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Spectrum of the Laplacian with weights

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Abstract
Given a compact Riemannian manifold (M, g) and two positive functions ρ and σ, we are interested in the eigenvalues of the Dirichlet energy functional weighted by σ, with respect to the L^2 inner product weighted by ρ. Under some regularity conditions on ρ and σ, these eigenvalues are those of the operator $-\rho^{-1}\mathrm{div}(\sigma\nabla u)$ with Neumann conditions on the boundary if $\partial M \neq \emptyset$. We investigate the effect of the weights on eigenvalues and discuss the existence of lower and upper bounds under the condition that the total mass is preserved.

Keywords Eigenvalue · Laplacian · Density · Cheeger inequality · Upper bounds

Mathematics Subject Classification 35P15 · 58J50

1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$, possibly with nonempty boundary. We designate by $\{\lambda_k(M, g)\}_{k\geq 0}$ the nondecreasing sequence of eigenvalues of the Laplacian on (M, g) under Neumann conditions on the boundary if $\partial M \neq \emptyset$. The min-max principle tells us that these eigenvalues are variationally defined by

$$\lambda_k(M, g) = \inf_{E\in S_{k+1}} \sup_{u\in E\setminus\{0\}} \frac{\int_M |\nabla u|^2 v_g}{\int_M u^2 v_g},$$

where S_k is the set of all k-dimensional vector subspaces of $H^1(M)$ and v_g is the Riemannian volume element associated with g.

The relationships between the eigenvalues $\lambda_k(M, g)$ and the other geometric data of (M, g) constitute a classical topic of research that has been widely investigated in recent

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decades (the monographs [3,4,7,24,35] are among basic references on this subject). In the present work, we are interested in eigenvalues of "weighted" energy functionals with respect to "weighted" L^2 inner products. Our aim is to investigate the interplay between the geometry of (M, g) and the effect of the weights.

Therefore, let ρ and σ be two positive continuous functions on M and consider the Rayleigh quotient

$$R_{(g,\rho,\sigma)}(u) = \frac{\int_M |\nabla u|^2 \sigma \, v_g}{\int_M u^2 \rho \, v_g}.$$

The corresponding eigenvalues are given by

$$\mu_k^g(\rho, \sigma) = \inf_{E \in S_{k+1}} \sup_{u \in E \setminus \{0\}} R_{(g,\rho,\sigma)}(u). \tag{1}$$

Under some regularity conditions on ρ and σ, $\mu_k^g(\rho, \sigma)$ is the kth eigenvalue of the problem

$$-\operatorname{div}(\sigma \nabla u) = \mu \rho u \quad \text{in } M \tag{2}$$

with Neumann conditions on the boundary if $\partial M \neq \emptyset$. Here, ∇ and div are the gradient and the divergence associated with the Riemannian metric g. When there is no risk of confusion, we will simply write $\mu_k(\rho, \sigma)$ for $\mu_k^g(\rho, \sigma)$.

Notice that the numbering of eigenvalues starts from zero. It is clear that the infimum of $R_{(g,\rho,\sigma)}(u)$ is achieved by constant functions; hence, $\mu_0^g(\rho, \sigma) = 0$ and

$$\mu_1^g(\rho, \sigma) = \inf_{\int_M u \rho v_g = 0} R_{(g,\rho,\sigma)}(u). \tag{3}$$

One obviously has $\mu_k^g(1, 1) = \lambda_k(M, g)$. When $\sigma = 1$, the eigenvalues $\mu_k(\rho, 1)$ correspond to the situation where M has a non-necessarily constant mass density ρ and describe, in dimension 2, the vibrations of a non-homogeneous membrane (see [24,31] and the references therein). The eigenvalues $\mu_k(1, \sigma)$ are those of the operator $\operatorname{div}(\sigma \nabla u)$ associated with a conductivity σ on M (see [24, Chapter 10] and [2]). In the case where $\rho = \sigma$, the eigenvalues $\mu_k(\rho, \rho)$ are those of the Witten Laplacian L_ρ (see [12] and the references therein). Finally, when σ and ρ are related by $\sigma = \rho^{\frac{n-2}{n}}$, the corresponding eigenvalues $\mu_k^g(\rho, \rho^{\frac{n-2}{n}})$ are exactly those of the Laplacian associated with the conformal metric $\rho^{\frac{2}{n}} g$, that is, $\mu_k^g(\rho, \rho^{\frac{n-2}{n}}) = \lambda_k(M, \rho^{\frac{2}{n}} g)$.

Our goal in this paper is to investigate the behavior of $\mu_k^g(\rho, \sigma)$, especially in the most significant cases mentioned above, under normalizations that we will specify in the sequel, but which essentially consist in the preservation of the total mass. The last case, corresponding to conformal changes of metrics, has been widely investigated in recent decades (see for instance [9,22,23,26,28,29,33,34]), and most of the questions we will address in this paper are motivated by results established in the conformal setting. These questions can be listed as follows:

1. Can one redistribute the mass density ρ (resp. the conductivity σ) so that the corresponding eigenvalues become as small as desired?
2. Can one redistribute ρ and/or σ so that the eigenvalues become as large as desired?
3. If Question (1) (resp. (2)) is answered positively, what kind of constraints can one impose in order to get upper or lower bounds for the eigenvalues?
4. If Question (1) (resp. (2)) is answered negatively, what are the geometric quantities that bound the eigenvalues?

5. If the eigenvalues are bounded, what can one say about their extremal values?
6. Is it possible, in some specific situations, to compute or to have sharp estimates for the first positive eigenvalues?

In a preliminary section, we deal with some technical issues concerning the possibility of relaxing the conditions of regularity and positivity of the densities. In the process, we prove a 2-dimensional convergence result (Theorem 2.1) which completes a theorem that Colin de Verdière had established in dimension $n \geq 3$. Question (1) is discussed at the beginning of Sect. 3 where we show that it is possible to fix one of the densities ρ and σ and vary the other one, among densities preserving the total mass, in order to produce arbitrarily small eigenvalues (Theorem 3.1). This leads us to get into Question (3) that we tackle by establishing the following Cheeger-type inequality (Theorem 3.2):

$$\mu_1(\rho, \sigma) \geq \frac{1}{4} h_{\sigma,\sigma}(M) h_{\rho,\sigma}(M),$$

where $h_{\sigma,\sigma}(M)$ and $h_{\rho,\sigma}(M)$ are suitably defined isoperimetric constants, in the spirit of what is done in [27].

Whenever a Cheeger-type inequality is proved, a natural question is to investigate a possible reverse inequality under some geometric restrictions (see [6] and the introduction of [32] for a general presentation of this issue). It turns out that in the present situation, such a reverse inequality cannot be obtained without additional assumptions on the densities. Indeed, we prove that on any given Riemannian manifold, there exists families of densities such that the associated Cheeger constants are as small as desired while the corresponding eigenvalues are uniformly bounded from below (Theorem 3.3).

Questions (2) and (4) are addressed in Sect. 4. A. Savo and the authors have proved in [12] that the first positive eigenvalue $\mu_1(\rho, \rho)$ of the Witten Laplacian is not bounded above as ρ runs over densities of fixed total mass. In Proposition 4.1, we prove that, given a Riemannian metric g_0, we can find a metric g, within the set of metrics conformal to g_0 and of the same volume as g_0, and a density ρ, among densities of fixed total mass with respect to g_0, so that $\mu_1^g(\rho, 1)$ is as large as desired. The same also holds for $\mu_1^g(1, \sigma)$.

However, if instead of requiring that the total mass of the densities is fixed with respect to g_0, we assume that it is fixed with respect to g, and then the situation changes completely. Indeed, Theorem 4.1 gives the following estimate when M is a domain of a complete Riemannian manifold $(\tilde{M}, g_0)$ whose Ricci curvature satisfies $\mathrm{Ric}_{g_0} \geq -(n-1)$ (including the case $M = \tilde{M}$ if $\tilde{M}$ is compact): For every metric g conformal to g_0 and every density ρ on M with $\int_M \rho v_g = |M|_g$, one has

$$\mu_k^g(\rho, 1) \leq \frac{1}{|M|_g^{\frac{2}{n}}} \left(A_n k^{\frac{2}{n}} + B_n |M|_{g_0}^{\frac{2}{n}} \right), \tag{4}$$

where $|\,.\,|_g$ and $|\,.\,|_{g_0}$ denote the Riemannian volumes with respect to g and g_0, respectively, and A_n and B_n are two constants that depend only on the dimension n.

A direct consequence of this theorem is the following inequality satisfied by any density ρ on (M, g) with $\int_M \rho v_g = |M|_g$:

$$\mu_k^g(\rho, 1) \leq A_n \left(\frac{k}{|M|_g} \right)^{\frac{2}{n}} + B_n \mathrm{ric}_0, \tag{5}$$

where ric_0 is a positive number such that $\mathrm{Ric}_g \geq -(n-1)\mathrm{ric}_0\, g$ (see Corollary 4.1).

Regarding the eigenvalues $\mu_k^g(1,\sigma)$, we are able to prove an estimate of the same type as (5): For every positive density σ on (M,g) with $\int_M \sigma v_g = |M|_g$, one has (Theorem 4.2)

$$\mu_k^g(1,\sigma) \le A_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}} + B_n \mathrm{ric}_0, \tag{6}$$

where A_n and B_n are two constants that depend only on the dimension n. It is worth noting that although the estimates (5) and (6) are similar, their proofs are of different nature. That is why we were not able to decide whether a stronger estimate such as (4) holds for $\mu_k^g(1,\sigma)$.

When M is a bounded domain of a manifold $(\tilde{M},\tilde{g})$ of nonnegative Ricci curvature (e.g., $\mathbb{R}^n$), the inequalities (5) and (6) give the following estimates that can be seen as extensions of Kröger's inequality [30]: $\mu_k^g(\rho,1) \le A_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}}$ and $\mu_k^g(1,\sigma) \le A_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}}$, provided that $\int_M \rho v_g = |M|_g$ and $\int_M \sigma v_g = |M|_g$. Notice that if we follow Kröger's approach, then we get an upper bound of $\mu_k^g(\rho,1)$ which involves the gradient of ρ and the integral of $\frac{1}{\rho}$ (see [16]).

According to (5) and (6), it is natural to introduce the following *extremal eigenvalues* on a given Riemannian manifold (M,g):

$$\mu_k^*(M,g) = \sup_{\fint_M \rho v_g = 1} \mu_k^g(\rho,1) \quad \text{and} \quad \mu_k^{**}(M,g) = \sup_{\fint_M \sigma v_g = 1} \mu_k^g(1,\sigma).$$

In Sect. 5, we investigate the qualitative properties of these quantities in the spirit of what we did in [9] for the *conformal spectrum*, thereby providing some answers to Question (5). For example, when M is of dimension 2, we have the following lower estimate (see [9, Corollary 1]):

$$\mu_k^*(M,g) \ge 8\pi \frac{k}{|M|_g}.$$

This means that, given any Riemannian surface (M,g), endowed with the constant mass distribution $\rho = 1$ (whose eigenvalues can be very close to zero), it is always possible to redistribute the mass density ρ so that the resulting eigenvalue $\mu_k^g(\rho,1)$ is greater or equal to $8\pi \frac{k}{|M|_g}$.

It turns out that this phenomenon is specific to the dimension 2. Indeed, we prove (Theorem 5.1) that on any compact manifold M of dimension $n \ge 3$, there exists a one-parameter family of Riemannian metrics g_ε of volume 1 such that

$$\mu_k^*(M,g_\varepsilon) \le Ck\varepsilon^{\frac{n-2}{n}},$$

where C is a constant that does not depend on ε. This means that in dimension $n \ge 3$, there exist geometric situations that generate very small eigenvalues, regardless of how the mass density is distributed.

Regarding the extremal eigenvalues $\mu_k^{**}(M,g)$, a similar result is proved (Theorem 5.2) which is, moreover, also valid in dimension 2.

Note, however, that it is possible to construct examples of Riemannian manifolds (M,g) with very small eigenvalues (for the constant densities), for which $\mu_k^*(M,g)$ and $\mu_k^{**}(M,g)$) are sufficiently large (see Proposition 5.2).

The last part of the paper (Sect. 6) is devoted to the study of the first extremal eigenvalues μ_1^* and μ_1^{**}. We give sharp estimates of these quantities for some standard examples or under strong symmetry assumptions.

2 Preliminary results

This section is dedicated to some preliminary technical results. The reason is that in order to construct examples and counterexamples, it is often more convenient to use densities that are non-smooth or which vanish somewhere in the manifold. The key arguments used in the proof of these results rely on the method developed by Colin de Verdière in [14].

Let (M, g) be a compact Riemannian manifold, possibly with boundary.

Proposition 2.1 *Let $\rho \in L^\infty(M)$ and $\sigma \in C^0(M)$ be two positive densities on M. For every $N \in \mathbb{N}^*$, there exist two sequences of smooth positive densities ρ_p and σ_p such that $\forall k \leq N$,*

$$\mu_k(\rho_p, \sigma_p) \to \mu_k(\rho, \sigma)$$

as $p \to \infty$.

Proof Using standard density results, let ρ_p and σ_p be two sequences of smooth positive densities such that, ρ_p converges to ρ in $L^2(M)$ and σ_p converges uniformly toward σ. Assume furthermore that $\frac{1}{2}\inf\rho \leq \rho_p \leq 2\sup\rho$ almost everywhere and that (replacing σ_p by $\sigma_p + \|\sigma_p - \sigma\|_\infty$ if necessary) $\sigma \leq \sigma_p$ on M. Then, the sequence of quadratic forms $q_p(u) = \int_M |\nabla u|^2 \sigma_p v_g$ together with the sequence of norms $\|u\|_p^2 = \int_M u^2 \rho_p v_g$ satisfies the assumptions of Theorem I.8 of [14] which enables us to conclude. □

Let M_0 be a domain in M with C^1-boundary, and let ρ be a positive bounded function on M_0. In order to state the next result, let us introduce the following quadratic form defined on $H^1(M_0)$:

$$Q_0(u) = \int_{M_0} |\nabla u|^2 v_g + \int_{M\setminus M_0} |\nabla H(u)|^2 v_g,$$

where $H(u)$ is the harmonic extension of u to $M\setminus M_0$, with Neumann condition on $\partial M\setminus\partial M_0$ if $\partial M\setminus\partial M_0 \neq \emptyset$ (i.e., $H(u)$ is harmonic on $M\setminus M_0$, coincides with u on $\partial M_0\setminus\partial M$, and $\frac{\partial H(u)}{\partial \nu} = 0$ on $\partial M\setminus\partial M_0$. The function $H(u)$ minimizes $\int_{M\setminus M_0} |\nabla v|^2 v_g$ among all functions v on $M\setminus M_0$ which coincide with u on $\partial M_0\setminus\partial M$). We denote by $\gamma_k(M_0, \rho)$ the eigenvalues of this quadratic form with respect to the inner product of $L^2(M_0, \rho v_g)$ associated with ρ, that is,

$$\gamma_k(M_0, \rho) = \inf_{E\in S^0_{k+1}} \sup_{u\in E\setminus\{0\}} \frac{\int_{M_0} |\nabla u|^2 v_g + \int_{M\setminus M_0} |\nabla H(u)|^2 v_g}{\int_{M_0} u^2 \rho\, v_g},$$

where S^0_k is the set of all k-dimensional vector subspaces of $H^1(M_0)$.

Proposition 2.2 *Let $M_0 \subset M$ be a domain with C^1-boundary, and let $\rho \in L^\infty(M_0)$ be a positive density with* $\operatorname{ess\,inf}_{M_0} \rho > 0$. *Define, for every $\varepsilon > 0$, the density $\rho_\varepsilon \in L^\infty(M)$ by*

$$\rho_\varepsilon(x) = \begin{cases} \rho(x) & \text{if } x \in M_0 \\ \varepsilon & \text{otherwise.} \end{cases}$$

Then, for every positive k, $\mu_k(\rho_\varepsilon, 1)$ converges to $\gamma_k(M_0, \rho)$ as $\varepsilon \to 0$.

Proof The eigenvalues $\mu_k(\rho_\varepsilon, 1)$ are those of the quadratic form $q(u) = \int_M |\nabla u|^2 v_g$, $u \in H^1(M)$, with respect to the inner product $\|u\|_\varepsilon^2 = \int_M u^2 \rho_\varepsilon v_g$. Set $M_\infty = M\setminus M_0$ and $\Gamma = \partial M_0 \cap \partial M_\infty = \partial M_0\setminus\partial M$. We identify $H^1(M)$ with the space $\mathcal{H}_\varepsilon = \{v = (v_0, v_\infty) \in H^1(M_0) \times H^1(M_\infty) : \ v_{\infty\restriction\Gamma} = \sqrt{\varepsilon}\, v_{0\restriction\Gamma}\}$ through the map $\Psi_\varepsilon(u) = (u_{\restriction M_0}, \sqrt{\varepsilon}\, u_{\restriction M_\infty})$.

We endow $\mathcal{H}_\varepsilon$ with the inner product given by $\|(v_0, v_\infty)\|_\rho^2 = \int_{M_0} v_0^2 \rho v_g + \int_{M_\infty} v_\infty^2 v_g$ and consider the quadratic form $q_\varepsilon(v_0, v_\infty) = \int_{M_0} |\nabla v_0|^2 v_g + \frac{1}{\varepsilon}\int_{M_\infty} |\nabla v_\infty|^2 v_g$, so that, for every $u \in H^1(M)$

$$\|\Psi_\varepsilon(u)\|_\rho = \|u\|_\varepsilon \quad \text{and} \quad q_\varepsilon(\Psi_\varepsilon(u)) = q(u).$$

Therefore, the eigenvalues of the quadratic form $q : H^1(M) \to \mathbb{R}$ with respect to $\| \ \|_\varepsilon$ (i.e., $\mu_k^g(\rho_\varepsilon, 1)$) coincide with those of $q_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}$ with respect to $\| \ \|_\rho$.

The space $\mathcal{H}_\varepsilon$ decomposes into the direct sum $\mathcal{H}_\varepsilon = \mathcal{K}_0^\varepsilon \oplus \mathcal{K}_\infty^\varepsilon$ with $\mathcal{K}_0^\varepsilon = \{(v_0, v_\infty) \in \mathcal{H}_\varepsilon \ : \ v_\infty$ is harmonic, and $\frac{\partial v_\infty}{\partial \nu} = 0$ on $\partial M \backslash \partial M_0$ if $\partial M \backslash \partial M_0 \neq \emptyset\}$, and $\mathcal{K}_\infty^\varepsilon = \{(v_0, v_\infty) \in \mathcal{H}_\varepsilon \ : \ v_0 = 0\}$ (Indeed, $v = (v_0, v_\infty) = (v_0, \sqrt{\varepsilon}H(v_0)) + (0, v_\infty - \sqrt{\varepsilon}H(v_0))$). These two subspaces are q_ε-orthogonal and, denoting by $\lambda_1(M_\infty)$ the first eigenvalue of M_∞ under Dirichlet boundary conditions on Γ and Neumann boundary conditions on $\partial M_\infty \backslash \Gamma$, we have, for every $v = (0, v_\infty) \in \mathcal{K}_\infty$,

$$q_\varepsilon(v) = \frac{1}{\varepsilon}\int_{M_\infty} |\nabla v_\infty|^2 v_g \geq \frac{1}{\varepsilon}\lambda_1(M_\infty)\int_{M_\infty} v_\infty^2 v_g = \frac{1}{\varepsilon}\lambda_1(M_\infty)\|v\|_\rho^2.$$

Theorem I.7 of [14] then implies that, given any integer $N > 0$, *the N first eigenvalues $\mu_k(\rho_\varepsilon, 1)$ of q_ε on $\mathcal{H}_\varepsilon$ are, for sufficiently small ε, as close as desired to the eigenvalues of the restriction of q_ε on $\mathcal{K}_0^\varepsilon$.*

We still have to compare the eigenvalues of q_ε on $\mathcal{K}_0^\varepsilon$, that we denote $\gamma_k(\varepsilon)$, with the eigenvalues $\gamma_k(M_0, \rho)$ of Q_0 on $L^2(M_0, \rho v_g)$. For this, we make use of Theorem I.8 of [14]. Indeed, $\mathcal{K}_0^\varepsilon$ can be identified to $H^1(M_0)$ through $\Psi_\varepsilon^0 : u \in H^1(M_0) \mapsto (u, \sqrt{\varepsilon}H(u)) \in \mathcal{K}_0^\varepsilon$, which satisfies $\|\Psi_\varepsilon^0(u)\|_\varepsilon^2 = \int_{M_0} u^2 \rho v_g + \varepsilon \int_{M_\infty} H(u)^2 v_g$ and $q_\varepsilon(\Psi_\varepsilon^0(u)) = Q_0(u) = \int_{M_0} |\nabla u|^2 v_g + \int_{M_\infty} |\nabla H(u)|^2 v_g$. Hence, we are led to compare, on $L^2(M_0)$, the eigenvalues of the quadratic form Q_0 with respect to the following two scalar products: $\|u\|_\rho^2 = \int_{M_0} u^2 \rho v_g$ and $\|u\|_\varepsilon^2 = \int_{M_0} u^2 \rho v_g + \varepsilon \int_{M_\infty} H(u)^2 v_g$.

Now, since $H(u)$ is a harmonic extension of $u_{\restriction \Gamma}$ to M_∞, there exists a constant C, which does not depend on ε, such that $\int_{M_\infty} H(u)^2 v_g \leq C \int_\Gamma u^2 v_{\bar{g}}$, where $\bar{g}$ is the metric induced on Γ by g. Indeed, let η be the solution in M_∞ of $\Delta \eta = -1$ with $\eta_{\restriction \Gamma} = 0$ and $\frac{\partial \eta}{\partial \nu} = 0$ on $\partial M_\infty \backslash \Gamma$. Observe that we have $\eta \geq 0$ (maximum principle and Hopf lemma) and, since $\int_{M_\infty} g(\nabla(\eta H(u)), \nabla H(u)) v_g = 0$, $\int_{M_\infty} g(\nabla \eta, \nabla H(u)^2) v_g = -2\int_{M_\infty} \eta |\nabla H(u)|^2 v_g \leq 0$. Thus,

$$\int_{M_\infty} H(u)^2 v_g = -\int_{M_\infty} H(u)^2 \Delta\eta \, v_g = \int_{M_\infty} g(\nabla \eta, \nabla H(u)^2) v_g + \int_\Gamma u^2 \frac{\partial \eta}{\partial \nu} v_{\bar{g}} \leq c \int_\Gamma u^2 v_{\bar{g}},$$

where c is an upper bound of $\frac{\partial \eta}{\partial \nu}$ on Γ. On the other hand, $\int_\Gamma u^2 v_{\bar{g}}$ is controlled by $\|u\|^2_{H^{\frac{1}{2}}(\Gamma)}$ which in turn is controlled (using boundary trace inequalities in M_0) by $\|u\|^2_{H^1(M_0)}$. Finally, there exists a constant C (which depends on $\operatorname{ess\,inf}_{M_0} \rho$ but not on ε) such that $\int_{M_\infty} H(u)^2 v_g \leq C(\int_{M_0} u^2 \rho v_g + \int_{M_0} |\nabla u|^2 v_g)$ and, then

$$\|u\|_\varepsilon^2 \leq C(\|u\|_\rho^2 + Q_0(u)).$$

Since $\|u\|_\varepsilon^2$ converges to $\|u\|_\rho^2$ as $\varepsilon \to 0$, this implies, according to [14, Theorem I.8] (see also [25, Remark 2.14]), that, for sufficiently small ε, the N first eigenvalues $\gamma_k(\varepsilon)$ of Q_0 with respect to $\| \ \|_\varepsilon$ are as close as desired to those, $\gamma_k(M_0, \rho)$, of Q_0, with respect to $\| \ \|_\rho$. □

Recall that in dimension 2, one has

$$\mu_k^g(\rho, 1) = \lambda_k(M, \rho g). \tag{7}$$

An immediate consequence of Proposition 2.2 is the following result which completes Theorem III.1 of Colin de Verdière [14].

Theorem 2.1 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$ and let $M_0 \subset M$ be a domain with boundary of class C^1. Let g_ε be a family of Riemannian metrics on M, with $g_\varepsilon = g$ on M_0 and $g_\varepsilon = \varepsilon g$ outside M_0. Let $k \geq 1$.*

1. *(Theorem III.1 of [14]) If $n \geq 3$, then $\lambda_k(M, g_\varepsilon)$ converges to $\lambda_k(M_0, g)$ as $\varepsilon \to 0$*
2. *If $n = 2$, then $\lambda_k(M, g_\varepsilon)$ converges to $\gamma_k(M_0, 1)$ as $\varepsilon \to 0$.*

From Propositions 2.1 and 2.2, we can deduce the following two corollaries:

Corollary 2.1 *Let $\rho \in L^\infty(M_0)$ be a positive density on a domain $M_0 \subset M$ with boundary of class C^1. There exists a family of smooth positive densities ρ_ε on M such that $\int_M \rho_\varepsilon v_g$ tends to $\int_{M_0} \rho v_g$ and, for every $k \in \mathbb{N}^*$, $\mu_k(\rho_\varepsilon, 1)$ converges to $\gamma_k(M_0, \rho)$ as $\varepsilon \to 0$.*

Corollary 2.2 *Let (M, g) be a compact manifold possibly with boundary, and let $M_0 \subset M$ be a domain with boundary of class C^1. For every integer $k > 0$ and every $\varepsilon > 0$, there exists a positive smooth density ρ_ε on M such that $\int_M \rho_\varepsilon v_g = |M|_g$ and*

$$\mu_k(\rho_\varepsilon, 1) \geq \frac{|M_0|_g}{|M|_g} \lambda_k(M_0, g) - \varepsilon.$$

Proof Let ρ be the density on M_0 defined by $\rho = \frac{|M|_g}{|M_0|_g}$. We apply Corollary 2.1 taking into account that $\gamma_k(M_0, \rho) = \frac{|M_0|_g}{|M|_g} \gamma_k(M_0, 1) \geq \frac{|M_0|_g}{|M|_g} \lambda_k(M_0, g)$. □

Remark 2.1 In dimension 2, it is clear from (7) that the problem of minimizing or maximizing $\mu_k^g(\rho, 1)$ w.r.t. ρ is equivalent to the problem of minimizing or maximizing $\lambda_k(M, g)$ w.r.t. conformal deformations of the metric g. In dimension $n \geq 3$, the two problems are completely different. To emphasize this difference, observe that, given a positive constant c, one has

$$\inf_{\rho \leq c} \mu_k^g(\rho, 1) \geq \frac{1}{c} \mu_k^g(1, 1) = \frac{1}{c} \lambda_k(M, g) > 0$$

while

$$\inf_{\rho \leq c} \lambda_k(M, \rho g) = 0.$$

Indeed, let B_j, $j \leq k+1$ be a family of mutually disjoint balls in M and consider the density ρ_ε which is equal to c on each B_j and equal to ε elsewhere. According to [14, Theorem III.1], $\lambda_k(M, \rho_\varepsilon g)$ converges as $\varepsilon \to 0$ to the $(k+1)$th Neumann eigenvalue of the union of balls which is zero.

3 Bounding the eigenvalues from below

3.1 Nonexistence of "density-free" lower bounds

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$, possibly with boundary, and denote by $[g]$ the set of all Riemannian metrics g' on M which are conformal to g with

$|M|_{g'} = |M|_g$. It is well known that $\lambda_k(M, g')$ can be as small as desired when g' varies within $[g]$, i.e., $\inf_{g' \in [g]} \lambda_k(M, g) = 0$ (Cheeger dumbbells). Since $\mu_k^g(\rho, \rho^{\frac{n-2}{n}}) = \lambda_k(M, \rho^{\frac{2}{n}} g)$, this property is equivalent to

$$\inf_{\int_M \rho v_g = |M|_g} \mu_k^g(\rho, \rho^{\frac{n-2}{n}}) = 0. \tag{8}$$

Let us denote by $\mathcal{R}_0$ the set of positive smooth functions ϕ on M satisfying $⨍_M \phi v_g = 1$, where

$$⨍_M \phi v_g = \frac{1}{|M|_g} \int_M \phi v_g.$$

The following theorem shows that $\mu_k(\rho, \sigma)$ is not bounded below when one of the densities ρ, σ is fixed and the second one is varying within $\mathcal{R}_0$. We also deal with the case $\sigma = \rho^p$, $p \geq 0$, which includes (8) and the case of the Witten Laplacian.

Theorem 3.1 *For every positive integer k, one has,* $\forall p > 0$

i. $\inf_{\rho \in \mathcal{R}_0} \mu_k(\rho, 1) = 0$

ii. $\inf_{\sigma \in \mathcal{R}_0} \mu_k(1, \sigma) = 0$

iii. $\inf_{\rho \in \mathcal{R}_0} \mu_k(\rho, \rho^p) = 0.$

Proof of Theorem 3.1 (i) In dimension 2, one has $\mu_k(\rho, 1) = \lambda_k(M, \rho g)$ and the problem is equivalent to that of deforming conformally the metric g into a metric ρg whose kth eigenvalue is as small as desired. The existence of such a deformation is well known.

Assume now that the dimension of M is at least 3. Let us choose a point x_0 in M. The Riemannian volume of a geodesic ball $B(x, r)$ of radius r in M is asymptotically equivalent, as $r \to 0$, to $\omega_n r^n$, where ω_n is the volume of the unit ball in the n-dimensional Euclidean space. Therefore, there exists $\varepsilon_0 \in (0, 1)$ sufficiently small and $N \in \mathbb{N}$ so that, for every $r < \frac{\varepsilon_0}{N}$ and every $x \in B(x_0, \varepsilon_0)$,

$$\frac{1}{2}\omega_n r^n \leq |B(x, r)| \leq 2\omega_n r^n. \tag{9}$$

Fix a positive integer k and let $\delta = \frac{n-2}{4}$ so that $\delta < \frac{n}{2} - 1$. One can choose $N \in \mathbb{N}$ sufficiently large so that, for every $\varepsilon < \frac{\varepsilon_0}{N}$, the ball $B(x_0, \varepsilon)$ contains k mutually disjoint balls of radius $2\varepsilon^{\frac{n}{2}-\delta}$ (indeed, since $\frac{n}{2} - \delta > 1$, $2\varepsilon^{\frac{n}{2}-\delta}$ is very small compared to ε as the latter tends to zero). We consider a smooth positive density ρ_ε such that $\rho_\varepsilon = \frac{1}{\varepsilon^n}$ inside $B(x_0, \varepsilon)$, $\rho_\varepsilon = \varepsilon$ in $M \backslash B(x_0, 2\varepsilon)$, and $\rho_\varepsilon \leq \frac{1}{\varepsilon^n}$ elsewhere. Thanks to (9), one has

$$\int_M \rho_\varepsilon v_g \leq \frac{1}{\varepsilon^n} |B(x_0, 2\varepsilon)|_g + \varepsilon |M|_g \leq 2^{n+1}\omega_n + \varepsilon |M|_g.$$

For simplicity, we set $\alpha = \frac{n}{2} - \delta = \frac{n+2}{4}$ and denote by $x_1, \ldots, x_k$ the centers of k mutually disjoint balls of radius $2\varepsilon^\alpha$ contained in $B(x_0, \varepsilon)$.

For each $i \leq k$, we denote f_i the function which vanishes outside $B(x_i, 2\varepsilon^\alpha)$, equals 1 in $B(x_i, \varepsilon^\alpha)$, and $f_i(x) = 2 - \frac{1}{\varepsilon^\alpha} d_g(x, x_i)$ for every x in the annulus $B(x_i, 2\varepsilon^\alpha) \backslash B(x_i, \varepsilon^\alpha)$. The norm of the gradient of f_i vanishes everywhere except inside the annulus where we have $|\nabla f_i| = \frac{1}{\varepsilon^\alpha}$. Thus, using (9),

$$\int_M f_i^2 \rho_\varepsilon v_g \geq \frac{1}{\varepsilon^n} \int_{B(x_i, \varepsilon^\alpha)} f_i^2 v_g = \frac{|B(x_i, \varepsilon^\alpha)|}{\varepsilon^n} \geq \frac{1}{2}\omega_n \varepsilon^{n(\alpha-1)}$$

and

$$\int_M |\nabla f_i|^2 v_g \le \frac{|B(x_i, 2\varepsilon^\alpha)|}{\varepsilon^{2\alpha}} \le 2^{n+1}\omega_n \varepsilon^{\alpha(n-2)}.$$

Thus,

$$R_{(g,\rho_\varepsilon,1)}(f_i) \le 2^{n+2}\varepsilon^{n-2\alpha} = 2^{n+2}\varepsilon^{\frac{n-2}{2}}.$$

In conclusion, we have

$$\mu_k(\rho_\varepsilon, 1) \le 2^{n+2}\varepsilon^{\frac{n-2}{2}}$$

and

$$\mu_k\left(\frac{\rho_\varepsilon}{\fint_M \rho_\varepsilon v_g}, 1\right) = \mu_k(\rho_\varepsilon, 1)\fint_M \rho_\varepsilon v_g \le 2^{n+2}\left(\frac{2^{n+1}\omega_n}{|M|_g}\varepsilon^{\frac{n-2}{2}} + \varepsilon^{\frac{n}{2}}\right).$$

Letting ε tend to zero we get the result.

(ii) The proof is similar to the previous one. For ε sufficiently small, we may assume that there exist $k+1$ mutually disjoint balls $B(x_i, \varepsilon^2)$ inside a ball $B(x_0, \varepsilon)$ and consider any function $\sigma_\varepsilon \in \mathcal{R}_0$ such that $\sigma_\varepsilon = \varepsilon^5$ inside $B(x_0, \varepsilon)$. For each $i \le k+1$, let f_i be the function which vanishes outside $B(x_i, 2\varepsilon^2)$, equals 1 in $B(x_i, \varepsilon^2)$, and $f_i(x) = 2 - \frac{1}{\varepsilon^2}d_g(x, x_i)$ in $B(x_i, 2\varepsilon^2)\backslash B(x_i, \varepsilon^2)$. As before,

$$\int_M f_i^2 v_g \ge \int_{B(x_i,\varepsilon^2)} f_i^2 \mathrm{d}x \ge |B(x_i, \varepsilon^2)| \ge \frac{1}{2}\omega_n \varepsilon^{2n}$$

and

$$\int_M |\nabla f_i|^2 \sigma_\varepsilon v_g \le \frac{1}{\varepsilon^4}\int_{B(x_i,2\varepsilon^2)} \sigma_\varepsilon v_g \le \varepsilon |B(x_i, 2\varepsilon^2)| \le 2^{n+1}\omega_n\varepsilon^{2n+1}.$$

Thus,

$$\mu_k(1, \sigma_\varepsilon) \le \max_{i \le k+1} \frac{\int_M |\nabla f_i|^2 \sigma_\varepsilon v_g}{\int_M f_i^2 v_g} \le 2^{n+2}\varepsilon.$$

(iii) For sufficiently small ε, let $B(x_i, 4\varepsilon)$, $i \le k+1$, be $k+1$ mutually disjoint balls of radius 4ε in M. As before, we can assume that, $\forall r \le 4\varepsilon$, $\frac{1}{2}\omega_n r^n \le |B(x_i, r)| \le 2\omega_n r^n$. We define ρ_ε to be equal to $\frac{1}{\varepsilon^n}$ on each of the balls $B(x_i, \varepsilon)$ and equal to ε^n in the complement of $\cup_{i \le k} B(x_i, 2\varepsilon)$. For every $i \le k+1$, the function f_i defined to be equal to 1 on $B(x_i, 2\varepsilon)$ and $f_i(x) = 2 - \frac{1}{2\varepsilon}d_g(x, x_i)$ in the annulus $B(x_i, 4\varepsilon)\backslash B(x_i, 2\varepsilon)$ and zero in the complement of $B(x_i, 4\varepsilon)$ satisfies

$$\int_M f_i^2 \rho_\varepsilon v_g \ge \int_{B(x_i,\varepsilon)} f_i^2 \rho_\varepsilon \mathrm{d}x = \frac{1}{\varepsilon^n}|B(x_i, \varepsilon)| \ge \frac{1}{2}\omega_n.$$

On the other hand, $\forall p > 0$,

$$\int_M |\nabla f_i|^2 \rho_\varepsilon^p v_g = \varepsilon^{pn}\int_{B(x_i,4\varepsilon)\backslash B(x_j,2\varepsilon)} |\nabla f_i|^2 v_g = \varepsilon^{pn}\frac{1}{4\varepsilon^2}|B(x_i, 4\varepsilon)| \le 2^{2n-1}\omega_n\varepsilon^{(p+1)n-2}.$$

Thus,

$$\mu_k(\rho_\varepsilon, \rho_\varepsilon^p) \le \max_{i \le k+1} \frac{\int_M |\nabla f_i|^2 \sigma_\varepsilon v_g}{\int_M f_i^2 v_g} \le 2^{2n}\varepsilon^{(p+1)n-2}.$$

Regarding $\fint_M \rho_\varepsilon v_g$, it is clear that it is bounded both from above and from below by positive constants that are independent of ε, which enables us to conclude. □

3.2 Cheeger-type inequality

Theorem 3.1 tells us that it is necessary to involve other quantities than the total mass in order to get lower bounds for the eigenvalues. Our next theorem gives a lower estimate which is modeled on Cheeger's inequality, with suitably defined isoperimetric constants, as was done by Jammes for Steklov eigenvalues [27].

Let (M, g) be a compact Riemannian manifold, possibly with boundary. The classical Cheeger constant is defined by

$$h(M) = \inf_{|D|_g \le \frac{1}{2}|M|_g} \frac{|\partial D \backslash \partial M|_g}{|D|_g} = \inf_{D \subset M} \frac{|\partial D \backslash \partial M|_g}{\min\{|D|_g, |M|_g - |D|_g\}}.$$

Given two positive densities ρ and σ on M, we introduce the following Cheeger-type constant:

$$h_{\rho,\sigma}(M) = \inf_{|D|_\sigma \le \frac{1}{2}|M|_\sigma} \frac{|\partial D \backslash \partial M|_\sigma}{|D|_\rho}$$

with $|D|_\sigma$ (resp. $|\partial D \backslash \partial M|_\sigma$) is the n-volume of D (resp. the $(n-1)$-volume of $\partial D \backslash \partial M$) with respect to the measure induced by σv_g.

Theorem 3.2 *One has*

$$\mu_1(\rho, \sigma) \ge \frac{1}{4} h_{\sigma,\sigma}(M) h_{\rho,\sigma}(M).$$

Proof The proof follows the same general outline as the original proof by Cheeger (see [5,8]). We give here a complete proof in the case where M is a closed manifold. The proof in the case $\partial M \ne \emptyset$ can be done analogously. Let f be a Morse function such that the σ-volume of its positive nodal domain $\Omega_+(f) = \{f > 0\}$ is less or equal to half the σ-volume of M. For every $t \in (0, \sup f)$ excepting a finite number of values, the set $f^{-1}(t)$ is a regular hypersurface of M. We denote by v_g^t the measure induced on $f^{-1}(t)$ by v_g and set $P_\sigma(t) = \int_{f^{-1}(t)} \sigma v_g^t$. The level sets of f are denoted $\Omega(t) = \{f > t\}$ and we set $V_\sigma(t) = \int_{\Omega(t)} \sigma v_g$ and $V_\rho(t) = \int_{\Omega(t)} \rho v_g$. Using the co-area formula, one gets

$$\int_{\Omega_+(f)} |\nabla f| \sigma v_g = \int_0^{+\infty} P_\sigma(t) \mathrm{d}t.$$

On the other hand, the same co-area formula gives

$$V_\rho(t) = \int_t^{+\infty} ds \int_{f^{-1}(s)} \frac{\rho}{|\nabla f|} v_g^s.$$

Thus,

$$V_\rho'(t) = -\int_{f^{-1}(t)} \frac{\rho}{|\nabla f|} v_g^t.$$

Now,

$$\int_{\Omega_+(f)} f \rho v_g = \int_0^{+\infty} \mathrm{d}t \int_{f^{-1}(t)} \frac{f\rho}{|\nabla f|} v_g^t = \int_0^{+\infty} t dt \int_{f^{-1}(t)} \frac{\rho}{|\nabla f|} v_g^t = -\int_0^{+\infty} t V_\rho'(t) \mathrm{d}t$$

which gives after integration by parts

$$\int_{\Omega_+(f)} f\rho\, v_g = \int_0^{+\infty} V_\rho(t)\mathrm{d}t.$$

Similarly, one has

$$\int_{\Omega_+(f)} f\sigma\, v_g = \int_0^{+\infty} V_\sigma(t)\mathrm{d}t.$$

Since $P_\sigma(t) \geq h_{\sigma,\sigma}(M)V_\sigma(t)$ and $P_\sigma(t) \geq h_{\rho,\sigma}(M)V_\rho(t)$, we deduce

$$\int_{\Omega_+(f)} |\nabla f|\sigma\, v_g \geq \max\left\{h_{\sigma,\sigma}(M)\int_{\Omega_+(f)} f\sigma\, v_g\ ,\ h_{\rho,\sigma}(M)\int_{\Omega_+(f)} f\rho\, v_g\right\}.$$

Using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\int_{\Omega_+(f)} |\nabla f|^2\sigma\, v_g &\geq \frac{1}{4}\frac{\left(\int_{\Omega_+(f)} |\nabla f^2|\sigma\, v_g\right)^2}{\int_{\Omega_+(f)} f^2\sigma\, v_g}\\ &\geq \frac{1}{4}\frac{h_{\sigma,\sigma}(M)h_{\rho,\sigma}(M)\int_{\Omega_+(f)} f^2\sigma\, v_g \int_{\Omega_+(f)} f^2\rho\, v_g}{\int_{\Omega_+(f)} f^2\sigma\, v_g}\\ &= \frac{1}{4}h_{\sigma,\sigma}(M)h_{\rho,\sigma}(M)\int_{\Omega_+(f)} f^2\rho\, v_g. \qquad (10)\end{aligned}$$

Now, let $m \in \mathbb{R}$ be such that $|\{f > m\}|_\sigma = |\{f < m\}|_\sigma = \frac{1}{2}|M|_\sigma$ (such an m is called a median of f for σ). Applying (10) to $f - m$ and $m - f$, we get

$$\int_{\{f>m\}} |\nabla f|^2\sigma\, v_g \geq \frac{1}{4}h_{\sigma,\sigma}(M)h_{\rho,\sigma}(M)\int_{\{f>m\}} (f-m)^2\rho\, v_g$$

and

$$\int_{\{f<m\}} |\nabla f|^2\sigma\, v_g \geq \frac{1}{4}h_{\sigma,\sigma}(M)h_{\rho,\sigma}(M)\int_{\{f<m\}} (f-m)^2\rho\, v_g.$$

Summing up we obtain

$$\int_M |\nabla f|^2\sigma\, v_g \geq \frac{1}{4}h_{\sigma,\sigma}(M)h_{\rho,\sigma}(M)\int_M (f-m)^2\rho\, v_g.$$

Since $\int_M (f-m)^2\rho\, v_g = \int_M f^2\rho\, v_g + m^2|M|_\rho - 2m\int_M f\rho\, v_g$, we deduce that, for every f such that $\int_M f\rho\, v_g = 0$,

$$\int_M |\nabla f|^2\sigma\, v_g \geq \frac{1}{4}h_{\sigma,\sigma}(M)h_{\rho,\sigma}(M)\int_M f^2\rho\, v_g$$

which, thanks to (3), implies the desired inequality. □

Remark 3.1 In dimension 2, Theorem 3.2 can be restated as follows: If (M, g) is a compact Riemannian surface, then

$$\lambda_1(M, g) \geq \frac{1}{4}\sup_{g'\in[g]} h_{g',g'}(M)h_{g,g'}(M), \qquad (11)$$

where $h_{g,g'}(M) = \inf_{|D|_{g'} \leq \frac{1}{2}|M|_{g'}} \frac{|\partial D|_{g'}}{|D|_g}$. Indeed, for any $g' \in [g]$ there exists a positive $\rho \in C^\infty(M)$ such that $g = \rho g'$. Thus, $\lambda_1(M, g) = \mu_1^{g'}(\rho, 1)$ and (11) follows from Theorem 3.2. This inequality can be seen as an improvement in Cheeger's inequality since the right-hand side is obviously bounded below by $h_{g,g}(M)^2$. Notice that in [6], Buser gives an example of a family of metrics on the 2-torus such that the Cheeger constant goes to zero, while the first eigenvalue is bounded below. The advantage of (11) is that its right-hand side does not go to zero for Buser's example.

A natural question is to investigate a possible reverse inequality of Buser's type (see [6,32]). The following theorem provides a negative answer to this question.

Theorem 3.3 *Let (M, g) be a compact Riemannian manifold, possibly with boundary.*

i. *There exists a family of positive densities σ_ε, $\varepsilon > 0$, on M with $⨍_M \sigma_\varepsilon v_g = 1$ and such that $h_{1,\sigma_\varepsilon}(M) h_{\sigma_\varepsilon,\sigma_\varepsilon}(M)$ goes to zero with ε, while $\mu_1(1, \sigma_\varepsilon)$ stays bounded below by a constant C which does not depend on ε.*
ii. *There exists a family of positive densities ρ_ε, $\varepsilon > 0$, on M with $⨍_M \rho_\varepsilon v_g = 1$ and such that $h_{\rho_\varepsilon,1}(M)$ goes to zero with ε, while $\mu_1(\rho_\varepsilon, 1)$ stays bounded below by a constant C which does not depend on ε.*

Proof We start by proving the result for the unit ball $B^n \subset \mathbb{R}^n$ and then explain how to deduce it for any compact Riemannian manifold. For every $r \in (0, 1)$, we denote by $B(r)$ the ball of radius r centered at the origin and by A_r the annulus $B^n \backslash B(r)$. In the sequel, whenever we integrate over a Euclidean set, the integration is implicitly made with respect to the standard Lebesgue's measure.
Proof of (i): For every $\varepsilon \in (0, \frac{1}{2})$, we define a smooth nonincreasing radial density σ_ε on B^n such that $\sigma_\varepsilon = \frac{1}{\varepsilon^{1+a}}$, with $a \in (0, 1)$ (e.g., $a = \frac{1}{2}$) inside $B^n(\varepsilon)$ and $\sigma_\varepsilon = b_\varepsilon$ in $B^n \backslash B(2\varepsilon)$, where b_ε is chosen so that $\int_{B^n} \sigma_\varepsilon = \omega_n$, the volume of B^n. We then have

$$\int_{B(\varepsilon)} \sigma_\varepsilon = \omega_n \varepsilon^{n-1-a} \quad \text{and} \quad \int_{A_{2\varepsilon}} \sigma_\varepsilon = \omega_n (1 - 2^n \varepsilon^n) b_\varepsilon.$$

Since $\int_{B^n} \sigma_\varepsilon = \omega_n$ and $b_\varepsilon \leq \sigma_\varepsilon \leq \varepsilon^{-1-a}$ on $B(2\varepsilon) \backslash B(\varepsilon)$, we have

$$\omega_n \varepsilon^{n-1-a} + b_\varepsilon \omega_n (1 - \varepsilon^n) \leq \omega_n \leq \omega_n 2^n \varepsilon^{n-1-a} + b_\varepsilon \omega_n (1 - 2^n \varepsilon^n),$$

that is,

$$\frac{1 - 2^n \varepsilon^{n-1-a}}{1 - 2^n \varepsilon^n} \leq b_\varepsilon \leq \frac{1 - \varepsilon^{n-1-a}}{1 - \varepsilon^n}. \tag{12}$$

Now, the Cheeger constant $h_{\sigma_\varepsilon,\sigma_\varepsilon}(B^n)$ satisfies

$$h_{\sigma_\varepsilon,\sigma_\varepsilon}(B^n) \leq \frac{|\partial B(2\varepsilon)|_{\sigma_\varepsilon}}{|B(2\varepsilon)|_{\sigma_\varepsilon}} \leq \frac{|\partial B(2\varepsilon)|_{\sigma_\varepsilon}}{|B(\varepsilon)|_{\sigma_\varepsilon}} = \frac{n b_\varepsilon \omega_n (2\varepsilon)^{n-1}}{\omega_n \varepsilon^{n-1-a}} \leq n 2^{n-1} \varepsilon^a.$$

On the other hand, for $r_0 = \left(\frac{1}{4}\right)^{\frac{1}{n}}$ we have $|B(r_0)|_{\sigma_\varepsilon} < \omega_n(\varepsilon^{n-1-a} + \frac{1}{4} b_\varepsilon) < \frac{1}{2}\omega_n$ when ε is sufficiently small, so that

$$h_{1,\sigma_\varepsilon}(B^n) \leq \frac{|\partial B(r_0)|_{\sigma_\varepsilon}}{|B(r_0)|} = \frac{n \omega_n r_0^{n-1} b_\varepsilon}{\omega_n r_0^n} \leq 4^{\frac{1}{n}} n.$$

Hence, the product $h_{1,\sigma_\varepsilon}(B^n)h_{\sigma_\varepsilon,\sigma_\varepsilon}(B^n)$ tends to zero as $\varepsilon \to 0$. Regarding the first positive eigenvalue $\mu_1(1,\sigma_\varepsilon)$, if f is a corresponding eigenfunction, then $\int_{B^n} f = 0$ and

$$\mu_1(1,\sigma_\varepsilon) = \frac{\int_{B^n} |\nabla f|^2 \sigma_\varepsilon}{\int_{B^n} f^2} \geq b_\varepsilon \frac{\int_{B^n} |\nabla f|^2}{\int_{B^n} f^2} \geq b_\varepsilon \lambda_1(B^n, g_E)$$

with $b_\varepsilon \geq \frac{1}{2}$ for sufficiently small ε according to (12).

Now, given a Riemannian manifold (M, g), we fix a point x_0 and choose $\delta > 0$ so that the geodesic ball $B(x_0, \delta)$ is 2-quasi-isometric to the Euclidean ball of radius δ. In the Riemannian manifold $(M, \frac{1}{\delta^2} g)$, the ball $B(x_0, 1)$ is 2-quasi-isometric to the Euclidean ball B^n. We define σ_ε in $B(x_0, 1)$ as the pullback of the function σ_ε constructed above, and extend it by b_ε in $M \backslash B(x_0, 1)$. Because of (12), we easily see that $⨍_M \sigma_\varepsilon v_g$ stays bounded independently from ε. We can also check that $h_{1,\sigma_\varepsilon}(M)$ and $h_{\sigma_\varepsilon,\sigma_\varepsilon}(M)$ have the same behavior as before and that (since $\sigma_\varepsilon \geq b_\varepsilon \geq \frac{1}{2}$) the eigenvalue $\mu_1^{\delta^{-2}g}(1,\sigma_\varepsilon)$ is bounded from below by $\frac{1}{2}\lambda_1(M, \delta^{-2}g)$ which is a positive constant C independent of ε. Thus, $\mu_1^g(1,\sigma_\varepsilon) = \delta^2 \mu_1^{\delta^{-2}g}(1,\sigma_\varepsilon) \geq C\delta^2$.

Proof of (ii) As before we define the density $\rho_\varepsilon \in L^\infty(B^n)$, $\varepsilon \in (0, \frac{1}{2})$, by

$$\rho_\varepsilon = \begin{cases} \frac{1}{\varepsilon^{1+a}} & \text{if } x \in B(\varepsilon) \\ b_\varepsilon = \frac{1-\varepsilon^{n-1-a}}{1-\varepsilon^n} & \text{if } x \in B^n \backslash B(\varepsilon) \end{cases} \tag{13}$$

so that $\int_{B^n} \rho_\varepsilon \mathrm{d}x = \omega_n$ and $b_\varepsilon < 1$. The corresponding Cheeger constant satisfies

$$h_{\rho_\varepsilon,1} \leq \frac{|\partial B(\varepsilon)|}{|B(\varepsilon)|_{\rho_\varepsilon}} = \frac{n\omega_n \varepsilon^{n-1}}{\omega_n \varepsilon^{n-1-a}} = n\varepsilon^a,$$

which goes to zero as $\varepsilon \to 0$.

To prove that the first positive Neumann eigenvalue $\mu_1(\rho_\varepsilon, 1)$ is uniformly bounded below we will first prove that the first Dirichlet eigenvalue $\lambda_1(\rho_\varepsilon)$ satisfies

$$\lambda_1(\rho_\varepsilon) \geq \frac{1}{4}\lambda^*, \tag{14}$$

where λ^* is the first Dirichlet eigenvalue of the Laplacian on B^n. Indeed, let f be a positive eigenfunction associated to $\lambda_1(\rho_\varepsilon)$. Such a function is necessarily a nonincreasing radial function and it satisfies (with $b_\varepsilon \leq 1$)

$$\lambda_1(\rho_\varepsilon) = \frac{\int_{B(\varepsilon)} |\nabla f|^2 + \int_{A_\varepsilon} |\nabla f|^2}{\int_{B(\varepsilon)} f^2 \rho_\varepsilon + \int_{A_\varepsilon} f^2 \rho_\varepsilon} \geq \frac{\int_{B(\varepsilon)} |\nabla f|^2 + \int_{A_\varepsilon} |\nabla f|^2}{\varepsilon^{-1-a} \int_{B(\varepsilon)} f^2 + \int_{A_\varepsilon} f^2}. \tag{15}$$

For convenience, we assume that $f(\varepsilon) = 1$.

If we denote by $\nu(A_\varepsilon)$ the first eigenvalue of the mixed eigenvalue problem on the annulus A_ε, with Dirichlet conditions on the outer boundary and Neumann conditions on the inner boundary, then it is well known that $\nu(A_\varepsilon)$ converges to λ^* as $\varepsilon \to 0$ (see[1]). Thus, using the min-max, we will have for sufficiently small ε,

$$\int_{A_\varepsilon} |\nabla f|^2 \geq \nu(A_\varepsilon) \int_{A_\varepsilon} f^2 \geq \frac{1}{2}\lambda^* \int_{A_\varepsilon} f^2. \tag{16}$$

On the other hand, since $f - 1$ vanishes along $\partial B(\varepsilon)$, its Rayleigh quotient is bounded below by $\frac{1}{\varepsilon^2}\lambda^*$, the first Dirichlet eigenvalue of $B(\varepsilon)$. Thus,

$$\int_{B(\varepsilon)} |\nabla f|^2 \geq \frac{1}{\varepsilon^2}\lambda^* \int_{B(\varepsilon)} (f-1)^2 \geq \frac{1}{\varepsilon^2}\lambda^* \left(\int_{B(\varepsilon)} f^2 - 2 \int_{B(\varepsilon)} f \right) \tag{17}$$

with

$$\int_{B(\varepsilon)} f \leq \left(\omega_n \varepsilon^n \int_{B(\varepsilon)} f^2\right)^{\frac{1}{2}}.$$

Thus, if $\omega_n \varepsilon^n \leq \frac{1}{16}\int_{B(\varepsilon)} f^2$, then (17) yields

$$\int_{B(\varepsilon)} |\nabla f|^2 \geq \frac{1}{2\varepsilon^2}\lambda^* \int_{B(\varepsilon)} f^2 > \frac{1}{2}\lambda^* \varepsilon^{-1-a} \int_{B(\varepsilon)} f^2$$

which, combined with (16) and (15), implies (14).

Assume now that $\omega_n \varepsilon^n \geq \frac{1}{16}\int_{B(\varepsilon)} f^2$ and let us prove the following:

$$\int_{A_\varepsilon} |\nabla f|^2 \geq \begin{cases} \frac{n(n-2)}{16\varepsilon^{1-a}}\ \varepsilon^{-1-a}\int_{B(\varepsilon)} f^2 & \text{if } n \geq 3 \\ \frac{1}{8\varepsilon^{1-a}\ln(1/\varepsilon)}\ \varepsilon^{-1-a}\int_{B(\varepsilon)} f^2 & \text{if } n = 2 \end{cases} \tag{18}$$

which implies for sufficiently small ε,

$$\int_{A_\varepsilon} |\nabla f|^2 \geq \frac{1}{2}\lambda^* \varepsilon^{-1-a} \int_{B(\varepsilon)} f^2 \tag{19}$$

enabling us to deduce (14) from (15) and (16). Indeed, since $f(\varepsilon) = 1$ and $f(1) = 0$, one has $\int_\varepsilon^1 f' = -1$. Therefore, applying the Cauchy–Schwarz inequality to the product $f' = \left(f' r^{(n-1)/2}\right) r^{-(n-1)/2}$, we get

$$\frac{1}{n\omega_n}\int_{A_\varepsilon} |\nabla f|^2 = \int_\varepsilon^1 f'^2 r^{n-1} \geq \left(\int_\varepsilon^1 f'\right)^2 \left(\int_\varepsilon^1 \frac{1}{r^{n-1}}\right)^{-1} \geq \frac{1}{\int_\varepsilon^1 \frac{1}{r^{n-1}}}$$

with

$$\int_\varepsilon^1 \frac{1}{r^{n-1}} = \begin{cases} \frac{1}{n-2}\left(\frac{1}{\varepsilon^{n-2}} - 1\right) < \frac{1}{n-2}\frac{1}{\varepsilon^{n-2}} & \text{if } n \geq 3 \\ \ln(1/\varepsilon) & \text{if } n = 2 \end{cases}. \tag{20}$$

Therefore,

$$\int_{A_\varepsilon} |\nabla f|^2 \geq \begin{cases} n(n-2)\omega_n \varepsilon^{n-2} & \text{if } n \geq 3 \\ \frac{2\pi}{\ln(1/\varepsilon)} & \text{if } n = 2 \end{cases} \tag{21}$$

which gives (18) since $\omega_n \varepsilon^n \geq \frac{1}{16}\int_{B(\varepsilon)} f^2$.

Let us check now that the first positive Neumann eigenvalue is also uniformly bounded from below. Indeed, let f be a Neumann eigenfunction with $\Delta f = -\mu_1(\rho_\varepsilon, 1)\rho_\varepsilon f$. If f is radial, then $\mu_1(\rho_\varepsilon, 1) \geq \lambda_1(\rho_\varepsilon) \geq \frac{1}{4}\lambda^*$ (there exists $r_0 < 1$ with $f(r_0) = 0$ so that f is a Dirichlet eigenfunction on the ball $B(r_0)$). If f is not radial, then, up to averaging (or assuming that f is orthogonal to radial functions), one can assume w.l.o.g. that $\int_{\mathbb{S}^{n-1}(r)} f\,\mathrm{d}\theta = 0$ for every $r < 1$. Thus, $\int_{\mathbb{S}^{n-1}(r)} |\nabla^0 f|^2 \mathrm{d}\theta \geq \frac{n-1}{r^2}\int_{\mathbb{S}^{n-1}(r)} f^2 \mathrm{d}\theta$, where $\nabla^0 f$ is the tangential part of ∇f. Hence,

$$\int_{B^n} |\nabla f|^2 = \int_0^1 r^{n-1} dr \int_{\mathbb{S}^{n-1}(r)} |\nabla f|^2 \mathrm{d}\theta \geq (n-1)\int_0^1 r^{n-1} dr \int_{\mathbb{S}^{n-1}(r)} \left(\frac{f}{r}\right)^2 \mathrm{d}\theta$$
$$= (n-1)\int_{B^n} \left(\frac{f}{r}\right)^2 \geq (n-1)\int_{B^n} f^2 \rho_\varepsilon$$

since $\rho_\varepsilon(r) \le \frac{1}{r^2}$ everywhere. Thus, in this case, $\mu_1(\rho_\varepsilon, 1) \ge n-1$. Finally,

$$\mu_1(\rho_\varepsilon, 1) \ge \min(n-1, \frac{1}{4}\lambda^*).$$

As before, this construction can be implemented in any Riemannian manifold (M, g), using a quasi-isometry argument, Proposition 2.2 and Corollary 2.1. □

A relevant problem is to know if a Buser's type inequality can be obtained in this context under assumptions on the volume of balls with respect to σ and ρ.

4 Bounding the eigenvalues from above

4.1 Unboundedness of eigenvalues if only one parameter among g, ρ, σ is fixed

Let (M, g_0) be a compact Riemannian manifold, possibly with boundary. Our first observation in this section is that the eigenvalues $\mu_k^g(\rho, \sigma)$ are not bounded from above when one quantity among $g \in [g_0]$, $\rho \in \mathcal{R}_0$, $\sigma \in \mathcal{R}_0$ is fixed and the two others are varying (here $\mathcal{R}_0 = \{\phi \in C^\infty(M) \ : \phi > 0 \text{ and } ⨍_M \phi\, v_{g_0} = 1\}$).

Let us first recall that the authors and Savo have proved in [12] that on any compact Riemannian manifold (M, g_0) there exists a sequence of densities $\rho_j \in \mathcal{R}_0$ such that $\mu_1^{g_0}(\rho_j, \rho_j)$ tends to $+\infty$ with j. In particular,

$$\sup_{⨍_M \rho v_{g_0}=1, ⨍_M \sigma v_{g_0}=1} \mu_1^{g_0}(\rho, \sigma) \ge \sup_{⨍_M \rho v_{g_0}=1} \mu_1^{g_0}(\rho, \rho) = +\infty. \tag{22}$$

A natural subsequent question is: Can one construct examples of $g \in [g_0]$ and $\rho \in \mathcal{R}_0$ (resp. $\sigma \in \mathcal{R}_0$) so that $\mu_1^g(\rho, 1)$ (resp. $\mu_1^g(1, \sigma)$) is as large as desired ?

Proposition 4.1 *Let (M, g_0) be a compact Riemannian manifold, possibly with boundary. Then,*

$$\sup_{g\in[g_0],\, \rho\in\mathcal{R}_0} \mu_1^g(\rho, 1) = +\infty \tag{23}$$

and

$$\sup_{g\in[g_0],\, \sigma\in\mathcal{R}_0} \mu_1^g(1, \sigma) = +\infty. \tag{24}$$

Proof To prove (23), the idea is to deform both the metric and the density so that $\rho_\varepsilon v_{g_\varepsilon}$ becomes everywhere small. Indeed, let V be an open set of M with $|V|_{g_0} \ge \frac{1}{10}|M|_{g_0}$. For every $\varepsilon \in (0, 1)$, we consider a continuous density ρ_ε such that $\rho_\varepsilon = \varepsilon$ on V, $\varepsilon \le \rho_\varepsilon \le 2$ everywhere on M, and $⨍_M \rho_\varepsilon v_{g_0} = 1$. Define $g_\varepsilon = \phi_\varepsilon^2 g_0$ with

$$\phi_\varepsilon^n = \frac{|M|_{g_0}}{\int_M \rho_\varepsilon^{-1} v_{g_0}} \frac{1}{\rho_\varepsilon}$$

so that $|M|_{g_\varepsilon} = \int_M \phi_\varepsilon^n v_{g_0} = |M|_{g_0}$ (here n denotes the dimension of M). Now, we observe that

$$\frac{1}{\varepsilon}|M|_{g_0} \ge \int_M \rho_\varepsilon^{-1} v_{g_0} \ge \int_V \rho_\varepsilon^{-1} v_{g_0} = \frac{1}{\varepsilon}|V|_{g_0} \ge \frac{1}{10\varepsilon}|M|_{g_0}.$$

Thus,

$$\phi_\varepsilon^n \le \frac{10\varepsilon}{\rho_\varepsilon}$$

and, since $\rho_\varepsilon \leq 2$,

$$\phi_\varepsilon^n \geq \frac{\varepsilon}{\rho_\varepsilon} \geq \frac{\varepsilon}{2}.$$

Now, for any smooth function u on M one has (with $\frac{\varepsilon}{2} \leq \phi_\varepsilon^n \leq \frac{10\varepsilon}{\rho_\varepsilon}$)

$$\frac{\int_M |\nabla u|^2 v_{g_\varepsilon}}{\int_M u^2 \rho_\varepsilon v_{g_\varepsilon}} = \frac{\int_M |\nabla u|^2 \phi_\varepsilon^{n-2} v_{g_0}}{\int_M u^2 \rho_\varepsilon \phi_\varepsilon^n v_{g_0}} \geq \frac{1}{2^{\frac{n-2}{n}} 10 \varepsilon^{\frac{2}{n}}} \frac{\int_M |\nabla u|^2 v_{g_0}}{\int_M u^2 v_{g_0}}.$$

Therefore,

$$\mu_1^{g_\varepsilon}(\rho_\varepsilon, 1) \geq \frac{1}{2^{\frac{n-2}{n}} 10 \varepsilon^{\frac{2}{n}}} \mu_1^{g_0}(1, 1)$$

which tends to infinity as ε goes to zero.

To prove (24) we first observe that, for any positive density σ, one has, $\forall u \in C^2(M)$,

$$R_{(\sigma g_0, 1, \sigma)}(u) = R_{(g_0, \sigma^{\frac{n}{2}}, \sigma^{\frac{n}{2}})}(u).$$

Thus,

$$\mu_k^{\sigma g_0}(1, \sigma) = \mu_k^{g_0}(\sigma^{\frac{n}{2}}, \sigma^{\frac{n}{2}}).$$

According to [12], there exists on M a sequence σ_j of positive densities such that $\int_M \sigma_j^{\frac{n}{2}} v_{g_0} = |M|_{g_0}$ and $\mu_k^{g_0}(\sigma_j^{\frac{n}{2}}, \sigma_j^{\frac{n}{2}})$ tends to infinity with j. We set $g_j = \sigma_j g_0 \in [g_0]$. Hölder's inequality implies that

$$\int_M \sigma_j v_{g_0} \leq \left(\int_M \sigma_j^{\frac{n}{2}} v_{g_0} \right)^{\frac{2}{n}} |M|_{g_0}^{1-\frac{2}{n}} = |M|_{g_0}.$$

Setting $\sigma_j' = \frac{\sigma_j}{⨍_M \sigma_j v_{g_0}} \in \mathcal{R}_0$, we get

$$\mu_k^{g_j}(1, \sigma_j') = \frac{1}{⨍_M \sigma_j v_{g_0}} \mu_k^{\sigma_j g_0}(1, \sigma_j) \geq \mu_k^{\sigma_j g_0}(1, \sigma_j) = \mu_k^{g_0}(\sigma_j^{\frac{n}{2}}, \sigma_j^{\frac{n}{2}})$$

which proves that $\mu_k^{g_j}(1, \sigma_j')$ tends to infinity with j. □

4.2 Upper bounds for $\mu_k(\rho, 1)$ and $\mu_k(1, \sigma)$

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$, possibly with boundary. According to the result by Hassannezhad [23], one has, when M is a closed manifold,

$$\lambda_k(M, g) \leq \frac{1}{|M|_g^{\frac{2}{n}}} \left(A_n k^{\frac{2}{n}} + B_n V([g])^{\frac{2}{n}} \right), \tag{25}$$

where A_n and B_n are two constants which only depend on n, and $V([g])$ is a conformally invariant geometric quantity defined as follows:

$$V([g]) = \inf\{|M|_{g_0} \ : \ g_0 \text{ is conformal to } g \text{ and } Ric_{g_0} \geq -(n-1) g_0\},$$

where Ric_{g_0} is the Ricci curvature of g_0. Now, for every positive ρ such that $\fint_M \rho v_g = 1$, we have $V([\rho^{\frac{2}{n}} g]) = V([g])$, $|M|_{\rho^{\frac{2}{n}} g} = |M|_g$ and $\lambda_k(M, \rho^{\frac{2}{n}} g) = \mu_k^g(\rho, \rho^{\frac{n-2}{n}})$. Hence, inequality (25) implies that for every positive ρ such that $\fint_M \rho v_g = 1$,

$$\mu_k^g(\rho, \rho^{\frac{n-2}{n}}) \le \frac{1}{|M|_g^{\frac{2}{n}}} \left(A_n k^{\frac{2}{n}} + B_n V([g])^{\frac{2}{n}} \right). \tag{26}$$

This estimate is in contrast to what happens for the Witten Laplacian where we have $\sup_{\fint_M \rho v_g = 1} \mu_1^g(\rho, \rho) = +\infty$ (see [12]).

Our aim in this section is to discuss the boundedness of $\mu_k^g(\rho, \sigma)$ in the two remaining important cases: $\mu_k^g(\rho, 1)$ and $\mu_k^g(1, \sigma)$. In [12, Theorem 2.1], it has been shown that the use of the GNY (Grigor'yan–Netrusov–Yau) method [22] leads to the following estimate

$$\mu_k^g(\rho, 1) \fint_M \rho v_g \le C([g]) \left(\frac{k}{|M|_g} \right)^{\frac{2}{n}}, \tag{27}$$

where $C([g])$ is a constant which only depends on the conformal class of the metric g.

This approach fails in the dual situation where σ is varying, while ρ is fixed. Indeed, the GNY method leads to an upper bound of $\mu_k^g(1, \sigma)$ in terms of the $L^{\frac{n-2}{n}}$-norm of σ (instead of the L^1-norm). However, using the techniques developed by Colbois and Maerten in [13], it is possible to obtain an inequality of the form

$$\mu_k^g(1, \sigma) \le C(M, g) \left(\frac{k}{|M|_g} \right)^{\frac{2}{n}} \fint_M \sigma v_g, \tag{28}$$

where $C(M, g)$ is a geometric constant which does not depend on σ (unlike (27), this method of proof does not allow us to obtain a conformally invariant constant instead of $C(M, g)$).

In what follows, we will establish inequalities of the type (26) for $\mu_k(\rho, 1)$ and $\mu_k(1, \sigma)$.

Theorem 4.1 *Let M be a bounded open domain possibly with boundary of class C^1 of a complete Riemannian manifold $(\tilde{M}, \tilde{g}_0)$ of dimension $n \ge 2$ (with $\tilde{M} = M$ if $\partial M = \emptyset$). Assume that $Ric_{\tilde{g}_0} \ge -(n-1)\tilde{g}_0$ and let $g_0 = \tilde{g}_0|_M$. For every metric g conformal to g_0 and every density ρ with $\fint_M \rho v_g = 1$, one has*

$$\mu_k^g(\rho, 1) \le \frac{1}{|M|_g^{\frac{2}{n}}} \left(A_n k^{\frac{2}{n}} + B_n |M|_{g_0}^{\frac{2}{n}} \right), \tag{29}$$

where A_n and B_n are two constants which depend only on the dimension n.

In the particular case where (M, g) is a compact manifold without boundary, we can apply Theorem 4.1 with $M = \tilde{M}$ and get immediately the following estimate which extends (25):

$$\mu_k^g(\rho, 1) \le \frac{1}{|M|_g^{\frac{2}{n}}} \left(A_n k^{\frac{2}{n}} + B_n V([g])^{\frac{2}{n}} \right). \tag{30}$$

On the other hand, if $\tilde{g}$ is a metric on $\tilde{M}$ and if ric_0 is a positive number such that $\mathrm{Ric}_{\tilde{g}} \ge -(n-1)\mathrm{ric}_0\, \tilde{g}$, then the metric $\tilde{g}_0 = \mathrm{ric}_0 \tilde{g}$ satisfies $\mathrm{Ric}_{\tilde{g}_0} \ge -(n-1)\tilde{g}_0$ and $|M|_{g_0} = \mathrm{ric}_0^{n/2} |M|_g$, where $g = \tilde{g}|_M$ and $g_0 = \tilde{g}_0|_M$. Thus, we get

Corollary 4.1 *Let M be a bounded open domain possibly with boundary of class C^1 of a complete Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $n \geq 2$ (with $\tilde{M} = M$ if $\partial M = \emptyset$) and let $g = \tilde{g}|_M$. For every density ρ with $\fint_M \rho v_g = 1$, one has*

$$\mu_k^g(\rho, 1) \leq A_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}} + B_n ric_0, \tag{31}$$

where $ric_0 > 0$ is such that $Ric_{\tilde{g}} \geq -(n-1) ric_0\, \tilde{g}$. In particular, $\forall k \geq |M|_g ric_0^{\frac{n}{2}}$,

$$\mu_k^g(\rho, 1) \leq C_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}} \tag{32}$$

with $C_n = A_n + B_n$.

Inequalities (30) and (31) are conceptually much stronger than (27), especially since they lead to a Kröger-type inequality (32) for every k exceeding an explicit geometric threshold, independent of ρ (it is well known that if the Ricci curvature is not nonnegative, then an inequality like (32) cannot hold for every k, see [13, Remark 1.2(iii)]).

Theorem 4.2 *Let M be a bounded open domain possibly with boundary of class C^1 of a complete Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $n \geq 2$ (with $\tilde{M} = M$ if $\partial M = \emptyset$) and let $g = \tilde{g}|_M$. For every positive density σ on M with $\fint_M \sigma v_g = 1$, one has*

$$\mu_k^g(1, \sigma) \leq A_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}} + B_n ric_0, \tag{33}$$

where $ric_0 > 0$ is such that $Ric_{\tilde{g}} \geq -(n-1) ric_0\, \tilde{g}$ and where A_n and B_n are two constants which depend only on n. In particular, $\forall k \geq |M|_g ric_0^{\frac{n}{2}}$,

$$\mu_k^g(1, \sigma) \leq C_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}} \tag{34}$$

with $C_n = A_n + B_n$.

Proof of Theorem 4.1 We consider the metric measured space (M, d_0, ν) where d_0 is the restriction to M of the Riemannian distance on $(\tilde{M}, \tilde{g}_0)$, and $\nu = \rho v_g$. Since $\mathrm{Ric}_{g_0} \geq -(n-1) g_0$, the space (M, d_0, ν) satisfies a $(2, N; 1)$-covering property for some fixed N (see [23]). Therefore, we can apply Theorem 2.1 of [23] and find a family of $3(k+1)$ pairs of sets (F_j, G_j) of M with $F_j \subset G_j$, such that the G_j's are mutually disjoint and $\nu(F_j) \geq \frac{\nu(M)}{c^2(k+1)}$, where $c = c(n)$ is a constant that depends only on n. Moreover, each pair (F_j, G_j) satisfies one of the following properties:

- F_j is an annulus A of the form $A = \{r < d_0(x, a) < R\}$, and $G_j = 2A = \{\frac{r}{2} < d_0(x, a) < 2R\}$, with outer radius $2R$ less than 1,
- F_j is an open set $V \subset M$ and $G_j = V^{r_0} = \{x \in M\ ;\ d_0(x, V) < r_0\}$, with $r_0 = \frac{1}{1600}$.

Let us start with the case where F_j is an annulus $A = A(a, r, R) = \{r < d_0(x, a) < R\}$ and $G_j = 2A$. To such an annulus, we associate the function u_A supported in $2A = \{\frac{r}{2} < d_0(x, a) < 2R\}$ and such that

$$u_A(x) = \begin{cases} \frac{2}{r} d_0(x, a) - 1 & \text{if } \frac{r}{2} \leq d_0(x, a) \leq r \\ 1 & \text{if } x \in A \\ 2 - \frac{1}{R} d_0(x, a) & \text{if } R \leq d_0(x, a) \leq 2R \end{cases}. \tag{35}$$

Since u_A is supported in $2A$ we get, using Hölder's inequality and the conformal invariance of $|\nabla^g u_A|^n v_g$,

$$\int_M |\nabla^g u_A|^2 v_g = \int_{2A} |\nabla^g u_A|^2 v_g \le \left(\int_{2A} |\nabla^g u_A|^n v_g\right)^{\frac{2}{n}} \left(\int_{2A} v_g\right)^{1-\frac{2}{n}}$$
$$= \left(\int_{2A} |\nabla^{g_0} u_A|^n v_{g_0}\right)^{\frac{2}{n}} |2A|_g^{1-\frac{2}{n}}.$$

Since

$$|\nabla^{g_0} u_A| \overset{a.e.}{=} \begin{cases} \frac{2}{r} & \text{if } \frac{r}{2} \le d_0(x,a) \le r \\ 0 & \text{if } r \le d_0(x,a) \le R \\ \frac{1}{R} & \text{if } R \le d_0(x,a) \le 2R \end{cases},$$

we get

$$\int_{2A} |\nabla^{g_0} u_A|^n v_{g_0} \le \left(\frac{2}{r}\right)^n |B(a,r)|_{g_0} + \left(\frac{1}{R}\right)^n |B(a,2R)|_{g_0} \le 2^{n+1}\Gamma(g_0),$$

where

$$\Gamma(g_0) = \sup_{x\in M, t\in(0,1)} \frac{|B(x,t)|_{g_0}}{t^n}$$

(here $B(x,t)$ stands for the ball of radius t centered at x in (M, d_0)). Notice that since $\mathrm{Ric}_{\tilde{g}_0} \ge -(n-1)\tilde{g}_0$, the constant $\Gamma(g_0)$ is bounded above by a constant that depends only on n (Bishop–Gromov inequality). Hence,

$$\int_M |\nabla^g u_A|^2 v_g \le C(n)|2A|_g^{1-\frac{2}{n}},$$

where $C(n) \ge 2^{n+1}\Gamma(g_0)$. On the other hand, we have

$$\int_M u_A^2 \rho\, v_g \ge \int_A \rho\, v_g = \nu(A) \ge \frac{\nu(M)}{c^2(k+1)}.$$

Thus,

$$R_{(g,\rho,1)}(u_A) = \frac{\int_M |\nabla^g u_A|^2 v_g}{\int_M u_A^2 \rho\, v_g} \le A_n \frac{|2A|_g^{1-\frac{2}{n}}}{\nu(M)}(k+1)$$

for some constant A_n.

Now, in the second situation, where F_j is an open set V and $G_j = V^{r_0}$, we introduce the function u_V defined to be equal to 1 inside V, 0 outside V^{r_0} and proportional to the d_0-distance to the outer boundary in $V^{r_0}\backslash V$. We have, since $u_V = 1$ in V and $|\nabla^{g_0} u_V|$ is equal to $\frac{1}{r_0}$ almost everywhere in $V^{r_0}\backslash V$ and vanishes in V and in $M\backslash V^{r_0}$,

$$\int_M u_V^2 \rho\, v_g \ge \int_V \rho\, v_g = \nu(V) \ge \frac{\nu(M)}{c^2(k+1)}$$

and

$$\int_M |\nabla^g u_V|^2 v_g \le \left(\int_{V^{r_0}} |\nabla^g u_V|^n v_g\right)^{\frac{2}{n}} |V^{r_0}|_g^{1-\frac{2}{n}} = \left(\int_{V^{r_0}} |\nabla^{g_0} u_V|^n v_{g_0}\right)^{\frac{2}{n}} |V^{r_0}|_g^{1-\frac{2}{n}}$$
$$\le \frac{|V^{r_0}|_{g_0}^{\frac{2}{n}} |V^{r_0}|_g^{1-\frac{2}{n}}}{{r_0}^2}.$$

Thus,

$$R_{(g,\rho,1)}(u_V) \leq B_n \frac{|V^{r_0}|_{g_0}^{\frac{2}{n}} |V^{r_0}|_g^{1-\frac{2}{n}}}{\nu(M)}(k+1),$$

where $B_n = \frac{c^2}{r_0^2}$ is a constant which depends only on n.

In conclusion, to each pair (F_j, G_j) we associate a test function u_j supported in G_j and satisfying either $R_{(g,\rho,1)}(u_j) \leq A_n \frac{|G_j|_g^{1-\frac{2}{n}}}{\nu(M)}(k+1)$ or $R_{(g,\rho,1)}(u_j) \leq B_n \frac{|G_j|_{g_0}^{\frac{2}{n}} |G_j|_g^{1-\frac{2}{n}}}{\nu(M)}(k+1)$, that is,

$$R_{(g,\rho,1)}(u_j) \leq A_n \frac{|G_j|_g^{1-\frac{2}{n}}}{\nu(M)}(k+1) + B_n \frac{|G_j|_{g_0}^{\frac{2}{n}} |G_j|_g^{1-\frac{2}{n}}}{\nu(M)}(k+1).$$

Now, observe that since $\sum_{j\leq 3(k+1)} |G_j|_{g_0} \leq |M|_{g_0}$ and $\sum_{j\leq 3(k+1)} |G_j|_g \leq |M|_g$, there exist at least $k+1$ sets among $G_1, \ldots, G_{3(k+1)}$ satisfying both $|G_j|_{g_0} \leq \frac{|M|_{g_0}}{k+1}$ and $|G_j|_g \leq \frac{|M|_g}{k+1}$. This leads to a subspace of $k+1$ disjointly supported functions u_j whose Rayleigh quotients are such that

$$\begin{aligned} R_{(g,\rho,1)}(u_j) &\leq A_n \frac{|G_j|_g^{1-\frac{2}{n}}}{\nu(M)}(k+1) + B_n \frac{|G_j|_{g_0}^{\frac{2}{n}} |G_j|_g^{1-\frac{2}{n}}}{\nu(M)}(k+1) \\ &\leq A_n \frac{{|M|_g}^{1-\frac{2}{n}}}{\nu(M)}(k+1)^{\frac{2}{n}} + B_n \frac{|M|_{g_0}^{\frac{2}{n}}}{\nu(M)} |M|_g^{1-\frac{2}{n}} \end{aligned}$$

with $\nu(M) = \int_M \rho v_g = |M|_g$. The desired inequality then immediately follows thanks to (1). □

Proof of Theorem 4.2 First, observe that it suffices to prove the theorem when $\mathrm{ric}_0 = 1$ (i.e., $\mathrm{Ric}_{\tilde{g}} \geq -(n-1)\tilde{g}$). Indeed, the Riemannian metric $\tilde{g}_0 = \mathrm{ric}_0 \tilde{g}$ satisfies $\mathrm{Ric}_{\tilde{g}_0} \geq -(n-1)\tilde{g}_0$ and $|M|_{g_0} = (\mathrm{ric}_0)^{n/2}|M|_g$, with $g_0 = \tilde{g}_0|M$. Hence, the inequality

$$\mu_k^{g_0}(1,\sigma) \leq A_n \left(\frac{k}{|M|_{g_0}}\right)^{\frac{2}{n}} + B_n$$

implies

$$\mu_k^g(1,\sigma) = \mathrm{ric}_0 \mu_k^{g_0}(1,\sigma) \leq \mathrm{ric}_0 \left(A_n \left(\frac{k}{|M|_{g_0}}\right)^{\frac{2}{n}} + B_n\right) = A_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}} + B_n \mathrm{ric}_0.$$

Therefore, assume that $\mathrm{ric}_0 = 1$ and consider the metric measured space (M, d, v_g) where d is the restriction to M of the Riemannian distance of $(\tilde{M}, \tilde{g})$. The proof relies on the method developed by Colbois and Maerten [13] as presented in Lemma 2.1 of [11]. Applying the Bishop–Gromov theorem, we deduce that there exist two constants, C_n and N_n, depending only on n, such that, $\forall x \in M$ and $\forall r \leq 1$,

- $|B(x,r)|_g \leq C_n r^n$
- $B(x, 4r)$ can be covered by N_n balls of radius r

where $B(x,r)$ stands for the ball in M of radius r with respect to the distance d.

Let k_0 be the smallest integer such that $2(k_0+1) > \frac{|M|_g}{4C_nN_n^2}$. For every $k \geq k_0$, we define r_k by

$$r_k^n = \frac{|M|_g}{8C_nN_n^2(k+1)} \leq 1$$

which means that, $\forall x \in M$,

$$|B(x, r_k)|_g \leq C_n r_k^n \leq \frac{|M|_g}{8N_n^2(k+1)}.$$

Thus, we can apply Lemma 2.1 of [11] and deduce the existence of $2(k+1)$ measurable subsets $A_1, \ldots, A_{2(k+1)}$ of M such that, $\forall i \leq 2(k+1)$, $|A_i|_g \geq \frac{|M|_g}{4N_n(k+1)}$ and, for $i \neq j$, $d(A_i, A_j) \geq 3r_k$. To each set A_j, we associate the function f_j supported in $A_j^{r_k} = \{x \in M : d(x, A_j) < r_k\}$ and defined to be equal to 1 inside A_j and proportional to the distance to the outer boundary in $A_j^{r_k} \backslash A_j$. The length of the gradient $|\nabla^g f_j|$ is then equal to $\frac{1}{r_k}$ almost everywhere in $A_j^{r_k} \backslash A_j$ and vanishes elsewhere, so that we get

$$R_{(g,1,\sigma)}(f_j) = \frac{\int_{A_j^{r_k}} |\nabla^g f_j|^2 \sigma v_g}{\int_{A_j^{r_k}} f_j^2 v_g} \leq \frac{\frac{1}{{r_k}^2}\int_{A_j^{r_k}} \sigma v_g}{|A_j|_g} \leq \frac{4N_n}{{r_k}^2} \frac{\int_{A_j^{r_k}} \sigma v_g}{|M|_g}(k+1)$$

which gives, after replacing r_k by its explicit value,

$$R_{(g,1,\sigma)}(f_j) \leq A_n \frac{\int_{A_j^{r_k}} \sigma v_g}{|M|_g^{1+\frac{2}{n}}}(k+1)^{1+\frac{2}{n}}.$$

for some constant A_n. Now, since $\sum_{j \leq 2(k+1)} \int_{A_j^{r_k}} \sigma v_g \leq \int_M \sigma v_g$, there exist at least $k+1$ sets among the A_j's such that $\int_{A_j^{r_k}} \sigma v_g \leq \frac{\int_M \sigma v_g}{k+1}$. This leads to a subspace of $k+1$ disjointly supported functions f_j whose Rayleigh quotients are such that

$$R_{(g,1,\sigma)}(f_j) \leq A_n \frac{\int_M \sigma v_g}{|M|_g^{1+\frac{2}{n}}}(k+1)^{\frac{2}{n}}.$$

Consequently, we have thanks to (1), for all $k \geq k_0$,

$$\mu_k^g(1,\sigma) \leq A_n \frac{\int_M \sigma v_g}{|M|_g^{1+\frac{2}{n}}}(k+1)^{\frac{2}{n}} = A_n \left(\frac{k+1}{|M|_g}\right)^{\frac{2}{n}}$$

since we have assumed that $\int_M \sigma v_g = |M|_g$. On the other hand, for every $k \leq k_0$, one obviously has (since $k_0 + 1 \leq \frac{|M|_g}{4C_nN_n^2}$)

$$\mu_k^g(1,\sigma) \leq \mu_{k_0}^g(1,\sigma) \leq A_n \left(\frac{k_0+1}{|M|_g}\right)^{\frac{2}{n}} \leq A_n \left(\frac{1}{4C_nN_n^2}\right)^{\frac{2}{n}}.$$

Denoting by B_n the latter constant we obtain, for every $k \geq 0$,

$$\mu_k^g(1,\sigma) \leq A_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}} + B_n.$$

□

5 Extremal eigenvalues

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$, possibly with boundary. In [9], we introduced the following conformally invariant quantities that we named "conformal eigenvalues": For every $k \in \mathbb{N}$, $\lambda_k^c(M, [g])$ is defined as the supremum of $\lambda_k(M, g')$ when g' runs over all metrics of unit volume which are conformal to g (or, equivalently, $\lambda_k^c(M, [g]) = \sup \lambda_k(M, g')|M|_{g'}^{\frac{2}{n}}$ when g' runs over all metrics conformal to g). Thus, we can write

$$\lambda_k^c(M, [g]) = \sup_{\int_M \rho\, v_g = 1} \lambda_k(M, \rho^{\frac{2}{n}} g) = \sup_{\int_M \rho\, v_g = 1} \mu_k^g(\rho, \rho^{\frac{n-2}{n}}).$$

We investigated in [9] some of the properties of the conformal eigenvalues such as the existence of a universal lower bound, and proved that

$$\lambda_k^c(M, [g]) \geq \lambda_k^c(\mathbb{S}^n, [g_s]) \geq n\alpha_n^{\frac{2}{n}} k^{\frac{2}{n}}, \tag{36}$$

where $\alpha_n = (n+1)\omega_{n+1}$ is the volume of the standard sphere. Moreover, we proved that the gap between two consecutive conformal eigenvalues satisfies the following estimate:

$$\lambda_{k+1}^c(M, [g])^{\frac{n}{2}} - \lambda_k^c(M, [g])^{\frac{n}{2}} \geq n^{\frac{n}{2}} \alpha_n. \tag{37}$$

Actually, these properties were established in the context of closed manifolds. However, they remain valid in the context of bounded domains, under Neumann boundary conditions, without the need to change anything to the proofs. In this regard, we can point out the following curious phenomenon that all bounded Euclidean domains have the same conformal spectrum.

Proposition 5.1 *For every bounded domain $\Omega \subset \mathbb{R}^n$ with C^1-boundary, one has*

$$\lambda_k^c(\Omega, [g_E]) = \lambda_k^c(B^n, [g_E]),$$

where g_E is the Euclidean metric.

For $k = 1$, we have $\lambda_1^c(\Omega, [g_E]) = n\alpha_n^{\frac{2}{n}}$ (see Corollary 6.1).

Proof Let us first observe that if Ω is a proper subset of Ω', then $\lambda_k^c(\Omega, [g_E]) \leq \lambda_k^c(\Omega', [g_E])$. Indeed, given a metric $g = f g_E$ conformal to g_E on Ω, we extend it to a metric g' on Ω', conformal to g_E. For every $\varepsilon > 0$, we multiply g' by the function f_ε which is equal to 1 on Ω and equal to ε on $\Omega' \backslash \Omega$ and apply Theorem 2.1. In dimension $n \geq 3$, this theorem tells us that $\lambda_k(\Omega', f_\varepsilon g')$ converges to $\lambda_k(\Omega, g)$. Since the volume of $(\Omega', f_\varepsilon g')$ converges to the volume of (Ω, g), we deduce that $\lambda_k(\Omega, g)|\Omega|_g^{2/n} \leq \lambda_k^c(\Omega', [g_E])$. In dimension 2, we obtain that $\lambda_k(\Omega', f_\varepsilon g')$ converges to the kth eigenvalue of the quadratic form $\int_\Omega |\nabla u|^2 v_g + \int_{\Omega' \backslash \Omega} |\nabla H(u)|^2 v_g$. This quadratic form is clearly larger than the Dirichlet energy $\int_\Omega |\nabla u|^2 v_g$ on Ω so that its k-th eigenvalue is bounded below by $\lambda_k(\Omega, g)$. Again, this implies that $\lambda_k(\Omega, g) \leq \lambda_k^c(\Omega', [g_E])$.

Now, since Ω is open and bounded, there exist two positive radii r_1 and r_2 so that

$$B^n(r_1) \subset \Omega \subset B^n(r_2),$$

where $B^n(r_1)$ and $B^n(r_2)$ are two concentric Euclidean balls. Using the observation above we get

$$\lambda_k^c(B^n(r_1), [g_E]) \leq \lambda_k^c(\Omega, [g_E]) \leq \lambda_k^c(B^n(r_2), [g_E]).$$

Since the balls $B^n(r_1)$ and $B^n(r_2)$ are homothetic to the unit ball B^n, one necessarily has $\lambda_k^c(B^n(r_1), [g_E]) = \lambda_k^c(B^n(r_2), [g_E]) = \lambda_k^c(B^n, [g_E])$ which enables us to conclude. □

As a consequence of the upper bounds given in the previous section, it is natural to introduce the following extremal eigenvalues:

$$\mu_k^*(M, g) = \sup_{\fint_M \rho\, v_g = 1} \mu_k^g(\rho, 1) = \sup_\rho \mu_k^g(\rho, 1) \fint_M \rho\, v_g$$

$$\mu_k^{**}(M, g) = \sup_{\fint_M \sigma v_g = 1} \mu_k^g(1, \sigma) = \sup_\sigma \frac{\mu_k^g(1, \sigma)}{\fint_M \sigma v_g}.$$

A natural question is whether properties such as (36) and (37) may occur for $\mu_k^*(M, g)$ and $\mu_k^{**}(M, g)$. Observe that these quantities are not invariant under metric scaling since

$$\mu_k^*(M, r^2 g) = r^{-2}\mu_k^*(M, g) \quad and \quad \mu_k^{**}(M, r^2 g) = r^{-2}\mu_k^{**}(M, g).$$

Hence, we will assume that the volume of the manifold is fixed.

In the particular case of manifolds (M, g) of dimension 2, one has for every ρ, $\mu_k^g(\rho, 1) = \lambda_k(M, \rho g)$. Thus,

$$\mu_k^*(M, g) = \frac{\lambda_k^c(M, g)}{|M|_g} \tag{38}$$

and we deduce from (36) and (37) that any 2-dimensional Riemannian manifold (M, g) satisfies

$$\mu_k^*(M, g) \geq \frac{8\pi k}{|M|_g}$$

and

$$\mu_{k+1}^*(M, g) - \mu_k^*(M, g) \geq \frac{8\pi}{|M|_g}.$$

The following theorem shows that the 2-dimensional case is in fact exceptional. Indeed, it turns out that any compact manifold of dimension $n \geq 3$ can be deformed in such a way that $\mu_k^*(M, g)$ becomes as small as desired.

Theorem 5.1 *Let M be a compact manifold of dimension $n \geq 3$. There exists on M a one-parameter family of metrics g_ε, $\varepsilon > 0$, of volume 1 such that*

$$\mu_k^*(M, g_\varepsilon) \leq Ck\,\varepsilon^{\frac{n-2}{n}},$$

where C is a constant which does not depend on ε or k.

Similarly, we have the following result for the supremum with respect to σ.

Theorem 5.2 *Let M be a compact manifold of dimension $n \geq 2$. There exists on M a one-parameter family of metrics g_ε, $\varepsilon > 0$, of volume 1 such that*

$$\mu_k^{**}(M, g_\varepsilon) \leq Ck^2 \varepsilon^{2\frac{n-1}{n}},$$

where C is a constant which depends only on n.

The proofs of these theorems rely on the construction below. It is worth noticing that the one-parameter family of metrics g_ε we will exhibit can be chosen within a fixed conformal class. Actually, we start with a Riemannian metric g_0 on M that we conformally deform in the neighborhood of a point.

The construction. We start with a metric g_0 on M and choose a sufficiently small open set $V \subset M$ so that g_0 is 2-quasi-isometric to a flat metric in V. Since the eigenvalues corresponding to two quasi-isometric metrics are "comparable," we can assume w.l.o.g. that the metric g_0 is flat inside V. Therefore, there exists a positive δ so that V contains a flat (Euclidean) ball of radius δ. After a possible dilation, we can assume that $\delta = 1$. We deform this unit Euclidean ball into a long capped cylinder (i.e., an Euclidean cylinder of radius δ closed by a spherical cap). This construction is standard and is explained, for example, in [20, pp. 3856–3857]. We can even do it through a conformal deformation of g_0, as explained in [10, pp. 718–719]. Therefore, we obtain a family of Riemannian manifolds (M, g_ε) so that M is the union of three parts

$$M = M_0 \cup C \cup S_0^n$$

with

- M_0 is an open subset of M and g_ε does not vary with ε on M_0,
- (C, g_ε) is isometric to the cylinder $[0, \frac{1}{\varepsilon}] \times \mathbb{S}^{n-1}$ of length $\frac{1}{\varepsilon}$ (with $0 < \varepsilon \leq 1$),
- S_0^n is a round hemisphere of radius 1 which closes the end of the cylinder C and $g_\varepsilon|_{S_0^n}$ is the round metric (and is independent of ε).

The only varying parameter in this construction is the length $\frac{1}{\varepsilon}$ of the cylinder (C, g_ε). Notice that the volume of (M, g_ε) is not equal to 1, but we will make a suitable scaling at the end of the proof.

In order to bound the eigenvalues $\mu_k^{g_\varepsilon}(\rho, 1)$ from above, we will use the GNY method [22]. To this end, we need a uniform control (w.r.t. ε) of the *packing constant* (see [22, Definition 3.3 and Theorem 3.5]) and of the volume growth of balls in (M, g_ε). This will be done in the following lemmas. For this purpose, we introduce the connected open subset $\tilde{M}_0 \subset M$ obtained as the union of M_0 and the part of the cylinder which corresponds to $(0, 3d_0) \times \mathbb{S}^{n-1} \subset [0, \frac{1}{\varepsilon}] \times \mathbb{S}^{n-1}$, where d_0 is the diameter of M_0.

Lemma 5.1 (volume growth of balls) *There exist two positive constants C_1 and C_2, independent of ε, such that, for every ball $B_\varepsilon(x, r)$ in (M, g_ε) we have*

$$|B_\varepsilon(x, r)|_{g_\varepsilon} \leq \begin{cases} C_1 r^n & \text{if } r \leq 2d_0 \\ C_2 r & \text{if } r \geq 2d_0 \end{cases} \tag{39}$$

Proof If $B_\varepsilon(x, r) \cap M_0 = \emptyset$, then $B_\varepsilon(x, r)$ is isometric to a geodesic ball of radius r of the capped cylinder and an obvious calculation shows that (39) holds true with two constants C_1 and C_2 independent of ε (in fact, we can compare the volume of $B_\varepsilon(x, r)$ with the volume of $(-r, r) \times \mathbb{S}^{n-1}$ to get $|B_\varepsilon(x, r)|_{g_\varepsilon} \leq Ar$ for some positive A). If $B_\varepsilon(x, r) \cap M_0 \neq \emptyset$ and $r < 2d_0$, then $B_\varepsilon(x, r)$ is contained in $\tilde{M}_0$. Hence, there exists a constant C, depending only on $\tilde{M}_0$, such that $|B_\varepsilon(x, r)|_{g_\varepsilon} \leq Cr^n$. If $B_\varepsilon(x, r) \cap M_0 \neq \emptyset$ and $r \geq 2d_0$, then $B_\varepsilon(x, r)$ is contained in the union of a ball $B(x_0, 2d_0) \subset \tilde{M}_0$ centered at a point $x_0 \in M_0$ and a ball of radius $r' \leq r$ contained in the cylindrical part. Thus, $|B_\varepsilon(x, r)|_{g_\varepsilon} \leq C2^n d_0^n + Ar \leq C_2 r$ for some positive C_2 which does not depend on ε. □

Lemma 5.2 *There exists a constant N, independent of ε, such that any ball of radius $r > 0$ in (M, g_ε) can be covered by N balls of radius $\frac{r}{2}$.*

Proof Let $B_\varepsilon(x,r)$ be a ball of radius r in (M,g_ε). If $B_\varepsilon(x,r)\cap M_0=\emptyset$, then, since $(M\backslash M_0, g_\varepsilon)$ is isometric to the capped cylinder whose Ricci curvature is everywhere non-negative, $B_\varepsilon(x,r)$ can be covered by N_E balls of radius $\frac{r}{2}$, where N_E is the packing constant of the Euclidean space $\mathbb{R}^n$ (Bishop–Gromov theorem).

Assume that $B_\varepsilon(x,r)\cap M_0\neq\emptyset$. If $r<2d_0$, then $B_\varepsilon(x,r)$ is contained in $\tilde{M}_0$. Thus, $B_\varepsilon(x,r)$ can be covered by $N(\tilde{M}_0)$ balls of radius $\frac{r}{2}$, where $N(\tilde{M}_0)$ is the packing constant of $\tilde{M}_0$. If $r\geq 2d_0$, then $B_\varepsilon(x,r)$ is contained in the union of a ball $B_\varepsilon(x_0,2d_0)\subset\tilde{M}_0$ centered at a point $x_0\in M_0$ and a ball of radius $r'\leq r$ contained in the capped cylinder. Again, $B_\varepsilon(x,r)$ can be covered by $N_E+N(\tilde{M}_0)$ balls of radius $\frac{r}{2}$. □

Proof of Theorem 5.1 Let ρ be a positive density on M with $⨍_M \rho v_{g_\varepsilon}=1$. Applying [22, Theorem 3.5] to the metric measured space $(M,d_\varepsilon,\rho v_{g_\varepsilon})$, where d_ε is the Riemannian distance associated to g_ε, we deduce the existence of $k+1$ annuli $A_1,\dots,A_{k+1}$ such that $\int_{A_j}\rho v_{g_\varepsilon}\geq\frac{|M|_{g_\varepsilon}}{Ck}$ and $2A_1,\dots 2A_{k+1}$ are mutually disjoint. Here, C should depend on the packing constant of (M,g_ε), but since the latter is dominated independently of ε, thanks to Lemma 5.2, we can assume that C is independent of ε.

To each annulus of the form $A=B_\varepsilon(x,R)\backslash B_\varepsilon(x,r)$, we associate a function u_A defined as in (35). We obtain

$$R_{(g_\varepsilon,\rho,1)}(u_A)=\frac{\int_{2A}|\nabla^\varepsilon u_A|^2_{g_\varepsilon}v_{g_\varepsilon}}{\int_{2A}u_A^2 v_{g_\varepsilon}}\leq\frac{\frac{4}{r^2}|B_\varepsilon(x,r)|_{g_\varepsilon}+\frac{1}{R^2}|B_\varepsilon(x,2R)|_{g_\varepsilon}}{\int_A\rho v_{g_\varepsilon}}.$$

Using Lemma 5.1, we get for every $r>0$,

$$\frac{1}{r^2}|B_\varepsilon(x,r)|_{g_\varepsilon}\leq\begin{cases}C_1r^{n-2}\leq C_1d_0^{n-2} & \text{if } r\leq 2d_0\\ \frac{C_2}{r}\leq\frac{C_2}{2d_0} & \text{if } r\geq 2d_0\end{cases}. \tag{40}$$

Therefore, there exists a constant C' which depends on C_1, C_2 and d_0 (but independent of ε), such that

$$R_{(g_\varepsilon,\rho,1)}(u_A)\leq\frac{C'}{\int_A\rho v_{g_\varepsilon}}.$$

Consequently, the $k+1$ annuli $A_1,\dots,A_{k+1}$ provide $k+1$ disjointly supported functions satisfying $R_{(g_\varepsilon,\rho,1)}(u_{A_j})\leq\frac{C'}{\int_{A_j}\rho v_{g_\varepsilon}}\leq\frac{CC'k}{|M|_{g_\varepsilon}}$. Thus,

$$\mu_k^{g_\varepsilon}(\rho,1)\leq C''\frac{k}{|M|_{g_\varepsilon}}.$$

In order to obtain a family of metrics of volume 1, we set $g'_\varepsilon=\frac{1}{|M|_{g_\varepsilon}^{2/n}}g_\varepsilon$. Hence, for any ρ such that $⨍_M\rho\, v_{g'_\varepsilon}=⨍_M\rho\, v_{g_\varepsilon}=1$, we have

$$\mu_k^{g'_\varepsilon}(\rho,1)=|M|_{g_\varepsilon}^{2/n}\mu_k^{g_\varepsilon}(\rho,1)\leq C''\frac{k}{|M|_{g_\varepsilon}^{1-\frac{2}{n}}}.$$

But $|M|_{g_\varepsilon}\geq|C|_{g_\varepsilon}\geq\frac{n\omega_n}{\varepsilon}$. Thus,

$$\mu_k^*(M,g'_\varepsilon)\leq Ck\varepsilon^{1-\frac{2}{n}}.$$

□

Proof of Theorem 5.2 Let (M, g_ε) be as in the construction above and let σ be such that $\int_M \sigma v_{g_\varepsilon} = |M|_{g_\varepsilon}$. The cylindrical part (C, g_ε) of (M, g_ε) can be decomposed into $2(k+1)$ small cylinders $C_j \approx [\frac{j}{2(k+1)\varepsilon}, \frac{j+1}{2(k+1)\varepsilon}] \times \mathbb{S}^{n-1}$, $j = 0, ..., 2k+1$, of length $\frac{1}{2(k+1)\varepsilon}$. At least $(k+1)$ cylinders among $C_0, \dots, C_{2k+1}$ have a measure with respect to σ which is less or equal to $\frac{|M|_{g_\varepsilon}}{k+1}$. To each such C_j, we associate a function f with support in C_j and which is defined in C_j, through the obvious identification between C_j and $[0, \frac{1}{2(k+1)\varepsilon}] \times \mathbb{S}^{n-1}$, as follows: $\forall (t, z) \in [0, \frac{1}{2(k+1)\varepsilon}] \times \mathbb{S}^{n-1} \approx C_j$,

$$f(t, z) = \begin{cases} 6(k+1)\varepsilon t & \text{if } 0 \le t \le \frac{1}{6(k+1)\varepsilon} \\ 1 & \text{if } \frac{1}{6(k+1)\varepsilon} \le t \le \frac{2}{6(k+1)\varepsilon} \\ -6(k+1)\varepsilon t + 3 & \text{if } \frac{2}{6(k+1)\varepsilon} \le t \le \frac{3}{6(k+1)\varepsilon}. \end{cases} \tag{41}$$

We have

$$\int_M f^2 v_{g_\varepsilon} \ge \int_{[\frac{1}{6(k+1)\varepsilon}, \frac{2}{6(k+1)\varepsilon}] \times \mathbb{S}^{n-1}} f^2 \, v_E = \frac{n\omega_n}{6(k+1)\varepsilon},$$

where v_E is the standard product measure. On the other hand, the norm of the gradient of f is supported in C_j and is dominated by $6(k+1)\varepsilon$. Thus,

$$\int_M |\nabla^\varepsilon f|^2_{g_\varepsilon} \sigma v_{g_\varepsilon} \le (6(k+1)\varepsilon)^2 \int_{C_j} \sigma v_{g_\varepsilon} \le (6(k+1)\varepsilon)^2 \frac{|M|_{g_\varepsilon}}{k+1} = 36(k+1)\varepsilon^2 |M|_{g_\varepsilon}$$

and the Rayleigh quotient of f satisfies

$$R_{(g_\varepsilon, 1, \sigma)}(f) \le \frac{216(k+1)^2 \varepsilon^3 |M|_{g_\varepsilon}}{n\omega_n}.$$

Consequently, the $k+1$ chosen cylinders provide $k+1$ disjointly supported functions satisfying the last inequality, which yields

$$\mu_k^{g_\varepsilon}(1, \sigma) \le C|M|_{g_\varepsilon}(k+1)^2 \varepsilon^3$$

with $C = \frac{216}{n\omega_n}$. Setting $g'_\varepsilon = \frac{1}{|M|_{g_\varepsilon}^{\frac{2}{n}}} g_\varepsilon$, we get

$$\mu_k^{g'_\varepsilon}(1, \sigma) = |M|_{g_\varepsilon}^{\frac{2}{n}} \mu_k^{g_\varepsilon}(1, \sigma) \le C\varepsilon^3 |M|_{g_\varepsilon}^{1+\frac{2}{n}} (k+1)^2$$

with $|M|_{g_\varepsilon} = |\tilde{M}_0|_g + |C|_{g_\varepsilon} + \frac{1}{2} n\omega_n \le \frac{A}{\varepsilon}$ for some constant A. Thus,

$$\mu_k^{**}(M, g'_\varepsilon) \le C' \varepsilon^{2-\frac{2}{n}} (k+1)^2.$$

□

Remark 5.1 The same type of construction used in the proof of Theorems 5.1 and 5.2 allows us to prove the existence of a family of bounded domains $\Omega_\varepsilon \subset \mathbb{R}^n$ of volume 1 such that $\mu_k^*(\Omega_\varepsilon, g_E)$ (resp. $\mu_k^{**}(\Omega_\varepsilon, g_E)$) goes to zero with ε. This is to be compared with the result of Proposition 5.1.

We end this section with the following proposition in which we show how to produce examples of manifolds (M, g_ε) of fixed volume for which the ratio $\frac{\mu_1^*(M, g_\varepsilon)}{\lambda_1(M, g_\varepsilon)}$ (resp. $\frac{\mu_1^{**}(M, g_\varepsilon)}{\lambda_1(M, g_\varepsilon)}$) tends to infinity as $\varepsilon \to 0$.

Proposition 5.2 *Let M be a compact manifold and let A be a positive constant.*

i. *There exists a family of metrics* g_ε *of volume 1 on* M *and a constant* $A > 0$ *such that* $\forall \varepsilon \in (0,1)$, $\lambda_1(M, g_\varepsilon) \le \varepsilon$ *while* $\mu_1^*(M, g_\varepsilon) \ge A$.
ii. *There exists a family of metrics* g_ε *of volume 1 on* M *and a constant* $A > 0$ *such that,* $\forall \varepsilon \in (0,1)$, $\lambda_1(M, g_\varepsilon) \to 0$ *while* $\mu_1^{**}(M, g_\varepsilon) \ge A$.

Proof (i) Let us start with a Riemannian metric g of volume one on M such that an open set V of M is isometric to the Euclidean ball of volume $\frac{1}{2}$. By a standard argument (Cheeger Dumbbell construction), one can deform the metric g outside V into a metric g_ε of volume 1 such that $\lambda_1(M, g_\varepsilon) \le \varepsilon$. Applying Corollary 2.2 with $M_0 = V$, we get $\mu_1^*(M, g_\varepsilon) \ge |V|_{g_\varepsilon}\lambda_1(V, g_\varepsilon) = \frac{1}{2}\lambda_1(V, g)$. Since $\lambda_1(V, g) = (2\omega_n)^{\frac{2}{n}}\lambda_1(B^n, g_E)$, where B^n is the unit Euclidean ball, we get the desired inequality with $A = \frac{1}{2}(2\omega_n)^{\frac{2}{n}}\lambda_1(B^n, g_E)$.

(ii) Let g be a Riemannian metric on M such that an open subset V of M is isometric to the capped cylinder $C = (-2, 2) \times \mathbb{S}^{n-1}$ closed by a spherical cap. We will deform the metric g inside V so that (M, g_ε) looks like a Cheeger dumbbell (thus $\lambda_1(M, g_\varepsilon) \to 0$ as $\varepsilon \to 0$) and associate with g_ε a family of densities such that $\mu_1^{g_\varepsilon}(1, \sigma_\varepsilon) \ge A > 0$. Indeed, the metric on the cylinder $C = (-2, 2) \times \mathbb{S}^{n-1}$ is given in coordinates $(t, x) \in (-2, 2) \times \mathbb{S}^{n-1}$ by $g_\varepsilon(t, x) = \mathrm{d}t^2 + \gamma_\varepsilon^2(t) g_{\mathbb{S}^{n-1}}$ with $\gamma_\varepsilon(-t) = \gamma_\varepsilon(t)$ and

$$\gamma_\varepsilon(t) = \begin{cases} \varepsilon & \text{if } t \in [0, \frac{1}{2}] \\ \in (\varepsilon, 1) & \text{if } t \in [\frac{1}{2}, 1] \\ 1 & \text{if } t \in [1, 2) \end{cases}. \tag{42}$$

We do not change the metric g outside V. We endow (M, g_ε) with the density σ_ε given by $\sigma_\varepsilon(t, x) = \frac{1}{\gamma_\varepsilon(t)^{n-1}}$ on the cylinder C and extended by 1 outside C.

It is well known that $\lambda_1(M, g_\varepsilon) \to 0$ as $\varepsilon \to 0$. Let us study $\mu_1^{g_\varepsilon}(1, \sigma_\varepsilon)$. One has for every $f \in C^\infty(M)$

$$\begin{aligned}\int_M |\nabla^\varepsilon f|^2_{g_\varepsilon}\sigma_\varepsilon v_{g_\varepsilon} &= \int_{M\setminus C} |\nabla f|^2_g v_g + \int_{-2}^{2}\mathrm{d}t \int_{\mathbb{S}^{n-1}} |\nabla^\varepsilon f|^2_{g_\varepsilon}\sigma_\varepsilon(t)\gamma_\varepsilon(t)^{n-1} v_{\mathbb{S}^{n-1}} \\ &= \int_{M\setminus C} |\nabla f|^2_g v_g + \int_{-2}^{2}\mathrm{d}t \int_{\mathbb{S}^{n-1}} |\nabla^\varepsilon f|^2_{g_\varepsilon} v_{\mathbb{S}^{n-1}},\end{aligned}$$

where $v_{\mathbb{S}^{n-1}}$ denotes the volume form on the sphere $\mathbb{S}^{n-1}$. Now, observe that $|\nabla^\varepsilon f|^2_{g_\varepsilon}$ can be estimated as follows:

$$|\nabla^\varepsilon f|^2_{g_\varepsilon} = \left(\frac{\partial f}{\partial t}\right)^2 + |\nabla_0 f|^2\gamma_\varepsilon(t)^{-2} \ge \left(\frac{\partial f}{\partial t}\right)^2 + |\nabla_0 f|^2 = |\nabla f|^2_g,$$

where $\nabla_0 f$ is the tangential part of the gradient of f w.r.t. $\mathbb{S}^{n-1}$. Therefore,

$$\int_M |\nabla^\varepsilon f|^2_{g_\varepsilon}\sigma_\varepsilon v_{g_\varepsilon} \ge \int_{M\setminus C} |\nabla f|^2_g v_g + \int_{-2}^{2}\mathrm{d}t \int_{\mathbb{S}^{n-1}} |\nabla f|^2_g v_{\mathbb{S}^{n-1}} = \int_M |\nabla f|^2_g v_g.$$

On the other hand (since $\gamma_\varepsilon(t)^2 \le 1$)

$$\int_M f^2 v_{g_\varepsilon} \le \int_M f^2 v_g.$$

In conclusion, for every $f \in C^\infty(M)$, one has

$$R_{(g_\varepsilon,1,\sigma_\varepsilon)}(f) \ge R_{(g,1,1)}(f).$$

It follows, thanks to the min-max principle, that

$$\mu_1^{g_\varepsilon}(1, \sigma_\varepsilon) \geq \lambda_1(M, g).$$

The last point is to suitably rescale g_ε and σ_ε. For this purpose, just observe that $\int_M \sigma_\varepsilon v_{g_\varepsilon} = |M|_g$ and $\frac{1}{2}|M|_g \leq |M|_{g_\varepsilon} \leq |M|_g$. □

6 Examples

In this section, we describe situations in which we can compute or give explicit estimates for the first extremal eigenvalues. Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$, possibly with a nonempty boundary.

Proposition 6.1 *Assume that there exists a conformal map ϕ from (M, g) to the standard n-dimensional sphere $\mathbb{S}^n$. Then,*

$$\lambda_1^c(M, g) = n\alpha_n^{\frac{2}{n}} \tag{43}$$

and

$$\mu_1^*(M, g) \leq n\left(\frac{\alpha_n}{|M|_g}\right)^{\frac{2}{n}}, \tag{44}$$

where α_n is the volume of the unit Euclidean n-sphere. Moreover, if $n = 2$, then the equality holds in (44).

Notice that when (M, g) is the standard sphere $\mathbb{S}^n$, then equality holds in (44) (see Corollary 6.3).

Proof of Propositon 6.1 Let us first prove (44). Let ρ be a density on M with $\int_M \rho v_g = 1$. Given any nonconstant map $\phi = (\phi_1, \cdots, \phi_{n+1}) : (M, g) \to \mathbb{S}^n$, a standard argument tells us that there exists a conformal diffeomorphism $\gamma \in Conf(\mathbb{S}^n)$ such that $\psi = \gamma \circ \phi$ satisfies $\int_M \psi_j \rho\, v_g = 0$, $j = 1 \ldots, n+1$ (see for instance [21, Proposition 4.1.5]). Thus, $\forall j \leq n+1$,

$$\mu_1(\rho, 1)\int_M \psi_j^2 \rho\, v_g \leq \int_M |\nabla\psi_j|^2 v_g$$

(see (3)) and, summing up w.r.t. j,

$$\mu_1(\rho, 1)\int_M \rho\, v_g \leq \int_M |d\psi|^2 v_g \leq \left(\int_M |d\psi|^n v_g\right)^{\frac{2}{n}} |M|_g^{1-\frac{2}{n}}.$$

Since $\psi = \gamma \circ \phi$ is a conformal map, $\int_M |d\psi|^n v_g$ is nothing but $n^{\frac{n}{2}}$ times the volume of $\psi(M) \subset \mathbb{S}^n$ with respect to the standard metric g_s of $\mathbb{S}^n$ (indeed, $\psi^* g_s = \frac{1}{n}|d\psi|^2 g$). Therefore,

$$\mu_1(\rho, 1)\fint_M \rho v_g \leq n|\psi(M)|_{g_s}^{\frac{2}{n}} |M|_g^{-\frac{2}{n}} \leq n\left(\frac{\alpha_n}{|M|_g}\right)^{\frac{2}{n}}$$

which proves (44).

Using the same arguments, we can prove the inequality $\lambda_1^c(M, g) \leq n\alpha_n^{\frac{2}{n}}$. The reverse inequality follows from [9, Theorem A]. □

It is well known that the Euclidean space $\mathbb{R}^n$ and the hyperbolic space $\mathbb{H}^n$ are conformally equivalent to open parts of the sphere $\mathbb{S}^n$. This leads to the following corollary.

Corollary 6.1 *Let Ω be a bounded domain of the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ or the sphere $\mathbb{S}^n$, endowed with the induced metric g_s. One has*

$$\lambda_1^c(\Omega, g_s) = n\alpha_n^{\frac{2}{n}}$$

and

$$\mu_1^*(\Omega, g_s) \leq n\left(\frac{\alpha_n}{|\Omega|}\right)^{\frac{2}{n}}.$$

Moreover, the following equality holds in dimension 2: $\mu_1^*(\Omega, g_s) = \lambda_1^c(\Omega, g_s)|\Omega|^{-1} = \frac{8\pi}{|\Omega|}$.

Remark 6.1 Let D be the unit disk in $\mathbb{R}^2$ and let $\rho_t = \frac{4t}{(t^2|z|^2+1)^2}$. Then,

$$\mu_1^*(D, g_E) = \lim_{t\to\infty} \mu_1^{g_E}\Big(\frac{\rho_t}{\fint_D \rho_t \mathrm{d}x}, 1\Big) = 8.$$

Indeed, the map $\phi_t(z) = \frac{1}{t^2|z|^2+1}(2tz, t^2|z|^2-1)$ identifies $(D, \frac{4t}{(t^2|z|^2+1)^2} g_E)$ with a spherical cap C_t in $\mathbb{S}^2$ whose radius goes to π as $t \to \infty$. Hence, $\mu_1^{g_E}(\rho_t, 1)\int_D \rho_t \mathrm{d}x = \mu_1(C_t)|C_t|$ which converges to 8π as $t \to \infty$.

Proposition 6.2 *Assume that there exists a map $\phi : (M, g) \to \mathbb{S}^p$ from (M, g) to the standard p-dimensional sphere $\mathbb{S}^p$ satisfying both $\int_M \phi v_g = 0$ and $|d\phi|^2 \leq \Lambda$ for some positive constant Λ. Then,*

$$\mu_1^{**}(M, g) \leq \Lambda. \tag{45}$$

Proof One has, for every $j \leq p+1$,

$$\mu_1(1, \sigma)\int_M \phi_j^2 v_g \leq \int_M |\nabla\phi_j|^2 \sigma v_g$$

and, summing up w.r.t. j,

$$\mu_1(1, \sigma)|M|_g \leq \int_M |d\phi|^2 \sigma v_g \leq \Lambda \int_M \sigma v_g$$

which implies (45). □

If (M, g) is a compact homogeneous Riemannian manifold, and if $\phi_1, \ldots, \phi_p$ is an L^2-orthonormal basis of the first eigenspace of the Laplacian, then both $\sum_{i\leq p} \phi_i^2$ and $|d\phi|^2 = \sum_{i\leq p} |d\phi_i|^2$ are constant on M. This enables us to apply Proposition 6.2 and get the following

Corollary 6.2 *Let (M, g) be a compact homogeneous Riemannian manifold. Then,*

$$\mu_1^{**}(M, g) = \mu_1(M, g).$$

In other words, on a compact homogeneous Riemannian manifold, $\mu_1(1, \sigma)$ is maximized when σ is constant.

Example 6.1 In [19], it is proved that if $\Gamma = \mathbb{Z}e_1 + \mathbb{Z}e_2 \subset \mathbb{R}^2$ is a lattice such that $|e_1| = |e_2|$, then the corresponding flat metric g_Γ on the torus $\mathbb{T}^2$ satisfies $\mu_1^c(\mathbb{T}^2, g_\Gamma) = \lambda_1(\mathbb{T}^2, g_\Gamma)|\mathbb{T}^2|_{g_\Gamma}$. A higher-dimensional version of this result was also established in [18]. Since a flat torus is a 2-dimensional homogeneous Riemannian manifold, we have the following equalities

$$\lambda_1^c(\mathbb{T}^2, g_\Gamma)|\mathbb{T}^2|_{g_\Gamma}^{-1} = \mu_1^*(\mathbb{T}^2, g_\Gamma) = \mu_1^{**}(\mathbb{T}^2, g_\Gamma) = \lambda_1(\mathbb{T}^2, g_\Gamma).$$

Nevertheless, whereas we always have $\mu_1^{**}(\mathbb{T}^2, g_\Gamma) = \mu_1(\mathbb{T}^2, g_\Gamma)$, it follows from [9, Theorem A] that when the length ratio $|e_2|/|e_1|$ of the vectors e_1 and e_2 is sufficiently far from 1, then $\mu_1^*(\mathbb{T}^2, g_\Gamma) = \lambda_1^c(\mathbb{T}^2, g_\Gamma)|\mathbb{T}^2|_{g_\Gamma}^{-1} > \lambda_1(\mathbb{T}^2, g_\Gamma)$.

Recall that a map $\phi = (\phi_1, \cdots, \phi_{p+1}) : (M, g) \to \mathbb{S}^p$ is harmonic if and only if its components $\phi_1, \cdots, \phi_{p+1}$ satisfy

$$\Delta_g \phi_j = -|d\phi|^2 \phi_j, \quad j = 1 \cdots, p+1.$$

The stress-energy tensor of a map ϕ is a symmetric covariant 2-tensor defined for every tangent vector field X on M by: $S_\phi(X, X) = \frac{1}{2}|d\phi|^2|X|_g^2 - |d\phi(X)|^2$. In [15, Theorem 3.1], it is proved that if the stress-energy tensor of a harmonic map ϕ is nonnegative, then for every conformal diffeomorphism γ of the sphere $\mathbb{S}^p$ one has

$$\int_M |d(\gamma \circ \phi)|^2 v_g \le \int_M |d\phi|^2 v_g.$$

Moreover, the strict inequality holds if γ is not an isometry and if S_ϕ is positive definite at some point. Observe that if $\phi : (M, g) \to \mathbb{S}^p$ is a conformal map or a horizontally conformal map, then S_ϕ is nonnegative (see [15]).

Proposition 6.3 *Assume that there exists a harmonic map $\phi : (M, g) \to \mathbb{S}^p$ with nonnegative stress-energy tensor. Then,*

$$\mu_1^*(M, g) \le \fint_M |d\phi|^2 v_g. \tag{46}$$

Proof Let ρ be a positive density on M. As before, we know that there exists $\gamma \in Conf(\mathbb{S}^n)$ such that $\psi = \gamma \circ \phi$ satisfies $\int_M \psi_j \rho \, v_g = 0$, $j = 1 \ldots, n+1$. Thus,

$$\mu_1(\rho, 1) \int_M \psi_j^2 \rho \, v_g \le \int_M |\nabla \psi_j|^2 v_g$$

and, summing up w.r.t. j,

$$\mu_1(\rho, 1) \int_M \rho \, v_g \le \int_M |d(\gamma \circ \phi)|^2 v_g \le \int_M |d\phi|^2 v_g$$

which implies (46). □

A particular case of Proposition 6.3 is when there exists a harmonic map $\phi : (M, g) \to \mathbb{S}^p$ which is homothetic. In this case, $S_\phi = \frac{n-2}{n}|d\phi|^2 g$ and $|d\phi|^2$ is constant and coincides with an eigenvalue $\lambda_k(M, g)$ for some $k \ge 1$. For example, if (M, g) is a compact isotropy irreducible homogeneous space (e.g., a compact rank-one symmetric space) and if $\phi_1, \ldots, \phi_p$ is an L^2-orthonormal basis of the first eigenspace of the Laplacian, then $\phi = \left(\frac{|M|_g}{p}\right)^{\frac{1}{2}} (\phi_1, \ldots, \phi_p)$ is a harmonic map from (M, g) to $\mathbb{S}^p$ which is homothetic and satisfies $|d\phi|^2 = \lambda_1(M, g)$. Proposition 6.3 then implies that $\mu_1^*(M, g) = \lambda_1(M, g)$. On the other hand, the second author and Ilias [17] proved that in this situation we also have $\lambda_1^c(M, g) = \lambda_1(M, g)|M|_g^{\frac{2}{n}}$. Consequently, we have the following

Corollary 6.3 *Let (M, g) be a compact isotropy irreducible homogeneous space. Then,*

$$\lambda_1^c(M, g)|M|_g^{-\frac{2}{n}} = \mu_1^*(M, g) = \mu_1^{**}(M, g) = \lambda_1(M, g).$$

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