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Tamed symplectic cones of compact Hermitian-symplectic manifolds

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Abstract In this paper we consider the tamed symplectic cone of a compact Hermitian-symplectic manifold. First, we prove a Poincaré duality theorem for the tamed symplectic cone. Second, we study the stability of Hermitian-symplectic metrics under the deformation of complex structures and show that the tamed symplectic cones are invariant under the parallel transport with respect to the Gauss–Manin connection.

Keywords Hermitian-symplectic manifold · Tamed symplectic cone · Positive current · Gauss–Manin connection

Mathematics Subject Classification 53C15 · 53C55 · 53B35

1 Introduction

Let (M, J) be an almost complex manifold of dimension $2n$ with almost complex structure J. Recall that a symplectic form on M is a closed non-degenerate 2-form $\Omega \in H^2_{dR}(M, \mathbb{R})$ (i.e., $d\Omega = 0$ and $\Omega^n \neq 0$). We say that a symplectic form Ω on (M, J) is *tamed* by J if $\Omega(v, Jv) > 0$ for every non-zero smooth vector field $v \in \Gamma(TM)$, or equivalently, the $(1, 1)$-part of Ω is strictly positive with respect to J. We say that a symplectic form Ω is *compatible* with J if Ω is J-tamed and $\Omega(u, v) = \Omega(Ju, Jv)$ for any non-zero smooth vector fields $u, v \in \Gamma(TM)$.

In dimension 4, there is a well known question proposed by Donaldson [3]: if J is an almost complex structure on a compact 4-manifold M and J is tamed by a symplectic form,

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does there exist a symplectic form compatible with J? Although there are some progressions recently (cf. [10,16]), it is still a widely open problem. Motivated by this question, Li and Zhang [10] studied the relation between the tamed symplectic cones and the compatible symplectic cones of almost complex manifolds, and they obtained some interesting results; see [10] for more details.

The purpose of this paper is to study some aspects of the structure of the tamed symplectic cones of compact Hermitian-symplectic manifolds. Recall that a *Hermitian-symplectic structure* on a complex manifold $X = (M, J)$ is a J-tamed symplectic form Ω on X. We say that a complex manifold is *Hermitian-symplectic* if it admits a Hermitian-symplectic structure. In [14], Streets–Tian constructed a parabolic flow for Hermitian-symplectic metric, which is analogous to the Kähler–Ricci flow. They showed that a compact complex surface admits a Hermitian-symplectic structure if and only if it is Kähler, and thus proposed the following question: does there exist a complex manifold, of complex dimension ≥ 3, which admits a Hermitian-symplectic structure but not Kähler (see [16, Question 1.7], the same question is formulated, independently by Li and Zhang [10, page 678])? Since then there is a lot of progression; see for examples [5,6,17], however the question of Streets–Tian is still open.

The study of this paper is greatly inspired by the work of Demailly and Paun [2] on the Kähler cones, of Fu and Xiao [8] on the relations of Kähler cones and balanced cones, and of Xiao [19] on the strongly Gauduchon cones. Similarly, we consider a cohomology cone generated by all tamed symplectic forms on a compact Hermtian-symplectic manifold. Such a cohomology cone is called the *tamed symplectic cone* of a compact Hermtian-symplectic manifold (see [10, Definition 1.1]). We also consider the *dual tamed symplectic cone* which is generated by some positive (1, 1)-currents. One of the most important foundation in the work of Fu and Xiao [8,19] is the intrinsic characterization of the special Hermitian metrics by using positive currents (cf. [9,11,13]). Historically, Sullivan [15] was the first one who gave the intrinsic characterization for admitting Hermitian-symplectic metrics by using the positive currents and the technicalities of Hahn–Banach theorem. Since then the intrinsic characterization for special Hermitian metrics in terms of positive currents has been extensively investigated; see for examples [4,9,11–13] .

This paper is organized as follows: in Sect. 2, we briefly recall some well known facts about the positive currents. In Sect. 3, the first feature of the tamed symplectic cone of a compact Hermitian-symplectic manifold is given, that is, the Poincaré duality property of the tamed symplectic cone (see Theorem 7). Finally, in Sect. 4, we will study the stability of Hermitian-symplectic metrics under the deformation of complex structures, and prove that the tamed symplectic cones are invariant under the parallel transport with respect to the Gauss–Manin connection (see Theorem 10).

2 Positive currents

In this section we briefly remind on some basic definitions and facts on the positive currents. Our main reference is Demailly's e-book [1, Chapter III].

Assume now that X is always a compact complex manifold of complex dimension n, which means that $X = (M, J)$ is a compact smooth $2n$-manifold M with a complex structure J. Let us first fix some notations: (1) $\mathfrak{D}^k_{\mathbb{R}}(X)$ is the space of real-valued k-forms with the standard C^∞-topology; (2) $\mathfrak{D}^k_{\mathbb{C}}(X)$ is the space of complex-valued k-forms with the standard C^∞-topology; (3) $\mathfrak{D}^{p,q}(X)$ is the space of complex-valued (p, q)-forms with the standard C^∞-topology, where the standard C^∞-topology means the natural topology of the order-∞

jet spaces of smooth sections of bundle, and their topological dual: (a) the topological dual $\mathfrak{D}'^{\mathbb{R}}_k(X)$ of $\mathfrak{D}^k_{\mathbb{R}}(X)$ is the *space of real k-currents*; (b) the topological dual $\mathfrak{D}'^{\mathbb{C}}_k(X)$ of $\mathfrak{D}^k_{\mathbb{C}}(X)$ is the *space of complex k-currents*; (c) the topological dual $\mathfrak{D}'_{p,q}(X)$ of $\mathfrak{D}^{p,q}(X)$ is the *space of (p,q)-currents*. It is important to notice that the decomposition of de Rham forms

$$\mathfrak{D}^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \mathfrak{D}^{p,q}(X)$$

gives a corresponding decomposition

$$\mathfrak{D}'^{\mathbb{C}}_k(X) = \bigoplus_{p+q=k} \mathfrak{D}'_{p,q}(X)$$

of currents. By the duality, one can define the adjoint exterior operators $d, \bar{\partial}, \partial$ on currents. Of course, these operators continue to satisfy the standard identities: $d^2 = \bar{\partial}^2 = \partial^2 = 0$ and $d = \bar{\partial} + \partial$. It is a classical result of de Rham that there exist isomorphisms

$$H^k(\mathfrak{D}'^{\mathbb{C}}_\bullet(X), d) \cong H^{2n-k}_{dR}(X, \mathbb{C}) \quad \text{and} \quad H^k(\mathfrak{D}'^{\mathbb{R}}_\bullet(X), d) \cong H^{2n-k}_{dR}(X, \mathbb{R})$$

for any $k \in \mathbb{N}$.

An element of $\mathfrak{D}'_{p,q}(X)$ is called a *current of bi-dimension (p,q)* on X. A current of bi-dimension (p,q) can be identified with a $(n-p, n-q)$-forms with coefficients distributions. We say that a current T of bi-dimension (p,q) is said *of order* 0 if its coefficients are measures, and a current T of bi-dimension (p,p) is *real* if $T(\phi) = T(\bar{\phi})$ for any complex-valued (p,p)-form $\phi \in \mathfrak{D}^{p,p}(X)$. If $T \in \mathfrak{D}'_{p,p}(X)$ is real, then one may write

$$T = \sigma_{n-p} \sum_{I,\bar{K}} T_{I\bar{K}} dz_I \wedge d\bar{z}_K$$

where $\sigma_{n-p} := \frac{i^{(n-p)^2}}{2^{n-p}}$, $T_{I\bar{K}}$ are distributions on X such that $T_{K\bar{I}} = \bar{T}_{I\bar{K}}$ and I, K are multi-indices of length $n-p$, $I = (i_1, \ldots, i_{n-p})$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_{n-p}}$.

Definition 1 We say that a real current $T \in \mathfrak{D}'_{p,p}(X)$ is a *positive (p,p)-current* if

$$T(\sigma_p \phi^1 \wedge \cdots \wedge \phi^p \wedge \bar{\phi}^1 \wedge \cdots \wedge \bar{\phi}^p) \geq 0$$

for any $\phi^i \in \mathfrak{D}^{1,0}(X)$, $i = 1, \ldots, p$.

It is not difficult to see that positive currents are of order 0 and real.

3 Tamed symplectic cone and its Poincaré duality

In this section, we shall discuss the tamed symplectic cone of a compact Hermitian-symplectic manifold and prove the Poincaré duality theorem for the tamed symplectic cone.

Definition 2 Let (X, g) be a complex Hermitian manifold of complex dimension n, and let ω be its fundamental 2-form (i.e., $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$). We say that ω is *Hermitian-symplectic* if $\partial\omega = \bar{\partial}\eta$ for some ∂-closed $(2,0)$-form η.

It is important to notice that a Hermitian-symplectic structure is equivalent to the existence of a unique Kähler form which is Hermitian-symplectic (cf. [5, Proposition 2.1]), and hence Hermitian-symplectic structure is a special strong KT metric (cf. [7]). In fact, if a real 2-form

$\Omega := \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}$ is a Hermitian-symplectic structure, then $d\Omega = 0$ implies that $\partial\Omega^{1,1} = \bar{\partial}(-\Omega^{2,0})$ and $\partial(-\Omega^{2,0}) = 0$. Suppose ω is a Hermitian-symplectic metric, that is there is a ∂-closed $(2,0)$-form α such that $\partial\omega = \bar{\partial}\alpha$. If we define a 2-form $\Omega := \omega - \alpha - \bar{\alpha}$, then $d\Omega = 0$ and $\Omega^{1,1} := \omega$ is strictly positive definite. Therefore, we may view Hermitian-symplectic structure and Hermitian-symplectic metric as the same thing.

Next let us recall the definition of the tamed symplectic cones of compact Hermitian-symplectic manifolds.

Definition 3 Let X be a compact Hermitian-symplectic manifold of complex dimension n. The *tamed symplectic cone* $\mathcal{K}^t(X)$ of X is given by

$$\mathcal{K}^t(X) := \{[\Omega] \in H^2_{dR}(X, \mathbb{R}) \mid \Omega \text{ is a tamed symplectic form}\},$$

where $H^2_{dR}(X, \mathbb{R})$ is the 2-th de Rham cohomology of X.

Notice that the tamed symplectic cone $\mathcal{K}^t(X)$ is an open convex cone in the de Rham cohomology group. In fact, let ϕ be a real d-closed 2-form and decomposes $\phi := \phi^{2,0} + \phi^{1,1} + \phi^{0,2}$. In the locally coordinate $\{z_1, \ldots, z_n\}$, we may write $\phi^{1,1} = \frac{i}{2}\sum_{i,j}\phi^{1,1}_{jk}dz_j \wedge d\bar{z}_k$, where $[\phi^{1,1}_{jk}]$ is a Hermitian matrix. If Ω is a tamed symplectic form, then $\Omega^{1,1} > 0$ and we write $\Omega^{1,1} = \frac{i}{2}\sum_{i,j}\Omega^{1,1}_{jk}dz_j \wedge d\bar{z}_k$, where $[\Omega^{1,1}_{jk}]$ is a positive Hermitian matrix. By linear algebra, for enough small $\varepsilon > 0$, $[\Omega^{1,1}_{jk} + \varepsilon\phi^{1,1}_{jk}]$ is still strictly positive. Since X is compact, we can deform Ω to another tamed symplectic form $\Omega + \varepsilon\phi$ for any direction of ϕ. Indeed, we have $d(\Omega + \varepsilon\phi) = 0$, and hence the tamed symplectic cone $\mathcal{K}^t(X)$ is an open convex cone.

In [15], Sullivan gave an intrinsic characterization of Hermitian-symplectic metrics on compact complex manifolds by using positive currents. More precisely, he obtained

Proposition 4 (Sullivan [15]) *Let X be a compact complex manifold of complex dimension n. Then X is Hermitian-symplectic if and only if there is no non-zero positive $(1, 1)$-current T on X which is a boundary.*

This proposition allow us to define a meaningful cone which is generated by some d-closed positive currents:

Definition 5 Let X be a compact Hermitian-symplectic manifold of complex dimension n. We define

$$\mathcal{C}(X) := \{[T] \in H^{2n-2}_{dR}(X, \mathbb{R}) \mid T \text{ is a positive } (1,1)\text{−current}\}.$$

to be the *dual tamed symplectic cone* of X.

Lemma 6 *The dual tamed symplectic cone $\mathcal{C}(X)$ is a closed convex cone.*

Proof Fix a tamed symplectic form Ω and a norm $\| - \|$ in $H^{2n-2}_{dR}(X, \mathbb{R})$. It is sufficient to prove that for a given positive $(1, 1)$-current sequences $\{[T_m]\}$ such that $\lim[T_m] = \tau \in H^{2n-2}_{dR}(X, \mathbb{R})$, there exists a positive $(1, 1)$-current T that represents τ.

For any $T'_m \in [T_m]$, and $T'_m - T_m = dS$ for some current S, the Stokes formula implies that

$$0 = \int_X d(S \wedge \Omega) = \int_X dS \wedge \Omega \pm \int_X S \wedge d\Omega.$$

Since $d\Omega = 0$, thus $\int_X dS \wedge \Omega = 0$. This implies that $\int_X T'_m \wedge \Omega = \int_X T_m \wedge \Omega$, that means $\|T_m\|_{\text{mass}}$ is independent of the representation of the class $[T_m]$. Therefore, by assumption $\lim[T_m] = \tau$, the mass of T_m,

$$\|T_m\|_{\text{mass}} = \int_X T_m \wedge \Omega = [T_m] \cdot [\Omega] \leq c$$

for some $c = c(\tau)$ is a uniform positive constant. By the weak compactness of positive currents, there exists a convergent subsequence $\{T_{m_j}\}$ of $\{T_m\}$, such that T_{m_j} approaches to T, where T is a positive $(1, 1)$-current and represents τ. Hence, $\mathcal{C}(X)$ is closed. □

The Poincaré duality says that there exists a natural duality between $H^{2n-2}_{dR}(X, \mathbb{R})$ and $H^2_{dR}(X, \mathbb{R})$. This Poincaré duality makes sure that there is a duality between the tamed symplectic cone and the dual tamed symplectic cone of X. More precisely, we have

Theorem 7 *Suppose that X is a compact Hermitian-symplectic manifold of complex dimension n. Then $\mathcal{C}(X)^\vee = \overline{\mathcal{K}^t(X)}$.*

Proof We follow the same strategy as in [19, Proposition 3.1]. By the positivity of the two cones, it is not difficult to see that $\overline{\mathcal{K}^t(X)} \subset \mathcal{C}(X)^\vee$.

In the following, we will show the interior of $\mathcal{C}(X)^\vee$ is a subset of $\mathcal{K}^t(X)$ and thus we shall obtain $\mathcal{C}(X)^\vee \subset \overline{\mathcal{K}^t(X)}$.

The proof is divided into two cases as follows:

Case 1 $\mathcal{C}(X) \neq \{[0]\}$. Given two norms, denote by the same notation $\| - \|$, on $H^{2n-2}_{dR}(X, \mathbb{R})$ and $H^2_{dR}(X, \mathbb{R})$. For any interior point $o \in \mathcal{C}(X)^\vee$, after a small perturbation τ we still have $o+\tau \in \mathcal{C}(X)^\vee$. Here we choose a perturbation $\tau := -[\Omega]$, for $\Omega \in \mathcal{K}^t(X)$ and $\|[\Omega]\| < \epsilon$ for some $\epsilon > 0$ is chosen small enough such that $o + \tau \in \mathcal{C}(X)^\vee$. Therefore, for any $\varsigma \in \mathcal{C}(X)$, we have $(o + \tau) \cdot \varsigma \geq 0$. By the linearity, we may restrict ς to the $\| - \|$-unit sphere. Thus, by the compactness of the intersection of $\mathcal{C}(X)$ and the $\| - \|$-unit sphere, we obtain that the function $\varsigma \mapsto -\tau \cdot \varsigma = [\Omega] \cdot \varsigma$ arrives its maximum max > 0 on the intersection of $\mathcal{C}(X)$ and the $\| - \|$-unit sphere. Here the positivity of $[\Omega] \cdot \varsigma$ is guaranteed by $\Omega \in \mathcal{K}^t(X) \subset \mathcal{C}(X)^\vee$.

In summary, for any interior point $o \in \mathcal{C}(X)^\vee$, $(o + \tau) \cdot \varsigma \geq 0$, i.e., $o \cdot \varsigma \geq -\tau \cdot \varsigma$, we have

$$o \cdot \varsigma \geq \max > 0,$$

for ς in the intersection of $\mathcal{C}(X)$ and the $\| - \|$-unit sphere. □

Lemma 8 *Suppose ϖ is a real d-closed smooth 2-form. If $[\varpi] \cdot [T] \geq 0$ for all d-closed positive $(1, 1)$-currents T, and $[\varpi] \cdot [T] = 0$ if and only if $T = 0$, then $[\varpi] \in H^2_{dR}(X, \mathbb{R})$ is a Hermitian-symplectic class.*

Proof Fix a Hermitian metric ω on X, we denote by $\|T\|_{\text{mass}}$ the mass of a positive $(1, 1)$-current T, which is given by

$$\|T\|_{\text{mass}} := \int_X T \wedge \omega.$$

To prove this lemma, we will use the technical of Hahn–Banach theorem and thus we need to define two subsets of $\mathfrak{D}_2'^{\mathbb{R}}$ associated to ϖ as follows:

$$E_1(\varpi) = \{T \in \mathfrak{D}_2'^{\mathbb{R}} \mid dT = 0 \text{ and } [\varpi] \cdot [T] = 0\};$$
$$E_2(\varpi) = \{T \in \mathfrak{D}_2'^{\mathbb{R}} \mid T \geq 0 \text{ and } \|T\|_{\text{mass}} = 1\}.$$

It is not difficult to see that the sets $E_1(\varpi)$ and $E_2(\varpi)$ are disjoint and $E_1(\varpi)$ contains the subspace $d\mathfrak{D}_2'^{\mathbb{R}}$. Furthermore, $E_1(\varpi)$ is a closed subspace (this is clear from the definition of

$E_1(\varpi)$) and $E_2(\varpi)$ is a compact convex subset. By the Hahn–Banach theorem, we obtain that there exists a real 2-form Ω such that

$$\Omega|_{E_1(\varpi)} = 0 \quad \text{and} \quad \Omega|_{E_2(\varpi)} > 0.$$

Now we may view Ω as a linear functional on $\mathfrak{D}_2^{'\mathbb{R}}$. Then $\Omega|_{E_1(\varpi)} = 0$ implies $d\Omega = 0$, and $\Omega|_{E_2(\varpi)} > 0$ implies that $\Omega^{1,1}$ is strictly positive. Hence Ω is a tamed symplectic form.

By the definition of Ω, we see that Ω is decided by $E_1(\varpi)$ and $E_2(\varpi)$. In fact, we will see that ϖ and Ω are much more related. Since Ω is non-degenerate and X is compact, so $\int_X \Omega^n \neq 0$ and thus $[\Omega]$ is not 0 in $H^2_{dR}(X, \mathbb{R})$; otherwise it will be contradicting to Stoke's theorem. Next we consider the quotient map

$$p : \{T \in \mathfrak{D}_2^{'\mathbb{R}} \mid dT = 0\} \to H^{2n-2}_{dR}(X, \mathbb{R}).$$

If we view $[\varpi]$ and $[\Omega]$ as linear functional on $H^{2n-2}_{dR}(X, \mathbb{R})$, then we have

$$\ker[\varpi] = p(E_1(\varpi)) \subset \ker[\Omega] \subseteq H^{2n-2}_{dR}(X, \mathbb{R}).$$

Hence, $[\Omega] \neq 0$ implies that $\dim \ker[\varpi] = \dim \ker[\Omega] = b_{2n-2} - 1$, where $b_{2n-2} = \dim H^{2n-2}_{dR}(X, \mathbb{R})$ is the $(2n-2)$-th Betti number of X. Therefore, $\ker[\varpi] = \ker[\Omega]$, and $[\varpi] = \kappa[\Omega]$ for some positive constant $\kappa > 0$. So we get that $[\varpi]$ is a Hermitian-symplectic class. □

By Lemma 8, we have the interior of $\mathcal{C}(X)^\vee$ is a subset of $\mathcal{K}^t(X)$.

Case 2 $\mathcal{C}(X) = \{[0]\}$

We know that $\mathcal{C}(X) = \{[0]\}$ is equivalent to that any d-closed positive $(1, 1)$-current is d-exact. Thus if $T \in \mathcal{C}(X)$, then $T = dS$ for some current S. Since our complex manifold X is Hermitian-symplectic, by Proposition 4, we have $T = 0$. Then the conditions of Lemma 8 is automatically hold for any smooth form in $H^2_{dR}(X, \mathbb{R})$.

Combining with *Cases 1* and 2, we obtain that the interior of $\mathcal{C}(X)^\vee$ is a subset of $\mathcal{K}^t(X)$. This concludes the proof of Theorem 7. □

4 Deformation invariant of tamed symplectic cones

In this section we will study the behavior of tamed symplectic cones under the parallel transport with respect to the Gauss–Manin connection.

Suppose that $\mathfrak{X}$ and B are complex manifolds, and we say that a holomorphic map $\pi : \mathfrak{X} \to B$ is a *complex analytic family* of complex manifolds if π is a proper holomorphic submersion. Then each fibre $X_b := \pi^{-1}(b)$, $b \in B$, is a compact complex manifold. We consider the sheaf $H^k_{\mathbb{R}} := R^k\pi_*\mathbb{R}$ the k-th higher direct image of $\mathbb{R}$, where $\mathbb{R}$ is the constant sheaf of stalk $\mathbb{R}$ over $\mathfrak{X}$ and $R^k\pi_*$ is the k-th right derived functor of the direct image functor. In fact, the sheaf $H^k_{\mathbb{R}}$ is a local system, over B, which is locally isomorphic to the constant sheaf of stalk $H^k(X_0, \mathbb{R})$.

For the local system $H^k_{\mathbb{R}}$, let us define a connection Δ associated to $H^k_{\mathbb{R}}$ (cf. [18, Section 9.2]): $\Delta : \mathcal{H}^k \to \mathcal{H}^k \otimes \mathcal{A}_B$, where $\mathcal{H}^k = H^k_{\mathbb{R}} \otimes \mathcal{C}^\infty_B$ and $\mathcal{A}_B$ is the sheaf of smooth differential 1-forms. For $h = \sum_i \alpha_i h_i \in \mathcal{H}^k$ in a basis h_i of a local trivialisation of $H^k_{\mathbb{R}}$, we define

$$\Delta h := \sum_i h_i \otimes d\alpha_i.$$

From the definition, it is not difficult to see that the curvature of the connection Δ is 0, i.e., the connection Δ is flat.

Definition 9 The flat connection $\Delta : \mathcal{H}^k \to \mathcal{H}^k \otimes \mathcal{A}_B$ on the vector bundle associated to the local system $H^k_{\mathbb{R}}$ is called the *Gauss–Manin connection* of the local system $H^k_{\mathbb{R}}$.

In [2], Demailly and Paun proved that for any complex analytic family $\pi : \mathfrak{X} \to B$ of compact Kähler manifolds, the Kähler cone of a very general fibre X_b is invariant by parallel transport under the $(1, 1)$-component of the Gauss–Manin connection (see [2, Theorem 0.9]). Recently, Xiao [19] obtained a similar result for strongly Gauduchon cones under the parallel transport with respect to the Gauss–Manin connection (see [19, Proposition 4.1]). Along the same line, we obtain an analogous result for Hermitian-symplectic manifolds.

Theorem 10 *Let $\pi : \mathfrak{X} \to B$ be a complex analytic family of compact Hermitian-symplectic manifolds. Then the tamed symplectic cones $\mathcal{K}^t(X_b) \subset H^2_{dR}(X_b, \mathbb{R})$ are invariant under the parallel transport with respect to the Gauss–Manin connection Δ of the local system $H^2_{\mathbb{R}}$.*

Proof First, we notice that the Hermitian-symplectic metric is stable under the small deformation of complex structures (cf. [20, Proposition 2.4]). In fact, by a theorem of Ehresmann (cf. [18, Theorem 9.3]), we know that $\mathfrak{X}$ is locally isomorphic to $X_b \times B$ for each point $b \in B$, that is, there exists a neighborhood B_b of b in B such that

$$\mathfrak{X}|_{\pi^{-1}(B_b)} \cong X_b \times B_b.$$

Fixed a reference point $0 \in B$ such that $X_0 = X$ is a compact Hermitian-symplectic manifold, and let $\varphi_b : X_b \to X$ be a diffeomorphism which varies smoothly with $b \in B_0$ and $\varphi_0 = id$. Suppose Ω is a tamed symplectic form on $X = X_0$, i.e., $d\Omega = 0$ and its $(1, 1)$-component $\Omega^{1,1}$ is strictly positive definite. Denote by $\Omega_b = \varphi_b^*\Omega$, and thus Ω_b is a real 2-form and $d\Omega_b = d\varphi_b^*\Omega = \varphi_b^* d\Omega = 0$. We decompose

$$\Omega_b := \Omega_b^{2,0} + \Omega_b^{1,1} + \Omega_b^{0,2}.$$

Since Ω_b is real, and thus $\Omega_b^{1,1}$ is real and approaches to $\Omega^{1,1}$ as b approaches to 0. Therefore, $d\Omega_b = 0$ and $\Omega_b^{1,1}$ ia strictly positive definite for b is sufficiently close to 0, and consequently Ω_b is also a tamed symplectic form.

Next, to finish the proof, we follow the same strategy as in [2, Theorem 0.9] and [19, Proposition 4.1]: let

$$c : [0, 1] \to B, \ s \mapsto c(s)$$

be a smooth curve in B, and let $\sigma(s) \in H^2_{dR}(X_{c(s)}, \mathbb{R})$ be a sequence of real 2-cohomology classes along the curve c such that there is an ordinary differential equation (ODE)

$$\nabla_{c'(s)}\sigma(s) = 0 \tag{1}$$

with an initial value $\sigma(0) = [\Omega_0]$ for some $\Omega_0 \in \mathcal{K}^t(X_{c(0)})$. By the theory of ODE, the above Eq. (1) gives us an injective map $\sigma(0) \mapsto \sigma(s)$. For small enough s, we have $\sigma(s) \in \mathcal{K}^t(X_{c(t)})$: In fact, we know that the fibers $H^2_{dR}(X_{\sigma(s)}, \mathbb{R})$ are locally constant, i.e., $H^2_{dR}(X_{\sigma(s)}, \mathbb{R}) \cong H^2_{dR}(X_{\sigma(0)}, \mathbb{R})$. Then the solution $\sigma(s)$ of the Eq. (1) may be viewed in $H^2_{dR}(X_{\sigma(0)}, \mathbb{R})$. For any norm $\| - \|$ in $H^2_{dR}(X_{\sigma(0)}, \mathbb{R})$, since $\sigma(s)$ is smooth along s, then we have $\lim_{s\to 0} \|\sigma(s) - \sigma(0)\| = 0$. We see that the quotient map from the Fréchet space of d-closed smooth 2-forms to $H^2_{dR}(X_{\sigma(0)}, \mathbb{R})$ is surjective and open. Hence, for any given $\epsilon > 0$, there is a smooth representative Ω_s satisfying $\|\Omega_s - \Omega_0\|_F < \epsilon$ for small enough s, where $\| - \|_F$ is the Fréchet norm. Therefore, Ω_s is a Hermitian-symplectic structure, i.e., $\sigma(s) \in \mathcal{K}^t(X_{c(t)})$.

Consequently, for small enough s, there is an injective map $\mathcal{K}^t(X_{c(0)}) \mapsto \mathcal{K}^t(X_{c(s)})$. Similarly, there is an injective map $\mathcal{K}^t(X_{c(t)}) \mapsto \mathcal{K}^t(X_{c(0)})$, that means $\mathcal{K}^t(X_{c(s)}) = \mathcal{K}^t(X_{c(0)})$ for small enough s. This completes the proof. □

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