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The existence of J-holomorphic curves in almost Hermitian manifolds

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Abstract In this paper, we investigate the existence of J-holomorphic curves on almost Hermitian manifolds. Let (M, g, J, F) be an almost Hermitian manifold and $f : \Sigma \to M$ be an injective immersion. We prove that if the L_p functional has a critical point or a stable point in the same almost Hermitian class, then the immersion is J-holomorphic.

Keywords Almost Hermitian manifold · J-holomorphic curve · J-compatible 2-form · L_p-functional

Mathematics Subject Classification 53B35 · 53D15

1 Introduction

Let (M, J) be a closed almost complex $2n$-manifold and Σ be a closed real surface. We call a smooth immersion $f : \Sigma \to M$ J-holomorphic if $J_{f(p)}$ maps $f_{*p}(T_p\Sigma)$ onto itself for any point $p \in \Sigma$. Under what condition an immersion is J-holomorphic is an interesting question in differential geometry. Recently, Arezzo and Sun [1] gave a variational characterization of J-holomorphic curves in almost Kähler manifold (M, g, J, ω). More precisely, they consider the change of the area functional according to the change of the symplectic form on M in the fixed cohomology class (with fixed immersion f and fixed almost complex structure J on M). Let

$$\tilde{\mathcal{H}} \triangleq \left\{\rho \in C^\infty(M, \mathbb{R}) : \omega_\rho \triangleq \omega + dd_J^c\rho \ \text{ tames } \ J\right\},$$

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which is clearly a nonempty open subset of $C^\infty(M, \mathbb{R})$. Here, d^c_J is the twisted exterior differential defined by $d^c_J = (-1)^p JdJ$ acting on p-forms, in particular $d^c_J\rho(X) = -d\rho(JX)$. To each $\rho \in \tilde{\mathcal{H}}$ we can associate a Riemannian metric g_ρ on M defined by $g_\rho(X, Y) = \frac{1}{2}(\omega_\rho(X, JY) + \omega_\rho(Y, JX))$. Let Σ be a closed real surface and $f : \Sigma \to M$ be a smooth immersion. Define

$$\mathcal{A}(\rho) = \text{Area}(f(\Sigma), f^*(g_\rho)) = \int_\Sigma \mathrm{d}\mu_\rho,$$

where $\mathrm{d}\mu_\rho$ is the volume form of the induced metric f^*g_ρ. We say that the area functional $\mathcal{A}$ has a critical point $\rho \in \tilde{\mathcal{H}}$ if for any $\phi(t) \in \tilde{\mathcal{H}}$ with $\phi(0) = \rho$, we have $\mathcal{A}'(0) = 0$. Their first result, Corollary 1.2 of [1], says that if the area functional has a critical point, then the injective immersion is J-holomorphic. We say that $\rho \in \tilde{\mathcal{H}}$ is a stable point for the area functional $\mathcal{A}$ if $\mathcal{A}''(0) \geq 0$ for any $\phi(t) \in \tilde{\mathcal{H}}$, $\phi(0) = \rho$. Furthermore, if J is compatible with ω_ρ, then we say that ρ is a compatible stable point. For the stable case, their second theorem (Theorem 3.2 of [1]) says that, if the area functional has a compatible stable point, then the injective immersion is also J-holomorphic. The area functional is a natural candidate to be considered because for a J-holomorphic immersion, the area functional is constant in the same cohomology class (Proposition 2.2 of [1]) so that every point is both a critical point and a stable point for a J-holomorphic curve. Immediately following, Arezzo and Sun [2] generalized the results in [1] to arbitrary dimension and codimension as well as current case.

In [12], J. Sun considered a family of more general functionals defined in terms of the Kähler angle. In order to ensure $\omega_\rho = \omega + dd^c_J\rho$ is a (1, 1)-form, Sun considered the general functionals on a compact Kähler manifold. Let (M, g, J, ω) be a compact Kähler manifold. Recall that the Kähler angle α of a surface Σ in M is defined by [3]

$$\omega|_\Sigma = \cos\alpha \mathrm{d}\mu_\Sigma, \tag{1.1}$$

where $\mathrm{d}\mu_\Sigma$ is the induced volume form on Σ. We call an immersion $f : \Sigma \to M$ **symplectic** if $\cos\alpha > 0$ and **Lagrangian** if $\cos\alpha \equiv 0$. As in [1], J. Sun fixed the immersion f and the complex structure J, and let the Kähler form vary in the fixed Kähler class. Then he defined a functional on $\tilde{\mathcal{H}}$ by

$$L_p(\rho) = \int_\Sigma \cos^p \alpha_\rho \mathrm{d}\mu_\rho, \tag{1.2}$$

where $\mathrm{d}\mu_\rho$ is the area form of the induced metric f^*g_ρ on Σ, α_ρ is the Kähler angle of the immersion f with respect to the Kähler form ω_ρ and associated Riemannian metric g_ρ. When $p < 0$ or p is not an integer, we assume the immersion to be symplectic in order to guarantee that the integral makes sense. When $p = 0$, L_0 is just the area functional. J. Sun proved in [12] that if the functional L_p has a critical point or a stable point in the fixed Kähler class, then the injective symplectic immersion is J-holomorphic.

In this paper, by considering the critical points and stable points of functional L_p, we investigate the existence of J-holomorphic curves in almost Hermitian manifolds. We prove that if the L_p has a critical point or a stable point, then the immersion is J-holomorphic.

2 Definitions and preliminaries

Let M be a closed oriented smooth $2n$-manifold. An almost complex structure on M is a differentiable endomorphism on the tangent bundle

$$J : TM \to TM \ \text{ with } \ J^2 = -id.$$

A manifold M with a fixed almost complex structure J is called an almost complex manifold denoted by (M, J). Suppose that M is an almost complex manifold with almost complex structure J, then for any $x \in M$, $T_x(M) \otimes_{\mathbb{R}} \mathbb{C}$ which is the complexification of $T_x(M)$, has the following decomposition (cf. [5]):

$$T_x(M) \otimes_{\mathbb{R}} \mathbb{C} = T_x^{1,0} + T_x^{0,1}, \tag{2.3}$$

where $T_x^{1,0}$ and $T_x^{0,1}$ are the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. A complex tangent vector is of type $(1, 0)$ (resp. $(0, 1)$) if it belongs to $T_x^{1,0}$ (resp. $T_x^{0,1}$). Let $T(M) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the tangent bundle. Similarly, let $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of the cotangent bundle T^*M. J can act on $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ as follows:

$$\forall \alpha \in T^*M \otimes_{\mathbb{R}} \mathbb{C}, \quad J\alpha(\cdot) = -\alpha(J\cdot).$$

Hence $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ has the following decomposition according to the eigenvalues $\pm\sqrt{-1}$:

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda_J^{1,0} \oplus \Lambda_J^{0,1}. \tag{2.4}$$

We can form exterior bundle $\Lambda_J^{p,q} = \Lambda^p \Lambda_J^{1,0} \otimes \Lambda^q \Lambda_J^{0,1}$. Let $\Omega_J^{p,q}(M)$ denote the space of C^∞ sections of the bundle $\Lambda_J^{p,q}$. Then we have a direct sum decomposition $\Omega^k(M) = \bigoplus_{p+q=k} \Omega_J^{p,q}(M)$. We denote the projections $\Omega^k(M) \to \Omega_J^{p,q}(M)$ by $\Pi^{p,q}$. The exterior differential operator acts on $\Omega_J^{p,q}$ as follows:

$$d\Omega_J^{p,q} \subset \Omega_J^{p-1,q+2} + \Omega_J^{p+1,q} + \Omega_J^{p,q+1} + \Omega_J^{p+2,q-1}. \tag{2.5}$$

Hence, d has the following decomposition:

$$d = A_J \oplus \partial_J \oplus \bar{\partial}_J \oplus \bar{A}_J, \tag{2.6}$$

where $A_J \triangleq \Pi^{p-1,q+2} \circ d$, $\partial_J \triangleq \Pi^{p+1,q} \circ d$, $\bar{\partial}_J \triangleq \Pi^{p,q+1} \circ d$ and $\bar{A}_J \triangleq \Pi^{p+2,q-1} \circ d$. Let α be a (p, q)-form. We have following formulas (cf. [10,11])

Proposition 2.1

$$\begin{aligned}
&\partial_J \alpha(\xi_1, \cdots, \xi_{p+1}, \bar{\eta}_1, \cdots, \bar{\eta}_q) \\
&\quad = \sum_{k=1}^{p+1} (-1)^{k+1} \xi_k \alpha(\xi_1, \cdots, \hat{\xi}_k, \cdots, \bar{\eta}_q) \\
&\qquad + \sum_{1 \le k < l \le p+1} (-1)^{k+1} \alpha([\xi_k, \xi_l], \xi_1, \cdots, \hat{\xi}_k, \cdots, \hat{\xi}_l, \cdots, \bar{\eta}_q) \\
&\qquad + \sum_{1 \le k \le p+1, 1 \le l \le q} (-1)^{k+l+p+1} \alpha([\xi_k, \bar{\eta}_l], \xi_1, \cdots, \hat{\xi}_k, \cdots, \hat{\bar{\eta}}_l, \cdots, \bar{\eta}_q), \\
&\bar{\partial}_J \alpha(\xi_1, \cdots, \xi_p, \bar{\eta}_1, \cdots, \bar{\eta}_{q+1}) \\
&\quad = \sum_{k=1}^{q+1} (-1)^{k+p+1} \bar{\eta}_k \alpha(\xi_1, \cdots, \hat{\bar{\eta}}_k, \cdots, \bar{\eta}_{q+1}) \\
&\qquad + \sum_{1 \le k < l \le q+1} (-1)^{k+1} \alpha([\bar{\eta}_k, \bar{\eta}_l], \xi_1, \cdots, \hat{\bar{\eta}}_k, \cdots, \hat{\bar{\eta}}_l, \cdots, \bar{\eta}_{q+1}) \\
&\qquad + \sum_{1 \le k \le p, 1 \le l \le q+1} (-1)^{k+l+p} \alpha([\xi_k, \bar{\eta}_l], \xi_1, \cdots, \hat{\xi}_k, \cdots, \hat{\bar{\eta}}_l, \cdots, \bar{\eta}_{q+1}),
\end{aligned}$$

$$A_J\alpha(\xi_1,\cdots,\xi_{p-1},\bar\eta_1,\cdots,\bar\eta_{q+2}) = \sum_{1\le k<l\le q+2}(-1)^{k+l}\alpha([\bar\eta_k,\bar\eta_l],\xi_1,\cdots,\hat{\bar\eta}_k,\cdots,\hat{\bar\eta}_l,\cdots,\bar\eta_{q+2})$$

and

$$\bar A_J\alpha(\xi_1,\cdots,\xi_{p+2},\bar\eta_1,\cdots,\bar\eta_{q-1}) = \sum_{1\le k<l\le p+2}(-1)^{k+l}\alpha([\xi_k,\xi_l],\xi_1,\cdots,\hat{\xi}_k,\cdots,\hat{\xi}_l,\cdots,\bar\eta_{q-1}),$$

where $\xi_1,\cdots,\xi_{p+2},\eta_1,\cdots,\eta_{q+2}$ *are vector fields of type* $(1,0)$.

It is easy to see that A_J and $\bar A_J$ are $\mathbb{R}$-linear operators of order 0. Recall that on an almost complex manifold (M,J), there exists Nijenhuis tensor $\mathcal{N}_J$ as follows:

$$4\mathcal{N}_J=[JX,JY]-[X,Y]-J[X,JY]-J[JX,Y], \tag{2.7}$$

where $X,Y\in TM$. By Newlander-Nirenberg Theorem (cf. [8]), $\mathcal{N}_J=0$ if and only if J is integrable, that is, J is a complex structure. Moreover, we have the following equivalent conditions (for details, see [5,9]):

1. J is integrable;
2. $d=\partial_J\oplus\bar\partial_J$;
3. $\bar\partial_J^2=0$, $\partial_J^2=0$;
4. If ξ and η are vector fields of type $(1,0)$, so is $[\xi,\eta]$.

Let (M,J) be an almost complex manifold and (Σ,j) be a Riemann surface. A smooth map $u:(\Sigma,j)\to(M,J)$ is called a J-holomorphic curve if the differential $\mathrm{d}u$ is a complex linear map with respect to j and J:

$$J\circ \mathrm{d}u=\mathrm{d}u\circ j. \tag{2.8}$$

Hence, $\bar\partial_J u(X)=\frac{1}{2}[\mathrm{d}u(X)+J(u)\mathrm{d}u(jX)]=0$ if u is a J-holomorphic curve. By a result of Nijenhuis and Woolf (cf. [9]), the local J-holomorphic curves in an almost complex manifold are always exist.

Theorem 2.2 *(cf.* [9]*) Let* (M,J) *be an almost complex manifold. Then to every point* x *of* M *and every complex tangent vector* $v\in T(M)\otimes_{\mathbb{R}}\mathbb{C}$*, there is a* J*-holomorphic curve passing through* x *with tangent vector* v *at* x.

Suppose that (M,J) is a closed almost complex $2n$-manifold. Let Σ be a closed real surface and $f:\Sigma\to M$ be a smooth immersion. By (2.8), the definition of J-holomorphic curve, we can naturally give the definition of J-holomorphic immersion.

Definition 2.3 Let (M,J) is a closed almost complex $2n$-manifold and Σ be a closed real surface. We call a smooth immersion $f:\Sigma\to M$ J-holomorphic if $J_{f(p)}$ maps $f_{*p}(T_p\Sigma)$ onto itself for any point $p\in\Sigma$.

It is well known that there always exists complex structure j on surface Σ, that is, (Σ,j) is a closed Riemann surface. By the definition of J-holomorphic curve, if immersion $f:(\Sigma,j)\to(M,J)$ is a J-holomorphic curve, then f is a J-holomorphic immersion. Conversely if $f:\Sigma\to M$ is a J-holomorphic immersion, then $f:(\Sigma,f^*J)\to(M,J)$ is a J-holomorphic curve.

A symplectic structure on a differentiable manifold is a nondegenerate closed 2-form $\omega\in\Omega^2$. A differentiable manifold with some fixed symplectic structure is called a symplectic manifold. Suppose (M, ω) is a closed symplectic manifold. An almost complex structure J is said to be tamed by ω when the bilinear form $\omega(\cdot, J\cdot)$ is positive definite. The almost complex structure J is said to be compatible with ω when this same bilinear form is also symmetric, that is, $\omega(\cdot, J\cdot) > 0$ and $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$. We also call ω a J-compatible symplectic structure. It is well known that there always exists ω-compatible almost complex structure J on (M, ω). Then we can define a J-invariant (J-compatible) Riemannian metric by $g(\cdot, \cdot) \triangleq \omega(\cdot, J\cdot)$. Such a quadruple (M, g, J, ω) is called an almost Kähler manifold. Recall that the energy of a smooth map $u : \Sigma \longrightarrow (M, g, J, \omega)$ is defined as the L^2-norm of the 1-form $\mathrm{d}u \in \Omega^1(\Sigma, u^*TM)$:

$$E_J(u) \triangleq \frac{1}{2}\int_\Sigma |\mathrm{d}u|_J^2 \mathrm{d}\mu_\Sigma.$$

Here, the norm of the (real) linear map $L \triangleq \mathrm{d}u(z) : T_z\Sigma \to T_{u(z)}M$ is defined by

$$|L|_J \triangleq |\xi|^{-1}\sqrt{|L(\xi)|_J^2 + |L(j_\Sigma \xi)|_J^2}$$

for $0 \neq \xi \in T_z\Sigma$, where $|L(\xi)|_J^2 = g(L(\xi), L(\xi))$. By Lemma 2.2.1 in [7],

$$E_J(u) = \int_\Sigma |\bar{\partial}_J u|_J^2 \mathrm{d}\mu_\Sigma + \int_\Sigma u^*\omega.$$

Hence J-holomorphic curve $u : (\Sigma, j) \longrightarrow (M, g, J, \omega)$ is a minimal surface with respect to the almost Kähler metric g. Under what condition a minimal surface is a J-holomorphic curve is an interesting question in differential geometry.

3 Critical point of L_p-functional

Suppose (M, J) is a closed almost complex $2n$-manifold. One can construct a J-invariant Riemannian metric g on M. Such a metric g is called an almost Hermitian metric for (M, J). This then in turn gives a J-compatible nondegenerate 2-form F by $F(X, Y) = g(JX, Y)$, called the fundamental 2-form. Such a quadruple (M, g, J, F) is called a closed almost Hermitian manifold. If $dF = 0$, then F will be written as ω and (M, g, J, ω) is called an almost Kähler manifold. By direct calculation, $F^n = n!\mathrm{d}\mu_g$, where $\mathrm{d}\mu_g$ is the volume form of M determined by g.

Proposition 3.1 (Wirtinger Inequality) *(we refer to* [4] *for a direct and simple proof) Suppose that (M, g, J, F) is a closed almost Hermitian $2n$-manifold. Let N be an oriented real smooth $2p$-submanifold in M, and let $\mathrm{d}\mu_N$ be the Riemannian volume form on N associated with the metric $g|_N$. Set*

$$\frac{1}{p!}F^p|_N = a\mathrm{d}\mu_N, \quad a \in C^\infty(N).$$

Then $|a| \leq 1$ and the equality holds if and only if N is an almost complex submanifold of M.

Hence, we can define the Kähler angle α for a surface Σ in almost Hermitian manifold (M, g, J, F) by

$$F|_\Sigma = \cos\alpha \mathrm{d}\mu_\Sigma. \tag{3.9}$$

Note that a smooth map $u : \Sigma \longrightarrow (M, g, J, F)$ (an almost Hermitian manifold) is a J-holomorphic curve if and only if it is conformal with respect to g, i.e. its differential preserves angles or, equivalently, it preserves inner products up to a common positive factor. By Wirtinger Inequality and the definition of Kähler angle, we can easily get the following Proposition,

Proposition 3.2 *Let (M, g, J, F) is a closed almost Hermitian $2n$-manifold. Then $f : \Sigma \to (M, J)$ is a J-holomorphic immersion if and only if $\sin\alpha \equiv 0$.*

Let (M, g, J, F) be an almost Hermitian $2n$-manifold. After a simple calculation, we can get the following properties:

$$d : \Omega^0 \longrightarrow \Omega^1, \;\; d = \partial_J + \bar{\partial}_J. \tag{3.10}$$

$$A_J \circ \partial_J + \bar{\partial}_J^2 + \bar{A}_J \circ \bar{\partial}_J + \partial_J^2 = 0 : \Omega^0 \longrightarrow (\Omega_J^{2,0} + \Omega_J^{0,2}). \tag{3.11}$$

$$\partial_J \circ \bar{\partial}_J + \bar{\partial}_J \circ \partial_J = 0 : \Omega^0 \longrightarrow \Omega^{1,1}. \tag{3.12}$$

$$d : \Omega^1 \longrightarrow \Omega^2, \;\; d = A_J + \partial_J + \bar{\partial}_J + \bar{A}_J. \tag{3.13}$$

By the above formulars, we get

Proposition 3.3 *Let (M, g, J, F) be an almost Hermitian $2n$-manifold. For any $\rho \in C^\infty(M, \mathbb{R})$, we have*

$$dd_J^c\rho = 2\sqrt{-1}\partial_J\bar{\partial}_J\rho + 2\sqrt{-1}(\bar{A}_J\bar{\partial}_J\rho - A_J\partial_J\rho).$$

Proof Firstly, by a simple calculation, we can get

$$dd_J^c\rho = 2\sqrt{-1}\partial_J\bar{\partial}_J\rho + \sqrt{-1}(\bar{A}_J\bar{\partial}_J\rho - \partial_J^2\rho) + \sqrt{-1}(\bar{\partial}_J^2\rho - A_J\partial_J\rho).$$

Since

$$\begin{aligned} d^2\rho &= d(\partial_J\rho + \bar{\partial}_J\rho) \\ &= \partial_J^2\rho + A_J\partial_J\rho + \bar{\partial}_J\partial_J\rho + \bar{\partial}_J^2\rho + \bar{A}_J\bar{\partial}_J\rho + \partial_J\bar{\partial}_J\rho \\ &= (\bar{\partial}_J\partial_J\rho + \partial_J\bar{\partial}_J\rho) + (\partial_J^2\rho + \bar{A}_J\bar{\partial}_J\rho) + (\bar{\partial}_J^2\rho + A_J\partial_J\rho) \\ &= 0, \end{aligned}$$

the corresponding individual components are equal to 0 respectively, that is, the $(1, 1)$-component $\bar{\partial}_J\partial_J\rho + \partial_J\bar{\partial}_J\rho = 0$; the $(2, 0)$-component $\partial_J^2\rho + \bar{A}_J\bar{\partial}_J\rho = 0$; the $(0, 2)$-component $\bar{\partial}_J^2\rho + A_J\partial_J\rho = 0$. Hence,

$$dd_J^c\rho = 2\sqrt{-1}\partial_J\bar{\partial}_J\rho + 2\sqrt{-1}(\bar{A}_J\bar{\partial}_J\rho - A_J\partial_J\rho).$$

□

Let (M, g, J, F) be an almost Hermitian $2n$-manifold. Let

$$\mathcal{H} \triangleq \{\rho \in C^\infty(M, \mathbb{R}) : F_\rho \triangleq F + dd_J^c\rho \;\text{ tames }\; J\},$$

which is clearly a nonempty open subset of $C^\infty(M, \mathbb{R})$. Given $\rho \in \mathcal{H}$, define

$$F_\rho = F + dd_J^c\rho. \tag{3.14}$$

In general, since J is not integrable, $dd_J^c\rho$ is not a $(1, 1)$-form. Thus, F_ρ is not a J-compatible 2-form. The associated almost Hermitian metric is given by

$$\begin{aligned} g_\rho(X, Y) &= \frac{1}{2}(F_\rho(X, JY) + F_\rho(Y, JX)) \\ &= \Pi^{1,1}(F_\rho)(X, JY) \\ &= (F + 2\sqrt{-1}\partial_J\bar{\partial}_J\rho)(X, JY). \end{aligned} \tag{3.15}$$

Obviously, $F+2\sqrt{-1}\partial_J\bar{\partial}_J\rho$ is a J-compatible 2-form and $(g_\rho, J, F+2\sqrt{-1}\partial_J\bar{\partial}_J\rho)$ is an almost Hermitian structure. Given the immersion $f:\Sigma\to M$, we have the induced metric and 2-form on Σ:

$$g'_\rho = f^*g_\rho, \quad F'_\rho = f^*(F+2\sqrt{-1}\partial_J\bar{\partial}_J\rho). \tag{3.16}$$

The cosine of the Kähler angle α_ρ is define by

$$F'_\rho = \cos\alpha_\rho \mathrm{d}\mu_{g'_\rho}. \tag{3.17}$$

Define the L_p-functional on $\mathcal{H}$ by

$$L_p(\rho) = \int_\Sigma \cos^p\alpha_\rho \mathrm{d}\mu_{g'_\rho}. \tag{3.18}$$

Definition 3.4 Given an immersion $F:\Sigma\to(M,g,J,F)$, we say that the functional L_p has a critical point $\rho\in\mathcal{H}$ if for any $\varphi(t)\in\mathcal{H}$ with $\varphi(0)=\rho$

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}L_p(\varphi(t)) = 0.$$

Stokes' theorem immediately gives the following

Proposition 3.5 *If $F:\Sigma\to(M,g,J,F)$ is a J-holomorphic immersion, then L_p is constant on $\mathcal{H}$.*

Proof By Proposition 3.2, for each $\rho\in\mathcal{H}$, we have $\cos^2\alpha_\rho\equiv 1$ on Σ. Without loss of generality, we may assume that $\cos\alpha_\rho\equiv 1$ on Σ since $\cos\alpha_\rho$ is smooth on Σ. Then, $L_p(\rho)=\int_\Sigma \mathrm{d}\mu_{g'_\rho}$ is just the area functional $\mathcal{A}(\rho)$. By Proposition 2.3 in [1], we get that L_p is constant on $\mathcal{H}$. □

By the above proposition, we will find that if $F:\Sigma\to(M,g,J,F)$ is a J-holomorphic immersion, then every $\rho\in\mathcal{H}$ is the critical point of L_p. Our interest is in which sense the converse holds. Choose a g'_0-orthonormal basis $\{e_1,e_2\}$ of $T_p\Sigma$, then

$$\cos\alpha_0 = F'_0(e_1,e_2) \tag{3.19}$$

and

$$\cos\alpha_\rho = \frac{F'_\rho(e_1,e_2)}{\sqrt{\det(g'_\rho(e_i,e_j))}}. \tag{3.20}$$

By (3.16),

$$F'_\rho = f^*(F+2\sqrt{-1}\partial_J\bar{\partial}_J\rho) = F'_0 + f^*(2\sqrt{-1}\partial_J\bar{\partial}_J\rho), \tag{3.21}$$

so that

$$F'_\rho(e_1,e_2) = \cos\alpha_0 + (2\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_1, f_*e_2). \tag{3.22}$$

Hence, by (3.20), we have

$$\cos\alpha_\rho = \frac{\cos\alpha_0 + (2\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_1, f_*e_2)}{\sqrt{\det(g'_\rho(e_i,e_j))}}. \tag{3.23}$$

Since $\{e_1,e_2\}$ is g'_0-orthonormal, by (3.15), we have

$$\begin{aligned} g'_\rho(e_i,e_j) &= g_\rho(f_*e_i, f_*e_j) \\ &= (F+2\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_i, Jf_*e_j) \\ &= g(f_*e_i, f_*e_j) + (2\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_i, Jf_*e_j) \\ &= \delta_{ij} + (2\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_i, Jf_*e_j). \end{aligned} \tag{3.24}$$

Therefore,

$$\begin{aligned}\det(g'_\rho) &= 1 + (2\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_1, Jf_*e_1) + (2\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_2, Jf_*e_2)\\ &\quad + 4(\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_1, Jf_*e_1)\cdot(\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_2, Jf_*e_2)\\ &\quad - 4[(\sqrt{-1}\partial_J\bar{\partial}_J\rho)(f_*e_1, Jf_*e_2)]^2. \end{aligned} \tag{3.25}$$

Choose a g-orthonormal frame $\{e_1, e_2, \cdots, e_{2n}\}$ of $T_{f(p)}M$ such that $\{e_1, e_2\}$ spans the tangent space $T_p\Sigma$ and $\{e_3, \cdots, e_{2n}\}$ spans the normal space of Σ. Here, we identify e_i with f_*e_i for simplicity. Then the almost complex structure J takes the form

$$J = \begin{pmatrix} (J_1)_{4\times 4} & 0_{4\times(2n-4)} \\ 0_{(2n-4)\times 4} & (J_2)_{(2n-4)\times(2n-4)} \end{pmatrix}, \tag{3.26}$$

where

$$J_1 = \begin{pmatrix} 0 & \cos\alpha_0 & \sin\alpha_0 & 0 \\ -\cos\alpha_0 & 0 & 0 & -\sin\alpha_0 \\ -\sin\alpha_0 & 0 & 0 & \cos\alpha_0 \\ 0 & \sin\alpha_0 & -\cos\alpha_0 & 0 \end{pmatrix}, \tag{3.27}$$

and J_2 satisfies $J_2^2 = -Id_{2n-4}$.

In [1], Arezzo and Sun have gotten the following useful result

$$\begin{aligned} dd^c_J\rho(X,Y) &= -(\nabla^2\rho)(X, JY) + (\nabla^2\rho)(Y, JX)\\ &\quad + \langle\nabla\rho, (\nabla_Y J)X - (\nabla_X J)Y\rangle_g, \end{aligned} \tag{3.28}$$

where ∇ is the Levi-Civita connection of g. By Proposition 3.3, we have

$$\begin{aligned} 2\sqrt{-1}\partial_J\bar{\partial}_J\rho(X,Y) &= dd^c_J\rho(X,Y) - 2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\rho - A_J\partial_J\rho)(X,Y)\\ &= -(\nabla^2\rho)(X, JY) + (\nabla^2\rho)(Y, JX)\\ &\quad + \langle\nabla\rho, (\nabla_Y J)X - (\nabla_X J)Y\rangle_g\\ &\quad - 2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\rho - A_J\partial_J\rho)(X,Y). \end{aligned} \tag{3.29}$$

Let $\varphi(t)$ be a variation coming from a 1-parameter deformation of $\varphi(0) = 0$ in $\mathcal{H}$ with $\dot{\varphi}(0) = \gamma$. By (3.23), the L_p-functional has the following representation

$$\begin{aligned} L_p(\varphi(t)) &= \int_\Sigma \cos^p\alpha_{\varphi(t)} \mathrm{d}\mu_{g'_{\varphi(t)}}\\ &= \int_\Sigma \Big[\frac{\cos\alpha_0 + (2\sqrt{-1}\partial_J\bar{\partial}_J\,\varphi(t))(f_*e_1, f_*e_2)}{\sqrt{\det(g'_{\varphi(t)}(e_i, e_j))}}\Big]^p \mathrm{d}\mu_{g'_{\varphi(t)}}. \end{aligned} \tag{3.30}$$

In the following part, we will compute the first variation of the L_p-functional. By (3.25), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det(g'_{\varphi(t)}) = (2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1, Je_1) + (2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_2, Je_2). \tag{3.31}$$

Then, by (3.23) and the above formualr, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\cos\alpha_{\varphi(t)}\\
&=(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1,e_2)-\frac{1}{2}\cos\alpha_0\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\det(g'_{\varphi(t)})\\
&=(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1,e_2)-\frac{1}{2}\cos\alpha_0[(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1,Je_1)\\
&\quad+(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_2,Je_2)].
\end{aligned}
\tag{3.32}
$$

With (3.26) and (3.29), by a direct computation,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\cos\alpha_{\varphi(t)}\\
&=\cos\alpha_0[(\nabla^2\gamma)(e_1,e_1)+(\nabla^2\gamma)(e_2,e_2)]+\sin\alpha_0[(\nabla^2\gamma)(e_1,e_4)+(\nabla^2\gamma)(e_2,e_3)]\\
&\quad+\langle\nabla\gamma,(\nabla_{e_2}J)e_1-(\nabla_{e_1}J)e_2\rangle_g-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma-A_J\partial_J\gamma)(e_1,e_2)\\
&\quad-\frac{1}{2}\cos\alpha_0\left\{(1+\cos^2\alpha_0)(\nabla^2\gamma)(e_1,e_1)+(1+\cos^2\alpha_0)(\nabla^2\gamma)(e_2,e_2)\right.\\
&\quad+2\sin\alpha_0\cos\alpha_0(\nabla^2\gamma)(e_2,e_3)+\sin^2\alpha_0(\nabla^2\gamma)(e_3,e_3)\\
&\quad\left.+2\sin\alpha_0\cos\alpha_0(\nabla^2\gamma)(e_1,e_4)+\sin^2\alpha_0(\nabla^2\gamma)(e_4,e_4)\right\}\\
&\quad-\frac{1}{2}\cos\alpha_0\{\langle\nabla\gamma,(\nabla_{Je_1}J)e_1-(\nabla_{e_1}J)Je_1\rangle_g-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma-A_J\partial_J\gamma)(e_1,Je_1)\}\\
&\quad-\frac{1}{2}\cos\alpha_0\{\langle\nabla\gamma,(\nabla_{Je_2}J)e_2-(\nabla_{e_2}J)Je_2\rangle_g-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma-A_J\partial_J\gamma)(e_2,Je_2)\}\\
&=\frac{1}{2}\sin^2\alpha_0\left\{\cos\alpha_0\left[(\nabla^2\gamma)(e_1,e_1)+(\nabla^2\gamma)(e_2,e_2)-(\nabla^2\gamma)(e_3,e_3)-(\nabla^2\gamma)(e_4,e_4)\right]\right.\\
&\quad\left.+2\sin\alpha_0\left[(\nabla^2\gamma)(e_1,e_4)+(\nabla^2\gamma)(e_2,e_3)\right]\right\}\\
&\quad+\langle\nabla\gamma,(\nabla_{e_2}J)e_1-(\nabla_{e_1}J)e_2\rangle_g-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma-A_J\partial_J\gamma)(e_1,e_2)\\
&\quad-\frac{1}{2}\cos\alpha_0\{\langle\nabla\gamma,(\nabla_{Je_1}J)e_1-(\nabla_{e_1}J)Je_1\rangle_g-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma-A_J\partial_J\gamma)(e_1,Je_1)\}\\
&\quad-\frac{1}{2}\cos\alpha_0\{\langle\nabla\gamma,(\nabla_{Je_2}J)e_2-(\nabla_{e_2}J)Je_2\rangle_g-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma-A_J\partial_J\gamma)(e_2,Je_2)\}.
\end{aligned}
\tag{3.33}
$$

Lemma 3.6 *Let (M,g,J,F) be an almost Hermitian manifold and $f:\Sigma\to M$ be an injective immersion such that $\cos\alpha_0>0$. Set $d: M\to\mathbb{R}$ any smooth extension from a tubular neighborhood of $f(\Sigma)$ to M of the distance function from $f(\Sigma)$, i.e. $d(q)=dist(q,f(\Sigma))$ for q sufficiently near $f(\Sigma)$. If*

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}L_p\left(F_\rho+tdd^c_J\left(\frac{d^2}{2}\right)\right)=0$$

for some $p\in\mathbb{Z}-\{1\}$ and $\rho\in\mathcal{H}$, then the immersion is J-holomorphic.

Proof Without loss of generality, we assume that $\rho\equiv 0$ so that $F_\rho=F$. Let $\varphi(t)$ be any curve in $\mathcal{H}$ such that $\varphi(0)=\rho\equiv 0$ and $\dot{\varphi}(0)=\gamma$. Fix a point $p\in\Sigma$ and take an orthonormal basis $\{e_1,e_2\}$ of $T_p\Sigma$ so that the complex structure J takes the form (3.26). By (3.24) and (3.31), it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{d}\mu_{g'_{\varphi(t)}} = \frac{1}{2}\sum_{i=1}^{2}(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_i, Je_i)\mathrm{d}\mu_{g'_0}.$$

Then, with (3.26) and (3.29), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{d}\mu_{g'_{\varphi(t)}} = \frac{1}{2}\{&(1+\cos^2\alpha_0)(\nabla^2\gamma)(e_1,e_1) + (1+\cos^2\alpha_0)(\nabla^2\gamma)(e_2,e_2)\\ &+2\sin\alpha_0\cos\alpha_0(\nabla^2\gamma)(e_2,e_3) + \sin^2\alpha_0(\nabla^2\gamma)(e_3,e_3)\\ &+2\sin\alpha_0\cos\alpha_0(\nabla^2\gamma)(e_1,e_4) + \sin^2\alpha_0(\nabla^2\gamma)(e_4,e_4)\\ &+\langle\nabla\gamma, (\nabla_{Je_1}J)e_1 - (\nabla_{e_1}J)Je_1 + (\nabla_{Je_2}J)e_2 - (\nabla_{e_2}J)Je_2\rangle_g\\ &-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma - A_J\partial_J\gamma)(e_1, Je_1)\\ &-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma - A_J\partial_J\gamma)(e_2, Je_2)\}\mathrm{d}\mu_{g'_0}.\end{aligned} \tag{3.34}$$

By (3.33) and (3.34),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}L_p(\varphi(t))\\ &= p\int_\Sigma \cos^{p-1}\alpha_0\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\cos\alpha_{\varphi(t)}\mathrm{d}\mu_{g'_0} + \int_\Sigma\cos^p\alpha_0\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{d}\mu_{g'_{\varphi(t)}}\\ &= \int_\Sigma \frac{p\cos^{p-1}\alpha_0\sin^2\alpha_0}{2}\left\{\cos\alpha_0\left[(\nabla^2\gamma)(e_1,e_1) + (\nabla^2\gamma)(e_2,e_2)\right.\right.\\ &\quad\left.-(\nabla^2\gamma)(e_3,e_3) - (\nabla^2\gamma)(e_4,e_4)\right]\\ &\quad\left.+2\sin\alpha_0\left[(\nabla^2\gamma)(e_1,e_4) + (\nabla^2\gamma)(e_2,e_3)\right]\right\}\mathrm{d}\mu_{g'_0}\\ &\quad+\int_\Sigma\frac{\cos^p\alpha_0}{2}\left\{(1+\cos^2\alpha_0)\left[(\nabla^2\gamma)(e_1,e_1)+(\nabla^2\gamma)(e_2,e_2)\right]\right.\\ &\quad+\sin^2\alpha_0\left[(\nabla^2\gamma)(e_3,e_3)+(\nabla^2\gamma)(e_4,e_4)\right]\\ &\quad\left.+2\sin\alpha_0\cos\alpha_0\left[(\nabla^2\gamma)(e_2,e_3)+(\nabla^2\gamma)(e_1,e_4)\right]\right\}\mathrm{d}\mu_{g'_0} + \Phi\\ &= \int_\Sigma\{\frac{1}{2}\cos^p\alpha_0(1+\cos^2\alpha_0 + p\sin^2\alpha_0)[(\nabla^2\alpha_0)(e_1,e_1) + (\nabla^2\alpha_0)(e_2,e_2)]\\ &\quad+\frac{1-p}{2}\sin^2\alpha_0\cos^p\alpha_0[(\nabla^2\alpha_0)(e_3,e_3)+(\nabla^2\alpha_0)(e_4,e_4)]\\ &\quad+\sin\alpha_0\cos^{p-1}\alpha_0(\cos^2\alpha_0 + p\sin^2\alpha_0)(\nabla^2\alpha_0)(e_2,e_3)\\ &\quad+\sin\alpha_0\cos^{p-1}\alpha_0(\cos^2\alpha_0+p\sin^2\alpha_0)(\nabla^2\alpha_0)(e_1,e_4)\}\mathrm{d}\mu_{g'_0} + \Phi,\end{aligned} \tag{3.35}$$

where

$$\begin{aligned}\Phi = p&\int_\Sigma\cos^{p-1}\alpha_0\langle\nabla\gamma, (\nabla_{e_2}J)e_1 - (\nabla_{e_1}J)e_2\rangle_g\mathrm{d}\mu_{g'_0}\\ &-p\int_\Sigma\cos^{p-1}\alpha_0 2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma - A_J\partial_J\gamma)(e_1,e_2)\mathrm{d}\mu_{g'_0}\\ &-\frac{p}{2}\int_\Sigma\cos^p\alpha_0\{\langle\nabla\gamma, (\nabla_{Je_1}J)e_1 - (\nabla_{e_1}J)Je_1\rangle_g\\ &\qquad\qquad -2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma - A_J\partial_J\gamma)(e_1, Je_1)\}\mathrm{d}\mu_{g'_0}\\ &-\frac{p}{2}\int_\Sigma\cos^p\alpha_0\{\langle\nabla\gamma, (\nabla_{Je_2}J)e_2 - (\nabla_{e_2}J)Je_2\rangle_g\end{aligned}$$

$$-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma - A_J\partial_J\gamma)(e_2, Je_2)\}\mathrm{d}\mu_{g_0'}$$
$$+\int_\Sigma \frac{\cos^p\alpha_0}{2}\langle\nabla\gamma, (\nabla_{Je_1}J)e_1 - (\nabla_{e_1}J)Je_1 + (\nabla_{Je_2}J)e_2 - (\nabla_{e_2}J)Je_2\rangle_g\mathrm{d}\mu_{g_0'}$$
$$-\int_\Sigma \cos^p\alpha_0\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma - A_J\partial_J\gamma)(e_1, Je_1)\mathrm{d}\mu_{g_0'}$$
$$-\int_\Sigma \cos^p\alpha_0\sqrt{-1}(\bar{A_J}\bar{\partial}_J\gamma - A_J\partial_J\gamma)(e_2, Je_2)\mathrm{d}\mu_{g_0'}.$$

We identify Σ with its image in M. Denote d the distance function of M from Σ with respect to the metric g, that is, for $q \in M$, $d(q) = dist_g(q, \Sigma)$. Recall that $\xi = \frac{1}{2}d^2$ is smooth in a neighborhood of Σ in M (cf. [6]). By Proposition 2.6 in [1], for any $p \in \Sigma$, the hessian $Hess(\xi)(p)$ represents the orthogonal projection on the normal space to Σ at p, that is, for each $X, Y \in T_pM$, we have

$$\nabla^2(\xi)(X, Y)(p) = \langle X^\perp, Y^\perp\rangle, \tag{3.36}$$

where $T_pM = T_p\Sigma \oplus N_p\Sigma$ and $X^\perp$ is the projection of X onto $N_p\Sigma$. Next, we will take special test function γ to be a smooth function on M such that $\gamma = \frac{1}{2}d^2$ in a neighborhood of Σ in M. Since $\{e_1, e_2\}$ is an orthonormal basis of $T_p\Sigma$, it is easy to see that $e_1^\perp = 0$ and $e_2^\perp = 0$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}L_p(\varphi(t))$$
$$= \int_\Sigma \frac{1-p}{2}\sin^2\alpha_0\cos^p\alpha_0[(\nabla^2\alpha_0)(e_3, e_3) + (\nabla^2\alpha_0)(e_4, e_4)]\mathrm{d}\mu_{g_0'} + \Phi. \tag{3.37}$$

It is well known that both ∇ and $\bar{A_J}\bar{\partial}_J - A_J\partial_J$ are $\mathbb{R}$-linear operators of order 1. So by the choice of $\gamma = \frac{1}{2}d^2$ and the definition of d, we can easily get $\Phi = \int_\Sigma(\cdot)\mathrm{d}\mu_{g_0'} = 0$. Then by (3.36) and (3.37), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}L_p(F + tdd_J^c(\frac{d^2}{2})) = (1-p)\int_\Sigma \sin^2\alpha_0\cos^p\alpha_0 d\mu_{g_0'} = 0. \tag{3.38}$$

On the other hand, by our assumption, $\cos\alpha_0 > 0$, $p \neq 1$ and $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}L_p(F + t2\sqrt{-1}\partial_J\bar{\partial}_J(\frac{d^2}{2})) = 0$. Therefore, we must have $\sin\alpha_0 \equiv 0$. By Proposition 3.2, to prove the theorem, it suffices to show that $\sin\alpha_0 \equiv 0$ on Σ. Hence, this completes the proof of Lemma 3.6.

Theorem 3.7 *Let (M, g, J, F) be an almost Hermitian manifold and $f : \Sigma \to M$ be an injective immersion such that $\cos\alpha_0 > 0$. If for some $p \in \mathbb{Z} - \{1\}$, the functional L_p has a critical point in $\mathcal{H}$, then the immersion is J-holomorphic.*

If $dF = 0$, then F will be written as ω and (M, g, J, ω) is called an almost Kähler manifold. The condition $\cos\alpha_0 > 0$ is just show that $f : \Sigma \to M$ is an injective symplectic immersion. Then Theorem 3.7 can be expressed as,

Corollary 3.8 *Let (M, g, J, ω) be an almost Kähler manifold and $f : \Sigma \to M$ be an injective symplectic immersion. If for some $p \in \mathbb{Z} - \{1\}$, the functional L_p has a critical point in $\mathcal{H}$, then the immersion is J-holomorphic.*

When $p = 0$, $L_0(\rho)$ is just the area functional $\mathcal{A}(\rho)$. Then the integrand of the right hand side of (3.38) becomes $\sin^2\alpha_0$. Hence, we get

Corollary 3.9 *(see [1]) Let (M, g, J, F) be an almost Hermitian manifold and $f : \Sigma \to M$ be an injective immersion. If the functional L_0 has a critical point in $\mathcal{H}$, then the immersion is J-holomorphic.*

Let (M, g, J, ω) be an almost Kähler manifold and $f : \Sigma \to M$ be an injective immersion. Suppose that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} L_p\left(\omega_\rho + tdd^c_J\left(\frac{d^2}{2}\right)\right) = 0$$

for some $p \in 2\mathbb{Z}^+$ and $\rho \in \mathcal{H}$. By (3.38), we have

$$\sin^2\alpha_0 \cos^p \alpha_0 \equiv 0$$

on Σ. Then we will obtain $\sin\alpha_0 \equiv 0$ or $\cos\alpha_0 \equiv 0$ on Σ. If $\sin\alpha_0 \equiv 0$, the immersion is J-holomorphic. If $\cos\alpha_0 \equiv 0$, the immersion is Lagrangian.

4 Stable point of L_p-functional

In light of our knowledge about the relationship between stable minimal surfaces and holomorphic curves, it is natural to look at special properties of the second variation of the functional L_p.

Definition 4.1 Given a symplectic immersion $F : \Sigma^2 \to (M, \bar{\omega}, J_M, \bar{g})$, we say that $\rho \in \mathcal{H}_p$ is a stable point for the functional L_p if for any $\varphi(t) \in \mathcal{H}$ with $\varphi(0) = \rho$

$$\frac{d^2}{dt^2}|_{t=0} L_p(\varphi(t)) \geq 0.$$

Let (M, g, J, F) be an almost Hermitian manifold and $f : \Sigma \to M$ be an injective immersion such that $\cos\alpha_0 > 0$ as in the previous section. Take any curve $\varphi(t) \in \mathcal{H}$ with $\varphi(0) = 0$, $\dot{\varphi}(0) = \gamma$ and $\ddot{\varphi}(0) = \zeta$. By (3.23),

$$L_p(\varphi(t)) = \int_\Sigma [\cos\alpha_0 + (2\sqrt{-1}\partial_J\bar{\partial}_J\varphi)(e_1, e_2)]^p \det(g'_\varphi)^{\frac{1-p}{2}}.$$

Then, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} L_p(\varphi(t)) = \int_\Sigma & p[\cos\alpha_0 + (2\sqrt{-1}\partial_J\bar{\partial}_J\varphi)]^{p-1} \det(g'_\varphi)^{\frac{1-p}{2}} \frac{\mathrm{d}}{\mathrm{d}t}(2\sqrt{-1}\partial_J\bar{\partial}_J\varphi)\\ & + \frac{1-p}{2}[\cos\alpha_0 + (2\sqrt{-1}\partial_J\bar{\partial}_J\varphi)]^p \det(g'_\varphi)^{\frac{-1-p}{2}} \frac{\mathrm{d}}{\mathrm{d}t}\det(g'_\varphi).\end{aligned}$$

Hence, combined the above formula, (3.25) and (3.31), the second variation formula for the functional L_p is given by

$$\begin{aligned}&\frac{d^2}{dt^2}|_{t=0} L_p(\varphi(t))\\ &= \int_\Sigma p(p-1)\cos^{p-2}\alpha_0 [\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(2\sqrt{-1}\partial_J\bar{\partial}_J\varphi(t))(e_1, e_2)]^2\\ &\quad + \frac{p(1-p)}{2}\cos^{p-1}\alpha_0 \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\det(g'_{\varphi(t)}) \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(2\sqrt{-1}\partial_J\bar{\partial}_J\varphi(t))(e_1, e_2)\\ &\quad + p\cos^{p-1}\alpha_0 \frac{d^2}{dt^2}|_{t=0}(2\sqrt{-1}\partial_J\bar{\partial}_J\varphi(t))(e_1, e_2)\end{aligned}$$

$$
\begin{aligned}
&+\frac{p(1-p)}{2}\cos^{p-1}\alpha_0\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(2\sqrt{-1}\partial_J\bar{\partial}_J\varphi(t))(e_1,e_2)\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\det(g'_{\varphi(t)})\\
&+\frac{(p-1)(p+1)}{4}\cos^p\alpha_0[\frac{d}{dt}|_{t=0}\det(g'_{\varphi(t)})]^2+\frac{1-p}{2}\cos^p\alpha_0\frac{d^2}{dt^2}|_{t=0}\det(g'_{\varphi(t)})\\
=&\int_\Sigma p(p-1)\cos^{p-2}\alpha_0[(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1,e_2)]^2\\
&+p(1-p)\cos^{p-1}\alpha_0(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1,e_2)\\
&\cdot[(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1,Je_1)+(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_2,Je_2)]\\
&+p\cos^{p-1}\alpha_0(2\sqrt{-1}\partial_J\bar{\partial}_J\zeta)(e_1,e_2)\\
&+\frac{(p-1)(p+1)}{4}\cos^p\alpha_0[(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1,Je_1)+(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_2,Je_2)]^2\\
&+\frac{1-p}{2}\cos^p\alpha_0\{(2\sqrt{-1}\partial_J\bar{\partial}_J\zeta)(e_1,Je_1)+(2\sqrt{-1}\partial_J\bar{\partial}_J\zeta)(e_2,Je_2)\}\\
&+\frac{1-p}{2}\cos^p\alpha_0\{2(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_1,Je_1)(2\sqrt{-1}\partial_J\bar{\partial}_J\gamma)(e_2,Je_2)\\
&-2[(2\sqrt{-1}\partial\bar{\partial}\gamma)(e_1,Je_2)]^2\}.
\end{aligned}
\tag{4.39}
$$

Lemma 4.2 *Let (M,g,J,F) be an almost Hermitian manifold and $f:\Sigma\to M$ be an injective immersion such that $\cos\alpha_0>0$ as above. Set $d\colon M\to\mathbb{R}$ any smooth extension from a tubular neighborhood of $f(\Sigma)$ to M of the distance function from $f(\Sigma)$, i.e. $d(q)=dist(q,f(\Sigma))$ for q sufficiently near $f(\Sigma)$. If*

$$\frac{d^2}{dt^2}|_{t=0}L_p\left(F_\rho+\frac{t^2}{2}dd^c_J\left(\frac{d^2}{2}\right)\right)=0$$

for some $p\in\mathbb{Z}-\{1\}$ and $\rho\in\mathcal{H}$, then the immersion is J-holomorphic.

Proof Without loss of generality, we assume that $\rho=0$. Moreover we take $\varphi(t)=\frac{t^2}{2}\zeta$ so that $\gamma=0$. Then formula (4.39) becomes

$$
\begin{aligned}
\frac{d^2}{dt^2}|_{t=0}L_p(\varphi(t))=&\int_\Sigma p\cos^{p-1}\alpha_0(2\sqrt{-1}\partial_J\bar{\partial}_J\zeta)(e_1,e_2)\\
&+\frac{1-p}{2}\cos^p\alpha_0(2\sqrt{-1}\partial_J\bar{\partial}_J\zeta)(e_1,Je_1)\\
&+\frac{1-p}{2}\cos^p\alpha_0(2\sqrt{-1}\partial_J\bar{\partial}_J\zeta)(e_2,Je_2).
\end{aligned}
\tag{4.40}
$$

By (3.26), (3.27) and (3.29), we have

$$
\begin{aligned}
2\sqrt{-1}\partial_J\bar{\partial}_J\zeta(e_1,e_2)=&-(\nabla^2\zeta)(e_1,Je_2)+(\nabla^2\zeta)(e_2,Je_1)\\
&+\langle\nabla\zeta,(\nabla_{e_2}J)e_1-(\nabla_{e_1}J)e_2\rangle_g\\
&-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\zeta-A_J\partial_J\zeta)(e_1,e_2)\\
=&\cos\alpha_0[(\nabla^2\zeta)(e_1,e_1)+(\nabla^2\zeta)(e_2,e_2)]\\
&+\sin\alpha_0[(\nabla^2\zeta)(e_1,e_4)+(\nabla^2\zeta)(e_2,e_3)]\\
&+\langle\nabla\zeta,(\nabla_{e_2}J)e_1-(\nabla_{e_1}J)e_2\rangle_g\\
&-2\sqrt{-1}(\bar{A_J}\bar{\partial}_J\zeta-A_J\partial_J\zeta)(e_1,e_2).
\end{aligned}
\tag{4.41}
$$

Now, we take ζ to be a smooth function on M so that $\zeta = \frac{d^2}{2}$ in a neighborhood of Σ. Then by (3.36) and the fact that both ∇ and $\bar{A_J}\bar{\partial}_J - A_J\partial_J$ are $\mathbb{R}$-linear operators of order 1, we have $2\sqrt{-1}\partial_J\bar{\partial}_J\zeta(e_1, e_2) = 0$ when restricting on Σ. Similarly, we have

$$(2\sqrt{-1}\partial_J\bar{\partial}_J\zeta)(e_1, Je_1) = (2\sqrt{-1}\partial_J\bar{\partial}_J\zeta)(e_2, Je_2) = \sin^2\alpha_0.$$

Therefore, we have

$$\frac{d^2}{dt^2}|_{t=0}L_p\left(F_\rho + \frac{t^2}{2}dd^c_J\left(\frac{d^2}{2}\right)\right) = (1-p)\int_\Sigma \cos^p\alpha_0\sin^2\alpha_0 = 0.$$

Then, we obtain $\sin\alpha_0 = 0$ since we have assumed that $\cos\alpha_0 > 0$ and $p \neq 1$. By Proposition 3.2, this proves the lemma.

If L_p has a stable point $\rho = 0$, then by Definition 4.1, we have

$$\frac{d^2}{dt^2}|_{t=0}L_p\left(F_\rho + \frac{t^2}{2}dd^c_J(\zeta)\right) \geq 0. \tag{4.42}$$

It easy to see that

$$\frac{d^2}{dt^2}|_{t=0}L_p\left(F_\rho - \frac{t^2}{2}dd^c_J(\zeta)\right) = -\frac{d^2}{dt^2}|_{t=0}L_p\left(F_\rho + \frac{t^2}{2}dd^c_J(\zeta)\right).$$

Replacing ζ by $-\zeta$ in (4.42), we can get $-\frac{d^2}{dt^2}|_{t=0}L_p(F_\rho + \frac{t^2}{2}dd^c_J(\zeta)) \geq 0$. That means

$$\frac{d^2}{dt^2}|_{t=0}L_p\left(F_\rho + \frac{t^2}{2}dd^c_J(\zeta)\right) = 0.$$

Then, with Lemma 4.2, we can easily get the following theorem

Theorem 4.3 *Let (M, g, J, F) be an almost Hermitian manifold and $f : \Sigma \to M$ be an injective immersion such that* $\cos\alpha_0 > 0$. *If the functional L_p ($p \in \mathbb{Z} - \{1\}$) has a stable point in $\mathcal{H}$, then the immersion is J-holomorphic.*

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