pi : \mathbb{E}(\kappa,\tau) \to \mathbb{M}^2(\kappa)$ and $\widetilde{\pi} : \mathbb{L}(\kappa, H) \to \mathbb{M}^2(\kappa)$, and denote by $\mathrm{Iso}_\xi^+(\mathbb{E}(\kappa,\tau))$ and $\mathrm{Iso}_\xi^+(\mathbb{L}(\kappa,H))$ the subgroups of $\mathrm{Iso}(\mathbb{E}(\kappa,\tau))$ and $\mathrm{Iso}(\mathbb{L}(\kappa,\tau))$ of direct Killing isometries leaving the unit Killing vector field ξ invariant, see [19].

For each $T \in \mathrm{Iso}_\xi^+(\mathbb{E}(\kappa,\tau))$ and $\widetilde{T} \in \mathrm{Iso}_\xi^+(\mathbb{L}(\kappa,H))$, there are unique direct isometries $\rho(T), \widetilde{\rho}(\widetilde{T}) \in \mathrm{Iso}^+(\mathbb{M}^2(\kappa))$ such that $\rho(T)\circ\pi = \pi\circ T$ and $\widetilde{\rho}(\widetilde{T})\circ\widetilde{\pi} = \widetilde{\pi}\circ\widetilde{T}$. Therefore, ρ and $\widetilde{\rho}$ define surjective group morphisms with kernel the subgroups of *vertical translations* (i.e., the isometries spanned by ξ), so there is an isomorphism

$$R : \frac{\mathrm{Iso}_\xi^+(\mathbb{E}(\kappa,\tau))}{\ker(\rho)} \longrightarrow \frac{\mathrm{Iso}_\xi^+(\mathbb{L}(\kappa,H))}{\ker(\widetilde{\rho})}. \tag{2.25}$$

Both the duality and the isomorphism R ignore vertical translations, so the following result makes sense. The proof can be found at [20].

Proposition 2.7 *Let $X : \Sigma \to \mathbb{E}(\kappa,\tau)$ and $\widetilde{X} : \Sigma \to \mathbb{L}(\kappa,H)$ be dual conformal immersions, and let $T \in \mathrm{Iso}_\xi^+(\mathbb{E}(\kappa,\tau))$. Then $T\circ X$ and $R(T)\circ\widetilde{X}$ are also dual.*

Remark 2.8 Proposition 2.7 generically describes the whole picture concerning Killing isometries [19], since the isometries T such that $T_*\xi = -\xi$, which project to isometries of $\mathrm{Iso}^-(\mathbb{M}^2(\kappa))$, change the sign of the mean curvature (we are considering the upward-pointing normal), and the duality does not apply.

If $H = 0$ or $\tau = 0$, then this is not actually an issue, and we can complete the picture by considering the following exceptional cases:

(1) If $H = 0$, then symmetries about horizontal geodesics in $\mathbb{E}(\kappa, \tau)$ correspond to mirror symmetries about vertical planes in $\mathbb{L}(\kappa, 0) = \mathbb{M}^2(\kappa) \times \mathbb{R}_1$.
(2) Likewise, if $\tau = 0$, mirror symmetries about vertical planes in $\mathbb{E}(\kappa, 0) = \mathbb{M}^2(\kappa) \times \mathbb{R}$ correspond to symmetries about horizontal geodesics in $\mathbb{L}(\kappa, H)$.
(3) If $H = \tau = 0$, then we get dual surfaces with zero mean curvature in product spaces, and mirror symmetries about horizontal planes in $\mathbb{E}(\kappa, \tau)$ correspond to mirror symmetries about horizontal planes in $\mathbb{L}(\kappa, \tau)$

In particular, for minimal surfaces in the Heisenberg group ($\kappa = H = 0$), Proposition 2.7 still holds true after extending the isomorphism R to an isomorphism

$$\overline{R} : \frac{\mathrm{Iso}(\mathrm{Nil}_3(\tau))}{\ker(\rho)} \longrightarrow \frac{\mathrm{Iso}_\xi(\mathbb{L}^3)}{\ker(\widetilde{\rho})}.$$

The following consequence of Proposition 2.7 will be helpful when analyzing some examples invariant under a 1-parameter group of isometries in the next section:

Corollary 2.9 *X is invariant under a subgroup of $\mathrm{Iso}_\xi^+(\mathbb{E}(\kappa, \tau))$ if and only if $\widetilde{X}$ is invariant under a subgroup of $\mathrm{Iso}_\xi^+(\mathbb{L}(\kappa, H))$, in which case both subgroups induce the same subgroup of $\mathrm{Iso}(\mathbb{M}^2(\kappa))$ via ρ and $\widetilde{\rho}$.*

2.4 Spacelike τ-surfaces with zero differential

Given $H, \kappa, \tau \in \mathbb{R}$, let $\widetilde{\Sigma}$ be a spacelike τ-surface immersed in $\mathbb{L}(\kappa, H)$ whose holomorphic differential identically vanishes. Then the dual H-surface Σ in $\mathbb{E}(\kappa, \tau)$ is nowhere vertical and has zero Abresch–Rosenberg differential . We deduce that Σ is a subset of one of the surfaces $S_{H,\kappa,\tau}$, $C_{H,\kappa,\tau}$ or $P_{H,\kappa,\tau}$ given by [10, Proposition 2.1]:

- The surface $S_{H,\kappa,\tau}$ is an H-sphere if $\kappa + 4H^2 > 0$, or an entire H-graph otherwise. In the former case, we have to restrict to half of $S_{H,\kappa,\tau}$ since we are considering the upward-pointing unit normal. The dual surface $\widetilde{S}_{H,\kappa,\tau}$ is rotationally invariant by Corollary 2.9. It is not difficult to integrate the twin relations (2.6) taking into account the explicit parameterization in [10], in order to get the global parameterization of $\widetilde{S}_{H,\kappa,\tau}$ given by

$$\phi(u, v) = \left(v\cos(u), v\sin(u), \int_0^v \frac{4\tau s\sqrt{1 - H^2 s^2}\,\mathrm{d}s}{(4 + \kappa s^2)\sqrt{1 + \tau^2 s^2}} \right). \tag{2.26}$$

  Although (2.26) defines an entire graph for all $H, \kappa, \tau \in \mathbb{R}$, it is spacelike if and only if $v \in [0, \min\{\frac{1}{|H|}, \frac{2}{\sqrt{-\kappa}}\})$ if $\kappa < 0$ or $v \in [0, \frac{1}{|H|})$ if $\kappa > 0$.
- If $\kappa + 4H^2 < 0$, then half of $C_{H,\kappa,\tau}$ is a screw-motion invariant H-surface with respect to the upward-pointing unit normal. The dual surface $\widetilde{C}_{H,\kappa,\tau}$ can be integrated explicitly, obtaining the global parameterization

$$\phi(u,v)=\left(v\cos(u),\, v\sin(u),\, \frac{-4H}{\kappa}u-\int_0^v\frac{16\tau\sqrt{\kappa^2s^2-16H^2}\,\mathrm{d}s}{\kappa s(4+\kappa s^2)\sqrt{\kappa^2s^2+16\tau^2}}\right). \tag{2.27}$$

This defines a properly embedded surface, which is spacelike if and only if $v\in[\frac{4|H|}{-\kappa},\frac{2}{\sqrt{-\kappa}})$. If $H\neq 0$, then $\widetilde{C}_{H,\kappa,\tau}$ is an embedded multigraph, otherwise it is a graphical annulus.

- As for the surface $P_{H,\kappa,\tau}$ with $\kappa+4H^2<0$, let us introduce the halfspace model $\mathbb{L}(\kappa,H)=\{(x,y,z)\in\mathbb{R}^3 : y>0\}$ with the Lorentzian metric

$$\frac{\mathrm{d}x^2+\mathrm{d}y^2}{-\kappa y^2}-\left(\mathrm{d}z+\frac{2H}{\kappa y}\,\mathrm{d}x\right)^2.$$

As $P_{H,\kappa,\tau}$ is invariant under 1-parameter groups of isometries $(x,y,z)\mapsto(x+t,y,z)$ and $(x,y,z)\mapsto(e^tx,e^ty,z+at)$, Corollary 2.9 ensures that the dual surface $\widetilde{P}_{H,\kappa,\tau}$ is invariant under $(x,y,z)\mapsto(x+t,y,z+rt)$ and $(x,y,z)\mapsto(e^tx,e^ty,z+st)$ for some $r,s\in\mathbb{R}$. This implies that $r=0$ and $\widetilde{P}_{H,\kappa,\tau}$ can be globally parameterized as $(u,v)\mapsto(u,v,s\log(v))$ up to vertical translations. The value of s can be obtained using the condition that the mean curvature is H, which leads to the parameterization

$$\phi(u,v)=\left(u,\,v,\,\frac{2\tau\sqrt{-\kappa-4H^2}}{\kappa\sqrt{-\kappa+4\tau^2}}\log(v)\right). \tag{2.28}$$

Remark 2.10 If $\tau=0$, then $\widetilde{S}_{H,\kappa,0}$ and $\widetilde{C}_{H,\kappa,0}$ become planes or certain helicoids. Therefore, the surfaces $S_H=S_{H,\kappa,0}$ and $C_H=C_{H,\kappa,0}$ with vanishing differential in $\mathbb{S}^2(\kappa)\times\mathbb{R}$ or $\mathbb{H}^2(\kappa)\times\mathbb{R}$ classified by Abresch and Rosenberg in [1] have very simple Lorentzian counterparts. Notice that all of these spacelike τ-surfaces are embedded, which contrasts with the possible non-embeddedness of H-spheres in Berger spheres $\mathbb{E}(\kappa,\tau)$ with $\kappa>0$ and $\tau\neq 0$, see [27].

The metric induced by $\mathbb{L}(\kappa,H)$ on the surfaces $\widetilde{S}_{H,\kappa,0}$ (with $4H^2+\kappa>0$) and $\widetilde{C}_{H,\kappa,0}$ (with $4H^2+\kappa<0$) is not complete. In fact, it is not difficult to show that there exist divergent curves of the form $\phi(t,0)$ contained in such surfaces and meeting their boundary with finite length.

Proposition 2.11 *Given $H,\kappa,\tau\in\mathbb{R}$, let $\widetilde{\Sigma}$ be a spacelike τ-surface immersed in $\mathbb{L}(\kappa,H)$ with zero differential. Then $\widetilde{\Sigma}$ is congruent to an open subset of $\widetilde{S}_{H,\kappa,\tau}$, $\widetilde{C}_{H,\kappa,\tau}$ (with $\kappa+4H^2<0$), or $\widetilde{P}_{H,\kappa,\tau}$ (with $\kappa+4H^2<0$).*

In particular, $\widetilde{\Sigma}$ is embedded and equivariant.

3 Curvature estimates for entire minimal graphs in $\mathrm{Nil}_3(\tau)$

We will begin by using the duality to get a nice formula relating the Hessian determinants of a minimal graph in $\mathrm{Nil}_3(\tau)$ and its dual τ-graph in $\mathbb{L}^3$. This will be the key to obtain our curvature estimates for entire minimal graphs in $\mathrm{Nil}_3(\tau)$.

Lemma 3.1 *Let $\Sigma_u\subset Nil_3(\tau)$ and $\Sigma_v\subset\mathbb{L}^3$ be dual graphs, and let us denote by $\widetilde{K}$ the Gauss curvature of Σ_v. Then*

$$u_{xy}^2-u_{xx}u_{yy}=\frac{v_{xy}^2-v_{xx}v_{yy}}{(1-v_x^2-v_y^2)^2}+\tau^2=\widetilde{K}+\tau^2. \tag{3.1}$$

Hence the following dual statements hold:

(a) *If* $\Sigma_u \subset Nil_3(\tau)$ *is an entire minimal graph, then*

$$0 \leq u_{xy}^2 - u_{xx}u_{yy} \leq \tau^2.$$

(b) *If* $\Sigma_v \subset \mathbb{L}^3$ *is an entire spacelike* τ*-graph, then*

$$-\tau^2 \leq \widetilde{K} \leq 0.$$

Proof Let $Q = Q_{1,-2i\tau}$ be the Abresch–Rosenberg differential of Σ_u, and let $\widetilde{Q} = \widetilde{Q}_{-i,0}$ be the Hopf differential of Σ_v, as in Remark 2.6. Taking squared moduli in the first equations of (2.22) and (2.23), and reducing the resulting expressions by means of (2.24), we reach the identities

$$\begin{aligned} |Q(e_1, e_1)|^2 &= 4(1+\alpha^2)^2\,\frac{u_{xy}^2 - u_{xx}u_{yy}}{\omega^4}, \\ |\widetilde{Q}(\widetilde{e}_1, \widetilde{e}_1)|^2 &= 4(1-\widetilde{\alpha}^2)^2\left(\frac{v_{xy}^2 - v_{xx}v_{yy}}{\widetilde{\omega}^4} + \tau^2\right). \end{aligned} \tag{3.2}$$

Since $Q(e_1, e_1) = \widetilde{Q}(\widetilde{e}_1, \widetilde{e}_1)$ by Theorem 2.3, and $\frac{1+\alpha^2}{\omega^2} = \frac{\omega^2-\beta^2}{\omega^2} = 1 - \widetilde{\alpha}^2$ by the twin relations, we deduce that Equation (3.1) holds true.

If Σ_v is an entire spacelike τ-graph in $\mathbb{L}^3$, then its second fundamental form $\widetilde{\sigma}$ satisfies $2\tau^2 \leq |\widetilde{\sigma}|^2 \leq 4\tau^2$, see [5,6,28]. Item (b) follows from the well-known identity $|\widetilde{\sigma}|^2 = 4\tau^2 + 2\widetilde{K}$, and item (a) is dual to (b) taking into account (3.1). □

Remark 3.2 Espinar and Rosenberg [11] defined the modulus of the Abresch–Rosenberg differential $Q\,\mathrm{d}\zeta^2$ of an H-surface Σ in $\mathbb{E}(\kappa, \tau)$, in terms of a complex conformal parameter $\zeta = r + is$, as the function $q = \frac{4|Q|^2}{\mu^4}$, where μ is the conformal factor such that the metric of Σ reads $\mu^2(\mathrm{d}r^2 + \mathrm{d}s^2)$. Hence q does not depend upon the choice of the parameter ζ. In the case of a minimal graph $\Sigma_u \subset \mathrm{Nil}_3(\tau)$, combining [11, Lemma 2.2], and Equations (3.1) and (3.4), it is not difficult to get

$$\frac{q}{4\tau^2\nu^4} = u_{xy}^2 - u_{xx}u_{yy},$$

which shows that the Hessian determinant is a geometric quantity for an entire minimal graph in $\mathrm{Nil}_3(\tau)$. On the other hand, the right-hand side of (3.1) is nothing but the modulus of the Hopf differential of Σ_v, so that Equation (3.1) can be understood as the equality between the moduli of the holomorphic differentials.

Lemma 3.3 *Let* $\Sigma_u \subset \mathbb{E}(\kappa, \tau)$ *be an entire* H*-graph with critical mean curvature. If* K *and* ν *denote the Gauss curvature and the angle function of* Σ_u*, then*

$$-4(H^2+\tau^2)\nu^2 \leq K \leq -3(H^2+\tau^2)\nu^4 < 0. \tag{3.3}$$

Proof Since Daniel correspondence [7] preserves K and ν, as well as the value of $H^2 + \tau^2$, we will assume that Σ_u is an entire minimal graph in $\mathrm{Nil}_3(\tau)$ with no loss of generality, and prove (3.3) with $H = 0$.

Let $\Sigma_v \subset \mathbb{L}^3$ be the dual entire τ-graph, and let $\Phi : \Sigma_u \to \Sigma_v$ be the conformal diffeomorphism given by (2.9). Since Φ induces a conformal factor ν^2 between Σ_u and Σ_v, see Equation (2.10), the Gauss curvature $\widetilde{K}$ of Σ_v satisfies $\widetilde{K}\nu^2 = K - \Delta\log(\nu)$, where the

Laplacian is computed on Σ_u (this identity and those hereafter only make sense through the diffeomorphism Φ).

The identity $\Delta\log(\nu)=\frac{1}{\nu}\Delta\nu-\frac{1}{\nu^2}\|\nabla\nu\|^2$, along with the fact that ν lies in the kernel of the stability operator of Σ_u (i.e., $\Delta\nu=2K\nu+4\tau^2\nu^3$), let us deduce that

$$\widetilde{K}=\frac{-K}{\nu^2}-4\tau^2+\frac{\|\nabla\nu\|^2}{\nu^4}. \tag{3.4}$$

The first inequality in the statement is a consequence of estimating $\widetilde{K}\leq 0$ and $\|\nabla\nu\|\geq 0$ in (3.4) in virtue of Lemma 3.1.

As for the second inequality, we will need a bound of $\|\nabla\nu\|^2$ from above. In the sequel, $\|\cdot\|_1$ and $\langle\cdot,\cdot\rangle_1$ will stand for the norm and inner product in $\mathbb{L}^3$, respectively. Also, ∇ and $\widetilde{\nabla}$ will denote the gradient operators in Σ_u and Σ_v, respectively. Using again that the conformal factor between Σ_u and Σ_v is ν^2, we obtain

$$\frac{1}{\nu^4}\|\nabla\nu\|^2=\frac{1}{\nu^2}\|\widetilde{\nabla}\nu\|_1^2=\nu^2\|\widetilde{\nabla}\tfrac{1}{\nu}\|_1^2. \tag{3.5}$$

The angle function of Σ_v is given by $\omega=\langle\widetilde{N},\partial_z\rangle_1$, where $\widetilde{N}$ is the upward-pointing unit normal to Σ_v, and satisfies $\omega=\frac{1}{\nu}$. Moreover, $\widetilde{\nabla}\omega=\widetilde{A}\widetilde{T}$, where $\widetilde{A}$ is the shape operator of Σ_v, and $\widetilde{T}=\partial_z+\omega\widetilde{N}$ is the tangent part to Σ_v of the (timelike) unit Killing vector field ∂_z in $\mathbb{L}^3$. Plugging this information in (3.5), we obtain

$$\frac{1}{\nu^4}\|\nabla\nu\|^2=\nu^2\|\widetilde{A}\widetilde{T}\|_1^2\leq\nu^2\|\widetilde{A}\|^2\|\widetilde{T}\|_1^2=(4\tau^2+2\widetilde{K})(1-\nu^2), \tag{3.6}$$

where we have used the identity $\|\widetilde{A}\|^2=4\tau^2+2\widetilde{K}$, as well as the fact that $\nu^2\|\widetilde{T}\|_1^2=\nu^2(-1+\omega^2)=1-\nu^2$. Combining (3.6) and (3.4), we reach the inequality

$$K\leq-4\tau^2\nu^4+\widetilde{K}\nu^2(1-2\nu^2). \tag{3.7}$$

Note that (3.7) holds for any (not necessarily entire) minimal graph in $\mathrm{Nil}_3(\tau)$, and our result will be a consequence of Lemma 3.1. Given $p\in\Sigma$, let us distinguish two cases. If $\nu(p)\geq\frac{1}{\sqrt{2}}$, then $1-2\nu(p)^2\leq 0$, so we have $\widetilde{K}\nu^2(1-2\nu^2)\leq-\tau^2\nu^2(1-2\nu^2)$ at p. If, on the contrary, $\nu(p)\leq\frac{1}{\sqrt{2}}$, then $1-2\nu(p)^2\geq 0$, and we get $\widetilde{K}\nu^2(1-2\nu^2)\leq 0$ at p. Hence (3.7) yields the estimate

$$K\leq\begin{cases}-2\tau^2\nu^4-\tau^2\nu^2 & \text{if } \nu\geq\frac{1}{\sqrt{2}},\\ -4\tau^2\nu^4 & \text{if } \nu\leq\frac{1}{\sqrt{2}}.\end{cases}$$

In both cases it follows that $K\leq-3\tau^2\nu^4$, so we are done. □

Observe that vertical planes are the only complete flat minimal surfaces in $\mathrm{Nil}_3(\tau)$, so they have finite total curvature, see [11, Theorem 3.1]. It is conjectured in [21] that they are the only complete orientable stable minimal surfaces with intrinsic quadratic area growth (i.e., such that the area of intrinsic metric balls grow at most quadratically with respect to their radius). The above estimate allows us to rewrite this conjecture from the point of view of curvature.

Theorem 3.4 *Any entire graph $\Sigma\subset\mathbb{E}(\kappa,\tau)$ with critical mean curvature has negative Gauss curvature. In particular, it has at least quadratic intrinsic area growth, and the following assertions are equivalent:*

(i) Σ *has quadratic intrinsic area growth.*
(ii) Σ *has finite total curvature.*

Proof The fact that Σ has at least quadratic area growth follows from (3.3) by comparing the area growth of Σ with the (quadratic) area growth of the Euclidean plane. Moreover, Li [17] showed that quadratic area growth plus negative curvature implies finite total curvature. The converse holds even if the curvature changes sign, as proved by Hartman [14]. □

Remark 3.5 Equation (3.3) does not give much information about the integrability of K, since ν^2 is never an integrable function on an entire minimal graph [22], and ν^4 can be either integrable or not, as we shall see in the next examples:

- **Umbrellas** Consider the rotationally invariant entire graph Σ_u with $u(x, y) = 0$. Its Gauss curvature and angle function are given by
$$K(r) = -\frac{3\tau^2 + 2\tau^4 r^2}{(1+\tau^2 r^2)^2}, \qquad \nu(r) = \frac{1}{\sqrt{1+\tau^2 r^2}},$$
where $r = (x^2 + y^2)^{1/2}$. Hence $K \notin L^1(\Sigma_u)$ but $\nu^4 \in L^1(\Sigma_u)$. There is a functional dependence between K and ν given by $K = -\tau^2\nu^4 - 2\tau^2\nu^2$.
- **Invariant surfaces** Complete minimal surfaces invariant under a 1-parameter group of translations were classified by Figueroa, Mercuri and Pedrosa as the family of entire graphs $\{\Sigma_{u_\theta} : \theta \in \mathbb{R}\}$, where
$$u_\theta(x, y) = -\tau xy + \frac{\sinh(\theta)}{4\tau}\left(2\tau x\sqrt{1+4\tau^2x^2} + \operatorname{arcsinh}(2\tau x)\right).$$
Their Gauss curvature and angle function can be computed as
$$K(x) = \frac{-4\tau^2}{\cosh^2(\theta)(1+4\tau^2x^2)^2}, \qquad \nu(x) = \frac{1}{\cosh(\theta)\sqrt{1+4\tau^2x^2}}.$$
In this case, $K \notin L^1(\Sigma_{u_\theta})$ and $\nu^4 \notin L^1(\Sigma_{u_\theta})$. The relation between K and ν for these surfaces reads $K = -4\tau^2\cosh^2(\theta)\nu^4$.

These examples also show that the exponents in (3.3) are sharp.

Remark 3.6 It is not true in general that minimal surfaces in $\mathrm{Nil}_3(\tau)$ have negative Gauss curvature. Note that Gauss equation is given by $K = \det(A) + \tau^2 - 4\tau^2\nu^2$ in this context, as a particular case of (1.1). Let us discuss a couple of examples showing that non-graphical surfaces may display different behaviors:

- **Catenoids** The 1-parameter family of catenoids in $\mathrm{Nil}_3(\tau)$ is given in terms of a parameter $E > 0$ by the function $u(x, y) = h(r)$, where $r = (x^2 + y^2)^{1/2}$ and $h : [E, \infty) \to \mathbb{R}$ is the solution to the ODE
$$h'(r) = \frac{E\sqrt{1+\tau^2r^2}}{\sqrt{r^2-E^2}}, \qquad h(E) = 0.$$
The Gauss curvature and the angle function can be computed as
$$K(r) = -\frac{E^2 + 3\tau^2r^4 + 2\tau^4r^6}{r^4(1+\tau^2r^2)^2} < 0, \quad \nu(r) = \frac{\sqrt{r^2-E^2}}{r\sqrt{1+\tau^2r^2}}.$$
It follows that K is not integrable for any $E > 0$.

- **Helicoids** The 1-parameter family of complete minimal helicoids parameterized by $(x, y) \in \mathbb{R}^2 \mapsto (x\cos(y), x\sin(y), ay)$ (for some $a > 0$) has constant Gauss curvature $\frac{2a\tau-1}{a^2}$ along the z-axis, so its sign depends on a.

As a conclusion, we will prove a simple necessary condition in order to have finite total curvature. Umbrellas are examples showing that it is not sufficient.

Proposition 3.7 *If $\Sigma \subset \mathbb{E}(\kappa, \tau)$ is a complete stable H-surface with finite total curvature, then both the Gauss curvature and the angle function tend to zero on any divergent sequence of points of Σ.*

Proof Let $\{p_n\} \subset \Sigma$ be a divergent sequence in Σ. Given $n \in \mathbb{N}$, consider Σ_n to be a translation of Σ taking p_n to the origin. Since the geometry of $\mathbb{E}(\kappa, \tau)$ is bounded and each Σ_n is stable, standard convergence arguments (see [26]) imply that $\{\Sigma_n\}$ subconverges to a complete H-surface Σ_∞. The convergence is in the $\mathcal{C}^k$-topology on compact subsets for all $k \geq 0$, so the angle function ν_∞ of Σ_∞ (as the limit of the positive angle functions of the surfaces Σ_n) is non-negative. Since ν_∞ lies in the kernel of the stability operator of Σ_∞, we infer that either ν_∞ never vanishes or ν_∞ is identically zero (see [23, Assertion 2.2]), i.e., Σ_∞ is either a complete multigraph or a vertical cylinder over a curve of geodesic curvature $\pm 2H$.

If $K(p_n)$ does not converge to zero, then the subsequence may be chosen so that Σ_∞ has Gauss curvature different from zero at the origin, so there is a relatively compact neighborhood $U_\infty \subset \Sigma_\infty$ such that $\int_{U_\infty} K_\infty \neq 0$. Since the convergence is in the $\mathcal{C}^k$-topology on compact subsets for all $k \geq 0$, we deduce that there exists a sequence of relatively compact open subsets $U_n \subset \Sigma$ such that $p_n \in U_n$ and $\int_{U_n} K$ becomes arbitrarily close to $\int_{U_\infty} K_\infty \neq 0$. This clearly implies that Σ does not have finite total curvature, contradicting the statement.

Now assume that the sequence $\{p_n\} \subset \Sigma$ is such that $\nu(p_n)$ converges to some value a. The complete H-surface Σ_∞ obtained as the limit of a subsequence must be flat by the previous argument for the Gauss curvature, so it must be a vertical cylinder by [11, Theorem 3.1]. Hence $a = 0$ since the angle function of a vertical cylinder vanishes identically, and we are done. □

Appendix A: The Bernstein problem

As pointed out in the introduction, we will give a shortcut in Fernández and Mira's solution to the Bernstein problem in $\mathrm{Nil}_3(\tau)$, plus a geometric description of the families of surfaces sharing the same Abresch–Rosenberg differential.

Let $\mathcal{E}(X, H)$ be the space of entire spacelike H-graphs in $X \in \{\mathrm{Nil}_3(\tau), \mathbb{L}^3\}$, up to vertical translations. The 4-dimensional group $\mathrm{Iso}_\xi(\mathrm{Nil}_3(\tau))$ acts on $\mathcal{E}(\mathrm{Nil}_3(\tau), 0)$ in the usual way, whereas on $\mathcal{E}(\mathbb{L}^3, \tau)$, we have the action of the 6-dimensional group $\mathrm{Iso}_\uparrow(\mathbb{L}^3)$ of isometries preserving the time orientation (which contains the 4-dimensional subgroup $\mathrm{Iso}_\xi(\mathbb{L}^3)$). Cheng and Yau [5] proved that a spacelike surface in $\mathbb{L}^3$ with constant mean curvature τ is an entire graph if and only if it is complete, and hence each element of $\mathrm{Iso}_\uparrow(\mathbb{L}^3)$ preserves the condition of being an entire graph.

On the one hand, Proposition 2.7 and Remark 2.8 yield an induced duality between $\mathcal{E}(\mathrm{Nil}_3(\tau), 0)/\mathrm{Iso}(\mathrm{Nil}_3(\tau))$ and $\mathcal{E}(\mathbb{L}^3, \tau)/\mathrm{Iso}_\xi(\mathbb{L}^3)$, and the Abresch–Rosenberg differential defines a map $\mathcal{Q}$ from $\mathcal{E}(\mathrm{Nil}_3(\tau), 0)/\mathrm{Iso}(\mathrm{Nil}_3(\tau))$ to the set **QD** of holomorphic quadratic differentials on the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ or the plane $\mathbb{C}$, excluding the zero differential in the latter case (notice that there is no entire parabolic minimal graphs in $\mathrm{Nil}_3(\tau)$ with

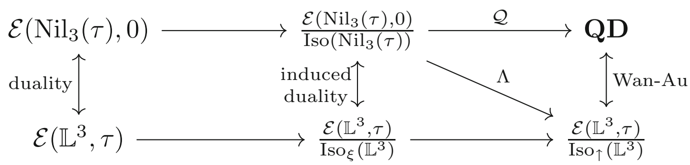

Fig. 1 Relations between entire graphs and quadratic differentials

zero Abresch–Rosenberg differential, for otherwise its dual would be totally umbilical, i.e., congruent to the hyperboloid Σ_v with $v(x, y) = (\tau^{-2} + x^2 + y^2)^{1/2}$, which is hyperbolic). On the other hand, Wan and Au [29,30] proved that the Hopf differential defines a bijection from $\mathcal{E}(\mathbb{L}^3, \tau)/\mathrm{Iso}_\uparrow(\mathbb{L}^3)$ to **QD**. Since the Hopf and the Abresch–Rosenberg differentials are dual by Theorem 2.3, we infer that $\mathcal{Q}$ induces a surjective map

$$\Lambda : \frac{\mathcal{E}(\mathrm{Nil}_3(\tau)), 0)}{\mathrm{Iso}(\mathrm{Nil}_3(\tau))} \longrightarrow \frac{\mathcal{E}(\mathbb{L}^3, \tau)}{\mathrm{Iso}_\uparrow(\mathbb{L}^3)}$$

such that the diagram in Figure 1 commutes.

Proposition A.1 (Bernstein problem) *For each $Q \in$ **QD**, there exists an entire minimal graph $\Sigma \subset Nil_3(\tau)$ with Abresch–Rosenberg differential Q. Another entire minimal graph $\Sigma' \subset Nil_3(\tau)$ has the same Abresch–Rosenberg differential as Σ if and only if the dual graphs of Σ and Σ' are congruent in $\mathbb{L}^3$.*

If $S \in \mathrm{Iso}_\uparrow(\mathbb{L}^3)$ and $\Sigma_u \in \mathcal{E}(\mathrm{Nil}_3(\tau), 0)$ with dual graph $\Sigma_v \in \mathcal{E}(\mathbb{L}^3, \tau)$, then $[\Sigma_u], [\Sigma_u]_S \in \mathcal{E}(\mathrm{Nil}_3(\tau), 0)/\mathrm{Iso}(\mathrm{Nil}_3(\tau))$ will denote the isometry class of Σ_u and of a dual of $S(\Sigma_v)$, respectively. Therefore, Proposition A.1 tells us that the set of isometry classes with the same Abresch–Rosenberg differential as Σ_u is given by

$$\Lambda^{-1}(\Lambda([\Sigma_u])) = \{[\Sigma_u]_S : S \in \mathrm{Iso}_\uparrow(\mathbb{L}^3)\}. \tag{A.1}$$

Nonetheless, this description is too redundant since $\mathrm{Iso}_\uparrow(\mathbb{L}^3)$ is 6-dimensional, and Proposition 2.7 implies that $[\Sigma_u]_S = [\Sigma_u]$ when S is a Killing isometry. Our goal is to prove that (A.1) still holds when one substitutes $\mathrm{Iso}_\uparrow(\mathbb{L}^3)$ with the 2-dimensional subgroup G spanned by the following two 1-parameter groups of isometries of $\mathbb{L}^3$:

- hyperbolic rotations, given in terms of a parameter $\theta \in \mathbb{R}$ by

$$\phi(x, y, z) = (x, y\cosh(\theta) + z\sinh(\theta), y\sinh(\theta) + z\cosh(\theta)); \tag{A.2}$$

- parabolic rotations, given in terms of a parameter $a \in \mathbb{R}$ by

$$\phi(x, y, z) = (x - ay + az, ax + (1 - \tfrac{a^2}{2})y + \tfrac{a^2}{2}z, ax - \tfrac{a^2}{2}y + (1 + \tfrac{a^2}{2})z). \tag{A.3}$$

The group G can be identified with a subgroup of isometries of the hyperbolic plane $\mathbb{H}^2 = \{(x, y, z) \in \mathbb{L}^3 : z = (1 + x^2 + y^2)^{1/2}\}$ fixing a point ∞ of the ideal boundary $\partial_\infty\mathbb{H}^2$

Lemma A.2 *For each $S \in Iso_\uparrow(\mathbb{L}^3)$, there exist unique $S_1 \in Iso_\xi(\mathbb{L}^3)$ and $S_2 \in G$ such that $S = S_1 \circ S_2$.*

Proof Let $S_3 \in \mathrm{Iso}_\xi(\mathbb{L}^3)$ be the translation mapping $S(0, 0, 0)$ to $(0, 0, 0)$, so $S_3 \circ S$ lies in the stabilizer of $(0, 0, 0)$ and induces an isometry of $\mathbb{H}^2$. Hence there exists $S_4 \in \mathrm{Iso}_\xi(\mathbb{L}^3)$ such that $S_4((S_3 \circ S)(\infty)) = \infty$. Then $S_1 = S_3^{-1} \circ S_4^{-1} \in \mathrm{Iso}_\xi(\mathbb{L}^3)$ and $S_2 = S_4 \circ S_3 \circ S \in G$ satisfy the conditions in the statement.

As for uniqueness, let us assume that $S_1, S_1' \in \mathrm{Iso}_\xi(\mathbb{L}^3)$ and $S_2, S_2' \in G$ are such that $S_1' \circ S_2' = S_1 \circ S_2$. Evaluating at $(0,0,0)$, we get that $S_1(0,0,0) = S_1'(0,0,0)$, so there is a translation S_5 such that $S_5 \circ S_1$ and $S_5 \circ S_1'$ lie in the stabilizer of the origin and induce rotations of $\mathbb{H}^2$ about $(0,0,1)$ coinciding at ∞. We deduce that $S_5 \circ S_1 = S_5 \circ S_1'$, from where $S_1' = S_1$, and hence $S_2' = S_2$. □

This factorization gives the desired parameterization of the level sets of Λ by means of G. Although the $[\Sigma_u]_S$ may depend upon the representative Σ_u in its isometry class, the family $\{[\Sigma_u]_S : S \in G\}$ does not change.

Theorem A.3 *The family of isometry classes of entire minimal graphs in $Nil_3(\tau)$ with the same Abresch–Rosenberg differential as $\Sigma_u \in \mathcal{E}(Nil_3(\tau), 0)$ is given by*

$$\Lambda^{-1}(\Lambda([\Sigma_u])) = \{[\Sigma_u]_S : S \in G\},$$

and depends on two real parameters unless the dual graph of Σ_u is invariant under some 1-parameter group of hyperbolic or parabolic rotations or screw motions.

Proof Let $S \in \mathrm{Iso}_\uparrow(\mathbb{L}^3)$ and factorize it as $S = S_1 \circ S_2$ with $S_1 \in \mathrm{Iso}_\xi(\mathbb{L}^3)$ and $S_2 \in G$ by means of Lemma A.2. Denote by Σ_v the dual of Σ_u and consider $T_1 \in \mathrm{Iso}_\xi(\mathrm{Nil}_3(\tau))$ such that $R(T_1) = S_1$, where R is defined by (2.25). Then Proposition 2.7 says that the dual of $S(\Sigma_v)$ is the image by T_1 of the dual of $S_2(\Sigma_v)$. This proves that $[\Sigma_u]_S = [\Sigma_u]_{S_2}$.

Given distinct $S, S' \in G$, we get that $[\Sigma_u]_S = [\Sigma_u]_{S'}$ if and only if $(S' \circ S^{-1})(\Sigma_v) = S_0(\Sigma_v)$ for some $S_0 \in \mathrm{Iso}_\xi(\mathbb{L}^3)$. Hence the 2-parameter family degenerates when Σ_v is invariant under a 1-parameter subgroup of $\mathrm{Iso}_\uparrow(\mathbb{L}^3)$ not contained in $\mathrm{Iso}_\xi(\mathbb{L}^3)$, i.e., a 1-parameter group of hyperbolic or parabolic rotations or screw motions. □

References

1. Abresch, U., Rosenberg, H.: A Hopf differential for constant mean curvature surfaces in $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^2 \times \mathbf{R}$. Acta Math. **193**, 141–174 (2004)
2. Abresch, U., Rosenberg, H.: Generalized Hopf differentials. Mat. Contemp. **28**, 1–28 (2005)
3. Araújo, H., Leite, M.L.: How many maximal surfaces do correspond to one minimal surface? Math. Proc. Camb. Phil. Soc. **146**(1), 165–175 (2009)
4. Calabi, E.: Examples of Bernstein problems for some non-linear equations. In: Proceedings of Symposia in Pure Mathematics, American Mathematics Society, Providence, RI **15**, 223–230 (1970)
5. Cheng, S.Y., Yau, S.T.: Maximal space-like hypersurfaces in the Lorentz–Minkowski spaces. Ann. Math. (2) **104**(3), 407–419 (1976)
6. Choi, H.I., Treibergs, A.: Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space. J. Differ. Geom. **32**, 775–817 (1990)
7. Daniel, B.: Isometric immersions into 3-dimensional homogeneous manifolds. Comment. Math. Helv. **82**(1), 87–131 (2007)
8. Daniel, B.: The Gauss map of minimal surfaces in the Heisenberg group. Int. Math. Res. Not. **2011**(3), 674–695 (2011)
9. Daniel, B., Hauswirth, L., Mira, P.: Constant mean curvature surfaces in homogeneous manifolds Notes of the 4th KIAS Workshop on Differential Geometry, Seoul (2009)
10. Domínguez-Vázquez, M., Manzano, J.M.: Isoparametric surfaces in $\mathbb{E}(\kappa,\tau)$-spaces. Preprint arXiv:1706.09394
11. Espinar, J.M., Rosenberg, H.: Complete constant mean curvature surfaces in homogeneous spaces. Comment. Math. Helv. **86**(3), 659–674 (2011)
12. Fernández, I., Mira, P.: Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space. Trans. Am. Math. Soc. **361**(11), 5737–5752 (2011)
13. Figueroa, C., Mercuri, F., Pedrosa, R.H.L.: Invariant surfaces of the Heisenberg groups. Ann. Mat. Pura Appl. (4) **177**, 173–194 (1999)
14. Hartman, P.: Parallel coordinates in the large. Am. J. Math. **86**(4), 705–727 (1964)

15. Lee, H.: Extensions of the duality between minimal surfaces and maximal surfaces. Geom. Dedicata **151**, 373–386 (2011)
16. Lee, H., Manzano, J. M.: Generalized Calabi's correspondence and complete spacelike surfaces. Asian J. Math. (to appear)
17. Li, P.: Complete surfaces of at most quadratic area growth. Comment. Math. Helv. **72**(1), 67–71 (1997)
18. López, R.: Differential Geometry of curves and surfaces in Lorentz-Minkowski space. Int. Electron. J. Geom. **7**(1), 44–107 (2014)
19. Manzano, J.M.: On the classification of Killing submersions and their isometries. Pac. J. Math. **270**(2), 367–692 (2014)
20. Manzano, J.M. : On the conformal duality between constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$ and $\mathbb{L}(\kappa, \tau)$. In: Proceedings of the *Young Researcher Workshop on Differential Geometry in Minkowski Space*, Universidad de Granada, pp. 89–118 (2017)
21. Manzano, J.M., Pérez, J., Rodríguez, M.: Parabolic stable surfaces with constant mean curvature. Calc. Var. Part. Differ. Equ. **42**(1–2), 137–152 (2011)
22. Manzano, J.M., Nelli, B.: Height and area estimates for constant mean curvature graphs in $\mathbb{E}(\kappa, \tau)$-spaces. J. Geom. Anal. **27**(4), 3441–3473 (2017)
23. Meeks, W.H., Pérez, J., Ros, A.: Stable constant mean curvature surfaces. In: Ji, L., Li, P., Schoen, R., Simon, L. (eds.) Handbook of Geometrical Analysis, vol. 1, pp 301–380. International Press (2008) ISBN: 978-1-57146-130-8
24. Meeks, W. H., Mira, P., Pérez, J., Ros, A.: Constant mean curvature spheres in homogeneous three-manifolds. Preprint arXiv:1706.09394
25. O'Neill, B.: Semi-Riemannian geometry. With applications to relativity. Pure and Applied Mathematics, **103**. Academic Press, Inc., New York (1983) ISBN: 0-12-526740-1
26. Rosenberg, H., Souam, R., Toubiana, E.: General curvature estimates for stable H-surfaces in 3-manifolds and applications. J. Differ. Geom. **84**(3), 623–648 (2010)
27. Torralbo, F.: Rotationally invariant constant mean curvature surfaces in homogeneous 3-manifolds. Differ. Geom. Appl. **28**(5), 593–607 (2010)
28. Treibergs, A.: Entire spacelike hypersurfaces of constant mean curvature in Minkowski space. Invent. Math. **66**(1), 39–56 (1982)
29. Wan, T.Y.: Constant mean curvature surface harmonic map and universal Teichmuller space. J. Differ. Geom. **35**, 643–657 (1992)
30. Wan, T.Y., Au, T.K.: Parabolic constant mean curvature spacelike surfaces. Proc. Am. Math. Soc. **120**, 559–564 (1994)