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Invariant metrics on unbounded strongly pseudoconvex domains with non-compact automorphism group

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Abstract We study invariant metrics on unbounded strongly pseudoconvex domains with non-compact automorphism group. The main result is that the corresponding Bergman and Kähler–Einstein metrics are metrically equivalent. We also determine the comparisons among invariant metrics, including the Carathéodory and Kobayashi pseudo-metrics additionally.

Keywords Automorphism groups · Invariant metrics and pseudo-distances · Strongly pseudoconvex domains

Mathematics Subject Classification Primary 32F45 · Secondary 32M05 · 32T15

1 Introduction

1.1 Backgrounds

The equivalence problem in complex differential geometry has been extensively studied in the literature regarding automorphism groups for bounded domains during the past decades, see for example [3, 15, 16] and the references therein. By an automorphism of a complex manifold Ω we mean a biholomorphic self-mapping $f : \Omega \to \Omega$. For the sake of the classification of

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domains in higher dimensions, people are interested in those whose automorphisms are non-compact in the compact-open topology. It was conjectured that any smoothly bounded domain with non-compact automorphism group should be biholomorphically equivalent to a domain of the form $\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : P(z', \bar{z}') + |z_n|^2 < 1\}$, where $P(z', \bar{z}') = \sum a_{JK} z'^J \bar{z}'^K$ is a polynomial with $a_{JK} = \bar{a}_{KJ}$. The situation is somehow the opposite for unbounded cases. For instance, there are abundant automorphisms of $\mathbb{C}^n$ when $n > 1$, and the structure of $\mathrm{Aut}(\mathbb{C}^n)$ is not well understood up to now.

If we focus our attention only on strongly pseudoconvex domains, then more is known about the automorphism groups comparatively. Let Ω be a C^2-smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$, or $\Omega \Subset M$ a relatively compact domain in a Stein manifold. Then it is already known that if the automorphism group $\mathrm{Aut}(\Omega)$ is non-compact then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$ due to the Wong–Rosay theorem [26,31]. Thus, the automorphism group of a bounded strongly pseudoconvex domain is either $SU(n, 1)$ or a compact Lie group. Conversely, it was shown by Bedford–Dadok [2] and Saerens–Zame [27], respectively, that for any compact Lie group G there is a bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ for which $\mathrm{Aut}(\Omega) = \mathrm{G}$. While this is not true any longer in the case when Ω is unbounded or non-hyperbolic in the sense of Kobayashi, for instance, see the example considered in the present paper.

Through the above inspections, we see that there are some essential differences between the relatively compact Stein domains with smooth boundary and the unbounded non-hyperbolic strongly pseudoconvex domains in a complex manifold. In the case that a domain $\Omega \subset \mathbb{C}^n$ is unbounded, or more generically, non-hyperbolic strongly pseudoconvex, we can no longer expect that the geometric and analytic properties of Ω are as good as the bounded cases. Then the following questions naturally arise: What are the similarity and disparity between the bounded and unbounded cases? To what extent the geometry on Ω can be interpreted in many ways as that of the bounded case, more precisely the Bergman geometry for the interest of complex analysis, for example, the intrinsically invariant metrics and the corresponding Bergman curvatures of Ω? Can one do further investigations on the biholomorphic invariants on Ω and some other complex differential geometric properties? These questions motivated the study concerning non-hyperbolic unbounded strongly pseudoconvex domains in this paper.

Let us mention one additional background related to this paper. Recently, it was shown by Harz–Shcherbina–Tomassini [13] that every strongly pseudoconvex domain Ω with smooth boundary in a complex manifold M admits a global defining function in the sense that there exists a smooth plurisubharmonic function $\varphi : U \to \mathbb{R}$ defined on an open neighborhood $U \subset M$ of $\overline{\Omega}$ such that $\Omega = \{\varphi < 0\}$, $d\varphi \neq 0$ on $\partial\Omega$ and φ is strongly plurisubharmonic near $\partial\Omega$. It is well known that if $\Omega \Subset M$ is a relatively compact domain in a Stein manifold with smooth strongly pseudoconvex boundary, one can even choose φ to be strongly plurisubharmonic in a neighborhood of $\overline{\Omega}$ due to Grauert [10]. Unfortunately, this is not true in general for either M is not Stein, or Ω is not relatively compact in M (cf. [13, Examples 1 and 2]). Actually, in [13], the authors proved that every strongly pseudoconvex domain Ω in a complex manifold M with smooth boundary admits a global defining function which is strongly plurisubharmonic precisely in the complement of the core $c(\Omega)$ of Ω, where by the core $c(\Omega)$ we mean the set of all points where every smooth and bounded from above plurisubharmonic function on Ω fails to be strongly plurisubharmonic. Due to this reason, the study of unbounded, or non-hyperbolic, strongly pseudoconvex domains in a complex manifold attracts lots of attentions recently (see [12,13,29] and the references therein).

The main goal of the present paper is to investigate the geometric and analytic properties of the following "typical" unbounded non-hyperbolic strongly pseudoconvex domain $D_{n,m}$ with non-compact automorphism group, which is defined by

$$D_{n,m} := \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\|^2 < e^{-\|z\|^2} \right\},$$

where $\|\cdot\|$ denotes the Euclidean norm.

The main features of our domain are as follows: We can say that $D_{1,1}$ is typical due to a result given by Kosiński (see [20, Theorem 3]), which describes all the possible proper holomorphic mappings between non-hyperbolic Reinhardt domains in $\mathbb{C}^2$. Consequently, we observe that for any non-hyperbolic strongly pseudoconvex Reinhardt domain Ω in $\mathbb{C}^2$ with smooth boundary, if the automorphism group $\mathrm{Aut}(\Omega)$ is non-compact, then Ω is biholomorphically equivalent to $D_{1,1}$. A direct computation shows that $D_{n,m}$ is strongly pseudoconvex (see Sect. 2.1) and contains the complex line $L := \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : w = 0\}$; hence $D_{n,m}$ is not Kobayashi hyperbolic. As in [13], we know that there does not exist a strongly plurisubharmonic function φ defined on a neighborhood of the closure of $D_{n,m}$ such that $D_{n,m} = \{\varphi < 0\}$, since by Liouville's theorem φ has to be constant on L. Recently, Kim–Ninh–Yamamori [17] determined the full automorphisms of $D_{n,m}$ and it turns out that $\mathrm{Aut}(D_{n,m})$ is non-compact (see Sect. 2.2). Furthermore, the automorphism group of $D_{n,m}$ shares some important properties with that of the Thullen domain $\{\|w\|^2 < (1 - \|z\|^2)^s\}$, $s > 0$. For a bounded domain $\Omega \subset \mathbb{C}^n$ with real analytic boundary, if the automorphism group $\mathrm{Aut}(\Omega)$ is non-compact, then Ω is biholomorphically equivalent to the Thullen domain [3]. In this sense, Thullen domain can be regarded as a model of bounded weakly pseudoconvex domains with non-compact automorphism group and real analytic boundary. We will see that $D_{n,m}$ plays the same rôle as the Thullen domain.

1.2 Organization of the paper

The paper is organized as follows: We prepare some basic facts about the $D_{n,m}$, including the strong pseudoconvexity, the non-compactness of automorphism groups, the Bergman kernel and metric in Sect. 2. As an application of the explicit formula of the Bergman kernel due to the second author [33], it is shown that the Bergman metric restricted on the line $L := \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : w = 0\}$ is Euclidean and consequently the Bergman metric of $D_{n,m}$ is complete.

In Sect. 3, we shall investigate the holomorphic sectional curvature and the Ricci curvature of the Bergman metric on $D_{1,1}$. It is well known that, for bounded strongly pseudoconvex domains in $\mathbb{C}^n$, the holomorphic sectional curvature tends to $-4/(n+1)$, and the Ricci curvature goes to -1 near the boundary. In this section, we prove that the boundary asymptotic behaviors of the above mentioned curvatures share the same properties. Moreover, the holomorphic sectional curvature is pinched between $-16/15$ and 0. We note that the holomorphic sectional curvature, the Ricci curvature and the scalar curvature of $D_{1,1}$ were considered in [32], among which some conclusions are not correct.

We will describe the Kähler–Einstein metric on $D_{1,1}$ in Sect. 4. Due to the pioneering work of Cheng–Yau [6], it is well known that for any C^2-smoothly bounded pseudoconvex domain there exists a unique complete Kähler–Einstein metric with Ricci curvature -1. Later, Mok–Yau [22] extended this result to an arbitrary domain of holomorphy (see [Section 3][22]). For any bounded homogeneous domain in $\mathbb{C}^n$, it is well known that the Bergman metric has constant Ricci curvature -1. For non-homogeneous bounded domains, it was Bland [5] who firstly described the Kähler–Einstein metric for the Thullen domain. Later, this result was generalized to certain Hartogs domains over the bounded symmetric domains, which are the so-called Cartan–Hartogs domains (see [30]). In this section, adopting the non-compact automorphism of $D_{1,1}$, we shall solve the Monge–Ampère equation and obtain the complete Kähler–Einstein metric on $D_{1,1}$. As an application, we prove that the Kähler–Einstein metric is equivalent to the Bergman metric on $D_{1,1}$ in Sect. 5. Then we shall expound the corresponding comparison theorem for $D_{n,m}$ in higher dimensions.

Finally, we shall investigate the behaviors of the Carathéodory and Kobayashi pseudo-metrics on $D_{n,m}$ L in Appendix. Further comparisons among these two pseudo-metrics and the Bergman metric on $D_{n,m}$ will be given by considering the explicit form of $\mathrm{Aut}(D_{n,m})$.

2 Preliminaries

2.1 Strong pseudoconvexity

We shall start by considering pseudoconvexity of domains in $\mathbb{C}^N$ $(N \geq 2)$. Let $\Omega \subset \mathbb{C}^N$ be a domain with C^2 boundary and p a point on the boundary $\partial\Omega$. A C^2 function $\rho : \mathbb{C}^N \to \mathbb{R}$ is called a *defining function* of Ω if $\Omega = \{z \in \mathbb{C}^N : \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega = \{z \in \mathbb{C}^N : \rho(z) = 0\}$. Let us denote by $\mathcal{T}_p(\partial\Omega)$ the complex tangent space to $\partial\Omega$ at p, that is,

$$\mathcal{T}_p(\partial\Omega) := \left\{ w = (w_1, \ldots, w_N) \in \mathbb{C}^N : \nabla_{(z)}\rho(p) \cdot w = \sum_{j=1}^{N} \frac{\partial\rho}{\partial z_j}(p) w_j = 0 \right\},$$

where $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$. Let us recall that $\partial\Omega$ is *pseudoconvex at* p if the *Levi form*

$$\sum_{j,k=1}^{N} \frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k}(p) w_j \overline{w}_k \geq 0, \quad \text{for all non-zero} \quad w \in \mathcal{T}_p(\partial\Omega). \tag{1}$$

A domain Ω is called *pseudoconvex* if $\partial\Omega$ is pseudoconvex at every $p \in \partial\Omega$. If the Levi form in (1) is strictly positive for every $p \in \partial\Omega$, then Ω is called a *strongly pseudoconvex* domain.

Consider a defining function ψ of $D_{n,m} \subset \mathbb{C}^n \times \mathbb{C}^m$ defined by

$$\psi(z, w) = \|w\|^2 - e^{-\|z\|^2}.$$

We choose a point $p = (z_0, w_0) := (z_{01}, \ldots, z_{0n}, w_{01}, \ldots, w_{0m})$ on the boundary $\partial D_{n,m}$. Then the complex Hessian of ψ at $p \in \partial D_{n,m}$ is

$$\mathcal{L}_\psi(p) = \begin{pmatrix} B & 0 \\ 0 & \mathrm{Id}_m \end{pmatrix},$$

where B is an $n \times n$ matrix with $e^{-\|z_0\|^2}(\delta_i^j - \overline{z_{0i}} z_{0j})$ in the (i, j) entry. For all non-zero $a = (a_1, a_2) := (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2m}) \in \mathcal{T}_p(\partial D_{n,m})$, we shall show that

$$\mathcal{L}_\psi(p)(a, a) := \begin{pmatrix} a_1 & a_2 \end{pmatrix} \mathcal{L}_\psi(p) \begin{pmatrix} {}^t\overline{a}_1 \\ {}^t\overline{a}_2 \end{pmatrix} > 0.$$

Since $a = (a_1, a_2)$ is a complex tangent vector to $\partial D_{n,m}$ at $p = (z_0, w_0)$, it follows from the Cauchy–Schwarz inequality that

$$\mathcal{L}_\psi(p)(a, a) = e^{-\|z_0\|^2} \|a_1\|^2 - e^{-\|z_0\|^2} \left| \sum_{i=1}^{n} \overline{z_{0i}} a_{1i} \right|^2 + \|a_2\|^2$$

$$
\begin{aligned}
&= e^{-\|z_0\|^2}\|a_1\|^2 - e^{\|z_0\|^2}\left|\sum_{\nu=1}^{m}\overline{w_{0\nu}}a_{2\nu}\right|^2 + \|a_2\|^2 \\
&\geq e^{-\|z_0\|^2}\|a_1\|^2 - e^{\|z_0\|^2}\|w_0\|^2\|a_2\|^2 + \|a_2\|^2 \\
&= e^{-\|z_0\|^2}\|a_1\|^2 - \|a_2\|^2 + \|a_2\|^2 \\
&= e^{-\|z_0\|^2}\|a_1\|^2 \\
&= \|w_0\|^2\|a_1\|^2.
\end{aligned} \tag{2}
$$

Aiming for a contradiction, suppose that $\mathcal{L}_\psi(p)(b, b) = 0$ for some non-zero $b = (b_1, b_2) \in \mathbb{C}^n \times \mathbb{C}^m$ which is a complex tangent vector to $\partial D_{n,m}$ at $p = (z_0, w_0)$. Then we see from (2) that at least one of b_1 and w_0 is a zero vector. However, the latter case cannot occur since the complex line $\{(z, 0) : z \in \mathbb{C}^n\}$ is completely contained in $D_{n,m}$. In the case when b_1 is a zero vector in $\mathbb{C}^n$, $\mathcal{L}_\psi(p)(b, b) = \|b_2\|^2 = 0$; hence $b_2 = 0$ which contradicts to the fact that b is chosen as a non-zero vector. This shows that each $D_{n,m}$ is a strongly pseudoconvex domain.

2.2 Non-compact automorphism group

Let Ω be a domain in $\mathbb{C}^n$. We denote by $\mathrm{Aut}(\Omega)$ the automorphism group of Ω which is the set of all biholomorphic self mappings of Ω equipped with the *compact-open topology*, equivalently the topology given by uniform convergence on compact subsets of Ω. Then $\mathrm{Aut}(\Omega)$ is *non-compact* if every orbit under the action of $\mathrm{Aut}(\Omega)$ is non-compact. Moreover, the non-compactness of $\mathrm{Aut}(\Omega)$ is equivalent to the existence of only one non-compact orbit. In the case of a bounded domain, the non-compactness of automorphism group is indeed equivalent to the existence of a *boundary orbit accumulation point*, which means that there exist points $q \in \partial\Omega$, $p \in \Omega$ and a sequence $\{f_j\} \subset \mathrm{Aut}(\Omega)$ such that $f_j(p) \to q$ as $j \to +\infty$ (cf. [11,15], and the references therein). On the contrary to the bounded case, for an unbounded strongly pseudoconvex domain, this equivalence does not hold generically: A generalization of Wong–Rosay theorem by Efimov [7] states that for a domain $\Omega \subset \mathbb{C}^n$ (not necessarily bounded) with C^2-smooth boundary, if Ω is strongly pseudoconvex at a boundary accumulation point $q \in \partial\Omega$, then Ω is biholomorphically equivalent to the unit ball in $\mathbb{C}^n$. On combining this theorem with the fact that $\mathrm{Aut}(D_{n,m})$ is non-compact and $D_{n,m}$ is not hyperbolic in the sense of Kobayashi, we prove the assertion. In this regard, $D_{n,m}$ can be considered as a remarkable prototype of unbounded strongly pseudoconvex domains with non-compact automorphism.

Now we shall explain the reason why $\mathrm{Aut}(D_{n,m})$ is indeed non-compact. Recently, Kim–Ninh–Yamamori [17] proved that $\mathrm{Aut}(D_{n,m})$ is generated by the following mappings:

$$
\begin{aligned}
r_U &: D_{n,m} \to D_{n,m}, \quad (z, w) \mapsto (Uz, w), \\
r_{U'} &: D_{n,m} \to D_{n,m}, \quad (z, w) \mapsto (z, U'w), \\
\tau_v &: D_{n,m} \to D_{n,m}, \quad (z, w) \mapsto (z + v, e^{-\langle z,v\rangle - \frac{1}{2}\|v\|^2} w),
\end{aligned} \tag{3}
$$

where $U \in U(n)$, $U' \in U(m)$, and $v \in \mathbb{C}^n$. Even though it is enough to show the existence of one non-compact orbit in order to demonstrate the non-compactness of $\mathrm{Aut}(D_{n,m})$, we shall explain the non-compactness of a real foliation of $D_{n,m}$ for its further geometric interpretation as follows: Precisely, for $(z, w) \in D_{n,m}$, we define an auxiliary function x by setting

$$
x = e^{\|z\|^2}\|w\|^2.
$$

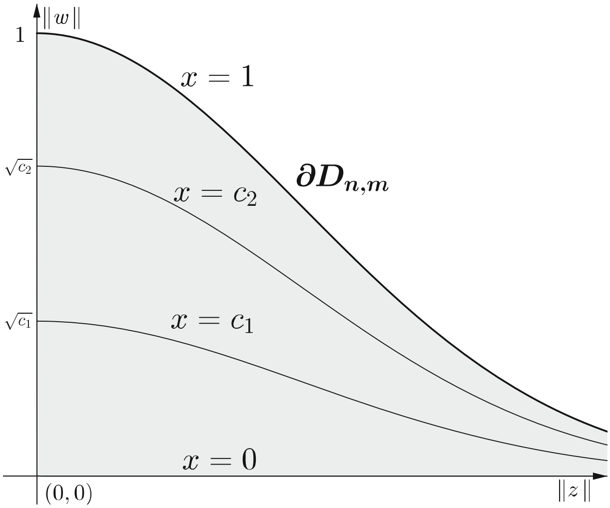

Fig. 1 Orbits of the automorphism group

Then we see from the definition of $D_{n,m}$ that $0 \leq x < 1$ in $D_{n,m}$ (see Fig. 1). Moreover, each leaf $x =$ constant is preserved by the automorphism group: Let us first fix a constant c such that $0 \leq c < 1$ and let φ be an automorphism of $D_{n,m}$ given by

$$\varphi(z, w) = r_U \circ r_{U'} \circ \tau_v(z, w) := (\varphi_1(z, w), \varphi_2(z, w))$$

for $(z, w) \in D_{n,m}$. Then, for any fixed point $(z, w) \in D_{n,m}$ such that $x = c$, we have

$$e^{\|\varphi_1(z,w)\|^2} \|\varphi_2(z, w)\|^2 = e^{\|z+v\|^2} e^{2\mathrm{Re}(-\langle z,v\rangle - \frac{1}{2}\|v\|^2)} \|w\|^2 = e^{\|z\|^2} \|w\|^2 = c.$$

Moreover, this computation is independent of the choice of $\varphi \in \mathrm{Aut}(D_{n,m})$, that is, $e^{\|\tilde{\varphi}_1(z,w)\|^2} \|\tilde{\varphi}_2(z, w)\|^2 = c$ if $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2) \in \mathrm{Aut}(D_{n,m})$ with $x = e^{\|z\|^2} \|w\|^2 = c$. Therefore, the non-compactness of the leaves $x =$ constant, conjunction with the invariance of each leaf under holomorphic automorphisms, yields the fact that $\mathrm{Aut}(D_{n,m})$ is non-compact.

2.3 Bergman kernel and Bergman metric

Let Ω be a domain in $\mathbb{C}^n$ and $A^2(\Omega)$ the space of square integrable holomorphic functions on Ω. Namely, the space $A^2(\Omega)$, which is called the Bergman space of Ω, is defined as

$$A^2(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f(z)|^2 \mathrm{d}V(z) < \infty \right\}.$$

The reproducing kernel K_Ω of $A^2(\Omega)$ is called the Bergman kernel of Ω. Let $\{\phi_k\}_{k\geq 0}$ be a complete orthonormal basis of $A^2(\Omega)$. Then the Bergman kernel is given by

$$K_\Omega(z, w) = \sum_{k=0}^{\infty} \phi_k(z) \overline{\phi_k(w)}. \tag{4}$$

Let us write

$$g_{i\overline{j}} = \frac{\partial^2 \log K_\Omega(z, z)}{\partial z_i \partial \overline{z}_j}.$$

Table 1 Form of $A_{n,m}(t)$

m/n	1	2	3
1	$1+t$	$2(2+t)$	$6(3+t)$
2	$1+4t+t^2$	$2(4+7t+t^2)$	$6(9+10t+t^2)$
3	$1+11t+11t^2+t^3$	$2(8+33t+18t^2+t^3)$	$6(27+67t+25t^2+t^3)$

We define the Bergman metric of Ω by

$$h_\Omega = \sum g_{i\overline{j}} \mathrm{d}z_i \otimes \mathrm{d}\overline{z}_j. \tag{5}$$

It is well known that the Bergman metric is biholomorphic invariant, yet it is quite difficult to compute the Bergman kernels explicitly in general. Fortunately, an explicit form of the Bergman kernel for our domain $D_{n,m}$ is known (see Theorem 3.1 and Remark 1 in [33]).

Theorem 2.1 *Let (z, w), (z', w') be arbitrary points in $D_{n,m}$. Then the Bergman kernel of $D_{n,m}$ is given as follows:*

$$\begin{aligned} K_{D_{n,m}}((z,w),(z',w')) &= \frac{e^{m\langle z,z'\rangle}}{\pi^{n+m}} \sum_{k=0}^{\infty} (k+1)_m (k+m)^n t^k \Big|_{t=e^{\langle z,z'\rangle}\langle w,w'\rangle} \\ &= \frac{e^{m\langle z,z'\rangle} A_{n,m}(t)}{\pi^{n+m}(1-t)^{n+m+1}} \Big|_{t=e^{\langle z,z'\rangle}\langle w,w'\rangle}, \end{aligned}$$

where $A_{n,m}$ is a polynomial defined by

$$A_{n,m}(t) = m! \sum_{j=0}^{n} (-1)^{n+j} (m+1)_j S(1+n, 1+j)(1-t)^{n-j}.$$

Here $(x)_m$ and $S(\cdot,\cdot)$ denote the Pochammer symbol and the Stirling number of the second kind, respectively.

The proof of this theorem is carried out using the Forelli–Rudin construction [21] (see also [25]) and an explicit form of the Fock–Bargmann kernel. For this reason, $D_{n,m}$ is called the Fock–Bargmann–Hartogs domain. We note that Springer [28] firstly computed the Bergman kernel of $D_{1,1}$ by constructing an explicit complete orthonormal basis of $A^2(D_{1,1})$. The domain $D_{1,1}$ is later called the Springer domain in [8]. The Table 1 gives explicit forms of the polynomial $A_{n,m}(t)$ for the first few cases.

Before explaining some properties of the polynomial $A_{n,m}(t)$, let us prepare some definitions. Let $a_{n,m}$ be the Eulerian number which is defined by

$$a_{n,m} := \sum_{r=1}^{m} (-1)^r \binom{n+1}{r} (m-r)^n.$$

We define the Eulerian polynomial $A_n(t)$ by

$$A_n(t) := \sum_{j=0}^{n-1} a_{n,j+1} t^j.$$

This polynomial is closely related to the zeta function $\zeta(s)$. The interested readers may consult Hirzebruch's paper [14] for further information of this polynomial. Let us now state some properties of the polynomial $A_{n,m}(t)$ [33, Lemma 4.3].

Lemma 2.2 *The polynomial $A_{n,m}$ satisfies the following recurrence relation*

$$A_{n,m+1}(t) = (n+m+1)A_{n,m}(t) + (1-t)A'_{n,m}(t),$$

with the initial condition $A_{n,1}(t) = A_{n+1}(t)$. Moreover, all coefficients of $A_{n,m}(t)$ are positive.

The recurrence relation in Lemma 2.2 is one of the main ingredients in showing the comparison theorem between the Bergman and Kähler–Einstein metrics for $D_{n,m}$.

We conclude this section with an explicit form of $(g_{i\bar{j}})$ for $D_{n,m}$ in (5). Denote by g the matrix $(g_{i\bar{j}})$ for $D_{n,m}$. Taking advantage of the invariance of the Bergman metric under biholomorphic mappings given in Eq. (3), we deduce that

$$g(z,w) = {}^t\,\mathrm{Jac}(\tau_{-z},(z,w))g(0,w^*)\overline{\mathrm{Jac}(\tau_{-z},(z,w))}.$$

Then, using Theorem 2.1, the entries of $g(z,w)$ can be written as follows: for $1 \le j, l \le n$ and $1 \le \lambda, \xi \le m$, we have

$$\begin{aligned} g_{j\bar{l}} &= (m + xF'(x))\delta_j^l \\ &\quad + \sum_{\sigma=1}^m \left(\sum_{k=1}^m \bar{z}_j z_l e^{\|z\|^2} w_k \bar{w}_\sigma \left(\widetilde{F_\sigma(x)}\delta_k^\sigma + (1-\delta_k^\sigma)F''(x)w_\sigma^* \bar{w}_k^* \right) \right), \qquad (6) \\ g_{j\overline{(n+\xi)}} &= \sum_{k=1}^m \bar{z}_j e^{\|z\|^2} w_k \left(\widetilde{F_\xi(x)}\delta_k^\xi + (1-\delta_k^\xi)F''(x)w_\xi^* \bar{w}_k^* \right) = \overline{g_{(n+\xi)\bar{j}}}, \\ g_{(n+\lambda)\overline{(n+\xi)}} &= e^{\|z\|^2}\left(\widetilde{F_\xi(x)}\delta_\lambda^\xi + (1-\delta_\lambda^\xi)F''(x)w_\xi^*\bar{w}_\lambda^* \right), \end{aligned}$$

where

$$\begin{cases} F(x) := \log\left(\left.\dfrac{A_{n,m}(t)}{(1-t)^{n+m+1}}\right|_{t=x} \right), \\ \widetilde{F_\sigma(x)} := F'(x) + |w_\sigma^*|^2 F''(x) \end{cases} \qquad (7)$$

for $1 \le \sigma \le m$, $w^* = (w_1^*, \ldots, w_m^*) := (e^{\frac{1}{2}\|z\|^2}w_1, \ldots, e^{\frac{1}{2}\|z\|^2}w_m)$ and $x = \|w^*\|^2$.

Example 2.1 We note that the associated entries of $g(z,w)$ for $D_{1,1}$ are given as follows:

$$\begin{aligned} g_{1\bar{1}} &= (1 + xF'(x)) + |z|^2 x \left(F'(x) + xF''(x)\right) \qquad (8) \\ &= S(x) + |z|^2 x S'(x), \\ g_{1\bar{2}} &= \bar{z}e^{|z|^2} w S'(x) = \overline{g_{2\bar{1}}}, \\ g_{2\bar{2}} &= e^{|z|^2} S'(x), \end{aligned}$$

where

$$\begin{cases} x := e^{|z|^2}|w|^2, \\ F(x) := \log \dfrac{(1+x)}{(1-x)^3}, \\ S(x) := 1 + xF'(x). \end{cases} \qquad (9)$$

Pflug and Zwonek considered the Bergman completeness of unbounded Hartogs domains with further assumptions at infinity(cf. [24, Theorem 6]). Their main ingredient for the proof was the *Kobayashi criterion* [19] together with pluricomplex Green function in pluripotential theory. In particular, $D_{1,1}$ is equipped with the Bergman completeness by utilizing their arguments.

3 Curvature estimates of Bergman metric

3.1 Holomorphic sectional curvature

We let R denote the curvature tensor of the Bergman metric which is given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{\alpha,\beta=1}^{2} g^{\alpha\bar{\beta}} \frac{\partial g_{i\bar{\beta}}}{\partial z_k} \frac{\partial g_{\alpha\bar{j}}}{\partial \bar{z}_l},$$

where $(g^{i\bar{j}})$ is the inverse matrix of $(g_{i\bar{j}})$. Then the holomorphic sectional curvature of the Bergman metric at $Z = (z_1, z_2) = (z, w)$ in the direction $X \in T_Z(D_{1,1})$ is defined by

$$H(Z; X) = \left(\sum_{\alpha,\beta} g_{\alpha\bar{\beta}}(Z) X_\alpha \overline{X_\beta} \right)^{-2} \sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}(Z) X_i \overline{X}_j X_k \overline{X}_l.$$

In ascertaining the holomorphic sectional curvature of the Bergman metric, we shall use the invariant property under biholomorphic mappings and the associated auxiliary function x for $D_{1,1}$. Recall the definition of τ_v from Sect. 2.3. As described above, for any fixed $(z_0, w_0) \in D_{1,1}$, the mapping τ_{-z_0} sends (z_0, w_0) to $(0, w^*)$ for $w^* = e^{\frac{1}{2}|z_0|^2} w_0$. Since a rotation in the w-coordinate belongs to $\mathrm{Aut}(D_{1,1})$, we could choose w^* to be real. This crucial observation, in conjunction with Theorem 2.1, implies that

$$H((z, w); X) = H((0, w^*); \mathrm{d}\tau_{-z}(X)) \tag{10}$$

which is a function of the auxiliary function x. Before going to the precise computation of the holomorphic sectional curvature of the Bergman metric on $D_{1,1}$, it is worth noting that the automorphism group of $D_{1,1}$ preserves the *leaves* $x =$ constant for $x = e^{|z|^2}|w|^2 \in [0, 1)$.

Let us define the differentials ∂ and $\bar{\partial}$ by setting

$$\partial := \frac{\partial}{\partial z}\mathrm{d}z + \frac{\partial}{\partial w}\mathrm{d}w \quad \text{and} \quad \bar{\partial} := \frac{\partial}{\partial \bar{z}}\mathrm{d}\bar{z} + \frac{\partial}{\partial \bar{z}}\mathrm{d}\bar{w}.$$

Instead of computing the values of $R_{i\bar{j}k\bar{l}}$ directly, we shall consider the following matrices of differential forms in order to determine $H((0, w^*); \mathrm{d}\tau_{-z}(X))$:

$$\begin{aligned} &\begin{pmatrix} \mathrm{d}z & \mathrm{d}w \end{pmatrix} g \begin{pmatrix} \mathrm{d}\bar{z} \\ \mathrm{d}\bar{w} \end{pmatrix} \Bigg|_{z=0}, \\ &-\partial\bar{\partial} g + (\partial g) g^{-1} \overline{(\partial g)}^T \Big|_{z=0}, \end{aligned} \tag{11}$$

where $g = (g_{i\bar{j}})$.

We now proceed to describe the matrices in (11) explicitly. Applying the chain rule to (8), we deduce that

$$\begin{aligned} \partial g_{1\bar{1}} = &\left(S'(x)\tfrac{\partial x}{\partial z} + \bar{z} x S'(x) + |z|^2 (S'(x) + x S''(x)) \tfrac{\partial x}{\partial z} \right) \mathrm{d}z \\ &+ \left(S'(x)\tfrac{\partial x}{\partial w} + |z|^2 (S'(x) + x S''(x)) \tfrac{\partial x}{\partial w} \right) \mathrm{d}w. \end{aligned} \tag{12}$$

This, conjunction with the setting (9), yields

$$\partial g_{1\bar{1}}\ |_{z=0} = S'(x)\bar{w}\mathrm{d}w. \tag{13}$$

Similar arguments to the remainings of entries in g show that

$$\partial g_{1\bar{2}}\big|_{z=0} = 0; \quad \partial g_{2\bar{1}}\big|_{z=0} = S'(x)\bar{w}\mathrm{d}z; \quad \partial g_{2\bar{2}}\big|_{z=0} = S''(x)\bar{w}\mathrm{d}w. \tag{14}$$

Taking the differential $\bar{\partial}$ to (12), we have

$$\bar{\partial}\partial g_{1\bar{1}}\big|_{z=0} = -\,\partial\bar{\partial} g_{1\bar{1}}\big|_{z=0} = -2xS'(z)\mathrm{d}z \wedge \mathrm{d}\bar{z} - \big(S'(x) + xS''(x)\big)\,\mathrm{d}w \wedge \mathrm{d}\bar{w}. \tag{15}$$

In addition to the above, we obtain

$$\begin{aligned}
\partial\bar{\partial} g_{1\bar{2}}\big|_{z=0} &= \big(S'(x) + xS''(x)\big)\,\mathrm{d}w \wedge \mathrm{d}\bar{z},\\
\partial\bar{\partial} g_{2\bar{1}}\big|_{z=0} &= \big(S'(x) + xS''(x)\big)\,\mathrm{d}z \wedge \mathrm{d}\bar{w},\\
\partial\bar{\partial} g_{2\bar{1}}\big|_{z=0} &= \big(S'(x) + xS''(x)\big)\,\mathrm{d}z \wedge \mathrm{d}\bar{z} + \Big(S''(x) + xS^{(3)}(x)\Big)\,\mathrm{d}w \wedge \mathrm{d}\bar{w}.
\end{aligned} \tag{16}$$

For convenience of exposition, we now rewrite the second matrix form in (11) as follows:

$$-\partial\bar{\partial} g + (\partial g)g^{-1}\overline{(\partial g)}^T\Big|_{z=0} = \big(\mathrm{d}z \ \ \mathrm{d}w\big)\begin{pmatrix} R_{1\bar{1}} & R_{1\bar{2}} \\ R_{2\bar{1}} & R_{2\bar{2}} \end{pmatrix}\begin{pmatrix} \mathrm{d}\bar{z} \\ \mathrm{d}\bar{w} \end{pmatrix}.$$

On combining (13) with (14), (15), and (16), we deduce that

$$\begin{aligned}
&R_{1\bar{1}}\mathrm{d}z \wedge \mathrm{d}\bar{z} + R_{1\bar{2}}\mathrm{d}z \wedge \mathrm{d}\bar{w} + R_{2\bar{1}}\mathrm{d}w \wedge \mathrm{d}\bar{z} + R_{2\bar{2}}\mathrm{d}w \wedge \mathrm{d}\bar{w}\\
&\quad = -2xS'(x)|\mathrm{d}z|^4 + 4\left(\frac{x(S'(x))^2}{S(x)} - S'(x) - xS''(x)\right)|\mathrm{d}z|^2|\mathrm{d}w|^2\\
&\qquad + \left(\frac{x(S''(x))^2}{S'(x)} - S''(x) - xS^{(3)}(x)\right)|\mathrm{d}w|^4.
\end{aligned} \tag{17}$$

Moreover, we have

$$\big(\mathrm{d}z\ \mathrm{d}w\big)\, g\begin{pmatrix} \mathrm{d}\bar{z} \\ \mathrm{d}\bar{w} \end{pmatrix}\Bigg|_{z=0} = S(x)|\mathrm{d}z|^2 + S'(x)|\mathrm{d}w|^2. \tag{18}$$

Using (17) and (18), we thus obtain the following:

Proposition 3.1 *The holomorphic sectional curvature $H((0, w^*); X)$ of $D_{1,1}$ is given by*

$$H((0, w^*); X) = \frac{Q(x; X)}{\big(S(x)|X_1|^2 + S'(x)|X_2|^2\big)^2},$$

where $F(x) = \log\dfrac{(1+x)}{(1-x)^3}$, $S(x) = 1 + xF'(x)$ and

$$\begin{aligned}
Q(x; X) = {}& -2xS'(x)|X_1|^4\\
&+ 4\left(\frac{x(S'(x))^2}{S(x)} - S'(x) - xS''(x)\right)|X_1|^2|X_2|^2\\
&+ \left(\frac{x(S''(x))^2}{S'(x)} - S''(x) - xS^{(3)}(x)\right)|X_2|^4
\end{aligned}$$

for $x = |w^*|^2 = e^{|z|^2}|w|^2$ and $X = (X_1, X_2) \in T_{(0,w^*)}(D_{1,1})$.

Now we shall estimate the holomorphic sectional curvature of the Bergman metric on $D_{1,1}$. For this purpose, we prepare the following lemma:

Lemma 3.2 *Let α, β, A, B and C be real numbers such that α, β, $C\alpha - B\beta$ and $A\beta - B\alpha$ are all positive. Set $f(X_1, X_2) = AX_1^2 + 2BX_1X_2 + CX_2^2$, $h(X_1, X_2) = \alpha X_1 + \beta X_2$. Then we have*

$$\max\{f(X_1, X_2) : X_1, X_2 \geq 0, h(X_1, X_2) = 1\} = \max\{A/\alpha^2, C/\beta^2\},$$
$$\min\{f(X_1, X_2) : X_1, X_2 \geq 0, h(X_1, X_2) = 1\} = \frac{AC - B^2}{A\beta^2 - 2B\beta\alpha + C\alpha^2}.$$

For the proof of this lemma, see [1, Lemma 4]. By virtue of Lemma 3.2, after normalizing the value of $\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} X_\alpha \overline{X_\beta}$, the lower and upper curvatures of $D_{1,1}$ are completely determined by rational functions induced from α, β, A, B and C. Precisely, the associated constants for $D_{1,1}$ are given as follows:

$$\begin{aligned}
&\alpha = S(x), \quad \beta = S'(x),\\
&A = -2xS'(x),\\
&B = 2\left(\frac{x(S'(x))^2}{S(x)} - S'(x) - xS''(x)\right),\\
&C = \left(\frac{x(S''(x))^2}{S'(x)} - S''(x) - xS^{(3)}(x)\right).
\end{aligned}$$

In determining the $\max\{A/\alpha^2, C/\beta^2\}$ firstly, we shall employ the monotonicity of A/α^2 and C/β^2. Namely, one can show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{A}{\alpha^2}\right) = -\frac{8(1-x^2)(1-x)^2}{(1+4x+x^2)^3} < 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{C}{\beta^2}\right) = -\frac{9(1-x^2)^3}{4(1+x+x^2)^4} < 0$$

for $0 \leq x < 1$. This relation tells us that the $\max\{A/\alpha^2, C/\beta^2\}$ occurs when $x = 0$. Thus, we see that $\max_{x=0}\{A/\alpha^2, C/\beta^2\} = \max\{0, -1/4\} = 0$.

We next consider the function $\dfrac{AC - B^2}{A\beta^2 - 2B\beta\alpha + C\alpha^2}$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{AC - B^2}{A\beta^2 - 2B\beta\alpha + C\alpha^2}\right) = \frac{8(1+x)(1-x)^3 t(x)}{3(1+4x+x^2)^3(5+14x+22x^2+14x^3+5x^4)^2} > 0$$

for $t(x) = 43 + 280x + 808x^2 + 1336x^3 + 1546x^4 + 1336x^5 + 808x^6 + 280x^7 + 43x^8$, it follows that

$$\min\left\{\frac{AC - B^2}{A\beta^2 - 2B\beta\alpha + C\alpha^2}\right\}_{0\leq x<1} = \left.\frac{AC - B^2}{A\beta^2 - 2B\beta\alpha + C\alpha^2}\right|_{x=0} = -\frac{16}{15}.$$

We also show the boundary limit of the holomorphic sectional curvature of the Bergman metric on $D_{1,1}$. Indeed, the leaf $x = 1$ clearly coincides with the boundary $\partial D_{1,1}$. Comparing the degrees of $(1 - x)$ in $H((z, w); X)$, we get

$$\begin{aligned}\lim_{(z,w)\to\partial D_{1,1}} H((z,w);X) &= \lim_{x\to 1}\frac{C}{\beta^2}\\ &= -\lim_{x\to 1}\frac{x^6+12x^5+15x^4+16x^3+15x^2+12x+1}{4\left(x^2+x+1\right)^3}\\ &= -\frac{2}{3}. \end{aligned} \tag{19}$$

Let us denote by $L(z,w)$ and $U(z,w)$ the lower and upper curvatures of $D_{1,1}$, respectively, that is,

$$L(z,w) := \min\left\{ H((z,w);X) : \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}(z,w)X_\alpha\overline{X_\beta} = 1,\ X \in T_{(z,w)}(D_{1,1})\right\},$$

$$U(z,w) := \max\left\{ H((z,w);X) : \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}(z,w)X_\alpha\overline{X_\beta} = 1,\ X \in T_{(z,w)}(D_{1,1})\right\}.$$

Altogether, we are now ready to expound our first main theorem:

Theorem 3.3 *Let $H((z,w);X)$ be the holomorphic sectional curvature of the Bergman metric at (z,w) in the direction $X \in T_{(z,w)}(D_{1,1})$. Then, for the auxiliary function x, we have:*

(i) $\lim_{(z,w)\to\partial D_{1,1}} H((z,w);X) = \lim_{x\to 1} H((0,w^*);d\tau_{-z}(X)) = -\frac{2}{3}$, *where* $\tau_v(z,w) = (z+v, e^{-z\bar{v}-\frac{1}{2}|v|^2}w)$ *for* $v\in\mathbb{C}$,
(ii) $L(z,w) = L(0,w^*)$ *and* $U(z,w) = U(0,w^*)$,
(iii) $L(z,w)$ *is strictly increasing with respect to* x,
(iv) $U(z,w)$ *is strictly decreasing with respect to* x*; hence,*
(v) $-\frac{16}{15} \le L(z,w) \le U(z,w) \le 0$.

Remark 1 Even though our domain $D_{1,1}$ is unbounded, this theorem indicates that an analog of a geometric property on bounded strongly pseudoconvex domains also holds for our unbounded domain $D_{1,1}$. This assertion is endorsed by (19) and the Klembeck's result [18] that the boundary limit of the curvature tensor of the Bergman metric on a bounded strongly pseudoconvex domain Ω approaches uniformly to a constant which depends only on the dimension of Ω.

3.2 Ricci curvature

For any bounded pseudoconvex domain with C^2-smooth boundary, Cheng and Yau [6] proved that there exists a unique complete Kähler–Einstein metric with Ricci curvature -1. Their profound result was later extended to more general classes of domains. For more details, we refer the reader to [4,9,22]. In this subsection, we shall focus attention to the Ricci curvature of the Bergman metric on $D_{1,1}$ which is unbounded but strongly pseudoconvex.

Let $g = (g_{i\bar{j}})$ be the 2×2-matrix corresponding to the Bergman metric on $D_{1,1}$ as defined in Eq. (8). Then, using the definitions of x and $S(x)$ in (9), we deduce that

$$\begin{aligned}\det(g(z,w)) &= e^{|z|^2} S(x)S'(x) \\ &= \frac{\pi^2(1-x)^3}{(1+x)} S(x)S'(x)K_{D_{1,1}}((z,w),(z,w)).\end{aligned} \tag{20}$$

Let $G(z,w) = \det(g(z,w))$. Then the *Ricci tensor* of the Bergman metric is given by

$$R_{\alpha\bar{\beta}} = -\frac{\partial^2 G(z_1,z_2)}{\partial z_\alpha \partial \bar{z}_\beta},$$

where $(z_1, z_2) = (z, w) \in D_{1,1}$. Taking the logarithm in (20), it follows that

$$\log G(z,w) = \log \pi^2 - F(x) + \log(S(x)S'(x)) + \log K_{D_{1,1}}((z,w),(z,w)),$$

where $F(x) = \log \frac{(1+x)}{(1-x)^3}$ defined in (9). For convenience of exposition, we define a function $T(x)$ by setting

$$\begin{aligned}T(x) &:= -F(x) + \log(S(x)S'(x)) \\ &= \log \tfrac{(x^2+4x+1)(x^2+x+1)}{(1+x)^4}.\end{aligned} \tag{21}$$

Then the Ricci tensor of the Bergman metric on $D_{1,1}$ can be rewritten as the following the form:

$$R_{\alpha\bar{\beta}} = -g_{\alpha\bar{\beta}} - \frac{\partial^2 T(x)}{\partial z_\alpha \partial \bar{z}_\beta},$$

where $1 \le \alpha, \beta \le 2$.

Now we shall investigate the boundary limit of the Ricci curvature. For this purpose, we first consider the square of the length of a vector X with respect to $\sum_{\alpha,\beta} R_{\alpha\bar{\beta}}(z,w)\mathrm{d}z_\alpha \otimes \mathrm{d}\bar{z}_\beta$ at the point $(z, w) \in D_{1,1}$, given by

$$\sum_{\alpha,\bar{\beta}} R_{\alpha\bar{\beta}}(z,w) X_i \overline{X}_j \tag{22}$$

for $X = (X_1, X_2) \in \mathbb{C}^2$. Note that the quantity $\sum_{\alpha,\beta} R_{\alpha\bar{\beta}}(z,w)\mathrm{d}z_\alpha \otimes \mathrm{d}\bar{z}_\beta$ is invariant under biholomorphic mappings. For this reason, after using the automorphism group $\mathrm{Aut}(D_{1,1})$, we shall compute the value in (22) only at the point $(0, w^*)$ where $w^* = e^{\frac{1}{2}|z|^2} w$. For a given vector X in (22), we let $\widetilde{X}$ denote the vector in $\mathbb{C}^2$ such that $\mathrm{d}\varphi(X) = \widetilde{X}$ where $\varphi(z,w) = (0, w^*)$ for an automorphism $\varphi \in \mathrm{Aut}(D_{1,1})$. By abuse of notation, we continue to write X for $\widetilde{X}$. Then it follows from the above settings (9) and (21) that

$$\sum_{\alpha,\beta} R_{\alpha\bar{\beta}}(0,w^*) X_i \widetilde{X}_j = -\left(S(x) + xT'(x)\right)|X_1|^2 - \left(S(x) + xT'(x)\right)'|X_2|^2.$$

This imposes the following relation:

$$\frac{\sum_{\alpha,\beta} R_{\alpha\bar{\beta}}(0,w^*) X_i \overline{X}_j}{\sum_{\alpha,\beta} g_{\alpha,\bar{\beta}}(0,w^*) X_i \overline{X}_j} = -1 - \frac{xT'(x)|X_1|^2 + \left(xT'(x)\right)'|X_2|^2}{S(x)|X_1|^2 + S'(x)|X_2|^2}. \tag{23}$$

Since $x \to 1$ as $(z,w) \to \partial D_{1,1}$, direct computation shows that

$$\lim_{x\to 1} S(x) = \lim_{x\to 1} S'(x) = \infty \quad \text{and} \quad \lim_{x\to 1} xT'(x) = \lim_{x\to 1} \left(xT'(x)\right)' = 0. \tag{24}$$

Then (24) yields the boundary limit on the left-hand side of (23)

$$\lim_{x\to 1}\frac{\sum_{\alpha,\beta} R_{\alpha\bar{\beta}}(0,w^*)X_i\overline{X}_j}{\sum_{\alpha,\beta} g_{\alpha,\bar{\beta}}(0,w^*)X_i\overline{X}_j}=-1$$

which is independent of the choice of X. Thus, we conclude that the Bergman metric on $D_{1,1}$ is "asymptotically" Ricci-negative Kähler–Einstein near the boundary $\partial D_{1,1}$.

4 Kähler–Einstein metric

In this section, we give a systematic description for the Kähler–Einstein metric and the associated sectional curvature on $D_{1,1}$ which is an unbounded non-hyperbolic strongly pseudoconvex domain. Our investigation about the Kähler–Einstein metric is motivated by the following problem due to Yau [34]: *classify pseudoconvex domains whose Bergman metrics are Kähler–Einstein*. This approach allows descendent to invoke the depiction of a domain in terms of the differential geometric properties of its Bergman metric. Concerning such a research, Nemirovski and Shafikov [23] proved that for a relatively compact strongly pseudoconvex domain Ω in $\mathbb{C}^2$, if the Bergman metric on Ω is Kähler–Einstein, then the domain is biholomorphically equivalent to the unit ball. In this regard, if it exists, the Kähler–Einstein metric on $D_{1,1}$ cannot be equal to its Bergman metric since the Kobayashi-hyperbolic property is invariant under biholomorphic mappings. Regarding this particular phenomenon contrary to the bounded case, it is meaningful to detect the disparity between these two biholomorphic invariant metrics.

Before comparing these metrics precisely, we shall construct the Kähler–Einstein metric on $D_{1,1}$ to carry conviction for its existence. Since $D_{1,1}$ possesses the non-compact automorphism group, it would be possible to deduce the associated auxiliary function x for $D_{1,1}$, even if it is unbounded. Moreover, the existence of the auxiliary function will save us the difficulty in solving the associated complex Monge–Ampére equation with a boundary condition. This speculation will be demonstrated below.

Let us consider the following complex Monge–Ampère equation with a boundary condition:

$$\begin{aligned}\det\left(\frac{\partial^2 h}{\partial z_i\partial\bar{z}_j}(Z)\right) &= e^{3h(Z)} \quad \text{for} \quad Z=(z_1,z_2)=(z,w)\in D_{1,1},\\ h(Z)&\to+\infty \quad \text{as} \quad Z\to\partial D_{1,1}.\end{aligned} \tag{25}$$

Then the complete Kähler–Einstein metric g^{KE}, if it exists, is generated by the unique solution h of (25):

$$g^{\text{KE}}:=\sum_{i,j}\frac{\partial^2 h}{\partial z_i\partial\bar{z}_j}\mathrm{d}z_i\otimes\mathrm{d}\bar{z}_j.$$

For $\tau_{-z}\in \mathrm{Aut}(D_{1,1})$ defined in Sect. 2, the invariance of the metric g^{KE} and the Monge–Ampère equation (25) imply

$$\begin{aligned}e^{3h(z,w)}&=|\det \mathrm{Jac}(\tau_{-z},(z,w))|^2e^{3h(0,w^*)}\\ &=e^{|z|^2}e^{3h(0,w^*)},\end{aligned} \tag{26}$$

where $|w^*|^2 = e^{|z|^2}|w|^2$. Taking the logarithm of (26), we deduce that

$$3h(z, w) = |z|^2 + 3h(0, w^*). \tag{27}$$

Let us recall the auxiliary function x defined by $x = |w^*|^2 \in [0, 1)$. Then (27) ensures that g^{KE} can be parametrized by $x \in [0, 1)$. For this reason, we denote by $H(x)$ the function defined by

$$H(x) = h(0, w^*).$$

Now we shall show that the Monge–Ampère equation (25) is equivalent to an ordinary differential equation for the function $H(x)$:

Proposition 4.1 *Let h be a C^2 function on $D_{1,1}$, which is a solution of the Monge–Ampère equation*

$$\det\left(\frac{\partial^2 h}{\partial z_i \partial \bar{z}_j}\right) = e^{3h}$$

and which generates an invariant form $\partial\bar{\partial}h$. Let

$$h(z, w) = \frac{1}{3}|z|^2 + H(x), \quad x = e^{|z|^2}|w|^2.$$

Then $H(x)$ satisfies the following differential equation on the interval $(0, 1)$:

$$\left(xH'(x) + \frac{1}{3}\right)\left(xH'(x)\right)' = e^{3H(x)},$$
$$xH'(x) \to 0 \quad as \quad x \to 0.$$

The boundary condition $h(z, w) \to \infty$ as $(z, w) \to \partial D_{1,1}$ implies

$$H(x) \to \infty \quad as \quad x \to 1. \tag{28}$$

Proof Applying the chain rule to the setting $x = e^{|z|^2}|w|^2$, one can see that

$$\begin{aligned} e^{3H(x)} &= e^{3h(z,w)}\big|_{(0,w^*)} \\ &= \det\left(\frac{\partial^2 h}{\partial z_i \partial \bar{z}_j}\right)\Bigg|_{(0,w^*)} \\ &= \det\begin{pmatrix} \frac{1}{3} + xH'(x) & 0 \\ 0 & H'(x) + xH''(x) \end{pmatrix} \\ &= \left(xH'(x) + \tfrac{1}{3}\right)\left(xH'(x)\right)'. \end{aligned} \tag{29}$$

The vanishing property of $xH'(x)$ as $x \to 0$, is derived by the fact that the function h is C^2. Indeed, since

$$\frac{\partial h}{\partial w^*}(0, w^*) = \bar{w}^* H'(x) = x^{1/2}H'(x),$$

the C^2-smoothness of h implies that

$$xH'(x) \to 0 \quad \text{as} \quad x \to 0.$$

In addition, since $x \to 1$ as $(z, w) \to \partial D_{1,1}$, we thus obtain

$$H(x) \to \infty \quad \text{as} \quad x \to 1$$

which completes the proof. □

From the previous proposition, (25) can be reformulated as the following ordinary differential equation:

$$e^{3H(x)} = G(x)G'(x) \tag{30}$$

with the initial condition $G(0) = \frac{1}{3}$, where $G(x) := \frac{1}{3} + xH'(x)$.

Instead of finding the solution of (25), we shall embody an implicit solution of (30) in what follows. Moreover, such a solution will be uniquely determined. Since $G(x)G'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}G^2(x)\right)$, the differential equation (30) yields

(i) $G(x)G'(x) > 0$ and $G^2(x)$ is strictly increasing,
(ii) $G(0) = \frac{1}{3}$ and $G'(0) > 0$; hence,
(iii) $G'(x) > 0$ on $[0, 1)$.

Taking the logarithmic derivatives of both sides of (30), we deduce that

$$\frac{(G(x)G'(x))'}{G(x)G'(x)} = 3H'(x) = \frac{3G(x) - 1}{x}. \tag{31}$$

The last equality clearly follows from the definition of $G(x)$ in (30). Using (31), we obtain

$$\begin{aligned}(xG(x)G'(x))' &= G(x)G'(x) + x(G(x)G'(x))' \\ &= G(x)G'(x) + (3G(x) - 1)G(x)G'(x) \\ &= 3G^2(x)G'(x).\end{aligned}$$

Then, integrating with the initial condition $G(0) = \frac{1}{3}$, we get

$$xG(x)G'(x) = G^3(x) - \frac{1}{27}. \tag{32}$$

The following lemma is the key to construct the generating function h for the Kähler–Einstein metric on $D_{1,1}$.

Lemma 4.2 *The differential equation*

$$\begin{cases} xG(x)G'(x) = G^3(x) - \frac{1}{27}, \\ G(0) = \frac{1}{3}, \\ G(x) \to \infty \quad as \quad x \to 1, \end{cases} \tag{33}$$

has a unique solution

$$G : [0, 1) \to [1/3, \infty).$$

Proof As noted above, since $G(0) = \frac{1}{3}$ and $G'(x) > 0$ on $[0, 1)$, a function $G^3(x) - 1/27$ is positive monotone increasing. Then its inverse function satisfies the following:

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}G} = \frac{G}{G^3 - \frac{1}{27}},$$
$$x \to 1 \quad \text{as} \quad G \to \infty.$$

The solution of the previous equation is given by

$$-\log x = \int_G^\infty \frac{y}{y^3 - \frac{1}{27}}\mathrm{d}y. \tag{34}$$

This relation shows that G can be presented as an implicit function of x. By computing the integral of the right-hand side in (34), we thus obtain

$$x = \frac{G(x) - \frac{1}{3}}{\sqrt{G^2(x) + \frac{1}{3}G(x) + \frac{1}{9}}} \exp\left(-\frac{\sqrt{3}}{2}\pi + \sqrt{3}\arctan\left(2\sqrt{3}\left(G(x) + \frac{1}{6}\right)\right)\right), \quad (35)$$

which proves the assertion. □

Now we are ready to state our second main result:

Theorem 4.3 *The generating function h for the Kähler–Einstein metric of*

$$D_{1,1} = \{(z, w) \in \mathbb{C} \times \mathbb{C} : \ |w|^2 < e^{-|z|^2}\}$$

is given by

$$h(z, w) = \frac{1}{3}|z|^2 + H(e^{|z|^2}|w|^2),$$

where

$$e^{3H} = GG'$$

and the function $G : [0, 1) \to [1/3, \infty)$ *is the solution of the differential equation* (33).

Remark 2 The fact that $\partial\bar{\partial}h$ defines a Kähler metric naturally follows from the relations $G(x) = \frac{1}{3} + xH'(x)$, $G(x)G'(x) > 0$, and $G(x) \geq \frac{1}{3}$ in Proposition 4.1. More precisely, we have

$$\begin{aligned}\partial\bar{\partial}h &= \partial\bar{\partial}\left(\tfrac{1}{3}|z|^2 + H(e^{|z|^2}|w|^2)\right)\\ &= G(x)\partial\bar{\partial}(|z|^2) + x(xH'(x))'\left(\tfrac{\mathrm{d}w}{w} + \partial(|z|^2)\right) \wedge \left(\tfrac{\mathrm{d}\bar{w}}{\bar{w}} + \bar{\partial}(|z|^2)\right)\\ &= G(x)\partial\bar{\partial}(|z|^2) + xG'(x)\left(\tfrac{\mathrm{d}w}{w} + \partial(|z|^2)\right) \wedge \left(\tfrac{\mathrm{d}\bar{w}}{\bar{w}} + \bar{\partial}(|z|^2)\right).\end{aligned}$$

Let us denote by $\widetilde{H}_{(z,w)}(X_1, X_2)$ the associated Hermitian metric at the point (z, w) in the direction $(X_1, X_2) \in \mathbb{C}^2$. Then we obtain

$$\widetilde{H}_{(z,w)}(X_1, X_2) = G(x)|X_1|^2 + xG'(x)\left|\frac{X_2}{w} + \bar{z}X_1\right|^2. \quad (36)$$

Since $G(x) > 0$ and $G'(x) > 0$ for $x \in [0, 1)$, $\widetilde{H}_{(z,w)}(X_1, X_2) \geq 0$ for all vectors X_1 and X_2. If $\widetilde{H}_{(z,w)}(X_1, X_2) = 0$, then $X_1 = X_2 = 0$ which proves the assertion.

4.1 Holomorphic sectional curvature

A similar argument to that in Sect. 3.1 shows that

$$\begin{aligned}&\widetilde{R}_{1\bar{1}}\mathrm{d}z \wedge \mathrm{d}\bar{z} + \widetilde{R}_{1\bar{2}}\mathrm{d}z \wedge \mathrm{d}\bar{w} + \widetilde{R}_{2\bar{1}}\mathrm{d}w \wedge \mathrm{d}\bar{z} + \widetilde{R}_{2\bar{2}}\mathrm{d}w \wedge \mathrm{d}\bar{w}\\ &= -2xG'(x)|dz|^4\\ &\quad + 4\left(\frac{x(G'(x))^2}{G(x)} - G'(x) - xG''(x)\right)|\mathrm{d}z|^2|\mathrm{d}w|^2\\ &\quad + \left(\frac{x(G''(x))^2}{G'(x)} - G''(x) - xG^{(3)}(x)\right)|\mathrm{d}w|^4,\end{aligned} \quad (37)$$

where

$$-\partial\bar{\partial}g^{\mathrm{KE}} + (\partial g^{\mathrm{KE}})(g^{\mathrm{KE}})^{-1}\overline{(\partial g^{\mathrm{KE}})}^T\Big|_{z=0} = \begin{pmatrix}\mathrm{d}z & \mathrm{d}w\end{pmatrix}\begin{pmatrix}\widetilde{R}_{1\bar{1}} & \widetilde{R}_{1\bar{2}}\\ \widetilde{R}_{2\bar{1}} & \widetilde{R}_{2\bar{2}}\end{pmatrix}\begin{pmatrix}\mathrm{d}\bar{z}\\ \mathrm{d}\bar{w}\end{pmatrix}.$$

Since $xG(x)G'(x) = G^3(x) - \frac{1}{27}$ and $G(x) > 0$ on $x \in [0, 1)$, we can force

$$xG'(x) = G^2(x) - \frac{1}{27G(x)}. \tag{38}$$

Substitute (38) into the second term in (37) to obtain that

$$\begin{aligned} 4\left(\frac{x(G'(x))^2}{G(x)} - G'(x) - xG''(x)\right) &= \frac{4G'(x)}{G(x)}(xG'(x)) - 4(xG'(x))' \\ &= -4G(x)G'(x)\left(1 + \frac{2}{27G^3(x)}\right). \end{aligned} \tag{39}$$

Considering the derivatives of (38) up to the second order, we get

$$\begin{aligned} xG''(x) &= 2G(x)G'(x) + \frac{G'(x)}{27G^2(x)} - G'(x); \\ xG^{(3)}(x) &= 2(G'(x))^2 + 2G(x)G''(x) + \frac{G''(x)}{27G^2(x)} - \frac{2(G'(x))^2}{27G^3(x)} - 2G''(x). \end{aligned} \tag{40}$$

Then it follows from (40) that the third term in (37) can be rewritten as

$$\begin{aligned} &\frac{x(G''(x))^2}{G'(x)} - G''(x) - xG^{(3)}(x) \\ &= xG''(x)\left(\frac{G''(x)}{G'(x)}\right) - G''(x) - xG^{(3)}(x) \\ &= -2(G'(x))^2\left(1 - \frac{1}{27G^3(x)}\right). \end{aligned} \tag{41}$$

On combining (39) with (41), we obtain

$$\begin{aligned} &\widetilde{R}_{1\bar{1}}\mathrm{d}z \wedge \mathrm{d}\bar{z} + \widetilde{R}_{1\bar{2}}\mathrm{d}z \wedge \mathrm{d}\bar{w} + \widetilde{R}_{2\bar{1}}\mathrm{d}w \wedge \mathrm{d}\bar{z} + \widetilde{R}_{2\bar{2}}\mathrm{d}w \wedge \mathrm{d}\bar{w} \\ &\quad = -2G^2(x)\left(1 - \tfrac{1}{27G^3(x)}\right)|\mathrm{d}z|^4 \\ &\qquad -4G(x)G'(x)\left(1 + \frac{2}{27G^3(x)}\right)|\mathrm{d}z|^2|\mathrm{d}w|^2 \\ &\qquad -2(G'(x))^2\left(1 - \tfrac{1}{27G^3(x)}\right)|\mathrm{d}w|^4. \end{aligned} \tag{42}$$

Moreover, we have

$$\left(\mathrm{d}z\ \mathrm{d}w\right) g^{\mathrm{KE}} \begin{pmatrix} \mathrm{d}\bar{z} \\ \mathrm{d}\bar{w} \end{pmatrix}\Bigg|_{z=0} = G(x)|\mathrm{d}z|^2 + G'(x)|\mathrm{d}w|^2. \tag{43}$$

Using (42) and (43), we are now ready to state the following:

Theorem 4.4 *Let $\widetilde{H}((z, w); X)$ be the holomorphic sectional curvature of the Kähler–Einstein metric at (z, w) in the direction $X \in T_{(z,w)}(D_{1,1})$. Then we have*

$$\widetilde{H}((z, w); X) \leq 0.$$

Proof To demonstrate $\widetilde{H}((z, w); X) \leq 0$, it is enough to show

$$\widetilde{H}((0, w^*); \widetilde{X}) \leq 0$$

such that $d\varphi(X) = \widetilde{X}$ where $\varphi(z, w) = (0, w^*) = (0, e^{\frac{1}{2}|z|^2} w)$ for an automorphism $\varphi \in \mathrm{Aut}(D_{1,1})$. Since $G(x)G'(x) > 0$ and $G(x) \geq \frac{1}{3}$ on $[0, 1)$, we thus have

$$-\frac{1}{2}\widetilde{H}((0, w^*); \widetilde{X}) = \left(1 - \frac{1}{27G^3(x)}\right) + \frac{\frac{2G'(x)}{9G^2(x)}|\widetilde{X}_1|^2|\widetilde{X}_2|^2}{\left(G(x)|\widetilde{X}_1|^2 + G'(x)|\widetilde{X}_2|^2\right)^2} \geq 0$$

for any $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2) \in T_{(0,w^*)}(D_{1,1})$. This finishes the proof. □

5 Comparison of the Bergman and Kähler–Einstein metrics

In this section, we first investigate the comparison of the Bergman and Kähler–Einstein metrics on $D_{1,1}$. For convenience of exposition, in the case of $D_{n,m}$ in higher dimensions, we will explain the details for the associated comparison theorem later on.

5.1 Comparison on $D_{1,1}$

As noted before, $D_{1,1}$ is unbounded and non-hyperbolic in the sense of Kobayashi. Moreover, it turned out that the Bergman metric on $D_{1,1}$ is not Kähler–Einstein in Sect. 3.2. To investigate the disparity of these two metrics, we restrict our detection to the behavior of $\frac{g_{\alpha\bar{\beta}}}{g^{\mathrm{KE}}_{\alpha\bar{\beta}}}$ only at the points $(0, w^*)$ for $|w^*|^2 = e^{|z|^2}|w|^2 = x$ and $(z, w) \in D_{1,1}$, using their biholomorphic invariant property. We denote by $(g_{\alpha\bar{\beta}})$ and $(g^{\mathrm{KE}}_{\alpha\bar{\beta}})$ the associated matrix representations describing the tensors corresponding to g and g^{KE}, respectively. Since $(g_{\alpha\bar{\beta}})$ and $(g^{\mathrm{KE}}_{\alpha\bar{\beta}})$ are forms of diagonal matrices at $(0, w^*)$, we consider only the fractions

$$\frac{g_{1\bar{1}}(0, w^*)}{g^{\mathrm{KE}}_{1\bar{1}}(0, w^*)} \quad \text{and} \quad \frac{g_{2\bar{2}}(0, w^*)}{g^{\mathrm{KE}}_{2\bar{2}}(0, w^*)}. \tag{44}$$

Before going to precise computation of the fractions in (44), it is worthy to note that these fractions are continuous functions for the variable x in a bounded interval $[0, 1)$. Applying (8), (9) and (35) to the first fraction in (44), we get

$$\lim_{x\to 0} \frac{g_{1\bar{1}}(0, w^*)}{g^{\mathrm{KE}}_{1\bar{1}}(0, w^*)} = \lim_{x\to 0} \frac{S(x)}{G(x)} = \frac{S(0)}{G(0)} = \frac{1}{\frac{1}{3}} = 3.$$

Since $G(0) = \frac{1}{3}$ and $G'(0)$ is of finite positive value in the proof of Lemma 4.2, the first derivative of (35) yields

$$G'(0) = \frac{e^{\frac{\sqrt{3}}{6}\pi}}{\sqrt{3}}. \tag{45}$$

Similar argument to the second fraction in (44), in conjunction with (45), implies

$$\lim_{x\to 0} \frac{g_{2\bar{2}}(0, w^*)}{g^{\mathrm{KE}}_{2\bar{2}}(0, w^*)} = \lim_{x\to 0} \frac{S'(x)}{G'(x)} = \lim_{x\to 0} \frac{F'(x) + xF''(x)}{G'(x)} = \frac{F'(0)}{G'(0)} < \infty.$$

To compute the associated limits in (44) as $x \to 1$, we first consider the following relation:

$$\lim_{x\to 1} \frac{g_{2\bar{2}}(0, w^*)}{g^{\mathrm{KE}}_{2\bar{2}}(0, w^*)} = \lim_{x\to 1} \frac{S'(x)}{G'(x)} = \lim_{x\to 1} \frac{\frac{4(x^2+x+1)}{(1-x^2)^2}}{G'(x)} = 12 \lim_{x\to 1} \frac{\frac{1}{(1-x^2)^2}}{G'(x)}.$$

Indeed, the latter equality follows from the fact that

$$\lim_{x\to 1} 4(x^2+x+1) = 12 \quad \text{and} \quad \lim_{x\to 1} \frac{\frac{1}{(1-x^2)^2}}{G'(x)} = \frac{1}{4}, \tag{46}$$

using a change of the variable x into a function for G and the implicit relation (35). More precisely, after using the Taylor's polynomial of degree 1 at the origin of exponential function and the product rule of limits in proving (46), we obtain

$$\begin{aligned}
\lim_{x\to 1} \frac{\frac{1}{(1-x^2)^2}}{G'(x)} &= \lim_{x\to 1} \frac{xG(x)\left(G^2(x)+\frac{1}{3}G(x)+\frac{1}{9}\right)^2}{\left(G^3(x)-\frac{1}{27}\right)\left(G^2(x)+\frac{1}{3}G(x)+\frac{1}{9}-(G(x)-\frac{1}{3})^2e^{\Phi(x)}\right)^2}\\
&= \lim_{G(x)\to+\infty} \frac{G(x)\left(G^2(x)+\frac{1}{3}G(x)+\frac{1}{9}\right)^2}{\left(G^3(x)-\frac{1}{27}\right)\left(G^2(x)+\frac{1}{3}G(x)+\frac{1}{9}-(G(x)-\frac{1}{3})^2e^{\Phi(x)}\right)^2}\\
&= \lim_{y\to+\infty} \frac{y\left(y^2+\frac{1}{3}y+\frac{1}{9}\right)^2}{\left(y^3-\frac{1}{27}\right)\left(y^2+\frac{1}{3}y+\frac{1}{9}-(y-\frac{1}{3})^2(1-\frac{1}{y})\right)^2}\\
&= \frac{1}{4},
\end{aligned}$$

where $\Phi(x) := -\sqrt{3}\pi + 2\sqrt{3}\arctan(2\sqrt{3}(\frac{1}{6}+G(x)))$. On combining (46) with the product rule of limits, we deduce that

$$\lim_{x\to 1} \frac{g_{2\bar{2}}(0,w^*)}{g^{\mathrm{KE}}_{2\bar{2}}(0,w^*)} = 12 \lim_{x\to 1} \frac{\frac{1}{(1-x^2)^2}}{G'(x)} = 3. \tag{47}$$

For the limit of the first fraction in (44), we shall utilize L'Hôpital's rule since $\lim_{x\to 1} S(x) = \lim_{x\to 1} G(x) = +\infty$. Namely, one can show that

$$\lim_{x\to 1} \frac{g_{1\bar{1}}(0,w^*)}{g^{\mathrm{KE}}_{1\bar{1}}(0,w^*)} = \lim_{x\to 1} \frac{S(x)}{G(x)} = \lim_{x\to 1} \frac{S'(x)}{G'(x)} = \lim_{x\to 1} \frac{g_{2\bar{2}}(0,w^*)}{g^{\mathrm{KE}}_{2\bar{2}}(0,w^*)} < +\infty.$$

Finally, through the above observation, together with the continuity of the fractions in (44), we state the following our third main theorem:

Theorem 5.1 *Let $D_{1,1}$ be as above. Then the Bergman and Kähler–Einstein metrics on $D_{1,1}$ are equivalent.*

5.2 General cases

We now consider the following complex Monge–Ampère equation with a boundary condition:

$$\begin{aligned}
\det\left(\frac{\partial^2 h}{\partial z_i \partial \bar{z}_j}(Z)\right) &= e^{(n+m+1)h(Z)} \quad \text{for} \quad Z \in D_{n,m},\\
h(Z) \to +\infty \quad &\text{as} \quad Z \to \partial D_{n,m},
\end{aligned} \tag{48}$$

where $Z = (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) = (z_1, \ldots, z_n, w_1, \ldots, w_m) = (z, w)$ is a local coordinate system on $D_{n,m}$. Denote by g^{KE} the complete Kähler–Einstein metric generated by the unique solution h of the previous equation, if it exists. Then, for $\tau_{-z} \in \mathrm{Aut}(D_{n,m})$ defined in Eq. (3), the biholomorphic invariance of the metric g^{KE} and the Monge–Ampère equation (48) imply

$$\begin{aligned}
e^{(n+m+1)h(z,w)} &= |\det\mathrm{Jac}(\tau_{-z},(z,w))|^2\, e^{(n+m+1)h(0,w^*)}\\
&= e^{m\|z\|^2} e^{(n+m+1)h(0,w^*)}.
\end{aligned} \tag{49}$$

Note that since $D_{n,m}$ is invariant under unitary transformations in z and w coordinates, we have

$$h(z, w) = h(\|z\|, \|w\|).$$

Taking the logarithm of (49), one can deduce that

$$(n+m+1)h(z, w) = m\|z\|^2 + (n+m+1)h(0, w^*). \tag{50}$$

Let us define a real-valued function $H(x)$ by setting

$$H(x) = h(0, w^*),$$

where $x = \|w^*\|^2 = e^{\|z\|^2}\|w\|^2 \in [0, 1)$ for $(z, w) \in D_{n,m}$. As noted before, each leaf $x =$ constant is invariant under the automorphism group. Applying the chain rule to the setting $x = e^{\|z\|^2}\|w\|^2$, we have

$$\begin{aligned} e^{(n+m+1)H(x)} &= e^{(n+m+1)h(z,w)}\Big|_{(z,w)=(0,w^*)} \\ &= \det\begin{pmatrix} \left(\dfrac{m}{n+m+1} + xH'(x)\right)\mathrm{Id}_n & \mathbf{O} \\ \mathbf{O} & A\cdot \mathrm{Id}_m \end{pmatrix}, \end{aligned} \tag{51}$$

where the (λ, ξ)-entry of the $m \times m$ matrix A is defined by

$$(A)_{(\lambda,\xi)} = H''(x)\bar{w}^*_\lambda w^*_\xi + H'(x)\delta^\xi_\lambda.$$

Then we obtain

$$e^{(n+m+1)H(x)} = \left(\frac{m}{n+m+1} + xH'(x)\right)^n \left(H'(x)\right)^{m-1}\left(xH'(x)\right)'. \tag{52}$$

A key step in obtaining (52) is the following:

$$\begin{aligned} \det\left(A\cdot \mathrm{Id}_m\right) &= \det\left(H'(x)\mathrm{Id}_m + \left(H'(x)\cdot\frac{H''(x)}{H'(x)}\bar{w}^*_\lambda w^*_\xi\right)_{m\times m}\right) \\ &= \left(H'(x)\right)^m \det\left(\mathrm{Id}_m + \left(\frac{H''(x)}{H'(x)}\bar{w}^*_\lambda w^*_\xi\right)_{m\times m}\right) \\ &= \left(H'(x)\right)^m\left(1 + \frac{H''(x)}{H'(x)}\|w^*\|^2\right) \\ &= \left(H'(x)\right)^m\left(1 + \frac{xH''(x)}{H'(x)}\right) \\ &= (H'(x))^{m-1}\left(xH'(x)\right)'. \end{aligned}$$

Multiplying both sides of (52) by x^{m-1}, we get

$$x^{m-1}e^{(n+m+1)H(x)} = \left(G(x) - \frac{m}{n+m+1}\right)^{m-1} G'(x)G^n(x), \tag{53}$$

where the function G is defined by $G(x) = \dfrac{m}{n+m+1} + xH'(x)$. If we denote by G_0 the function defined by

$$G_0(x) = G(x) - \frac{m}{n+m+1}, \tag{54}$$

then the relation (53) can be rewritten as

$$x^{m-1}e^{(n+m+1)H(x)} = G_0^{m-1}(x)G_0'(x)\left(G_0(x) + \frac{m}{n+m+1}\right)^n.$$

With these setting, we obtain the following theorem.

Theorem 5.2 *The generating function h for the Kähler–Einstein metric on $D_{n,m}$ is given by*

$$h(z,w)=\frac{m}{n+m+1}\|z\|^2+H\left(e^{\|z\|^2}\|w\|^2\right),$$

where $e^{(n+m+1)H(x)}=x^{1-m}G_0^{m-1}(x)G_0'(x)\left(G_0(x)+\frac{m}{n+m+1}\right)^n$ *and the function* $G_0(x):[0,1)\to[0,\infty)$ *satisfies*

$$\begin{cases} x\left(G_0(x)+\frac{m}{n+m+1}\right)^n G_0'(x)=G_0(x)S(G_0(x)),\\ G_0(0)=0,\\ G_0(x)\to+\infty \quad as\ x\to 1,\end{cases} \tag{55}$$

with

$$T^mS(T):=\int_0^T\{(n+m+1)t+m\}t^{m-1}\left(t+\frac{m}{n+m+1}\right)^n\mathrm{d}t. \tag{56}$$

Proof Combining the initial condition $G_0(0)=0$ with the boundary condition $G_0(x)\to\infty$ as $x\to1$, we have that $G_0(x)$ and $G(x)$ are strictly increasing functions where $G(x)$ is defined by

$$G(x)=\frac{m}{n+m+1}+xH'(x)=\frac{m}{n+m+1}+G_0(x). \tag{57}$$

For completeness of exposition, we first note that

$$G_0^{m-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(G_0(x)+\frac{m}{n+m+1}\right)^{n+1}\geq 0,$$

since

$$xG_0^{m-1}(x)\left(G_0(x)+\frac{m}{n+m+1}\right)^nG_0'(x)=x^me^{(n+m+1)H(x)}\geq 0.$$

Aiming for a contradiction, suppose that $G_0(x_0)<0$ for some point $x_0\in(0,1)$. Then, the continuity of G_0 and the assumption that $G(x)\to+\infty$ as $x\to1$ yield $G_0(\tilde{x}_0)=0$ for some point $\tilde{x}_0\in(0,1)$; hence, $e^{(n+m+1)H(\tilde{x}_0)}=0$, contrary to $e^{(n+m+1)H(x)}>0$ for all $x\in[0,1)$. A similar argument also holds for the case when $G(x_0)=0$ for some point $x_0\in(0,1)$. In addition to the above arguments, $G_0'(0)$ should be positive since

$$G(0)=\frac{m}{n+m+1}>0\quad\text{and}\quad G^n(x)\left(H'(x)\right)^{m-1}G_0'(x)=e^{(n+m+1)H(x)}>0. \tag{58}$$

Altogether, we obtain that for all $t\in(0,1)$,

$$G_0(0)=0,\ G_0(t)>0,\ G_0'(0)>0\quad\text{and}\quad G_0'(t)>0,$$

as desired.Now we may consider the inverse function of G satisfies the following:

$$\begin{cases}\frac{1}{x}\frac{dx}{dG}=\frac{\left(G_0+\frac{m}{n+m+1}\right)^n}{G_0S(G_0)}=\frac{G^n}{\left(G-\frac{m}{n+m+1}\right)S(G-\frac{m}{n+m+1})},\\ x\quad\to 1\quad\text{as}\quad G\to\infty,\end{cases} \tag{59}$$

where S is the polynomial of degree $n+1$ in the non-negative variable T defined by

$$T^mS(T)=\int_0^T\{(n+m+1)t+m\}t^{m-1}\left(t+\frac{m}{n+m+1}\right)^n\mathrm{d}t.$$

From its construction of the polynomial $S(G_0)$ and the binomial theorem, one can deduce that all the coefficients of $S(G_0)$ are positive. Applying the method of separation of variables to (59), the solution of (59) is given by

$$\log\frac{1}{x}=\int_{G(x)}^{\infty}\frac{y^n}{(y-\frac{m}{n+m+1})S(y-\frac{m}{n+m+1})}\mathrm{d}y=\int_{G_0(x)}^{\infty}\frac{\left(y+\frac{m}{n+m+1}\right)^n}{yS(y)}\mathrm{d}y. \tag{60}$$

In addition, an obvious modification of the argument in [30, Lemma 6] shows the uniqueness of the solution of (55). □

Remark 3 The Kähler condition for h naturally follows from the fact that for the case when $(z,w)=(z,0)$,

$$G(x)>0,\ G'(x)>0\quad\text{and}\quad\lim_{x\to 0}xH'(x)=0.$$

Since

$$\begin{aligned}\partial\bar\partial h&=G(x)\partial\bar\partial(\|z\|^2)\\&\quad+x\left(xH'(x)\right)'\left(\frac{\partial(\|w\|^2)}{\|w\|^2}+\partial(\|z\|^2)\right)\wedge\left(\frac{\bar\partial(\|w\|^2)}{\|w\|^2}+\bar\partial(\|z\|^2)\right)\\&\quad+\frac{xH'(x)}{\|w\|^4}\left(\bar\partial(\|w\|^2)\wedge\partial(\|w\|^2)+\|w\|^2\partial\bar\partial(\|w\|^2)\right),\end{aligned}$$

it follows that

$$\begin{aligned}\widetilde{H}_{(z,w)}(X)&=G(x)\sum_{k=1}^{n}|X_k|^2\\&\quad+x\left(xH'(x)\right)'\left|\frac{1}{\|w\|^2}\sum_{\lambda=1}^{m}\bar w_\lambda X_{n+\lambda}+\sum_{k=1}^{n}\bar z_k X_k\right|^2\\&\quad+\frac{xH'(x)}{\|w\|^4}\left(\left|\sum_{\lambda=1}^{m}\bar w_\lambda X_{n+\lambda}\right|^2+\|w\|^2\sum_{\lambda=1}^{m}|X_{n+\lambda}|^2\right),\end{aligned}$$

where $\widetilde{H}_{(z,w)}(X)$ is the associated hermitian metric at the point (z,w) in the direction $X=(X_1,\dots,X_n,X_{n+1},\dots,X_{n+m})\in\mathbb{C}^n\times\mathbb{C}^m$. Using $G(x)>0$ and $G'(x)>0$ for $x\in[0,1)$, one can force $\widetilde{H}_{(z,w)}(X)\geq 0$ for all vectors $X\in\mathbb{C}^n\times\mathbb{C}^m$. If $\widetilde{H}_{(z,w)}(\widetilde{X})=0$ for some point (z,w) and non-zero vector $\widetilde{X}\in\mathbb{C}^n\times\mathbb{C}^m$, then it follows that

$$\sum_{k=1}^{n}|\widetilde{X}_k|^2=0\quad\text{and}\quad\|w\|^2\sum_{\lambda=1}^{m}|\widetilde{X}_{n+\lambda}|^2=0.$$

These two conditions ensure that $\widetilde{H}_{(z,w)}(\widetilde{X})=0$ for some non-zero vector $\widetilde{X}\in\mathbb{C}^n\times\mathbb{C}^m$, only if $(z,w)=(z,0)$.

Now we note that

$$\partial\bar\partial h_{(z,0)}(\widetilde{X})=G(0)\sum_{k=1}^{n}|\widetilde{X}_k|^2+\left(\lim_{x\to 0}\frac{xH'(x)}{\|w\|^2}\right)\sum_{\lambda=1}^{m}|\widetilde{X}_{n+\lambda}|^2$$

for some non-zero vector $\widetilde{X} \in \mathbb{C}^n \times \mathbb{C}^m$. Since moreover $xH'(x) \geq 0$, $\lim_{x\to 0} xH'(x) = 0$ and $x = e^{\|z\|^2}\|w\|^2 \geq \|w\|^2$, we obtain

$$\lim_{x\to 0} \frac{xH'(x)}{\|w\|^2} \geq \lim_{x\to 0} \frac{xH'(x)}{x} = \lim_{x\to 0} \left(xH'(x)\right)' = G'(0) > 0 \tag{61}$$

after using the L'Hôpital's rule. Combining (61) with $G(0) = \frac{m}{n+m+1} > 0$, one can deduce that

$$\partial\bar{\partial}h_{(z,0)}(\widetilde{X}) = G(0)\sum_{k=1}^{n} |\widetilde{X}_k|^2 + \left(\lim_{x\to 0} \frac{xH'(x)}{\|w\|^2}\right)\sum_{\lambda=1}^{m} |\widetilde{X}_{n+\lambda}|^2 > 0$$

for all non-zero vectors $\widetilde{X} \in \mathbb{C}^n \times \mathbb{C}^m$. Then this contradicts the assumption that $\widetilde{H}_{(z,w)}(\widetilde{X}) = 0$ for some point (z, w) and non-zero vector $\widetilde{X} \in \mathbb{C}^n \times \mathbb{C}^m$.

The comparison between the Bergman and Kähler–Einstein metrics on $D_{n,m}$ is inherited from the limits of two fractions

$$\frac{m + xF'(x)}{G(x)} \quad \text{and} \quad \frac{F'(x) + xF''(x)}{G'(x)} \tag{62}$$

near the leaves $x = 0$ and $x = 1$. The corresponding limits for the previous fractions near $x = 0$ are relatively easy to compute. In calculating the associated limits to the case near $x = 1$, we require more elaborate approach as follows: As a first step, we observe that $F'(x)$ can be described using the polynomial $A_{n,m}(x)$ in Theorem 2.1 and its recurrence relation in Lemma 2.2. Namely, one can show that

$$F'(x) = \frac{A_{n,m+1}(x)}{(1-x)A_{n,m}(x)} = \frac{(n+m+1)A_{n,m}(x) + (1-x)A'_{n,m}(x)}{(1-x)A_{n,m}(x)}.$$

Since $\left|\lim_{x\to 1} \frac{A'_{n,m}(x)}{A_{n,m}(x)}\right| < \infty$, it follows that

$$F'(x) = \frac{n+m+1}{1-x}$$

as $x \to 1$. Then the fact that $G(x)$ tends to ∞ as $x \to 1$ would yield

$$\lim_{x\to 1} \frac{m + xF'(x)}{G(x)} = \lim_{x\to 1} \frac{m + \frac{(n+m+1)xA_{n,m}(x) + x(1-x)A'_{n,m}(x)}{(1-x)A_{n,m}(x)}}{G(x)} = \lim_{x\to 1} \frac{\frac{(n+m+1)x}{(1-x)}}{G(x)}.$$

For computational convenience, we shall consider

$$2(n+m+1)\lim_{x\to 0} \frac{\frac{1}{(1-x^2)}}{G(x)} \tag{63}$$

instead of $\lim_{x\to 1} \frac{\frac{(n+m+1)x}{(1-x)}}{G(x)}$, if it exists. Another crucial ingredient for the associated comparison theorem to $D_{n,m}$ is in the computation of

$$\lim_{x\to 1} \frac{F'(x) + xF''(x)}{G'(x)}.$$

In a similar fashion to $m + xF'(x)$, as $x \to 1$, we have

$$\begin{aligned} F'(x) + xF''(x) &= \frac{A_{n,m+1}(x)}{(1-x)A_{n,m}(x)} + \frac{xA'_{n,m+1}(x)}{(1-x)A_{n,m}(x)} + \frac{xA_{n,m+1}(x)}{(1-x)^2 A_{n,m}(x)} \\ &\quad - \frac{xA_{n,m+1}(x)A'_{n,m}(x)}{(1-x)A^2_{n,m}(x)} \\ &= \frac{n+m+1}{1-x} + \frac{S(1+n,n)x}{1-x} + \frac{(n+m+1)x}{(1-x)^2} \\ &\quad - \frac{(n+m+1)S(1+n,n)x}{1-x} \\ &= \frac{(n+m+1)(1+x)^2}{(1-x^2)^2} - \frac{S(1+n,n)(1+x)x}{(1-x^2)(n+m)}, \end{aligned}$$

where $S(\cdot,\cdot)$ is the Stirling number of the second kind. The reason why we have changed the degrees of two denominators in the previous relation is because of the following: up to constant multiple,

$$\frac{1}{(1-x^2)^2} = G'(x) = G^2(x) \tag{64}$$

as $x \to 1$. Combining (64) with the product rule of limits, we deduce that

$$\lim_{x\to 1} \frac{F'(x) + xF''(x)}{G'(x)} = 4(n+m+1) \lim_{x\to 1} \frac{\frac{1}{(1-x^2)^2}}{G'(x)}.$$

The following lemma, in conjunction with (55), concludes the proof of the specific relation (64).

Lemma 5.3 *Let x and $G(x)$ be as above. Then we obtain*

$$\exp\left(-\frac{1}{G(x) - \frac{m}{n+m+1}}\right) \le x \le \exp\left(-\frac{m}{(n+m+1)G(x)}\right). \tag{65}$$

Proof From the definition of $S(T)$ in Theorem 5.2, we have

$$\begin{aligned} &T^m S(T) \\ &= \int_0^T (n+m+1)t^m \left(t + \frac{m}{n+m+1}\right)^n \mathrm{d}t + \int_0^T mt^{m-1} \left(t + \frac{m}{n+m+1}\right)^n \mathrm{d}t \\ &= \sum_{k=0}^n \binom{n}{k} \frac{mT^{m+k+1}}{m+k+1} \left(\frac{m}{n+m+1}\right)^{n-k-1} + \sum_{k=0}^n \binom{n}{k} \frac{mT^{m+k}}{m+k} \left(\frac{m}{n+m+1}\right)^{n-k} \end{aligned} \tag{66}$$

Since both of two terms in the right-hand side of (66) are positive, it follows that

$$\begin{aligned} S(T) &\ge \sum_{k=0}^n \binom{n}{k} \frac{mT^{k+1}}{m+k+1} \left(\frac{m}{n+m+1}\right)^{n-k-1} \\ &\ge \sum_{k=0}^n \binom{n}{k} \frac{mT^{k+1}}{m+n+1} \left(\frac{m}{n+m+1}\right)^{n-k-1} \end{aligned}$$

$$= T\sum_{k=0}^{n}\binom{n}{k}T^k\left(\frac{m}{n+m+1}\right)^{n-k}$$
$$= T\left(T+\frac{m}{n+m+1}\right)^n.$$

This, in conjunction with (60), yields

$$\log\frac{1}{x} = \int_{G_0(x)}^{\infty}\frac{\left(y+\frac{m}{n+m+1}\right)^n}{yS(y)}\mathrm{d}y \le \int_{G_0(x)}^{\infty}\frac{1}{y^2}\mathrm{d}y = \frac{1}{G_0(x)} = \frac{1}{G(x)-\frac{m}{n+m+1}}$$

which is equivalent to the first inequality in (65). To prove the second inequality in (65), we first observe the following:

$$\begin{aligned}
&S(T)\\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{mT^{k+1}}{m+k+1}\left(\frac{m}{n+m+1}\right)^{n-k-1} + \sum_{k=0}^{n}\binom{n}{k}\frac{mT^k}{m+k}\left(\frac{m}{n+m+1}\right)^{n-k}\\
&\le \sum_{k=0}^{n}\binom{n}{k}T^{k+1}\left(\frac{m}{n+m+1}\right)^{n-k-1} + \sum_{k=0}^{n}\binom{n}{k}T^k\left(\frac{m}{n+m+1}\right)^{n-k}\\
&= \left(\frac{(n+m+1)T}{m}+1\right)\sum_{k=0}^{n}\binom{n}{k}T^k\left(\frac{m}{n+m+1}\right)^{n-k}\\
&= \frac{n+m+1}{m}\left(T+\frac{m}{n+m+1}\right)^{n+1}.
\end{aligned}$$

Then, combining this with (60), one can deduce that

$$\begin{aligned}
\log\frac{1}{x} &= \int_{G_0(x)}^{\infty}\frac{\left(y+\frac{m}{n+m+1}\right)^n}{yS(y)}\mathrm{d}y\\
&\ge \frac{m}{n+m+1}\int_{G_0(x)}^{\infty}\frac{1}{y\left(y+\frac{m}{n+m+1}\right)}\mathrm{d}y\\
&\ge \frac{m}{n+m+1}\int_{G_0(x)}^{\infty}\frac{1}{\left(y+\frac{m}{n+m+1}\right)^2}\mathrm{d}y\\
&= \frac{m}{(n+m+1)\left(G_0(x)+\frac{m}{n+m+1}\right)}\\
&= \frac{m}{(n+m+1)G(x)}.
\end{aligned}$$

The proof of the lemma is complete. □

Remark 4 The inequalities in Lemma 5.3 imply that

$$\frac{1}{1-\exp\left(-\frac{2}{G(x)-\frac{m}{n+m+1}}\right)} \le \frac{1}{(1-x^2)} \le \frac{1}{1-\exp\left(-\frac{2m}{(n+m+1)G(x)}\right)} \tag{67}$$

Then, using the Taylor's polynomial of degree 1 at the origin of exponential function and the fact that the positive function $G(x)$ tends to ∞ as $x\to 1$, (67) forces the fraction $\frac{m+xF'(x)}{G(x)}$

to satisfy

$$n+m+1 \leq \lim_{x\to 1} \frac{m+xF'(x)}{G(x)} \leq \frac{(n+m+1)^2}{m}. \tag{68}$$

What is more, the ordinary differential equation (55) in Theorem 5.2, together with (67) and (68), shows that

$$n+m+1 \leq \lim_{x\to 1} \frac{F'(x)+xF''(x)}{G'(x)} \leq \frac{(n+m+1)^3}{m^2}. \tag{69}$$

Throughout what follows, we denote by $(g_{\alpha\bar{\beta}})$ and $(g^{\mathrm{KE}}_{\alpha\bar{\beta}})$ the associated matrix representations describing the tensors corresponding to the Bergman and Kähler–Einstein metrics on $D_{n,m}$, respectively. Since $(g_{\alpha\bar{\beta}})$ and $(g^{\mathrm{KE}}_{\alpha\bar{\beta}})$ are forms of block diagonal matrices at $(0, w^*) \in D_{n,m}$, we consider only two kinds of fractions:

$$\frac{\sum_{k,l=1}^{n} g_{k\bar{l}}(0,w^*)X_k\overline{X_l}}{\sum_{k,l=1}^{n} g^{\mathrm{KE}}_{k\bar{l}}(0,w^*)X_k\overline{X_l}} \quad \text{and} \quad \frac{\sum_{\lambda,\xi=1}^{m} g_{(n+\lambda)\overline{(n+\xi)}}(0,w^*)X_{n+\lambda}\overline{X_{n+\xi}}}{\sum_{\lambda,\xi=1}^{m} g^{\mathrm{KE}}_{(n+\lambda)\overline{(n+\xi)}}(0,w^*)X_{n+\lambda}\overline{X_{n+\xi}}} \tag{70}$$

for all vectors $X = (X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m}) \in \mathbb{C}^n \times \mathbb{C}^m$.

Now we fix a vector $X = (X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m}) \in \mathbb{C}^n \times \mathbb{C}^m$. Then, for the fixed vector X, we define a vector $\widetilde{X}$ by setting

$$\widetilde{X} = (0, \ldots, 0, X_{n+1}, \ldots, X_{n+m}).$$

This specific construction of $\widetilde{X}$ guarantees the existence of a function $k(w^*, \widetilde{X})$ which depends on w^* and $\widetilde{X}$ such that

$$\sum_{\lambda=1}^{m} \bar{w}^*_\lambda X_{n+\lambda} = \langle \widetilde{X}, (0, w^*)\rangle = xk(w^*, \widetilde{X}).$$

Then the numerator and the denominator for the second fraction in (70) can be written as follows:

$$\begin{aligned}
&\sum_{\lambda,\xi=1}^{m} g_{(n+\lambda)\overline{(n+\xi)}} X_{n+\lambda}\overline{X_{n+\xi}} \\
&\quad = F'(x)\sum_{\lambda=1}^{m} |X_{n+\lambda}|^2 + F''(x)\left|\sum_{\lambda=1}^{m} \bar{w}^*_\lambda X_{n+\lambda}\right|^2 \\
&\quad = F'(x)\left(x\left|k(w^*,\widetilde{X})\right|^2 + \left\|\widetilde{X} - k(w^*,\widetilde{X})(0,w^*)\right\|^2\right) + F''(x)\left(x^2\left|k(w^*,\widetilde{X})\right|^2\right) \\
&\quad = x\left|k(w^*,\widetilde{X})\right|^2\left(xF'(x)\right)' + F'(x)\left\|\widetilde{X} - k(w^*,\widetilde{X})(0,w^*)\right\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\lambda,\xi=1}^{m} g^{\mathrm{KE}}_{(n+\lambda)\overline{(n+\xi)}}(0,w^*) X_{n+\lambda}\overline{X_{n+\xi}} \\
&\quad = H'(x)\sum_{\lambda=1}^{m} |X_{n+\lambda}|^2 + H''(x)\left|\sum_{\lambda=1}^{m} \bar{w}^*_\lambda X_{n+\lambda}\right|^2 \\
&\quad = x\left|k(w^*,\widetilde{X})\right|^2\left(xH'(x)\right)' + H'(x)\left\|\widetilde{X} - k(w^*,\widetilde{X})(0,w^*)\right\|^2
\end{aligned}$$

$$= x \left|k(w^*, \widetilde{X})\right|^2 G'(x) + H'(x) \left\|\widetilde{X} - k(w^*, \widetilde{X})(0, w^*)\right\|^2.$$

Combining these values with the facts that $F'(x) \geq 0$, $\left(xF'(x)\right)' > 0$, $H'(x) \geq 0$ and $G'(x) > 0$ for all $x \in [0, 1)$, we deduce that

$$\frac{x \left|k(w^*, \widetilde{X})\right|^2 \left(xF'(x)\right)'}{x \left|k(w^*, \widetilde{X})\right|^2 G'(x) + H'(x)\|\widetilde{X}\|^2} \leq \frac{\sum_{\lambda,\xi=1}^m g_{(n+\lambda)\overline{(n+\xi)}}(0, w^*) X_{n+\lambda}\overline{X_{n+\xi}}}{\sum_{\lambda,\xi=1}^m g^{\mathrm{KE}}_{(n+\lambda)\overline{(n+\xi)}}(0, w^*) X_{n+\lambda}\overline{X_{n+\xi}}} \leq \frac{x \left|k(w^*, \widetilde{X})\right|^2 \left(xF'(x)\right)' + F'(x)\|\widetilde{X}\|^2}{x \left|k(w^*, \widetilde{X})\right|^2 G'(x)}. \tag{71}$$

If $k(w^*, \widetilde{X}) = 0$, then (71) implies

$$\begin{aligned}\lim_{x\to 1} \frac{\sum_{\lambda,\xi=1}^m g_{(n+\lambda)\overline{(n+\xi)}}(0, w^*) X_{n+\lambda}\overline{X_{n+\xi}}}{\sum_{\lambda,\xi=1}^m g^{\mathrm{KE}}_{(n+\lambda)\overline{(n+\xi)}}(0, w^*) X_{n+\lambda}\overline{X_{n+\xi}}} &= \lim_{x\to 1} \frac{F'(x)\|\widetilde{X}\|^2}{H'(x)\|\widetilde{X}\|^2} \\ &= \lim_{x\to 1} \frac{xF'(x)\|\widetilde{X}\|^2}{\left(G(x) - \frac{m}{n+m+1}\right)\|\widetilde{X}\|^2} \\ &= \lim_{x\to 1} \frac{\left(m + xF'(x)\right)\|\widetilde{X}\|^2}{G(x)\|\widetilde{X}\|^2}\end{aligned} \tag{72}$$

since $\lim_{x\to 1} G(x) = \infty$. In addition, the limit of the first fraction in (70) as $x \to 1$ is

$$\lim_{x\to 1} \frac{\sum_{k,l=1}^n g_{k\bar{l}}(0, w^*) X_k \overline{X}_l}{\sum_{k,l=1}^n g^{\mathrm{KE}}_{k\bar{l}}(0, w^*) X_k \overline{X}_l} = \lim_{x\to 1} \frac{\left(m + xF'(x)\right)\sum_{k=1}^n |X_k|^2}{G(x)\sum_{k=1}^n |X_k|^2}. \tag{73}$$

Using (72) and (73), one can show that $\lim_{x\to 1} \dfrac{m + xF'(x)}{G(x)}$ determines the limits of two fractions in (70) as $x \to 1$, if $k(w^*, \widetilde{X}) = 0$.

Before going to the case when $k(w^*, \widetilde{X}) \neq 0$, it is worth noting that

$$F'(x) = G(x) \quad \text{and} \quad \left(xF'(x)\right)' = G^2(x) \tag{74}$$

up to constant multiple as $x \to 1$. Thus, if $k(w^*, \widetilde{X}) \neq 0$, then (71) and (74) imply

$$\frac{\sum_{\lambda,\xi=1}^m g_{(n+\lambda)\overline{(n+\xi)}}(0, w^*) X_{n+\lambda}\overline{X_{n+\xi}}}{\sum_{\lambda,\xi=1}^m g^{\mathrm{KE}}_{(n+\lambda)\overline{(n+\xi)}}(0, w^*) X_{n+\lambda}\overline{X_{n+\xi}}} = \lim_{x\to 1} \frac{\left(xF'(x)\right)'}{G'(x)}.$$

Altogether, we obtain that the limits of two fractions in (70) as $x \to 1$ are completely determined by

$$\lim_{x\to 1} \frac{m + xF'(x)}{G(x)} \quad \text{and} \quad \lim_{x\to 1} \frac{\left(xF'(x)\right)'}{G'(x)}.$$

Finally, combining this fact with (68) and (69), we conclude the following theorem.

Theorem 5.4 *Let $D_{n,m}$ be as above. Then the Bergman and Kähler–Einstein metrics are equivalent.*

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6 Appendix: Carathéodory and Kobayashi pseudo-metrics and pseudo-distances

As is widely known, the Carathéodory and Kobayashi pseudo-metrics squeeze all pseudo-differential metrics on complex manifolds satisfying the Schwarz lemma with respect to holomorphic mappings and coinciding with the Poincaré metric on the unit disc. By contrast, the Bergman metric does not admit the Schwarz lemma. For this reason, one can ask whether the Bergman metric is compared with some invariant metrics. For a bounded or Kobayashi-hyperbolic domain Ω in $\mathbb{C}^n$, it is well known that

$$C_\Omega \le K_\Omega \quad \text{and} \quad C_\Omega \le B_\Omega,$$

where C_Ω, K_Ω, and B_Ω are the Carathéodory pseudo-metric, the Kobayashi pseudo-metric, and the Bergman metric, respectively.

We shall focus our attention first on the behaviors of the Carathéodory and Kobayashi pseudo-distances along two different leaves on $D_{1,1}$. In addition, further comparisons among the Bergman distance and the above two pseudo-distances on $D_{1,1}$ will be given. Then we will explain the corresponding relation for the case of $D_{n,m}$ in higher dimensions.

Before moving to description of the behaviors of the Carathéodory and Kobayashi pseudo-metrics on $D_{1,1}$, let us briefly consider some basics on these two pseudo-metrics on an arbitrary domain Ω in $\mathbb{C}^n$. Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. Then we define the *Poincaré distance* $d_{\mathbb{D}}$ on $\mathbb{D}$ by setting

$$d_{\mathbb{D}}(a,b) = \frac{1}{2}\log\frac{|1-a\bar{b}|+|a-b|}{|1-a\bar{b}|-|a-b|} = \tanh^{-1}\left|\frac{a-b}{1-a\bar{b}}\right|$$

for all $a, b \in \mathbb{D}$. Given two complex spaces Ω_1 and Ω_2, let $\mathrm{Hol}(\Omega_1, \Omega_2)$ denote the set of all holomorphic mappings from Ω_1 into Ω_2. The *Carathéodory pseudo-distance* d^{C}_{Ω} between two points p and q in a domain $\Omega \subset \mathbb{C}^n$ is defined by

$$d^{\mathrm{C}}_{\Omega}(p,q) = \sup_f \{d_{\mathbb{D}}(f(p), f(q)) : \ f \in \mathrm{Hol}(\Omega, \mathbb{D})\}.$$

This Carathéodory pseudo-distance is closely related to the following pseudo-metric through the consideration of its integrated form: The *infinitesimal Carathéodory pseudo-metric* at a point $p \in \Omega$ and $\xi \in T_p\Omega$ is defined by

$$C_\Omega(p;\xi) = \sup_f \{|f_*(p)\xi| : \ f \in \mathrm{Hol}(\Omega,\mathbb{D}), \ f(p) = 0\},$$

where $f_*(p)$ denotes the $\mathbb{C}$-differential of f at p. Then, given two points $p, q \in \Omega$, the *integrated form of the infinitesimal Carathéodory pseudo-metric* is defined by

$$c_\Omega(p,q) = \inf_\gamma \int_0^1 C_\Omega(\gamma(t);\gamma'(t))\mathrm{d}t,$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = p$ and $\gamma(1) = q$. It has been known that $d^{\mathrm{C}}_{\Omega} \le c_{\Omega}$ holds for a domain Ω in $\mathbb{C}^n$.

To establish the dual concept of the Carathéodory pseudo-distance, we first define the *Lempert function* $\delta^{\mathrm{K}}_{\Omega}$ for Ω by setting

$$\delta^{\mathrm{K}}_{\Omega}(p, q) = \inf_h \{d_{\mathbb{D}}(a, b) : \ h \in \mathrm{Hol}(\mathbb{D}, \Omega), h(a) = p, h(b) = q\}, \quad \text{for } p, q \in \Omega.$$

Moreover, it is well known that the Lempert function is not a pseudo-distance because it does not satisfy the triangle inequality. As the largest pseudo-distance bounded by $\delta^{\mathrm{K}}_{\Omega}$, we now define the *Kobayashi pseudo-distance* d^{K}_{Ω} by setting

$$d^{\mathrm{K}}_{\Omega}(p, q) = \inf \sum_{j=1}^{N} \delta^{\mathrm{K}}_{\Omega}(p_{j-1}, p_j), \quad \text{for } p, q \in \Omega, \tag{75}$$

where the infimum is taken over all the possible chains of holomorphic discs from p to q. Then it follows obviously from the definitions above that

$$d^{\mathrm{K}}_{\Omega}(p, q) \le \delta^{\mathrm{K}}_{\Omega}(p, q) \le d_{\mathbb{D}}(a, b), \tag{76}$$

where $h(a) = p$ and $h(b) = q$ for $h \in \mathrm{Hol}(\mathbb{D}, \Omega)$.

Let us now consider the concept of the infinitesimal Kobayashi pseudo-metric due to H. L. Royden in 1971: For a domain Ω in $\mathbb{C}^n$, the *infinitesimal Kobayashi pseudo-metric* at a point $p \in \Omega$ and $\xi \in T_p\Omega$ is defined by

$$K_{\Omega}(p; \xi) = \inf_h \{|\lambda| : \ h \in \mathrm{Hol}(\mathbb{D}, \Omega), h(0) = p, h_*(0)\lambda = \xi\}.$$

Analogously to that of the Carathéodory pseudo-distance, given two points $p, q \in \Omega$, we define the *integrated form of the infinitesimal Kobayashi pseudo-metric* by setting

$$k_{\Omega}(p, q) = \inf_{\gamma} \int_0^1 K_{\Omega}(\gamma(t); \gamma'(t))\mathrm{d}t,$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = p$ and $\gamma(1) = q$. This concept is indeed identical to the Kobayashi pseudo-distance defined in (75). Altogether, one can reach the following comparison:

$$d^{\mathrm{C}}_{\Omega}(p, q) \le c_{\Omega}(p, q) \le k_{\Omega}(p, q) = d^{\mathrm{K}}_{\Omega}(p, q), \quad \text{for } p, q \in \Omega.$$

Now we turn into the investigation of the behaviors of the Carathéodory and Kobayashi pseudo-distances on the domain $D_{1,1}$ in $\mathbb{C}^2$. Since both of the Carathéodory and Kobayashi pseudo-distances are invariant under holomorphic automorphisms, we shall utilize the explicit form of $\mathrm{Aut}(D_{1,1})$ in determining the behaviors of these pseudo-metrics on $D_{1,1}$. As described in Sect. 2, $\mathrm{Aut}(D_{n,m})$ is generated by the following mappings:

$$\begin{aligned} r_U \ &: D_{n,m} \to D_{n,m}, \quad (z, w) \mapsto (Uz, w), \\ r_{U'} \ &: D_{n,m} \to D_{n,m}, \quad (z, w) \mapsto (z, U'w), \\ \tau_v \ &: D_{n,m} \to D_{n,m}, \quad (z, w) \mapsto (z + v, e^{-\langle z, v\rangle - \frac{1}{2}\|v\|^2} w), \end{aligned}$$

where $U \in U(n)$, $U' \in U(m)$, and $v \in \mathbb{C}^n$. Considering $\gamma_{U'}$ and τ_{-z_0} for the case of $D_{1,1}$, we have

$$\gamma_{U'} \circ \tau_{-z_0}(z, w) = (z - z_0, U' e^{z\bar{z}_0 - \frac{1}{2}|z_0|^2} w), \quad \text{for } (z, w) \in D_{1,1}.$$

Then, for each $(z_0, w_0) \in D_{1,1}$, we obtain

$$\gamma_{U'} \circ \tau_{-z_0}(0,0) = (-z_0, 0) \quad \text{and} \quad \gamma_{U'} \circ \tau_{-z_0}(z_0, w_0) = (0, U' e^{\frac{1}{2}|z_0|^2} w_0). \tag{77}$$

Combining (77) with the invariance of the Carathéodory pseudo-distance under biholomorphic mappings, one can deduce that

$$d^{\mathrm{C}}_{D_{1,1}}((0,0),(z_0,w_0)) = d^{\mathrm{C}}_{D_{1,1}}((-z_0,0),(0,w_0^*)),$$

where $|w_0^*|^2 = e^{|z_0|^2}|w_0|^2$. Since $d^{\mathrm{C}}_{D_{1,1}}$ vanishes identically along a complex line $L := \{(z,0) : z \in \mathbb{C}\} \subset D_{1,1}$, it follows from the triangle inequality that

$$\begin{aligned} d^{\mathrm{C}}_{D_{1,1}}((0,0),(z_0,w_0)) &= d^{\mathrm{C}}_{D_{1,1}}((z_0,0),(z_0,w_0)) \\ &= d^{\mathrm{C}}_{D_{1,1}}((0,0),(0,w_0^*)). \end{aligned} \tag{78}$$

The latter equality comes from the invariance of the Carathéodory pseudo-distance under the composition of the above automorphism $\gamma_{U'} \circ \tau_{-z_0}$ and a unitary transformation in the w-coordinate. The relation (78) tells us that the Carathéodory pseudo-distance between the leaf

$$\{(z,w) \in D_{1,1} : e^{|z|^2}|w|^2 = c \text{ for a fixed } c \in [0,1)\}$$

and the complex line L is constant with respect to the constant c.

Now we shall show that

$$d^{\mathrm{C}}_{D_{1,1}}((0,0),(0,w_0^*)) = d_{\mathbb{D}}(0,|w_0^*|) = d^{\mathrm{K}}_{D_{1,1}}((0,0),(0,w_0^*)). \tag{79}$$

For the proof, one starts with a holomorphic mapping $f : D_{1,1} \to \mathbb{D}$ defined by

$$f(z,w) = w.$$

Then it follows from the definition that

$$f(0,w_0^*) = w_0^* \quad \text{and} \quad f(0,0) = 0.$$

This, in conjunction with the definition of $d^{\mathrm{C}}_{D_{1,1}}$, yields

$$\begin{aligned} d^{\mathrm{C}}_{D_{1,1}}((0,0),(0,w_0^*)) &\geq d_{\mathbb{D}}(f(0,0), f(0,w_0^*)) \\ &= d_{\mathbb{D}}(0,w_0^*) \\ &= d_{\mathbb{D}}(0,|w_0^*|). \end{aligned} \tag{80}$$

Let us now consider a holomorphic disc $h : \mathbb{D} \to D_{1,1}$ defined by

$$h(w) = (0,w).$$

Since $d^{\mathrm{C}}_{D_{1,1}}(p,q) \leq d^{\mathrm{K}}_{D_{1,1}}(p,q)$ holds for all $p, q \in D_{1,1}$, (76) and (80) imply that

$$\begin{aligned} d^{\mathrm{C}}_{D_{1,1}}((0,0),(0,w_0^*)) &\leq d^{\mathrm{K}}_{D_{1,1}}((0,0),(0,w_0^*)) \\ &\leq \delta^{\mathrm{K}}_{D_{1,1}}((0,0),(0,w_0^*)) \\ &= \delta^{\mathrm{K}}_{D_{1,1}}(h(0),h(w_0^*)) \\ &\leq d_{\mathbb{D}}(0,w_0^*) \\ &= d_{\mathbb{D}}(0,|w_0^*|) \\ &\leq d^{\mathrm{C}}_{D_{1,1}}((0,0),(0,w_0^*)). \end{aligned}$$

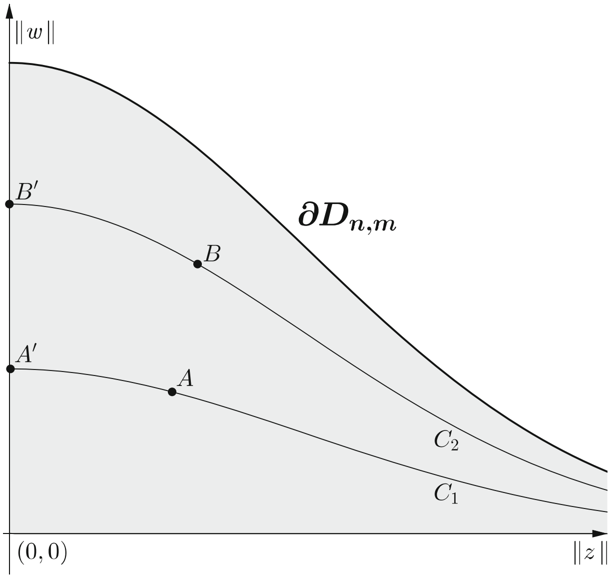

Fig. 2 Pseudo-distances between two leaves

This finishes the proof of Eq. (79).

Concerning the associated Carathéodory and Kobayashi pseudo-distances between the leaves C_1 and C_2 as shown in Fig. 2, one can get

$$d^{\mathrm{C}}_{D_{1,1}}(C_1, C_2) = d_{\mathbb{D}}(A', B') = d^{\mathrm{K}}_{D_{1,1}}(C_1, C_2). \tag{81}$$

Here is a proof of (81). Let us denote by O the origin $(0, 0)$ in $\mathbb{C}^2$. For any point B in the leaf C_2 given in Figure 2, (79) and the triangle inequality imply that

$$\begin{aligned} d^{\mathrm{C}}_{D_{1,1}}(O, C_1) + d^{\mathrm{C}}_{D_{1,1}}(C_1, B) &\geq d^{\mathrm{C}}_{D_{1,1}}(O, B) \\ &= d^{\mathrm{C}}_{D_{1,1}}(O, B') \\ &= d_{\mathbb{D}}(O, B') \\ &= d_{\mathbb{D}}(O, A') + d_{\mathbb{D}}(A', B'), \end{aligned} \tag{82}$$

where OB' is the geodesic with respect to the Poincaré metric. This, in conjunction with the fact that $d^{\mathrm{C}}_{D_{1,1}}(O, C_1) = d_{\mathbb{D}}(O, A')$ inherited from (78) and (79), yields

$$d^{\mathrm{C}}_{D_{1,1}}(C_1, B) \geq d_{\mathbb{D}}(A', B'). \tag{83}$$

Through observation of orbits of $\mathrm{Aut}(D_{1,1})$, we can assure the existence of a point A in the leaf C_1 with the same first component as the point B such that

$$d^{\mathrm{C}}_{D_{1,1}}(A, B) = d_{\mathbb{D}}(A', B'). \tag{84}$$

Then it follows from (84) that

$$d^{\mathrm{C}}_{D_{1,1}}(C_1, B) \leq d^{\mathrm{C}}_{D_{1,1}}(A, B) = d_{\mathbb{D}}(A', B'). \tag{85}$$

Since B was arbitrary, (83) and (85) conclude the verification of (81). Since we do not know the explicit form of geodesic in general, it leads to another highly intricate stage to compute the pseudo-distances between any two points in the leaves C_1 and C_2. In the present paper, we would not take such a kind of consideration.

We shall now investigate the relation among the Bergman distance, the Carathéodory and Kobayashi pseudo-distances on $D_{1,1}$. In determining the associated comparison theorem, we utilize the invariance of the Bergman metric under biholomorphic mappings. Note that, for each piecewise C^1 curve $\gamma(t) := (u(t), v(t)) : [0, 1] \to D_{1,1}$, there exists a piecewise C^1 curve $\tilde{\gamma}(t) := (0, \tilde{v}(t)) : [0, 1] \to D_{1,1}$ such that

$$\left(u'(t)\ v'(t)\right)\left(g_{i\bar{j}}(u,v)\right)\begin{pmatrix}\overline{u'(t)}\\ \overline{v'(t)}\end{pmatrix} = \left(0\ \tilde{v}'(t)\right)\begin{pmatrix}S(|\tilde{v}(t)|^2) & 0\\ 0 & S'(|\tilde{v}(t)|^2)\end{pmatrix}\begin{pmatrix}0\\ \overline{\tilde{v}'(t)}\end{pmatrix} \tag{86}$$

using Eq. (8) in Example 2.1. Then, (86) forces the Bergman metric to satisfy

$$\begin{aligned}\int_0^1 \sqrt{\left(u'(t)\ v'(t)\right)\left(g_{i\bar{j}}(u,v)\right)\begin{pmatrix}\overline{u'(t)}\\ \overline{v'(t)}\end{pmatrix}}\mathrm{d}t &= \int_0^1 \sqrt{S'(x)|\tilde{v}'(t)|^2}\mathrm{d}t \\ &\geq \int_0^1 \frac{2|\tilde{v}'(t)|}{(1+x)(1-x)}\mathrm{d}t \\ &\geq \int_0^1 \frac{|\tilde{v}'(t)|}{1-|\tilde{v}(t)|^2}\mathrm{d}t \\ &= d_{\mathbb{D}}(\tilde{v}(0), \tilde{v}(1)),\end{aligned} \tag{87}$$

where $x = |\tilde{v}(t)|^2$. Taking the infimum of the left-hand side of the inequality (87) over all the possible piecewise curves, one can deduce that the Bergman distance between two leaves in $D_{1,1}$ is always greater than or equal to the corresponding Carathéodory and Kobayashi pseudo-distances, that is,

$$B_{D_{1,1}} \geq C_{D_{1,1}} = K_{D_{1,1}}.$$

Moreover, if we take a curve $\gamma(t) = (u(t), v(t))$ along the complex line L, then the associated Bergman distance is nothing but

$$\int_0^1 |\tilde{v}'(t)|\mathrm{d}t$$

which is exactly the Euclidean length of $\gamma(t)$ in L.

Remark 5 Similar arguments as above can be made for the cases of $D_{n,m}$ in higher dimensions. To compute the Carathéodory pseudo-distance, we first consider the following mappings:

$$\begin{aligned}\tau_{-z} &: D_{n,m} \to D_{n,m}, \quad (z,w) \mapsto (0, w^*),\\ \psi_{w^*} &: D_{n,m} \to D_{n,m}, \quad (z,w) \mapsto (z, \underbrace{0,\ldots,0}_{m-1}, \tilde{w}^*_m),\\ P_{n,m} &: D_{n,m} \to \mathbb{D}, \quad (z,w) \mapsto w_m,\\ \mu_b &: \mathbb{D} \to \mathbb{D}, \quad a \mapsto \frac{a-b}{1-\bar{b}a},\end{aligned}$$

where $\psi_{w^*} \in \mathrm{Id}_n \times \mathcal{U}(m) \subset \mathrm{Aut}(D_{n,m})$. Let us define a holomorphic mapping $f : D_{n,m} \to \mathbb{D}$ by setting

$$f(p,q) = \mu_{\tilde{w}^*_m} \circ P_{n,m} \circ \psi_{w^*} \circ \tau_{-z}(p,q) \tag{88}$$

for a fixed point $(z, w) \in D_{n,m}$. Then (88) clearly forces f to satisfy $f(z, w) = 0$.

To get the associated Kobayashi pseudo-distance, we next consider a Möbius transformation $\tilde{\mu}_c : \mathbb{D} \to \mathbb{D}$ defined by $\tilde{\mu}_c = -\mu_c$. Adopting a unitary action on w-coordinate in $D_{n,m}$, we may choose a function $\varphi_{w^*} : \mathbb{D} \to D_{n,m}$ such that $\varphi_{w^*}(\tilde{w}_m^*) = (0, w^*)$ with $\|w^*\| = |\tilde{w}_m^*|$. Then we define a holomorphic mapping $h : \mathbb{D} \to D_{n,m}$ by setting

$$h(a) = \varphi_{w^*} \circ \tilde{\mu}_{\tilde{w}_m^*}(a). \tag{89}$$

In particular, h satisfies $h(0) = (0, w^*)$. These certain mappings given in (88) and (89) make it possible that the Carathéodory and Kobayashi pseudo-distances between two leaves in Fig. 2 are identical. Furthermore, the result of comparisons among the Bergman metric and the above two pseudo-metrics is essentially same as that of $D_{1,1}$.

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