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Zero varieties for the Nevanlinna class in weakly pseudoconvex domains of maximal type F in $\mathbb{C}^2$

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Abstract Let Ω be a bounded, uniformly totally pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary $b\Omega$. Assume that Ω is a domain admitting a maximal type F. Here, the condition maximal type F generalizes the condition of finite type in the sense of Range (Pac J Math 78(1):173–189, 1978; Scoula Norm Sup Pisa, pp 247–267, 1978) and includes many cases of infinite type. Let α be a d-closed $(1, 1)$-form in Ω. We study the Poincaré–Lelong equation

$$i\partial\bar{\partial}u = \alpha \quad \text{on } \Omega$$

in $L^1(b\Omega)$ norm by applying the $L^1(b\Omega)$ estimates for $\bar{\partial}_b$-equations in [11]. Then, we also obtain a prescribing zero set of Nevanlinna holomorphic functions in Ω.

Keywords Pseudoconvex domains · Poincaré–Lelong equation · Blaschke condition · Nevanlinna class · $\bar{\partial}_b$-operator · Henkin solution

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1 Introduction

Let Ω be a bounded domain in $\mathbb{C}^2$ with smooth boundary $b\Omega$, and let $\mathfrak{g}$ be a Nevanlinna holomorphic function on Ω. In pluripotential theory, it is well-known that the zero variety $Z(\Omega, \mathfrak{g})$ associated to $\mathfrak{g}$ on Ω satisfies the Blaschke condition. Naturally, we are interested in studying the converse, that is seeking geometric conditions on Ω so that any given analytic variety is defined as the zero set of a Nevanlinna holomorphic function.

We briefly recall the illustrious history of this problem. When Ω is the unit disk on the complex plane, a well-known fact in potential theory (e.g., [8,17]) says that if Ω satisfies

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the Blaschke condition, any analytic variety $M \subset \Omega$ is the zero variety of a Nevanlinna holomorphic function, or a bounded holomorphic function on Ω. Actually, this is true for all simply connected domains in the complex plane by the Riemann mapping theorem.

It is more difficult when we consider the problem in $\mathbb{C}^n$, for $n \geq 2$. The existence of a Nevanlinna holomorphic function determining a given positive divisor M on the unit ball in $\mathbb{C}^n$ is well-understood, see in [29]. This is also true under certain algebraic topology conditions when Ω is a strongly pseudoconvex domain, for instance, by Gruman [10], by Henkin [14] and Skoda [33] independently. Moreover, in [21], Laville showed that if Ω is star-shaped, then there exists a Nevanlinna function $\mathfrak{g}$ determining M and $\log|\mathfrak{g}| \in L^1(\Omega)$. Another positive result was obtained by Anderson [1] when Ω is a polydisc in $\mathbb{C}^n$. The problematic situation is if Ω is a weakly pseudoconvex domain. Existence results have been obtained on some special domains: on complex ellipsoids of finite type by Bonami and Charpentier [3]; on uniformly totally pseudoconvex domains of finite type in the sense of Range in $\mathbb{C}^2$ by Shaw [31]. The large class of uniformly totally pseudoconvex/ convex domains of finite type in the sense of Range introduced in [25,26] consists all balls, strongly pseudoconvex domains and complex ellipsoids, and convex domains with real analytic boundaries in $\mathbb{C}^2$. In this paper, we shall give an answer to this problem on a large class of pseudoconvex domains of infinite type.

The main results are the following theorems. The first is the L^p boundary regularity for solutions of the $\bar\partial$-equation.

Theorem 1.1 *Let Ω be a smooth bounded, uniformly totally pseudoconvex domain and admit maximal type F at all boundary points for some function F (see Definition (2.2)). Assume that $\bar\Omega$ has a Stein neighborhood basis. Let φ be a continuous $(0,1)$-form on $\overline{\Omega}$ and satisfy $\bar\partial\varphi = 0$ in the weak sense. Then there exists a function $u \in \Lambda^f(\overline{\Omega})$ such that*

$$\bar\partial u = \varphi,$$

where

$$f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}}{t}\mathrm{d}t\right)^{-1},$$

with F^* the inversion of F.

Moreover, we also have

(i) $||u||_{L^1(\Omega)} \leq C(||\varphi||_{L^1_{(0,1)}(\Omega)} + ||\varphi||_{L^1_{(0,1)}(b\Omega)})$;
(ii) $||u||_{L^p(b\Omega)} \leq C_p||\varphi||_{L^p_{(0,1)}(b\Omega)}$ for all $1 \leq p \leq +\infty$;
(iii) $||u||_{\Lambda^f_p(b\Omega)} \leq C_p||\varphi||_{L^p_{(0,1)}(b\Omega)}$ for all $1 \leq p \leq +\infty$.

Example 1.1 Let us define

$$\Omega^\infty = \left\{(z_1, z_2) \in \mathbb{C}^2 : \exp(1 + 2/s) \cdot \exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\right\}.$$

Let φ be a continuous $(0,1)$-form on $\overline{\Omega}$ and satisfy $\bar\partial\varphi = 0$ in the weak sense. Then there exists a function $u \in \Lambda^f(\overline{\Omega})$ such that

$$\bar\partial u = \varphi,$$

where $f(t) = \frac{1024^s(1-2s)}{2s}(|\ln t|)^{\frac{1}{2s}-1}$, for $0 < s < 1/2$.

Moreover, we have

(i) $||u||_{L^1(\Omega)} \leq C(||\varphi||_{L^1_{(0,1)}(\Omega)} + ||\varphi||_{L^1_{(0,1)}(b\Omega)})$;

(ii) $||u||_{L^p(b\Omega)} \leq C_p||\varphi||_{L^p_{(0,1)}(b\Omega)}$ for all $1 \leq p \leq +\infty$;
(iii) $||u||_{\Lambda^f_p(b\Omega)} \leq C_p||\varphi||_{L^p_{(0,1)}(b\Omega)}$ for all $1 \leq p \leq +\infty$.

Let $H^2(\Omega, \mathbb{R})$ be the DeRham cohomology of the second degree on Ω. The existence of solutions to the Poincaré–Lelong equation is our second result.

Theorem 1.2 *Let Ω be a smooth bounded, uniformly totally pseudoconvex domain and admit maximal type F at all boundary points for some function F. Assume that $\bar{\Omega}$ has a Stein neighborhood basis, and $H^2(\Omega, \mathbb{R}) = 0$. Let α be a positive d-closed, smooth $(1, 1)$-form on Ω. Then the Poincaré–Lelong equation*

$$i\partial\bar{\partial}u = \alpha$$

admits a solution u such that

(i) $u = \bar{u}$;
(ii) $||u||_{L^1(b\Omega)} + ||u||_{L^1(\Omega)} \leq C||\alpha||_{L^1_{(1,1)}(\Omega)}$.

Let $H^2(\Omega, \mathbb{Z})$ be the Čech cohomology group of the second degree with integer coefficients on Ω. The last result is about prescribing zeros of holomorphic functions in the Nevanlinna class on Ω.

Theorem 1.3 *Let Ω be a smooth bounded, uniformly totally pseudoconvex domain and admit maximal type F at all boundary points for some function F. Assume that $\bar{\Omega}$ has a Stein neighborhood basis and $H^2(\Omega, \mathbb{Z}) = 0$. If M is a finite area, positive divisor of Ω, then we have*

$$M = Z(\Omega, \mathfrak{g}),$$

for some Nevanlinna holomorphic function $\mathfrak{g}$ defined on Ω.

Following the same lines in the proof of Corollary 3.3 in [31], we get a boundary property for meromorphic functions in Nevanlinna class.

Corollary 1.4 *Let Ω be the same as in Theorem 1.3. Let $\mathfrak{g}$ be a meromorphic function in $\mathcal{N}(\Omega)$ such that the associated polar divisor $(M_\mathfrak{g})_\infty$ has finite area. Then there are two Nevanlinna holomorphic functions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on Ω such that $\mathfrak{g} = \mathfrak{g}_1/\mathfrak{g}_2$. Therefore, $\mathfrak{g}$ has non-tangential limit values almost everywhere on the boundary $b\Omega$.*

The paper is organized as follows: In Sect. 2, we shall introduce some geometric conditions on Ω and recall the main result of [11]. Basic definitions and facts from Lelong's theory are briefly recalled in Sect. 3. Sections 4, 5 and 6 are devoted to the proofs of the main theorems.

2 The tangential Cauchy–Riemann equation $\bar{\partial}_b u = \varphi$ on the boundary $b\Omega$

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary $b\Omega$. Let ρ be a smooth defining function for Ω such that $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on $b\Omega = \{z \in \mathbb{C}^2 : \rho(z) = 0\}$, and $\nabla\rho \perp b\Omega$. The pseudoconvexity means

$$\langle\partial\bar{\partial}\rho, L \wedge \bar{L}\rangle \geq 0 \quad \text{on } b\Omega,$$

where L is an any nonzero tangential holomorphic vector field. If the strict inequality holds on the boundary, Ω is called a strongly pseudoconvex domain.

It is well-known that there are some pseudoconvex domains not admitting any holomorphic support function, even of finite type. This phenomenon was established by Kohn and Nirenberg in [20]. Therefore, in this work, we only consider admissible domains enjoying the existence of holomorphic support functions, which were found by Range in [25].

Definition 2.1 Ω is said to be uniformly totally pseudoconvex at the point $P \in b\Omega$ if there are positive constants δ, c and a C^1 map $\Psi : U^\delta \times \Omega^\delta \to \mathbb{C}$ such that for all boundary points $\zeta \in b\Omega \cap B(P, \delta)$, the following properties are satisfied:

(1) $\Psi(\zeta, .)$ is holomorphic on Ω;
(2) $\Psi(\zeta, \zeta) = 0$, and $d_z\Psi|_{z=\zeta} \neq 0$;
(3) $\rho(z) > 0$ for all z with $\Psi(\zeta, z) = 0$ and $0 < |z - \zeta| < c$.
By multiplying ρ and Ψ by suitable non-zero functions of ζ, one may assume more
(4) $|\partial\rho(\zeta)| = 1$, and $\partial\rho(\zeta) = d_z\Psi|_{z=\zeta}$,

where $\Omega^\delta = \{z \in \mathbb{C}^2 : \rho(z) < \delta\}$, and $U^\delta = \Omega^\delta \setminus \Omega$.

Here, $M_\zeta = \{z : \Psi(\zeta, z) = 0\}$ is called the supporting analytic hypersurface for $b\Omega$ at $\zeta \in b\Omega$, i.e., near ζ, $\{z : \rho(z) \leq 0, \Psi(\zeta, z) = 0\} = \{\zeta\}$. The following observation on M_ζ is needed. Let Ω be uniformly totally pseudoconvex at $P \in b\Omega$. For any $\zeta \in b\Omega \cap B(P, \delta)$, we define the map $\psi_\zeta : B(P, \delta) \to \mathbb{C}^2$ by $\psi_\zeta(z) = w = (w_1, \Psi(\zeta, z))$ such that the Jacobian matrix of the map ψ_ζ at ζ is unitary. The existence of such maps is provided in [26]. Hence, after shrinking the neighborhood U of P, we could choose $c > 0$, $d > 0$ sufficiently small such that ψ_ζ maps $B(\zeta, c)$ biholomorphically onto the neighborhood $\psi_\zeta(B(\zeta, c)) \supset B(0, d)$ of 0 in $\mathbb{C}^2$ for all $\zeta \in b\Omega \cap U$. Moreover, the analytic hypersurface $M_\zeta = \{z \in B(\zeta, c) : \Psi(\zeta, z) = 0\}$ is mapped by ψ_ζ biholomorphically into $\{w \in \mathbb{C}^2 : w_2 = 0\}$.

Definition 2.2 Let $F : [0, \infty) \to [0, \infty)$ be a smooth, increasing function such that

(1) $F(0) = 0$;
(2) $\int_0^R |\ln F(r^2)| \mathrm{d}r < \infty$ for some $R > 0$;
(3) $\dfrac{F(r)}{r}$ is increasing.

Let $\Omega \subset \mathbb{C}^2$ be uniformly totally pseudoconvex at $P \in b\Omega$. Ω is called a domain admitting maximal type F at the boundary point $P \in b\Omega$ if there are positive constants c, c' such that for all $\zeta \in b\Omega \cap B(P, c')$, we have

$$\rho(z) \gtrsim F(|z_1 - \zeta_1|^2), \quad \text{for all } z \in B(\zeta, c) \text{ with } \Psi(\zeta, z) = 0.$$

Here and in what follows, the notations $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant, and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

Remark 2.3 (1) The Definition 2.2 is independent of the choice on holomorphic coordinates in a neighborhood of P and of the particular defining function ρ in Definition 2.1.
(2) The domain Ω is called a uniformly totally pseudoconvex domain and admit maximal type F if it has these above properties at every point $P \in b\Omega$, with the common function F. Actually, we could choose the common function F for all boundary points by the compactness of $b\Omega$,

For more discussions of uniformly total pseudoconvexity and its properties, the basic references are [25,30].

Some examples will be provided to show that Definition 2.2 generalizes all uniformly totally pseudoconvex domains of finite type and a class of convex domains of infinite type in the sense of Range.

Example 2.1 (1) Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$ with a strictly plurisubharmonic defining function ρ. We define

$$\Psi(\zeta, z) = \sum_{j=1}^{n} \frac{\partial\rho}{\partial\zeta_j}(z_j - \zeta_j) + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial\zeta_j\partial\zeta_k}(\zeta)(z_j - \zeta_j)(z_k - \zeta_k).$$

Let us define $F(t) = t$, then Ω is in this case uniformly totally pseudoconvex of the maximal type F.

(2) Let $\Omega \subset \mathbb{C}^2$ be pseudoconvex of strict finite type $m(p)$ at every point $p \in b\Omega$ as defined in [19], and generalized by Range [25,26], Shaw [30]. Let $m_0 := \sup_{p\in b\Omega} m(p) < \infty$ and $F(t) = t^{m_0/2}$. We define

$$\Psi(\zeta, z) = \sum_{s+t\le m_0} \frac{1}{s!t!}\frac{\partial^{s+t}\rho}{\partial\zeta_1^s\partial\zeta_2^k}(z_1 - \zeta_1)^s(z_2 - \zeta_2)^k.$$

Then Ω, in this case, is of the maximal type F.

(3) Let us define

$$\Omega^\infty = \left\{(z_1, z_2) \in \mathbb{C}^2 : \exp(1 + 2/s)\cdot\exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\right\}.$$

Then, for $0 < s < 1/2$, Ω^∞ is a convex domain admitting the maximal type $F(t) = \exp(\frac{-1}{32.t^s})$, see [36].

(4) Recently, in [15], the present author et al. have considered a class of smooth, bounded domains Ω with a global defining function ρ such that for any $P \in b\Omega$, there exist a coordinates $z_P = T_P(z)$ with the origin at P where T_P is a linear transformation, and function F_P such that

$$\begin{aligned}\Omega_P = T_P(\Omega) = \{z_P = (z_{P,1}, z_{P,2}) \in \mathbb{C}^2 : \rho(T_P^{-1}(z_P))\\ = F_P(|z_{P,1}|^2) + |z_{P,2} - 1|^2 < 0\}\end{aligned}$$

where $F_P : \mathbb{R} \to \mathbb{R}$ satisfies:

(i) $F_P(0) = 0$;
(ii) $F'_P(t)$, $F''_P(t)$, $F'''_P(t)$ and $(\frac{F_P(t)}{t})'$ are non-negative on $(0, \delta)$;

where d_P is the square of the diameter of Ω_P and δ is a small number.
This class of convex domains includes many examples of finite type as well as infinite type domains. Then, the support function is

$$\Psi(\zeta, z) = \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j - z_j).$$

By the properties of F, we have

$$\rho(z) \ge F(|z_1 - \zeta_1|^2) \quad \text{for } |\zeta_1| \ge |z_1 - \zeta_1|, \text{ with } \Psi(\zeta, z) = 0, \tag{2.1}$$

where $z = (z_1, z_2) \in \Omega$, $\zeta = (\zeta_1, \zeta_2) \in \{z \in \bar{\Omega} : \rho(z) \ge -2\delta\} \cap B(0, \frac{1}{2}\epsilon)$. Therefore, Ω is uniformly totally pseudoconvex of the maximal type F at the boundary point $(0, 0)$.

Let f be an increasing function such that $\lim_{t\to+\infty} f(t) = +\infty$. We define the f-Hölder space on $b\Omega$ by

$$\Lambda^f(b\Omega) = \left\{ u \in L^\infty(b\Omega) : ||u||_{L^\infty} + \sup_{\substack{x(.)\in\mathcal{C} \\ 0\le t\le 1}} f(t^{-1})|u(x(t)) - u(x(0))| < +\infty \right\},$$

where the class of curves $\mathcal{C}$ in $b\Omega$ is

$$\mathcal{C} = \left\{ x(t) : t \in [0,1] \to x(t) \in b\Omega,\ x(t) \text{ is } C^1 \text{ and } |x'(t)| \le 1 \right\}.$$

That means $\Lambda^f(b\Omega)$ consists all complex-valued functions u such that for each curve $x(.) \in \mathcal{C}$, the function $t \mapsto u(x(t)) \in \Lambda^f([0,1])$.

For $1 \le p < \infty$, the f-Besov space is denoted by

$$\Lambda^f_p(b\Omega) = \left\{ u \in L^p(b\Omega) : ||u||_{L^p} \right.$$
$$\left. + \sup_{0\le t\le 1} f(t^{-1}) \left[\left(\int_{b\Omega} |u(x(t)) - u(x(0))|^p \mathrm{d}x \right)^{1/p} \right] < +\infty \right\},$$

where the integral is taken in $x = x(t) \in \mathcal{C}$ over the boundary $b\Omega$. It is obvious that $\Lambda^f_\infty(b\Omega) = \Lambda^f(b\Omega)$. Note that for each $1 \le p \le \infty$, the notion of the f-Besov space $\Lambda^f_p(b\Omega)$ includes the standard Besov space $\Lambda^\alpha_p(b\Omega)$ by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha \le 1$. The boundary regularity in standard Besov spaces for the tangential Cauchy–Riemann equation was obtained by Shaw [30,31].

Now, let $\mathcal{A}_{(0,1)}(b\Omega)$ be the space of restrictions of $(0,1)$-forms in $\mathbb{C}^2$ to $b\Omega$. Let $\mathcal{B}_{(0,1)}(b\Omega)$ be the subspace of $\mathcal{A}_{(0,1)}(b\Omega)$ which is orthogonal to the ideal generated by $\bar\partial\rho$. Let τ be the projection from $\mathcal{A}_{(0,1)}(b\Omega)$ to $\mathcal{B}_{(0,1)}(b\Omega)$.

Let L be the unit holomorphic tangential vector field on $b\Omega$ and ω be its dual. The tangential Cauchy–Riemann equation $\bar\partial_b u = \varphi$, with $\varphi \in \mathcal{B}_{(0,1)}(b\Omega)$, is seeking a function u on $b\Omega$ such that $\bar L u = \phi$ in the sense of distributions, where $\tau(\phi\bar\omega) = \varphi$. In this sense, the tangential Cauchy–Riemann operator could be identified by $\bar L$. We refer the reader to Chen–Shaw's book [6] for a general theory of $\bar\partial_b$.

In [11], the present author has proved the global solvability for the tangential Cauchy–Riemann equations on the boundary $b\Omega$ in L^p-spaces.

Theorem 2.4 *Let Ω be a smooth bounded, uniformly totally pseudoconvex domain and admit maximal type F at all boundary points for some function F. Assume that $\bar\Omega$ has a Stein neighborhood basis. Let $\varphi \in L^p_{(0,1)}(b\Omega)$, $1 \le p \le \infty$ and φ satisfies the compatibility condition*

$$\int_{b\Omega} \varphi \wedge \alpha = 0,$$

for every $\bar\partial$-closed $(2,0)$-form α defined continuously up to $b\Omega$.

Let F^* be the inversion of F, and let

$$f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} \mathrm{d}t \right)^{-1}.$$

Then, there exists a function u defined on $b\Omega$ such that $\bar\partial_b u = \varphi$ on $b\Omega$, and

(1) $||u||_{\Lambda^f(b\Omega)} \leq C||\varphi||_{L^\infty_{(0,1)}(b\Omega)}$, *if* $p = \infty$;
(2) $||u||_{L^p(b\Omega)} \leq C_p||\varphi||_{L^p_{(0,1)}(b\Omega)}$, *if* $1 \leq p < \infty$, *where* $C_p > 0$ *independent on* φ;
(3) $||u||_{\Lambda^f_p(b\Omega)} \leq C_p||\varphi||_{L^p_{(0,1)}(b\Omega)}$, *for every* $1 \leq p \leq \infty$.

This result is applied to prove Theorems 1.1 and 1.2.

3 Lelong's theory

3.1 Cohomology groups

We briefly recall the definitions of the DeRham cohomology and the Čeck cohomology groups on Ω, see the Range's fundamental book [27] for more details.

Definition 3.1 The space of d-closed 2-forms on Ω is

$$Z_2(\Omega) = \{\omega \in C^\infty_2(\Omega) : \mathrm{d}\omega = 0\}$$

and the space of d-exact forms $B_2(\Omega) = dC^\infty_1(\Omega)$. Then, the quotient space

$$H(\Omega, \mathbb{R}) := \frac{Z_2(\Omega)}{B_2(\Omega)}$$

is called the DeRham cohomology group of the second degree on Ω. This space measures the obstruction to the solvability of the d-equation on Ω.

Let $\mathcal{U} = \{U_j;\ j \in J\}$ be an open cover of Ω. A 2-cochain f for $\mathcal{U}$ with integer coefficients is a map f which assigns to each 3-tuple $(j_0, j_1, j_2) \in J^3$ with

$$U(j_0, j_1, j_2) = U_{j_0} \cap U_{j_1} \cap U_{j_2} \neq \emptyset$$

a section

$$f(j_0, j_1, j_2) \in \Gamma(U(j_0, j_1, j_2), \mathbb{Z}),$$

where $\Gamma(U(j_0, j_1, j_2), \mathbb{Z})$ is the collection of all sections of $\mathbb{Z}$ over $U(j_0, j_1, j_2)$.

The set of all 2-cochains for $\mathcal{U}$ with integer coefficients is denoted by $C^2(\mathcal{U}, \mathbb{Z})$. This is an abelian group. The set $C^1(\mathcal{U}, \mathbb{Z})$, $C^3(\mathcal{U}, \mathbb{Z})$ and $C^4(\mathcal{U}, \mathbb{Z})$ are also defined similarly.

The coboundary map $\delta_2 : C^2(\mathcal{U}, \mathbb{Z}) \to C^3(\mathcal{U}, \mathbb{Z})$ is defined by

$$(\delta_2 f)(j_0, j_1, j_2, j_3) = \sum_{k=0}^{3} (-1)^k f(j_0, \ldots, \widehat{j_k}, \ldots, j_3)|_{U(j_0, j_1, j_2, j_3)},$$

where $\widehat{j_k}$ denotes the omission of the index j_k. We also have the similar definitions for δ_1, δ_3. We could verify straightforward that $\delta \circ \delta = 0$, where δ is one of δ_1, δ_2 or δ_3.

The kernel of δ_2 is called the group $Z^2(\mathcal{U}, \mathbb{Z})$, and the image of δ_1 in $C^2(\mathcal{U}, \mathbb{Z})$ is called the group $B^2(\mathcal{U}, \mathbb{Z})$.

Definition 3.2 The Čech cohomology group of the second degree of $\mathcal{U}$ with integer coefficients is

$$H^2(\mathcal{U}, \mathbb{Z}) := \frac{Z^2(\mathcal{U}, \mathbb{Z})}{B^2(\mathcal{U}, \mathbb{Z})}.$$

The direct limit

$$H^2(\Omega,\mathbb{Z}) := \lim_{\overrightarrow{\mathcal{U}}} H^2(\mathcal{U},\mathbb{Z})$$

is the set of all equivalence classes in the disjoint union $\bigcup_{\mathcal{U}} H^2(\mathcal{U},\mathbb{Z})$ over all open covers $\mathcal{U}$ of Ω. This abelian group is called the Čech cohomology group of the second degree on Ω with integer coefficients.

Definition 3.3 Let Ω be a bounded domain in $\mathbb{C}^2$. For each holomorphic function $\mathfrak{g}$ on Ω, the zero set $Z(\Omega,\mathfrak{g})$ of $\mathfrak{g}$ on Ω is given by

$$Z(\Omega,\mathfrak{g}) = \{(z_1,z_2)\in\Omega : \mathfrak{g}(z_1,z_2)=0\}.$$

The zero set in the above definition is a one complex dimensional analytic subvariety of Ω.

The following theorem is a fundamental result in the theory of several complex variables.

Theorem 3.4 (Cartan) *If the cohomology group $H^2(\Omega,\mathbb{Z}) = 0$, and M is a complex one-dimensional analytic subvariety of Ω, then*

$$M = Z(\Omega,\mathfrak{g})$$

for some holomorphic function $\mathfrak{g}$ defined on Ω.

3.2 Currents

Definition 3.5 We denote $\mathcal{D}_{(p,q)}(\Omega)$ be the space $C^\infty_{(p,q)}(\Omega)$ with Schwarz topology. Any continuous linear functional on the space $\mathcal{D}_{(p,q)}(\Omega)$ is called a current of bi-degree $(n-p, n-q)$ (or bi-dimension (p,q)) in Ω.

We equip the space of currents of bi-degree $(n-p,n-q)$ with a weak-topology as follows: a sequence T_j of currents of bi-degree $(n-p,n-q)$ converges to T if and only if $\lim_{j\to\infty} T_j(\phi) = T(\phi)$ for any $\phi\in\mathcal{D}_{(p,q)}(\Omega)$.

Let T be a current of bi-degree (p,p) in Ω. If we have

$$(T,\omega)\geq 0,$$

for any simple positive test form $\omega = i^p\omega_1\wedge\overline{\omega}_1\wedge\cdots\wedge\omega_p\wedge\overline{\omega}_p$, with ω_k's $\in C^\infty_{(1,0)}$, then T is called a positive current.

In particular, a $(1,1)$-current T is positive if for every compactly support $C^\infty_{(0,1)}$-form ω, we have

$$\int_\Omega T\wedge\left(\frac{\omega\wedge\bar{\omega}}{i}\right)\geq 0.$$

Note that if $T = \sum_{i,j=1}^2 T_{ij}\mathrm{d}z_i\wedge\mathrm{d}\bar{z}_j$ is a positive $(1,1)$-current, then $T_{ij} = -T_{ji}$, i.e., $T=\bar{T}$, and all coefficients are locally finite Borel measures. A positive and d-closed $(1,1)$-current is called a Lelong current. By Henkin's result [14], if T is a Lelong $(1,1)$-current, then

$$\int_\Omega |T(z)\wedge\partial\rho(z)\wedge\bar{\partial}\rho(z)|\mathrm{d}V(z)<\infty$$

and

$$\int_\Omega ||\rho(z)|^{1/2}T(z)\wedge\partial\rho(z)|\mathrm{d}V(z)+\int_\Omega ||\rho(z)|^{1/2}T(z)\wedge\bar{\partial}\rho(z)|\mathrm{d}V(z)<\infty.$$

For an increasing ordered multi-index J, we denote by J' the unique increasing multi-index such that $J \cup J' = \{1, 2, \ldots, n\}$ and $|J| + |J'| = n$. Let us denote by α_{JK} the form complementary to $\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K$, that is

$$\alpha_{JK} = \lambda \mathrm{d}z_{J'} \wedge \mathrm{d}\bar{z}_{K'},$$

where λ is chosen so that $\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K \wedge \alpha_{JK}$ equals to the volume form β_n in $\mathbb{C}^n$.

We could identify a current $T \in \mathcal{D}'_{(p,q)}(\Omega)$ with a $(n-p, n-q)$-form which has distributional coefficients, i.e.,

$$T = \sum_{|J|=n-p, |K|=n-q}' T_{JK} \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K.$$

The coefficients T_{JK} are defined by

$$(T_{JK}, \phi) = (T, \phi\alpha_{JK}).$$

Moreover, all T_{JK} are non-negative Radon measures if T is positive. For a current T with measure coefficients, we define

$$||T||_E = \sum_{|J|=n-p, |K|=n-q}' |T_{JK}|_E \quad \text{the norm of } T,$$

where $|T_{JK}|_E$ is the total variation of T_{JK} on a compact set E. We also define the wedge product of a current and a smooth form ω by setting

$$(T \wedge \omega, \phi) := (T, \omega \wedge \phi)$$

for any test form ϕ. If T is positive and ω is a positive $(1, 1)$-form, then $T \wedge \omega$ is positive as well. In particular, for a positive (p, p)-current T, and a $(n-p, n-p)$ simple form, the current $T \wedge \omega$ is a non-negative Borel measure. We differentiate currents according to the formula

$$(\mathrm{D}T, \phi) = -(T, \mathrm{D}\phi),$$

for a first order differential operator D.

3.3 Divisors

Definition 3.6 Let $M := \{M_j\}$ be a locally finite family of hypersurfaces of Ω. The formal sum

$$\sum_j a_j M_j,$$

with $a_j \in \mathbb{Z}$, is called a divisor of Ω. For a given divisor M of Ω, there are uniquely distinct irreducible hypersurfaces $\{M_j\}$ of Ω and $a_j \in \mathbb{Z}\backslash\{0\}$ such that we have the following irreducible decomposition

$$M = \sum_{a_j \neq 0} a_j M_j.$$

If $M = \sum_{a_j \neq 0} a_j M_j$ with $a_j > 0$ for all j, we call M to be a positive divisor of Ω, and write $M > 0$.

For example, let h be a holomorphic function on Ω. Then, the hypersurface $M_h := \{z \in \Omega : h = 0\}$ is a positive divisor, and

$$M_h = \sum_{a_j \neq 0} a_j M_j,$$

where $a_j > 0$ is the zero order of h on M_j. In this case, M_h is also called the zero divisor of Ω.

Conversely, for any positive divisor $M = \sum_{a_j \neq 0} a_j M_j$ of Ω, the vanishing of the second Čech cohomology group $H^2(\Omega, \mathbb{Z})$ induces the existence of a holomorphic function h on Ω such that $h = 0$ of order a_j on M_j, and $h(z) \neq 0$ for $z \notin M$. This is a consequence of Theorem 3.4.

More generally, a meromorphic function h on Ω is locally expressed by the ratio $h = h_1/h_2$ of two holomorphic functions h_1, h_2 with $h_2 \neq 0$. By this property, the zero hypersurface M_h is locally expressed by

$$M_h = (M_h)_0 + (M_h)_\infty := \sum_{a_j > 0} a_j M_j + \sum_{a_j < 0} a_j M_j,$$

where $(M_h)_0$ is called the zero divisor of Ω and $(M_h)_\infty$ is called the polar divisor of Ω associated to h.

The following theorem asserts that every divisor M_h locally associates to a closed (1, 1) positive current on Ω.

Theorem 3.7 (Poincaré–Lelong Formula [24]) *Let h be a non-zero, meromorphic function on Ω and let η be a 2-form of C^2 class on Ω with compact support. Then,*

$$\frac{1}{2\pi} \partial \bar{\partial} [\log |h|^2] = M_h,$$

that is

$$\int_{M_h} \eta = \frac{1}{2\pi} \int_\Omega \log |h|^2 \partial \bar{\partial} \eta = \frac{1}{2\pi} \int_\Omega \partial \bar{\partial} [\log |h|^2] \wedge \eta$$

in this sense of currents.

The following definitions and their properties could be found in [24,33].

Definition 3.8 Let $M = \sum_{a_j \neq 0} a_j M_j$ be a divisor of Ω and $d\delta$ be the surface measure on M. Then, M is said to have finite area if

$$\sum_{a_j \neq 0} a_j \int_{z \in M_j} \mathrm{d}\delta(z)$$

is finite. M is said to satisfy the Blaschke condition if

$$\sum_{a_j \neq 0} a_j \int_{z \in M_j} |\rho(z)| \mathrm{d}\delta(z)$$

is finite.

Definition 3.9 Let $\mathfrak{g}$ be a holomorphic function on Ω. Then $\mathfrak{g}$ is called a Nevanlinna holomorphic function on Ω if

$$\limsup_{\epsilon \to 0^+} \int_{b\Omega_\epsilon} \log^+ |\mathfrak{g}(z)| \mathrm{d}S_\epsilon(z)$$

is finite, where $\log^+ |\mathfrak{g}(z)| := \max\{\log |\mathfrak{g}(z)|, 0\}$. Here, for $\epsilon > 0$ small, $\Omega_\epsilon := \{z \in \Omega : \rho(z) < -\epsilon\}$, and dS_ϵ is the Lebesgue measure of $b\Omega_\epsilon$. The Nevanlinna class on Ω denoted by $\mathcal{N}(\Omega)$ is the collection of all Nevanlinna holomorphic functions on Ω.

Definition 3.10 A meromorphic function $\mathfrak{g}$ on Ω is said to belong to $\mathcal{N}(\Omega)$ if

$$\limsup_{\epsilon \to 0^+} \int_{b\Omega_\epsilon} \log^+ |\mathfrak{g}(z)| \mathrm{d}S_\epsilon(z)$$

is finite and the pole divisor of Ω associated to $\mathfrak{g}$ satisfying the Blaschke condition. In other words, let $\mathfrak{g} = \frac{\mathfrak{g}_1}{\mathfrak{g}_2}$ for two holomorphic functions $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_2 \neq 0$. The second condition means that we have $\int_\Omega (\partial\bar{\partial}|\mathfrak{g}_2|^2)(z)|\rho(z)|\mathrm{d}V(z)$ is finite by the Poincaré-Lelong Formula.

Theorem 3.11 (Henkin–Skoda Theorem) *Let Ω be a smooth bounded domain in $\mathbb{C}^n$, for $n \geq 2$. Let $\mathfrak{g}$ be a Nevanlinna holomorphic function on Ω, then the zero divisor $M_\mathfrak{g}$ of $\mathfrak{g}$ satisfies the Blaschke condition.*

Moreover, if Ω is strongly pseudoconvex, and M is a positive divisor of Ω and satisfies the Blaschke condition on Ω, then there exists a holomorphic function $\mathfrak{h} \in \mathcal{N}(\Omega)$ such that

$$Z(\Omega, \mathfrak{h}) = M.$$

4 Proof of Theorem 1.1

In this section, by applying Theorem 2.4, we prove the boundary L^p estimates in Theorem 1.1. The center of the proof is based on the construction of the $\bar{\partial}$-solution by Henkin–Skoda and Range (see [11,12,15,26,27,31,33] for more details).

Lemma 4.1 *Let Ω be a smooth bounded, uniformly totally pseudoconvex domain in $\mathbb{C}^2$. Assume that $\bar{\Omega}$ has a Stein neighborhood basis. Then there exists a C^1-function $\Phi(\zeta, z)$ on $U^\delta \times \Omega^\delta$, which is holomorphic in $z \in \Omega^\delta$ and satisfies*

(1) $\Phi(\zeta, \zeta) = 0$;
(2) $|\Phi(\zeta, z)| \geq A > 0$, *for all* $|\zeta - z| \geq c$;
(3) $\Phi(\zeta, z) = H(\zeta, z)\Psi(\zeta, z)$, *for all* $|\zeta - z| < c$;

where H is a C^1-function with $0 < A_0 \leq |H| \leq A_1 < \infty$.

This is a consequence of the fact that $\bar{\Omega}$ has a Stein neighborhood basis, see [26]. Recently, in [35], Straube has obtained the global Sobolev regularity of the $\bar{\partial}$-Neumann problem in a class of smooth bounded pseudoconvex domains admitting good Stein neighborhood bases. The global regularity does not hold if we merely assume the existence of a standard Stein neighborhood basis. The next lemma is the key in our analysis.

Lemma 4.2 *Let $\Omega \subset \mathbb{C}^2$ be a smooth bounded, uniformly totally pseudoconvex domain and admit maximal type F at $P \in b\Omega$. Assume that $\bar{\Omega}$ has a Stein neighborhood basis. Then* *there is a positive constant c such that the support function* $\Phi(\zeta, z)$ *satisfies the following estimate*

$$|\Phi(\zeta, z)| \gtrsim |\rho(z)| + |\operatorname{Im}\Phi(\zeta, z)| + F(|z-\zeta|^2), \tag{4.1}$$

for every $\zeta \in b\Omega \cap B(P, c)$*, and* $z \in \overline{\Omega}$*,* $|z-\zeta| < c$*.*

By Hefer's Theorem in [12], we obtain the following representation

$$\Phi(\zeta, z) = \langle P(\zeta, z), \zeta - z\rangle,$$

where $P(\zeta, z) = (p_1(\zeta, z), p_2(\zeta, z))$, and each p_j is C^1 in ζ and holomorphic in z. Here $P(\zeta, z)$ is called a Leray map which is holomorphic in z.

To construct the Henkin solution for the $\bar\partial$-equation, we recall the Bochner–Martinelli kernel for (0, 1)-forms to be

$$B(\zeta, z) = -\frac{1}{4\pi^2}\frac{(\bar\zeta_1 - \bar z_1)d\bar\zeta_2 - (\bar\zeta_2 - \bar z_2)d\bar\zeta_1}{|\zeta - z|^4},$$

and

$$L(\zeta, z) = -\frac{1}{4\pi^2}\frac{p_1(\zeta, z)\bar\partial_{\zeta,z}p_2(\zeta, z) - p_2(\zeta, z)\bar\partial_{\zeta,z}p_1(\zeta, z)}{\langle P(\zeta, z), \zeta - z\rangle^2},$$

and

$$\begin{aligned} R(\zeta, z, \lambda) &= -\frac{1}{4\pi^2}\big[\eta_1(\zeta, z, \lambda) \wedge (\bar\partial_{\zeta,z} + d_\lambda)\eta_2(\zeta, z, \lambda) \\ &\quad -\eta_2(\zeta, z, \lambda) \wedge (\bar\partial_{\zeta,z} + d_\lambda)\eta_1(\zeta, z, \lambda)\big], \end{aligned}$$

where

$$\eta_j(\zeta, z, \lambda) = \lambda\frac{\bar\zeta_j - \bar z_j}{|\zeta - z|^2} + (1-\lambda)\frac{p_j(\zeta, z)}{\langle P(\zeta, z), \zeta - z\rangle}, \quad \text{for } j = 1, 2 \text{ and } \lambda \in [0, 1].$$

The Bochner–Martinelli–Koppelman operators acting on $\varphi \in C^1_{(0,1)}(\bar\Omega)$ are

$$\begin{aligned} B_\Omega\varphi(z) &= \int_\Omega \varphi(\zeta) \wedge B(\zeta, z) \wedge d\zeta_1 \wedge d\zeta_2, \\ R_{b\Omega}\varphi(z) &= \int_{b\Omega}\int_0^1 \varphi(\zeta) \wedge R(\zeta, z, \lambda) \wedge d\zeta_1 \wedge d\zeta_2 \\ &= \int_{b\Omega} \varphi(\zeta) \wedge K(\zeta, z) \wedge d\zeta_1 \wedge d\zeta_2, \end{aligned} \tag{4.2}$$

for $z \in \Omega$, and where

$$K(\zeta, z) = -\frac{1}{4\pi^2}\frac{p_1(\zeta, z)(\bar\zeta_2 - \bar z_2) - p_2(\zeta, z)(\bar\zeta_1 - \bar z_1)}{\Phi(\zeta, z)|\zeta - z|^2}.$$

Lemma 4.3 (Henkin–Skoda Theorem) *Let* $\varphi \in C_{(0,1)}(\overline{\Omega})$*. Then, for* $z \in \Omega$*,*

$$u(z) = B_\Omega\varphi(z) + R_{b\Omega}\varphi(z)$$

is a solution of the equation $\bar\partial u = \varphi$ *on* Ω*. This solution is called the Henkin solution of the* $\bar\partial$*-equation.*

Proof of Theorem 1.1 Part 1: The existence in $\Lambda^f(\Omega)$.

For any f such that $0 < f(d^{-1}) < d^{-1}$, by Lemma 1.15 in [27], we always have

$$||B_\Omega\varphi||_{L^\infty(\Omega)} \lesssim ||\varphi||_{L^\infty(\Omega)} \quad \text{and} \quad ||B_\Omega\varphi||_{\Lambda^f(\Omega)} \lesssim ||\varphi||_{L^\infty(\Omega)}. \tag{4.3}$$

Hence, we only concentrate on the boundary term $R_{b\Omega}\varphi$. It is necessary to recall the General Hardy-Littlewood Lemma proved by Khanh [18]. □

Lemma 4.4 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$ and let $\delta_{b\Omega}(x)$ denote the distance function from x to the boundary $b\Omega$ of Ω. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\frac{G(t)}{t}$ is decreasing and the integral $\int_0^d \frac{G(t)}{t}\mathrm{d}t$ is finite for some sufficiently small $d > 0$. If $u \in C^1(\Omega)$ such that*

$$|\nabla u(x)| \lesssim \frac{G(\delta_{b\Omega})(x)}{\delta_{b\Omega}(x)} \quad \textit{for every}\, x \in \Omega, \tag{4.4}$$

then $u \in \Lambda^f(\Omega)$ in which $f(d^{-1}) := \left(\int_0^d \frac{G(t)}{t}\mathrm{d}t\right)^{-1}$.

By (4.2) and the calculus quotient rule, we have

$$\begin{aligned}|\nabla_z R_{b\Omega}\varphi(z)| &\leq ||\varphi||_{L^\infty}.\int_{b\Omega}|\nabla_z K(\zeta,z)|\mathrm{d}\sigma(\zeta)\\ &\lesssim ||\varphi||_{L^\infty}.\int_{b\Omega}\left(\frac{1}{|\Phi(\zeta,z)|.|\zeta-z|^2}+\frac{1}{|\Phi(\zeta,z)|^2.|\zeta-z|}\right)\mathrm{d}\sigma(\zeta).\end{aligned} \tag{4.5}$$

Now, for each fixed $z \in \Omega$, by the condition (2) in Lemma 4.1, it is enough to consider the integral (4.5) over $b\Omega \cap B(z,c)$. For convenience, we put

$$I_1(z) := \int_{b\Omega\cap B(z,c)} \frac{1}{|\Phi(\zeta,z)|.|\zeta-z|^2}\mathrm{d}\sigma(\zeta)$$

and

$$I_2(z) := \int_{b\Omega\cap B(z,c)} \frac{1}{|\Phi(\zeta,z)|^2.|\zeta-z|}\mathrm{d}\sigma(\zeta).$$

To estimate these integrals, we recall a real coordinate system $t = (t', t_3) = (t_1, t_2, t_2)$ introduced by Henkin, where

$$\begin{cases} t_1 = \mathrm{Re}\,(\zeta_1 - z_1),\\ t_2 = \mathrm{Im}\,(\zeta_1 - z_1),\\ t_3 = \mathrm{Im}\;\Phi(\zeta,z).\end{cases}$$

Since $|\zeta - z| \geq |t'| + |\rho(z)|$, we have

$$I_1(z) \lesssim \int_{|t|\leq R, t_3\geq 0} \frac{1}{(|\rho(z)| + t_3 + F(|t'|^2)).(|t'| + |\rho(z)|)^2}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3$$

and

$$I_2(z) \lesssim \int_{|t|\leq R, t_3\geq 0} \frac{1}{(|\rho(z)| + t_3 + F(|t'|^2))^2.|t'|}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3'.$$

Since $|\rho(z)| \approx \delta_{b\Omega}(z)$, after some simple calculations, we obtain

$$I_1(z) \lesssim |\ln(|\rho(z)|)|^2 \lesssim \frac{G(\delta_{b\Omega})(z)}{\delta_{b\Omega},(z)} \tag{4.6}$$

for any G satisfying Lemma 4.4.

Moreover, we also have

$$\begin{aligned} I_2(z) &\lesssim \int_0^R \frac{1}{|\rho(z)| + F(r^2)} \mathrm{d}r \\ &= \int_0^{\sqrt{F^*(|\rho(z)|)}} \frac{1}{|\rho(z)| + F(r^2)} \mathrm{d}r \\ &\quad + \int_{\sqrt{F^*(|\rho(z)|)}}^R \frac{1}{|\rho(z)| + F(r^2)} \mathrm{d}r, \end{aligned} \tag{4.7}$$

where F^* is the inversion of F.

The hypothesis that $\frac{F(r)}{r}$ is increasing implies

$$\frac{F(r^2)}{|\rho(z)|} \geq \frac{r^2}{F^*(|\rho(z)|)} \quad \text{for all } r \geq \sqrt{F^*(|\rho(z)|)},$$

and so

$$\int_{\sqrt{F^*(|\rho(z)|)}}^R \frac{1}{|\rho(z)| + F(r^2)} \mathrm{d}r \leq \frac{\pi}{4} \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

It is easy to see that

$$\int_0^{\sqrt{F^*(|\rho(z)|)}} \frac{1}{|\rho(z)| + F(r^2)} \mathrm{d}r \leq \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|},$$

and then we obtain

$$I_2(z) \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

The last step in this proof is to check the function $G(t) := \sqrt{F^*(t)}$ satisfies all conditions in Lemma 4.4. Then, by (4.3), we have

$$I_1(z) + I_2(z) \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|},$$

and by (4.6), $u \in \Lambda^f(\Omega)$ in which $f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} \mathrm{d}t\right)^{-1}$, for small $d > 0$.

Now, since $\sqrt{F^*(t)}$ is increasing and $\frac{\sqrt{F^*(t)}}{t}$ is decreasing, for some small $R > 0$, $|\ln(F(t^2))|$ is decreasing for all $0 \leq t \leq R$. Thus, by the hypothesis (2) of F, we have

$$|\ln F(\eta^2)|\eta \leq \int_0^\eta |\ln F(t^2)| \mathrm{d}t \leq \int_0^R |\ln F(t^2)| \mathrm{d}t < \infty,$$

for all $0 \leq \eta \leq R$. As a consequence, $\sqrt{F^*(t)}|\ln t|$ is finite for all $0 \leq t \leq \sqrt{F^*(R)}$ and $\lim_{t\to 0} t|\ln F(t^2)|$ is zero. These facts, and the second hypothesis of F imply

$$\int_0^d \frac{\sqrt{F^*(t)}}{t}\mathrm{d}t = \int_0^{\sqrt{F^*(d)}} y(\ln F(y^2))'\mathrm{d}y = \sqrt{F^*(d)}\ln d - \int_0^{\sqrt{F^*(d)}} (\ln F(y^2))\mathrm{d}y < \infty,$$

for $d > 0$ small enough.

This completes the proof of the first part.

Part 2: The estimates (i), (ii), (iii).

By Lemma 4.3, to prove the estimates in Theorem 1.1, we estimate $B_\Omega\varphi$ and $R_{b\Omega}\varphi$.

For the interior term $B_\Omega\varphi$.

Applying the following basic estimate

$$|B(\zeta, z)| \lesssim \frac{1}{|\zeta - z|^3},$$

the operator $B_\Omega\varphi$ is bounded from $L^1(\Omega) \to L^{\frac{4}{3}-\epsilon}(\Omega)$ for all small $\epsilon > 0$. Hence, for $\epsilon = 1/3$, in particular, we have

$$||B_\Omega\varphi||_{L^1(\Omega)} \lesssim ||\varphi||_{L^1_{(0,1)}(\Omega)}.$$

For the boundary term $R_{b\Omega}\varphi$.

We know that for each fixed ζ, the set of singularities of the kernel $K(\zeta, z)$ is the surface $\{z = \zeta\}$. Hence, for any ball $B(\zeta, \epsilon)$ centered at ζ, with radius ϵ, the following estimate

$$\int_{\Omega\setminus B(\zeta,\epsilon)} |K(\zeta, z)|\mathrm{d}V(z) \lesssim \int_{\Omega\setminus B(\zeta,\epsilon)} \frac{\mathrm{d}V(z)}{|\Phi(\zeta, z)|\cdot|\zeta - z|} \lesssim 1 \tag{4.8}$$

holds uniformly in $\zeta \in b\Omega$.

Therefore, the problematic point is to estimate the integral on the ball $B(\zeta, \epsilon)$ containing the singularities of $K(\zeta, z)$. Again, applying the Henkin setting up above, we recall a special real coordinate chart $(t', t_3, y) = (t_1, t_2, t_3, y)$ such that

$$\begin{cases} y &= |\rho(z)| \\ t_1 &= \mathrm{Re}(z_1 - \zeta_1) \\ t_2 &= \mathrm{Im}(z_1 - \zeta_1) \\ t_3 &= |\,\mathrm{Im}(\Phi(\zeta, z))|. \end{cases}$$

Thus, in this special coordinate chart, it follows from Lemma 4.2 that

$$|\Phi(\zeta, z)| \gtrsim y + t_3 + F(|t'|^2). \tag{4.9}$$

Then, for a sufficient large $R > 0$, we obtain

$$\begin{aligned}\int_{\Omega\cap B(\zeta,\epsilon)} |K(\zeta, z)|\mathrm{d}V(z) &\leq \int_{\Omega\cap B(\zeta,\epsilon)} \frac{\mathrm{d}V(z)}{|\Phi(\zeta, z)|\cdot|\zeta_1 - z_1|} \\ &\lesssim \int_{|(t,y)|\leq R} \frac{1}{(y + t_3 + F(|t'|^2))|t'|}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 dy \\ &\lesssim \int_{|t|\leq R} \frac{1}{(t_3 + F(|t'|^2))|t'|}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 \\ &\lesssim \int_{|t'|\leq R} \frac{\ln F(|t'|^2)}{|t'|}\mathrm{d}t_1\mathrm{d}t_2. \end{aligned} \tag{4.10}$$

Using the polar coordinates $(t_1, t_2) = r(\cos\theta, \sin\theta)$, we have

$$\int_{\Omega\cap B(\zeta,\epsilon)} |K(\zeta, z)|\mathrm{d}V(z) \lesssim \int_0^R \ln F(r^2)\mathrm{d}r \le C < \infty \tag{4.11}$$

uniformly in $\zeta \in b\Omega$.

Now, (4.8) and (4.11) imply

$$\begin{aligned} ||R_{b\Omega}\varphi||_{L^1(\Omega)} &\le \int_\Omega \int_{b\Omega} |K(\zeta, z)||\varphi(\zeta)|dS(\zeta)\mathrm{d}V(z) \\ &\le \int_{b\Omega} \left(\int_\Omega |K(\zeta, z)|\mathrm{d}V(z)|\varphi(\zeta)| \right) dS(\zeta) \\ &\lesssim \int_{b\Omega} |\varphi(\zeta)|\mathrm{d}S(\zeta) \\ &\lesssim ||\varphi||_{L^1(b\Omega)}. \end{aligned} \tag{4.12}$$

Finally, we have the first inequality

$$||u||_{L^1(\Omega)} \lesssim ||\varphi||_{L^1(\Omega)} + ||\varphi||_{L^1(b\Omega)}. \tag{4.13}$$

To estimate the boundary norms of u in (ii) and (iii), we convert the interior term $B_\Omega(\varphi)$ into a suitable boundary manner. This manner was introduced by Shaw in [31]. Let us define the following kernel

$$R^*(\zeta, z, \lambda) = R(z, \zeta, \lambda). \tag{4.14}$$

This kernel is well-defined on $(\zeta, z) \in \Omega \times U^\delta$. Then, we have

Lemma 4.5 ([31], page 414) *For $z \in b\Omega$, we have*

$$u(z) = R_{b\Omega}\varphi(z) - R^*_{b\Omega}\varphi(z),$$

where

$$R^*_{b\Omega}\varphi(z) = \int_{b\Omega} \int_0^1 \varphi(\zeta) \wedge R^*(\zeta, z, \lambda) \wedge \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2.$$

Now, for $z \in b\Omega$, let $\varphi(z) = \varphi_t(z) + \varphi_n(z)$, where φ_t defined on $b\Omega$ is the tangential part of φ, which is orthogonal to $\bar\partial\rho$, and $\varphi_n(z) = g(z)\bar\partial\rho(z)$ is the corresponding normal part, for a function g defined on $b\Omega$. And since $d\rho \perp b\Omega$, we have

$$\begin{aligned} R_{b\Omega}\varphi_n(z) &= \int_{b\Omega} g(\zeta)\bar\partial\rho(\zeta) \wedge K(\zeta, z) \wedge \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2 \\ &= \int_{b\Omega} g(\zeta)d\rho(\zeta) \wedge K(\zeta, z) \wedge \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2 \\ &= 0. \end{aligned} \tag{4.15}$$

That is $R_{b\Omega}\varphi(z) = R_{b\Omega}\varphi_t(z)$ for all $z \in b\Omega$. Similarly, we obtain $R^*_{b\Omega}\varphi(z) = R^*_{b\Omega}\varphi_t(z)$ for all $z \in b\Omega$.

Therefore, we have

$$u(z) = R_{b\Omega}\varphi_t(z) - R^*_{b\Omega}\varphi_t(z), \quad \text{for } z \in b\Omega, \tag{4.16}$$

where the right-hand side only depends on the tangential part of φ on the boundary $b\Omega$.

The right-hand side in (4.16) agrees with the term after the operator $\bar{\partial}_b$ in the formula (3.8) of Lemma 3.6 in [11]. That means u given by (4.16) solves the tangential Cauchy–Riemann

$$\bar{\partial}_b u = \varphi_t$$

on the boundary $b\Omega$.

Therefore, using the estimates (1), (2) and (3) in Theorem 2.4, we obtain (i) and (ii) in Theorem 1.1.

Hence, the first main theorem is completely proved.

5 Proof of Theorem 1.2

Solving the Poincaré–Lelong equation $i\partial\bar{\partial}u = \alpha$ is based on solutions to the d-equations on star-shaped domains and Theorem 1.1. Hence, we first assume that Ω is a star-shaped domain and contains the origin.

Let $\mathcal{K}$ be the Poincaré–Cartan homotopy operator defined in [7, page 36]. Let $\alpha = \sum_{ij} \alpha_{ij} dz_i \wedge d\bar{z}_j$ be a positive, smooth (1, 1)-form on Ω such that $d\alpha = 0$, then

$$\mathcal{K}\alpha(z) = \sum_j \left(\sum_i \int_0^1 t\alpha_{ij}(tz) \mathrm{d}tz_i \right) d\bar{z}_j - \sum_i \left(\sum_j \int_0^1 t\alpha_{ij}(tz) \mathrm{d}t\bar{z}_j \right) dz_i. \quad (5.1)$$

By Proposition 2.13.2 in [7], we have

$$d\mathcal{K}\alpha(z) = \alpha(z).$$

Because of the positivity of α, we obtain

$$\mathcal{K}\alpha(z) = \sum_j \left(\sum_i \int_0^1 t\alpha_{ij}(tz) \mathrm{d}tz_i \right) \mathrm{d}\bar{z}_j - \overline{\sum_j \left(\sum_i \int_0^1 t\alpha_{ij}(tz) \mathrm{d}tz_i \right) \mathrm{d}\bar{z}_j}. \quad (5.2)$$

In short, $\mathcal{K}\alpha(z) = \mathcal{F}(z) + \overline{\mathcal{F}(z)}$, where

$$\mathcal{F}(z) = \sum_j \left(\sum_i \int_0^1 t\alpha_{ij}(tz) \mathrm{d}tz_i \right) \mathrm{d}\bar{z}_j.$$

Moreover, as a consequence of the d-closed property of α,

$$\bar{\partial}\mathcal{F} = \partial\mathcal{F} = 0. \quad (5.3)$$

By a changing coordinates $b\Omega \times [0, 1] \to \Omega$, we also obtain

$$||\mathcal{F}||_{L^1(b\Omega)} \lesssim ||\alpha||_{L^1(\Omega)} \quad \text{and} \quad ||\mathcal{F}||_{L^1(\Omega)} \leq ||\alpha||_{L^1(\Omega)}. \quad (5.4)$$

Applying the estimates (5.3), (5.4) and the existence in Theorem 1.1, there is a function $v \in L^1(\bar{\Omega})$ solving the equation $\bar{\partial}v = \mathcal{F}$ on $\bar{\Omega}$, and satisfying

$$\begin{aligned} ||v||_{L^1(\Omega)} + ||v||_{L^1(b\Omega)} &\lesssim ||\mathcal{F}||_{L^1(\Omega)} + ||\mathcal{F}||_{L^1(b\Omega)} \\ &\lesssim ||\alpha||_{L^1(\Omega)}. \end{aligned} \quad (5.5)$$

Now, we define $u = \frac{v-\bar{v}}{i}$, then $u = \bar{u}$, and

$$||u||_{L^1(b\Omega)} + ||u||_{L^1(\Omega)} \lesssim ||\alpha||_{L^1(\Omega)},$$

and

$$\begin{aligned}\alpha &= d(\mathcal{K}\alpha) = \partial\mathcal{F} + \bar{\partial}\bar{\mathcal{F}} \\ &= \partial(\bar{\partial}v) + \bar{\partial}(\partial\bar{v}) \\ &= i\partial\bar{\partial}\left(\frac{v-\bar{v}}{i}\right) \\ &= i\partial\bar{\partial}u. \end{aligned} \tag{5.6}$$

Thus, the theorem is proved in the case that Ω is a star-shaped domain.

Generally, when Ω is a domain in $\mathbb{C}^2$ such that the DeRham cohomology of the second degree $H^2(\Omega, \mathbb{R}) = 0$, we could apply the well-known global construction of Weil [37] for $H^2(\Omega, \mathbb{R})$ to obtain the Poincaré-Cartan Lemma in a local sense. Then, Theorem 1.2 is proved.

6 Proof of Theorem 1.3

Applying a smooth approximation and the Poincaré–Lelong Formula, Theorem 1.3 follows from Theorems 1.1 and 1.2.

Indeed, by Theorem 3.7, let α_M be a d-closed $(1, 1)$ positive current associated with M. That means, for some holomorphic function h which has zero set M on Ω, we have

$$\alpha_M = \frac{1}{\pi}\partial\bar{\partial}[\log|h|]$$

in the sense of currents.

Let

$$V_\epsilon(z) = \log|h| * \chi_\epsilon(z)$$

be the smooth regularity of $\log|h(z)|$, where for each $\epsilon > 0$, and $\chi_\epsilon \in C_c^\infty(\mathbb{R})$ is a non-negative function such that χ_ϵ is supported on $[-\epsilon/2, \epsilon/2]$, and $\int_{\mathbb{R}} \chi_\epsilon(x)\mathrm{d}x = 1$. Then, V_ϵ is smooth on $\Omega_\epsilon = \{\rho(z) < -\epsilon\} \Subset \Omega$ and $V_\epsilon(z) \to \log|h(z)|$ as $\epsilon \to 0^+$.

For convenience, we also denote V_ϵ by the smooth extension of V_ϵ to a neighborhood of Ω, so $V_\epsilon(z) \to \log|h(z)|$ almost everywhere as $\epsilon \to 0^+$. Then the smooth regularity of α_M is $\alpha_\epsilon = \frac{1}{\pi}\partial\bar{\partial}V_\epsilon \in C^\infty_{(1,1)}(\bar{\Omega})$, and α_ϵ is also d-closed and positive. Moreover, $\alpha_\epsilon \to \alpha_M$ in the sense of currents. Thus, applying Theorem 1.2 to each $\pi\alpha_\epsilon$, we could seek a function u_ϵ such that

$$\begin{cases} u_\epsilon = \bar{u_\epsilon}, \\ \frac{1}{\pi}\partial\bar{\partial}u_\epsilon = \alpha_\epsilon, \\ ||u_\epsilon||_{L^1(b\Omega)} + ||u_\epsilon||_{L^1(\Omega)} \lesssim ||\alpha_\epsilon||_{L^1(\Omega)}. \end{cases}$$

As a consequence, for some constant $C > 0$, we have

$$\int_\Omega |u_\epsilon(z)|\mathrm{d}V(z) < C, \quad \text{uniformly in } \epsilon > 0. \tag{6.1}$$

The plurisubharmonicity implies that $\log|h(z)|$ is locally integrable. Hence, for any compact subset $K \subset \Omega$, we have

$$\int_K |V_\epsilon(z)|\mathrm{d}V(z) < C_K, \quad C_K > 0 \ \text{ depends only on } K. \tag{6.2}$$

Next, we define

$$g_\epsilon = u_\epsilon - V_\epsilon.$$

It is easy to see that g_ϵ is a pluriharmonic function on Ω. Since Ω is a domain, $g_\epsilon = \mathrm{Re}[G_\epsilon]$, where G_ϵ is holomorphic on Ω.

Using (6.1), (6.2) and Montel's Theorem applied to g_ϵ, there exists a subsequence $\{g_{\epsilon_n}\}$ of $\{g_\epsilon\}$ that converges to a pluriharmonic function g uniformly on every compact set of Ω, where $\lim_{n\to\infty} \epsilon_n = 0$. Moreover, we also have

$$g = \lim_{n\to\infty} g_{\epsilon_n} = \lim_{n\to\infty} \mathrm{Re}[G_{\epsilon_n}] = \mathrm{Re}[G],$$

for some holomorphic function G on Ω. Now, let $u = \log[|h|] + g = \log[|h|] + \mathrm{Re}[G] = \log[|he^G|]$, then we have

$$\begin{cases} \lim_{n\to\infty} u_{\epsilon_n} = u, & \text{in } L^1(\overline{\Omega}), \\ \frac{1}{\pi}\partial\bar\partial u = \alpha_M & \text{in the sense of currents}, \\ u \in L^1(\overline{\Omega}), & \text{by Theorem 1.2.} \end{cases}$$

On the other hand, let $\mathfrak{g}(z) = he^G(z)$ since $\frac{1}{\pi}\partial\bar\partial \log[|h|] = \frac{1}{\pi}\partial\bar\partial \log[|\mathfrak{g}|] = \alpha_M$, the zero set of $\mathfrak{g}$ is the same as the zero set of h. Finally, $\mathfrak{g} \in \mathcal{N}(\Omega)$ since $u = \log[|\mathfrak{g}|] \in L^1(\overline{\Omega})$. Thus, we complete the proof.

Remark. In the next paper, we will apply the present technique to construct a bounded holomorphic function which defines the given positive divisor in $\mathbb{C}^2$.

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