

Twistor spaces of Riemannian manifolds with even Clifford structures

Gerardo Arizmendi¹ · Charles Hadfield²

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Abstract In this paper, we introduce the twistor space of a Riemannian manifold with an even Clifford structure. This notion generalizes the twistor space of quaternion-Hermitian manifolds and weak-Spin(9) structures. We also construct almost complex structures on the twistor space for parallel even Clifford structures and check their integrability. Moreover, we prove that in some cases one can give Kähler and nearly Kähler metrics to these spaces.

Keywords Clifford structure · Twistor spaces · Almost complex structure · Nearly Kähler · Kähler · Grassmannian

1 Introduction

The notion of twistor space was first introduced by Roger Penrose in [13]. Following the ideas of Penrose, the twistor construction for a four-dimensional Riemannian manifold was developed in [2]. This was later generalized for even-dimensional manifolds in [11]. The twistor space Z of an even-dimensional Riemannian manifold admits a natural almost complex structure, and it is well known that such a twistor space is complex if and only if the manifold is self-dual for $\dim(M) = 4$ and locally conformally flat for $\dim(M) \geq 6$ [2, 11]. A converse theorem (the so-called reverse Penrose construction) in dimension 4 has been used to construct half-conformally flat Einstein manifolds.

In another generalization, the twistor space Z of quaternion-Kähler manifolds was defined in [14]. This is an \mathbb{S}^2 -bundle of pointwise Hermitian structures compatible with the quaternionic structure. It is well known that this bundle admits two almost complex structures \mathcal{J}

✉ Gerardo Arizmendi
gerardo@cimat.mx

Charles Hadfield
charles.hadfield@ens.fr

¹ Centro de Investigación en Matemáticas, P.O. Box 402, Guanajuato, GTO 36000, Mexico

² DMA, École Normale Supérieure, 45 rue d'Ulm, 75230 Paris cedex 05, France

and $\tilde{\mathcal{J}}$, one of which is always integrable and the other is never integrable [6]. Moreover, the manifold Z admits two Einstein metrics h and \tilde{h} such that (Z, \mathcal{J}, h) is Kähler–Einstein [3, 14] and $(Z, \tilde{\mathcal{J}}, \tilde{h})$ is nearly Kähler [1].

In [7], the twistor space was defined in the context of weak $\text{Spin}(9)$ structures on 16-dimensional Riemannian manifolds, which correspond to rank 9 even Clifford structures [9]. Additionally, this twistor construction was studied for \mathbb{R}^{16} , which carries a parallel flat even Clifford structure, the Cayley plane $F_4/\text{Spin}(9)$, which carries a parallel non-flat even Clifford structure, and $\mathbb{S}^1 \times \mathbb{S}^{15}$, which carries a non-parallel even Clifford structure. In the first two cases, the twistor space admits a Kähler metric and in the last case the twistor space is a complex manifold which does not admit a Kähler metric.

In this paper, we generalize these constructions to even Clifford structures of arbitrary rank $r \geq 3$. Our construction comprises the quaternion-Kähler case ($r = 3$) and the weak $\text{Spin}(9)$ case ($r = 9$). We construct a $\tilde{\text{Gr}}(2, r)$ -bundle of pointwise Hermitian structures. This bundle admits an almost complex structure and we prove theorems analogous to those in [1, 3, 14].

The paper is organized as follows. Section 2 is devoted to the notion of even Clifford structures. In Sect. 3, we explain the construction of the twistor space of Riemannian manifolds with even Clifford structures and its almost complex structures and check their integrability. The main results of the paper are Theorems 3.3 and 3.4 as well as Corollary 3.6.

2 Even Clifford structures

The definition of a rank r even Clifford structure (or Cl_r^0 structures) on a Riemannian manifold was given in [9]:

Definition 2.1 A rank r even Clifford structure on a Riemannian manifold (M, g) is an oriented rank r Euclidean bundle E over M together with a non-vanishing algebra bundle morphism, called a Clifford morphism, $\varphi : \text{Cl}^0(E) \rightarrow \text{End}(TM)$ which maps $\Lambda^2 E$ into the bundle of skew-symmetric endomorphisms $\text{End}^-(TM)$.

The prototype of the idea of locally defined endomorphisms on TM modelling some Clifford algebra $\text{Cl}^0(E, h)$ can be found already in [7, Section 2] in the context of $\text{Spin}(9)$ structures. This kind of structure also appears naturally in several other settings. For instance [9, Example 2.6 and 2.7] almost Hermitian structures are the same as rank 2, and quaternionic structures are the same as rank 3, even Clifford structures. In dimension 8, quaternionic Hopf structures are examples of rank 5 even Clifford structures [12, Example 4]. An even Clifford structure (M, g, E) is called parallel, if there exists a metric connection ∇^E on E such that φ is connection preserving, i.e.

$$\varphi(\nabla_X^E \sigma) = \nabla_X^g \varphi(\sigma)$$

for every tangent vector $X \in TM$ and section σ of $\text{Cl}^0(E)$. For instance, a manifold with a parallel Cl_2^0 structure is actually a Kähler manifold, while a manifold with a parallel Cl_3^0 structure is a quaternion-Kähler manifold. Manifolds with $\text{Spin}(7)$ holonomy correspond to eight-dimensional manifolds with Cl_7^0 parallel structures. Thus, one can hope this definition gives a more general framework in which to study these geometries.

Definition 2.2 A parallel even Clifford structure (M, E, ∇^E) is called flat if the connection ∇^E is flat.

Table 1 Manifolds with a flat even Clifford structure

| r | M | Dimension of M |
|-----------|-------------------------------|--|
| 2 | Kähler | $2m, m \geq 1$ |
| 3 and 4 | Hyper-Kähler | $4q, q \geq 1$ |
| 4 | Reducible hyper-Kähler | $4(q^+ + q^-), q^+ \geq 1, q^- \geq 1$ |
| Arbitrary | Cl_r^0 representation space | Multiple of $N_0(r)$ |

Table 2 Manifolds with a parallel non-flat even Clifford structure

| r | M | Dimension of M |
|-----|---|------------------|
| 2 | Kähler | $2m, m \geq 1$ |
| 3 | Quaternion-Kähler (QK) | $4q, q \geq 1$ |
| 4 | Product of two QK manifolds | $4(q^+ + q^-)$ |
| 5 | QK | 8 |
| 6 | Kähler | 8 |
| 7 | Spin(7) holonomy | 8 |
| 8 | Riemannian | 8 |
| 5 | $Sp(k + 2)/Sp(k) \times Sp(2)$ | $8k, k \geq 2$ |
| 6 | $SU(k + 4)/S(U(k) \times U(4))$ | $8k, k \geq 2$ |
| 8 | $SO(k + 8)/SO(k) \times SO(8)$ | $8k, k \geq 2$ |
| 9 | $\mathbb{O}P^2 = F_4/Spin(9)$ | 16 |
| 10 | $(\mathbb{C} \otimes \mathbb{O})P^2 = E_6/Spin(10) \cdot U(1)$ | 32 |
| 12 | $(\mathbb{H} \otimes \mathbb{O})P^2 = E_7/Spin(12) \cdot SU(2)$ | 64 |
| 16 | $(\mathbb{O} \otimes \mathbb{O})P^2 = E_8/Spin^+(16)$ | 128 |

The list of complete simply connected Riemannian manifolds M carrying a parallel rank r even Clifford structure was found in [9] and is given in Tables 1 and 2.

For the sake of simplicity, in Table 2 the non-compact duals of the compact symmetric spaces have been omitted. $N_0(r)$ denotes the dimension of the irreducible representations of Cl_r^0 . For further details on even Clifford structures, we refer to [8, 9].

3 The twistor space of an even Clifford structure

Let (M, g, E) be a manifold with even Clifford structure. Let φ denote the Clifford map. Given $x \in M$, let $J_{ij} := \varphi(e_i \cdot e_j)$ where $\{e_1, \dots, e_r\}$ is an orthonormal basis for E_x and \cdot denotes Clifford multiplication. For each x we consider the subspace Z_x of $\text{End}(T_x M)$ where

$$Z_x = \left\{ J = \sum_{1 \leq i < j \leq r} a_{ij} J_{ij} \mid J^2 = -\text{Id}_{T_x M}, \quad a_{ij} \in \mathbb{R} \right\}$$

and define the twistor space of the even Clifford structure to be the disjoint union

$$Z = \bigsqcup_{x \in M} Z_x.$$

We will denote by π the projection onto M . This is a bundle of pointwise orthogonal complex structures. For a parallel Cl_3^0 structure this coincides with the definition of the twistor space of a quaternion-Kähler manifold, where the fibre is homeomorphic to S^2 . It is not hard to see that for a Cl_4^0 structure the fibre of the twistor space is homeomorphic to $S^2 \times S^2$, which corresponds to the isomorphism between $Spin(4)$ and $Spin(3) \times Spin(3)$. In general, the fibre at each point is isomorphic to $\tilde{Gr}(2, r)$, the Grassmannian of oriented 2-planes in \mathbb{R}^r , as we see in the next lemma.

Lemma 3.1 *Let $A \in \mathfrak{spin}(r) \subset Cl_r^0$, then $A^2 = -1$ if and only if there exist $v_1, v_2 \in \mathbb{R}^r$ orthonormal vectors such that $A = v_1 \cdot v_2$.*

Proof Let $A \in \mathfrak{spin}(r) \subset Cl_r^0$ and $\{e_1, \dots, e_r\}$ be an orthonormal basis for \mathbb{R}^r , then under a change of basis, we can suppose that $A = \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} a_i e_{2i-1} \cdot e_{2i}$. The condition $A^2 = -1$ yields the equations

$$\sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} a_i^2 = 1, \quad a_i a_j = 0, \quad (i < j).$$

The solutions of these equations are the r -tuples $(\pm 1, 0, \dots, 0), (0, \pm 1, 0, \dots, 0), \dots, (0, \dots, \pm 1)$. Therefore, $A = \pm e_{2i-1} \cdot e_{2i}$ for some $1 \leq i \leq \lfloor \frac{r}{2} \rfloor$. Conversely, if $A = v_1 \cdot v_2$ with v_1 and v_2 orthonormal, then $A^2 = v_1 \cdot v_2 \cdot v_1 \cdot v_2 = -v_1^2 v_2^2 = -1$, which proves the assertion. \square

Remark Another way to prove this is using that an element A in $\Lambda^2 \mathbb{R}^r$ is decomposable if and only if $A \wedge A = 0$.

3.1 Almost complex structures on the twistor space

Consider $\tilde{Gr}(2, r)$ as a Hermitian-symmetric space. Its complex structure can be given using Clifford multiplication. Let $z \in \tilde{Gr}(2, r)$, which for a suitable frame can be written as $z = e_1 \wedge e_2 = e_1 \cdot e_2$. The tangent space $T_z \tilde{Gr}(2, r)$ can be identified with

$$\text{span}(e_i \cdot e_j \mid i \in \{1, 2\}, j \in \{3, \dots, r\}) = \left\{ \sum_{s=3}^r \alpha_s e_1 \cdot e_s + \beta_s e_2 \cdot e_s \mid \alpha_s, \beta_s \in \mathbb{R} \right\}.$$

The complex structure is then given by $\tilde{J}_z(v) = z \cdot v$.

The Levi-Civita connection on M induces a connection on Z . For each $S \in Z$, the connection gives a splitting $T_S Z = V_S \oplus H_S$ where $V_S = \ker(\pi_*)$ is isomorphic to $T_{\pi(S)} \tilde{Gr}(2, r)$, the isomorphism given by the differential of the Clifford map, and H_S , the horizontal subspace, is isomorphic to $T_{\pi(S)} M$. We recall the usual construction of almost complex structures on Z . Given $U \in V_S$ and $X \in H_S$ we define

$$\mathcal{J}(U + X)_S = \hat{J}(U) + \pi_*^{-1}(S\pi_*(X))$$

where $\hat{J}(U) = \varphi_* \tilde{J} \varphi_*^{-1}(U) = SU$.

For a rank 3 parallel even Clifford structure, this is the construction of the almost complex structure for quaternion-Kähler manifolds, so (Z, \mathcal{J}) is a complex manifold, see [3, 14]. For

a rank 4 parallel even Clifford structure, the manifold is locally a product of two quaternion-Kähler manifolds [9]. On the other hand, $\tilde{\text{Gr}}(2, 4)$ is isomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ as Kähler manifolds, so in this case the twistor space is the product of the twistor spaces of two quaternion-Kähler manifolds. In particular, it is a complex manifold.

From now on, we will suppose that (M, g) is a complete, simply connected Riemannian manifold, carrying a parallel even Clifford structure of rank $r \geq 5$. We treat the eight-dimensional case first. In this case, the rank should be 5, 6, 7 or 8 and the following holds:

- $r = 5$: in this case the manifold is known to be quaternion-Kähler [9]. The twistor space has fibre isomorphic to $\text{Sp}(2)/\text{U}(2)$ and has been considered in [5]. The twistor space is complex exactly when M is locally symmetric.
- $r = 6$: in this case the manifold is known to be Kähler [9]. The twistor space has fibre $\tilde{\text{Gr}}(2, 6)$ and has been considered in [5, 11]. The twistor is complex exactly when the Bochner tensor of M vanishes.
- $r = 7$: in this case the manifold has $\text{Spin}(7)$ holonomy. The twistor space has fibre $\text{SO}(7)/\text{SO}(5) \times \text{SO}(2)$. According to [5], the twistor space in this case is never complex.
- $r = 8$: in this case the manifold is Riemannian. The twistor fibre is isomorphic to $\text{SO}(8)/\text{U}(4)$ which is the usual fibre of the twistor space defined for even-dimensional Riemannian manifolds. As mentioned in the introduction, the twistor space is complex if and only if M is conformally flat.

Now we will assume that $n \neq 8$. The following is the analogous of Lemma 14.74 in [4].

Lemma 3.2 *Let M be a complete, simply connected Riemannian manifold of dimension $n \neq 8$ carrying a parallel even Clifford structure of rank $r > 4$, then for every $S \in Z$ and $X, Y \in T_{\pi(S)}M$, the curvature R of the Levi-Civita connection satisfies*

$$[R_{SX,SY}, S] - S[R_{SX,Y}, S] - S[R_{X,SY}, S] - [R_{X,Y}, S] = 0.$$

Proof It suffices to prove the proposition for $S = J_{12}$. If the parallel even Clifford structure is flat, then by Theorem 2.9 in [9] the manifold is flat, so $R(X, Y) = 0$ for all $X, Y \in T_{\pi(S)}M$. If the parallel even Clifford structure is not flat and $n \neq 8$, the proof of Proposition 2.10 in [9], explicitly Equation (15), implies the existence of a non-zero constant κ such that

$$[R_{X,Y}, J_{12}] = \kappa \sum_{s>2} g(J_{s1}X, Y)J_{s2} - g(J_{s2}X, Y)J_{s1}. \tag{3.1}$$

Using the properties of the endomorphisms J_{ij} , Lemma 2.4 [9], specifically, $J_{ij} \circ J_{ik} = J_{jk}$ for i, j, k mutually distinct, the result follows upon summing the following four calculations.

$$\begin{aligned} [R_{J_{12}X, J_{12}Y}, J_{12}] &= \kappa \sum_{s>2} -g(J_{s1}X, Y)J_{s2} + g(J_{s2}X, Y)J_{s1} \\ -J_{12}[R_{J_{12}X, Y}, J_{12}] &= \kappa \sum_{s>2} -g(J_{s2}X, Y)J_{s1} + g(J_{s1}X, Y)J_{s2} \\ -J_{12}[R_{X, J_{12}Y}, J_{12}] &= \kappa \sum_{s>2} -g(J_{s2}X, Y)J_{s1} + g(J_{s1}X, Y)J_{s2} \\ -[R_{X, Y}, J_{12}] &= \kappa \sum_{s>2} -g(J_{s1}X, Y)J_{s2} + g(J_{s2}X, Y)J_{s1} \end{aligned}$$

□

Theorem 3.3 *Let M be a complete, simply connected Riemannian manifold of dimension $n \neq 8$ carrying a parallel even Clifford structure of rank $r > 4$, then the almost complex structure \mathcal{J} on Z is integrable.*

Proof We proceed as in 14.68 of [4]. For an arbitrary vector field W , we let $\mathcal{V}(W)$ denote the vertical part of W and $\mathcal{H}(W)$ the horizontal part of W . Let $N_{\mathcal{J}}$ be the Nijenhuis tensor of \mathcal{J} . Let U and V be vertical vector fields and X and Y basic horizontal vector fields.

Let us first check that $N_{\mathcal{J}}(U, V) = 0$. Since U and V are vertical, $\mathcal{J}(U)$ and $\mathcal{J}(V)$ are also vertical vector fields. Thus, $N_{\mathcal{J}}(U, V) = N_{\hat{J}}(U, V) = 0$, since \hat{J} is a complex structure.

Now we will check that $N_{\mathcal{J}}(X, U) = 0$. From the two facts that the horizontal transport of the horizontal distribution respects \hat{J} , and that $[X, U]$ is vertical if U is, we obtain $[X, \mathcal{J}U] = \mathcal{J}[X, U]$. This reduces the Nijenhuis tensor to $N_{\mathcal{J}}(X, U) = \mathcal{J}([\mathcal{J}(X), U]) - [\mathcal{J}(X), \mathcal{J}(U)]$. The vertical part of this vanishes by noting that both terms in $\mathcal{V}([\mathcal{J}(X), \mathcal{J}(U)]) = \mathcal{J}(\mathcal{V}[J(X), U])$ are tensorial in X . Finally, for the horizontal part of $N_{\mathcal{J}}(X, U)$ observe first that $\pi_*([\mathcal{J}(X), U]) = -U\pi_*X$ from which we obtain

$$\begin{aligned} \pi_*(\mathcal{J}[\mathcal{J}(X), U]) &= \pi_*(\mathcal{J}\mathcal{H}[\mathcal{J}(X), U]) \\ &= \pi_*(\pi_*^{-1}S\pi_*[\mathcal{J}(X), U]) \\ &= -SU\pi_*X \end{aligned}$$

By the same reasoning

$$\begin{aligned} \pi_*([\mathcal{J}(X), \mathcal{J}(U)]) &= -\mathcal{J}(U)\pi_*X \\ &= -SU\pi_*X \end{aligned}$$

and so $N_{\mathcal{J}}(X, U) = 0$.

Finally, we check that $N_{\mathcal{J}}(X, Y) = 0$. This is done by considering the horizontal and vertical components separately. For the horizontal component, we consider S in Z with $\pi(S) = x$ as a section, also denoted S , of Z about x and demand that $\nabla S = 0$ at x . This gives a local almost complex structure on a neighbourhood of x which has an associated Nijenhuis tensor N_S . A direct calculation gives agreement, on the neighbourhood of x , between the two Nijenhuis tensors considered, explicitly,

$$\pi_*(N_{\mathcal{J}}(X, Y)_S) = N_S(\pi_*(X), \pi_*(Y)).$$

The tensor N_S is then seen to vanish at x as ∇ is torsion free and, at x , ∇S vanishes. Studying the vertical component, one recalls O’Neill’s formulas for Riemannian submersions (see Chapter 9, [4]). In particular, $\mathcal{V}[X, Y]_{\pi(S)} = -[R_{\pi_*X, \pi_*Y}, S]$, which implies $\mathcal{V}(N_{\mathcal{J}}(X, Y)) = 0$ precisely by Lemma 3.2. \square

Theorem 3.4 *The twistor space (Z, \mathcal{J}) of a complete, simply connected Riemannian manifold of dimension $n \neq 8$ with a parallel even Clifford structure of rank $r > 4$ and $\text{Ric} > 0$ admits a Kähler metric.*

Proof In this case, the manifold (M, g) is Einstein with $\text{Ric} = \kappa(n/4 + 2r - 4)$ (Proposition 2.10 [9]). Using the condition that $\text{Ric} > 0$, we choose a metric h on Z such that π is a Riemannian submersion with totally geodesic fibres isometric to $\tilde{G}r(2, r)$ with Kähler metric and $\text{Ric} = 2r\kappa$, so that the collection $\{J_{ij}\}$ forms a mutually orthogonal frame and $\|J_{ij}\|^2 = 1/\kappa$. Let U and V be vertical vector fields and X and Y basic horizontal vector fields. The theorem follows a similar argument to that given in 14.81 of [4]. We consider separately the four cases coming from $(\nabla_E \mathcal{J})F$ where E, F may be horizontal or vertical.

First we show $\nabla_U \mathcal{J} = 0$. Restricting to its action on a vertical field, we immediately get $(\nabla_U \mathcal{J})V = 0$ as the fibre is Kähler and totally geodesic. To prove $(\nabla_U \mathcal{J})X = 0$, it suffices to consider only the horizontal component (again as the fibres are totally geodesic). By appropriately choosing a local orthonormal frame for E , we may assume that $S = J_{12}$

and $U = \lambda J_{s1}$ with $s > 2$. The Koszul formula and the relationship between the vertical component of the Lie bracket and the curvature mentioned in the previous proof give, at S ,

$$\begin{aligned} 2h(\nabla_U X, Y) &= -h([X, Y], U) \\ &= \lambda h([R_{\pi_* X, \pi_* Y}, J_{12}], J_{s1}). \end{aligned}$$

Recalling Eq. 3.1, we deduce $h(\nabla_U X, Y) = -\frac{1}{2}\lambda g(J_{s2}\pi_* X, \pi_* Y)$. Using this result, $\pi_*(\nabla_U X)_S = -\frac{1}{2}\lambda J_{s2}\pi_* X$, we obtain

$$\begin{aligned} \pi_*(\mathcal{J}\nabla_U X)_S &= -\frac{1}{2}J_{12}\lambda J_{s2}\pi_* X \\ &= \frac{1}{2}U\pi_* X. \end{aligned}$$

Similarly, one proves that

$$\pi_*(\nabla_U \mathcal{J}X) = \frac{1}{2}U\pi_* X,$$

from which we conclude $\pi_*(\nabla_U \mathcal{J}X) = 0$.

Second, we show $\nabla_X \mathcal{J} = 0$. Recall O’Neill’s A tensor

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E} \mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E} \mathcal{V}F$$

where E and F are arbitrary vectors. We show, as an initial calculation, that $A_X(\mathcal{J}Y) = \mathcal{J}(A_X Y)$ and $\mathcal{J}(A_X U) = A_X(\mathcal{J}U)$. In our situation, we note the following decomposition into horizontal and vertical components

$$\begin{aligned} \nabla_X U &= \mathcal{V}\nabla_X U + A_X U \\ \nabla_X Y &= A_X Y + \mathcal{H}\nabla_X Y. \end{aligned}$$

By Proposition 9.24 in [4], we have $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$ so at $S = J_{12}$, we get $A_X Y = -\frac{1}{2}[R_{\pi_* X, \pi_* Y}, J_{12}]$ and the claim that $A_X(\mathcal{J}Y) = \mathcal{J}(A_X Y)$ is equivalent to

$$[R_{X', J_{12}Y'}, J_{12}] = J_{12}[R_{X', Y'}, J_{12}]$$

where for the sake of notation, we have denoted $X' = \pi_* X$ and $Y' = \pi_* Y$. Equation 3.1 gives the result as

$$\begin{aligned} J_{12}[R_{X', Y'}, J_{12}] &= J_{12} \left(\kappa \sum_{s>2} g(J_{s1}X', Y')J_{s2} - g(J_{s2}X', Y')J_{s1} \right) \\ &= \kappa \sum_{s>2} -g(J_{s1}X', Y')J_{s1} - g(J_{s2}X', Y')J_{s2} \end{aligned}$$

and similarly

$$\begin{aligned} [R_{X', J_{12}Y'}, J_{12}] &= \kappa \sum_{s>2} g(J_{s1}X', J_{12}Y')J_{s2} - g(J_{s2}X', J_{12}Y')J_{s1} \\ &= \kappa \sum_{s>2} -g(J_{s2}X', Y')J_{s2} - g(J_{s1}X', Y')J_{s1}. \end{aligned}$$

Table 3 Twistor spaces for low rank and dimension 8

| r | M | $\dim(M)$ | Fibre of Z | Type of Z |
|-----|---------------------------|----------------|--|--|
| 3 | QK manifold | $4n$ | \mathbb{S}^2 | Complex, Kähler if $\text{Ric} > 0$ |
| 4 | $M_1 \times M_2, M_i$ QK | $4(n_1 + n_2)$ | $\mathbb{S}^2 \times \mathbb{S}^2$ | Complex, Kähler if $\text{Ric}(M_i) > 0$ |
| 5 | QK | 8 | $\text{Sp}(2)/\text{U}(2)$ | Complex if locally symmetric |
| 6 | Kähler | 8 | $\text{U}(4)/\text{U}(2) \times \text{U}(2)$ | Complex if Bochner tensor $\equiv 0$ |
| 7 | $\text{Spin}(7)$ holonomy | 8 | $\tilde{\text{Gr}}(2, 7)$ | Not complex |
| 8 | Riemannian | 8 | $\text{SO}(8)/\text{U}(4)$ | Complex if Weyl tensor $\equiv 0$ |

For the second claim, we use the skew symmetry of $A, h(A_X Y, U) = -h(Y, A_X U)$ to obtain

$$\begin{aligned} h(\mathcal{J}A_X U, Y) &= h(U, A_X \mathcal{J}Y) \\ &= h(U, \mathcal{J}A_X Y) \\ &= h(A_X \mathcal{J}U, Y). \end{aligned}$$

Therefore, $\mathcal{J}(A_X U) = A_X(\mathcal{J}U)$.

We apply this result to $(\nabla_X \mathcal{J})U$ where

$$\begin{aligned} (\nabla_X \mathcal{J})U &= \nabla_X(\mathcal{J}U) - \mathcal{J}\nabla_X U \\ &= \mathcal{V}\nabla_X(\mathcal{J}U) + A_X \mathcal{J}U - \mathcal{J}\mathcal{V}\nabla_X U - \mathcal{J}A_X U \\ &= \mathcal{V}\nabla_X(\mathcal{J}U) - \mathcal{J}\mathcal{V}\nabla_X U \end{aligned}$$

Taking the inner product of each term with V and studying the respective Koszul formulas give the result that $(\nabla_X \mathcal{J})U = 0$. By a similar calculation for $(\nabla_X \mathcal{J})Y$,

$$(\nabla_X \mathcal{J})Y = \mathcal{H}\nabla_X(\mathcal{J}Y) - \mathcal{J}\mathcal{H}\nabla_X Y.$$

As this is horizontal, we may use a similar idea to that presented in the previous proof. Specifically, we consider $S \in Z$ with $x = \pi(S)$ as a section over a neighbourhood of x with $\nabla S = 0$ at x . Studying the appropriate Koszul formulas one concludes that, at x ,

$$(\nabla_X \mathcal{J})Y = (\nabla_{\pi_* X} S)\pi_* Y.$$

The result now follows since, at x ,

$$\begin{aligned} (\nabla_X \mathcal{J})Y &= \pi_*^{-1}(\nabla_{\pi_* X}(S\pi_* Y) - S\nabla_{\pi_* X}\pi_* Y) \\ &= \pi_*^{-1}((\nabla_{\pi_* X} S)\pi_* Y) = 0. \end{aligned}$$

□

Tables 3 and 4 summarize our findings.

For the non-compact dual spaces of these symmetric spaces, the twistor space is only complex as the negative curvature obstructs the construction of an appropriate metric on the fibres.

Finally, we use the following observation of Nagy [10] to construct nearly Kähler metrics on the twistor space. Consider a Riemannian submersion with totally geodesic fibres

$$F \rightarrow (Z, h) \rightarrow M$$

Table 4 Twistor spaces for higher rank

| r | M | $\dim(M)$ | Fibre of Z | Type of Z |
|-----|---|----------------|--|-------------|
| 5 | $\mathrm{Sp}(k+2)/(\mathrm{Sp}(k) \times \mathrm{Sp}(2))$ | $8k, k \geq 2$ | $\mathrm{Sp}(2)/\mathrm{U}(2)$ | Kähler |
| 6 | $\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(4))$ | $8k, k \geq 2$ | $\mathrm{U}(4)/\mathrm{U}(2) \times \mathrm{U}(2)$ | Kähler |
| 8 | $\mathrm{SO}(k+8)/(\mathrm{SO}(k) \times \mathrm{SO}(8))$ | $8k, k \geq 2$ | $\mathrm{SO}(8)/\mathrm{U}(4)$ | Kähler |
| 9 | $\mathrm{F}_4/\mathrm{Spin}(9)$ | 16 | $\tilde{\mathrm{Gr}}(2, 9)$ | Kähler |
| 10 | $\mathrm{E}_6/(\mathrm{Spin}(10) \cdot \mathrm{U}(1))$ | 32 | $\tilde{\mathrm{Gr}}(2, 10)$ | Kähler |
| 12 | $\mathrm{E}_7/(\mathrm{Spin}(12) \cdot \mathrm{SU}(2))$ | 64 | $\tilde{\mathrm{Gr}}(2, 12)$ | Kähler |
| 16 | $\mathrm{E}_8/\mathrm{Spin}^+(16)$ | 128 | $\tilde{\mathrm{Gr}}(2, 16)$ | Kähler |
| r | Cl_r^0 representation | $N_0(r)n$ | $\tilde{\mathrm{Gr}}(2, r)$ | Kähler |

and let $TZ = V \oplus H$ be the corresponding splitting of TZ . Suppose that Z admits a complex structure \mathcal{J} compatible with h and preserving V and H such that (Z, \mathcal{J}, h) is a Kähler manifold. Consider now the Riemannian metric on Z defined by

$$\tilde{h}(X, Y) = \frac{1}{2}h(X, Y) \quad \text{for } X, Y \in V,$$

$$\tilde{h}(X, Y) = h(X, Y) \quad \text{for } X, Y \in H.$$

The metric \tilde{h} admits a compatible almost complex structure $\tilde{\mathcal{J}}$ given by $\tilde{\mathcal{J}}|_V = -\mathcal{J}$ and $\tilde{\mathcal{J}}|_H = \mathcal{J}$. The next proposition is proved in [10].

Proposition 3.5 [10] *The manifold $(Z, \tilde{\mathcal{J}}, \tilde{h})$ is nearly Kähler.*

□

Corollary 3.6 *The twistor space Z of a Riemannian manifold with a parallel even Clifford structure of rank $r \geq 3$ and $\mathrm{Ric} > 0$, admits an almost complex structure $\tilde{\mathcal{J}}$ and a metric \tilde{h} such that $(Z, \tilde{\mathcal{J}}, \tilde{h})$ is nearly Kähler.*

□

In our case, using the definition of the almost complex structure, one can easily check that this almost complex structure is never integrable.

We conclude by pointing out that even though a classification of parallel even Clifford structures was given in [9], and one can try to deal with each of these cases separately, our approach does not rely on this classification (except for dimension 8 in which the curvature condition is not automatically satisfied). Furthermore, the constructions above can be studied in a more general context. One could check integrability conditions of these twistor spaces for manifolds with non-parallel even Clifford structures, as in [7]. In fact, for the twistor space to be complex, Lemma 3.2 should be satisfied for every S in the twistor space. One nice example is given by $\mathbb{S}^1 \times \mathbb{S}^{15}$, which admits a non-parallel Cl_0^0 structure but its twistor space is a complex manifold which cannot be Kähler since its first Betti number is odd.

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