Ann Glob Anal Geom (2016) 49:87–103
DOI 10.1007/s10455-015-9482-0

Self-adjoint extensions of differential operators on Riemannian manifolds

Ognjen Milatovic[1] · **Françoise Truc**[2]

Received: 3 July 2015 / Accepted: 27 October 2015 / Published online: 12 November 2015

Abstract We study $H = D^*D + V$, where D is a first order elliptic differential operator acting on sections of a Hermitian vector bundle over a Riemannian manifold M, and V is a Hermitian bundle endomorphism. In the case when M is geodesically complete, we establish the essential self-adjointness of positive integer powers of H. In the case when M is not necessarily geodesically complete, we give a sufficient condition for the essential self-adjointness of H, expressed in terms of the behavior of V relative to the Cauchy boundary of M.

Keywords Essential self-adjointness · Hermitian vector bundle · Higher-order differential operator · Riemannian manifold

Mathematics Subject Classification Primary 58J50 · 35P05; Secondary 47B25

1 Introduction

As a fundamental problem in mathematical physics, self-adjointness of Schrödinger operators has attracted the attention of researchers over many years now, resulting in numerous sufficient conditions for this property in $L^2(\mathbb{R}^n)$. For reviews of the corresponding results, see, for instance, the books [14,28].

The study of the corresponding problem in the context of a non-compact Riemannian manifold was initiated by Gaffney [15,16] with the proof of the essential self-adjointness

✉ Ognjen Milatovic
omilatov@unf.edu

Françoise Truc
francoise.truc@ujf-grenoble.fr

1 Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA

2 Unité mixte de recherche CNRS-UJF 5582, Institut Fourier, Grenoble University, BP 74, 38402 Saint Martin d'Hères Cedex, France

of the Laplacian on differential forms. About two decades later, Cordes (see Theorem 3 in [11]) proved the essential self-adjointness of positive integer powers of the operator

$$\Delta_{M,\mu} := -\frac{1}{\kappa}\left(\frac{\partial}{\partial x^i}\left(\kappa g^{ij}\frac{\partial}{\partial x^j}\right)\right) \tag{1.1}$$

on an n-dimensional geodesically complete Riemannian manifold M equipped with a (smooth) metric $g = (g_{ij})$ [here $(g^{ij}) = ((g_{ij})^{-1})$] and a positive smooth measure $d\mu$ [i.e. in any local coordinates $x^1, x^2, \ldots, x^n$ there exists a strictly positive C^∞-density $\kappa(x)$ such that $d\mu = \kappa(x)\, dx^1 dx^2 \ldots dx^n$]. Theorem 1 of our paper extends this result to the operator $(D^*D + V)^k$ for all $k \in \mathbb{Z}_+$, where D is a first order elliptic differential operator acting on sections of a Hermitian vector bundle over a geodesically complete Riemannian manifold, D^* is the formal adjoint of D, and V is a self-adjoint Hermitian bundle endomorphism; see Sect. 2.2 for details.

In the context of a general Riemannian manifold (not necessarily geodesically complete), Cordes (see Theorem IV.1.1 in [12], Theorem 4 in [11]) proved the essential self-adjointness of P^k for all $k \in \mathbb{Z}_+$, where

$$Pu := \Delta_{M,\mu} u + qu, \quad u \in C^\infty(M), \tag{1.2}$$

and $q \in C^\infty(M)$ is real-valued. Thanks to a Roelcke-type estimate (see Lemma 3.1 below), the technique of Cordes [12] can be applied to the operator $(D^*D + V)^k$ acting on sections of Hermitian vector bundles over a general Riemannian manifold. To make our exposition shorter, in Theorem 1 we consider the geodesically complete case. Our Theorem 2 concerns $(\nabla^*\nabla + V)^k$, where ∇ is a metric connection on a Hermitian vector bundle over a non-compact geodesically complete Riemannian manifold. This result extends Theorem 1.1 of [13] where Cordes showed that if (M, g) is non-compact and geodesically complete and P is semi-bounded from below on $C_c^\infty(M)$, then P^k is essentially self-adjoint on $C_c^\infty(M)$, for all $k \in \mathbb{Z}_+$.

For the remainder of the introduction, the notation $D^*D + V$ is used in the same sense as described earlier in this section. In the setting of geodesically complete Riemannian manifolds, the essential self-adjointness of $D^*D + V$ with $V \in L^\infty_{\text{loc}}$ was established in [20], providing a generalization of the results in [3,26,27,31] concerning Schrödinger operators on functions (or differential forms). Subsequently, the operator $D^*D + V$ with a singular potential V was considered in [5]. Recently, in the case $V \in L^\infty_{\text{loc}}$, the authors of [4] extended the main result of [5] to the operator $D^*D + V$ acting on sections of infinite-dimensional bundles whose fibers are modules of finite type over a von Neumann algebra.

In the context of an incomplete Riemannian manifold, the authors of [17,21,22] studied the so-called Gaffney Laplacian, a self-adjoint realization of the scalar Laplacian generally different from the closure of $\Delta_{M,d\mu}|_{C_c^\infty(M)}$. For a study of Gaffney Laplacian on differential forms, see [23].

Our Theorem 3 gives a condition on the behavior of V relative to the Cauchy boundary of M that will guarantee the essential self-adjointness of $D^*D + V$; for details see Sect. 2.3 below. Related results can be found in [6,24,25] in the context of (magnetic) Schrödinger operators on domains in $\mathbb{R}^n$, and in [10] concerning the magnetic Laplacian on domains in $\mathbb{R}^n$ and certain types of Riemannian manifolds.

Finally, let us mention that Chernoff [7] used the hyperbolic equation approach to establish the essential self-adjointness of positive integer powers of Laplace–Beltrami operator on differential forms. This approach was also applied in [2,8,9,18,19,30] to prove essential self-adjointness of second-order operators (acting on scalar functions or sections of Hermitian vector bundles) on Riemannian manifolds. Additionally, the authors of [18,19] used path integral techniques.

The paper is organized as follows. The main results are stated in Sect. 2, a preliminary lemma is proven in Sect. 3, and the main results are proven in Sects. 4–6.

2 Main results

2.1 The setting

Let M be an n-dimensional smooth, connected Riemannian manifold without boundary. We denote the Riemannian metric on M by g^{TM}. We assume that M is equipped with a positive smooth measure $\mathrm{d}\mu$, i.e. in any local coordinates $x^1,\ x^2,\dots,x^n$ there exists a strictly positive C^∞-density $\kappa(x)$ such that $\mathrm{d}\mu=\kappa(x)\,\mathrm{d}x^1\mathrm{d}x^2\dots\mathrm{d}x^n$. Let E be a Hermitian vector bundle over M and let $L^2(E)$ denote the Hilbert space of square integrable sections of E with respect to the inner product

$$(u,v)\;=\;\int_M\langle u(x),v(x)\rangle_{E_x}\,\mathrm{d}\mu(x),\tag{2.1}$$

where $\langle\cdot,\cdot\rangle_{E_x}$ is the fiberwise inner product. The corresponding norm in $L^2(E)$ is denoted by $\|\cdot\|$. In Sobolev space notations $W^{k,2}_{\mathrm{loc}}(E)$ used in this paper, the superscript $k\in\mathbb{Z}_+$ indicates the order of the highest derivative. The corresponding dual space is denoted by $W^{-k,2}_{\mathrm{loc}}(E)$.

Let F be another Hermitian vector bundle on M. We consider a first order differential operator $D\colon C^\infty_c(E)\to C^\infty_c(F)$, where C^∞_c stands for the space of smooth compactly supported sections. In the sequel, by $\sigma(D)$ we denote the principal symbol of D.

Assumption (A0) Assume that D is elliptic. Additionally, assume that there exists a constant $\lambda_0>0$ such that

$$|\sigma(D)(x,\xi)|\;\leq\;\lambda_0|\xi|,\quad\text{for all }x\in M,\ \xi\in T^*_xM,\tag{2.2}$$

where $|\xi|$ is the length of ξ induced by the metric g^{TM} and $|\sigma(D)(x,\xi)|$ is the operator norm of $\sigma(D)(x,\xi)\colon E_x\to F_x$.

Remark 2.1 Assumption (A0) is satisfied if $D=\nabla$, where $\nabla\colon C^\infty(E)\to C^\infty(T^*M\otimes E)$ is a covariant derivative corresponding to a metric connection on a Hermitian vector bundle E over M.

2.2 Schrödinger-type operator

Let $D^*\colon C^\infty_c(F)\to C^\infty_c(E)$ be the formal adjoint of D with respect to the inner product (2.1). We consider the operator

$$H\;=\;D^*D\;+\;V,\tag{2.3}$$

where $V\in L^\infty_{\mathrm{loc}}(\mathrm{End}E)$ is a linear self-adjoint bundle endomorphism. In other words, for all $x\in M$, the operator $V(x)\colon E_x\to E_x$ is self-adjoint and $|V(x)|\in L^\infty_{\mathrm{loc}}(M)$, where $|V(x)|$ is the norm of the operator $V(x)\colon E_x\to E_x$.

2.3 Statements of results

Theorem 1 *Let M, g^{TM}, and $\mathrm{d}\mu$ be as in Sect. 2.1. Assume that (M,g^{TM}) is geodesically complete. Let E and F be Hermitian vector bundles over M, and let $D\colon C^\infty_c(E)\to$*

$C_c^\infty(F)$ be a first order differential operator satisfying the Assumption (A0). Assume that $V \in C^\infty(\mathrm{End} E)$ and

$$V(x) \geq C, \quad \textit{for all } x \in M,$$

where C is a constant, and the inequality is understood in operator sense. Then H^k is essentially self-adjoint on $C_c^\infty(E)$, for all $k \in \mathbb{Z}_+$.

Remark 2.2 In the case $V = 0$, the following result related to Theorem 1 can be deduced from Chernoff (see Theorem 2.2 in [7]):

Assume that (M, g) is a geodesically complete Riemannian manifold with metric g. Let D be as in Theorem 1, and define

$$c(x) := \sup\{|\sigma(D)(x, \xi)| \colon |\xi|_{T_x^* M} = 1\}.$$

Fix $x_0 \in M$ and define

$$\widetilde{c}(r) := \sup_{x \in B(x_0, r)} c(x),$$

where $r > 0$ and $B(x_0, r) := \{x \in M \colon \mathrm{d}_g(x_0, x) < r\}$. Assume that

$$\int_0^\infty \frac{1}{\widetilde{c}(r)} \,\mathrm{d}r = \infty. \tag{2.4}$$

Then the operator $(D^* D)^k$ is essentially self-adjoint on $C_c^\infty(E)$ for all $k \in \mathbb{Z}_+$.

At the end of this section we give an example of an operator for which Theorem 1 guarantees the essential self-adjointness of $(D^* D)^k$, whereas Chernoff's result cannot be applied.

The next theorem is concerned with operators whose potential V is not necessarily semi-bounded from below.

Theorem 2 *Let M, g^{TM}, and $\mathrm{d}\mu$ be as in Sect. 2.1. Assume that (M, g^{TM}) is noncompact and geodesically complete. Let E be a Hermitian vector bundle over M and let ∇ be a Hermitian connection on E. Assume that $V \in C^\infty(\mathrm{End} E)$ and*

$$V(x) \geq q(x), \quad \textit{for all } x \in M, \tag{2.5}$$

where $q \in C^\infty(M)$ and the inequality is understood in the sense of operators $E_x \to E_x$. Additionally, assume that

$$((\Delta_{M,\mu} + q)u, u) \geq C\|u\|^2, \quad \textit{for all } u \in C_c^\infty(M), \tag{2.6}$$

where $C \in \mathbb{R}$ and $\Delta_{M,\mu}$ is as in (1.1) with g replaced by g^{TM}. Then the operator $(\nabla^*\nabla + V)^k$ is essentially self-adjoint on $C_c^\infty(E)$, for all $k \in \mathbb{Z}_+$.

Remark 2.3 Let us stress that non-compactness is required in the proof to ensure the existence of a positive smooth solution of an equation involving $\Delta_{M,\mu} + q$. In the case of a compact manifold, such a solution exists under an additional assumption; see Theorem III.6.3 in [12].

In our last result we will need the notion of Cauchy boundary. Let $d_{g^{\mathrm{TM}}}$ be the distance function corresponding to the metric g^{TM}. Let $(\widehat{M}, \widehat{d}_{g^{\mathrm{TM}}})$ be the metric completion of $(M, d_{g^{\mathrm{TM}}})$. We define the *Cauchy boundary* $\partial_C M$ as follows: $\partial_C M := \widehat{M} \backslash M$. Note that $(M, d_{g^{\mathrm{TM}}})$ is metrically complete if and only if $\partial_C M$ is empty. For $x \in M$ we define

$$r(x) := \inf_{z \in \partial_C M} \widehat{d}_{g^{\mathrm{TM}}}(x, z). \tag{2.7}$$

We will also need the following assumption:
Assumption (A1) Assume that $\widehat{M}$ is a smooth manifold and that the metric g^{TM} extends to $\partial_C M$.

Remark 2.4 Let N be a (smooth) n-dimensional Riemannian manifold without boundary. Denote the metric on N by g^{TN} and assume that (N, g^{TN}) is geodesically complete. Let Σ be a k-dimensional closed sub-manifold of N with $k < n$. Then $M := N\backslash\Sigma$ has the properties $\widehat{M} = N$ and $\partial_C M = \Sigma$. Thus, Assumption (A1) is satisfied.

Theorem 3 *Let M, g^{TM}, and $\mathrm{d}\mu$ be as in Sect. 2.1. Assume that (A1) is satisfied. Let E and F be Hermitian vector bundles over M, and let $D\colon C_c^\infty(E) \to C_c^\infty(F)$ be a first order differential operator satisfying the Assumption (A0). Assume that $V \in L^\infty_{\mathrm{loc}}(\mathrm{End}E)$ and there exists a constant C such that*

$$V(x) \geq \left(\frac{\lambda_0}{r(x)}\right)^2 - C, \quad \textit{for all } x \in M, \tag{2.8}$$

where λ_0 is as in (2.2), the distance $r(x)$ is as in (2.7), and the inequality is understood in the sense of linear operators $E_x \to E_x$. Then H is essentially self-adjoint on $C_c^\infty(E)$.

In order to describe the example mentioned in Remark 2.2, we need the following

Remark 2.5 As explained in [5], we can use a first-order elliptic operator $D\colon C_c^\infty(E) \to C_c^\infty(F)$ to define a metric on M. For $\xi, \eta \in T_x^*M$, define

$$\langle \xi, \eta\rangle \;=\; \frac{1}{m}\,\mathrm{Re}\;\mathrm{Tr}\left((\sigma(D)(x,\xi))^*\,\sigma(D)(x,\eta)\right), \quad m = \dim E_x, \tag{2.9}$$

where Tr denotes the usual trace of a linear operator. Since D is an elliptic first-order differential operator and $\sigma(D)(x,\xi)$ is linear in ξ, it is easily checked that (2.9) defines an inner product on T_x^*M. Its dual defines a Riemannian metric on M. Denoting this metric by g^{TM} and using elementary linear algebra, it follows that (2.2) is satisfied with $\lambda_0 = \sqrt{m}$.

Example 2.6 Let $M = \mathbb{R}^2$ with the standard metric and measure, and $V = 0$. Denoting respectively by $C_c^\infty(\mathbb{R}^2;\mathbb{R})$ and $C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ the spaces of smooth compactly supported functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, we define the operator $D\colon C_c^\infty(\mathbb{R}^2;\mathbb{R}) \to C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ by

$$D \;=\; \begin{pmatrix} a(x,y)\frac{\partial}{\partial x} \\ b(x,y)\frac{\partial}{\partial y}\end{pmatrix},$$

where

$$a(x,y) = (1-\cos(2\pi e^x))x^2 + 1;$$
$$b(x,y) = (1-\sin(2\pi e^y))y^2 + 1.$$

Since a, b are smooth real-valued nowhere vanishing functions in $\mathbb{R}^2$, it follows that the operator D is elliptic. We are interested in the operator

$$H \;:=\; D^*D \;=\; -\frac{\partial}{\partial x}\left(a^2\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial y}\left(b^2\frac{\partial}{\partial y}\right).$$

The matrix of the inner product on T^*M defined by D via (2.9) is $\mathrm{diag}(a^2/2, b^2/2)$. The matrix of the corresponding Riemannian metric g^{TM} on M is $\mathrm{diag}(2a^{-2}, 2b^{-2})$, so the metric itself is $ds^2 = 2a^{-2}\mathrm{d}x^2 + 2b^{-2}dy^2$ and it is geodesically complete (see Example 3.1 of [5]). Moreover, thanks to Remark 2.5, Assumption (A0) is satisfied. Thus, by Theorem 1 the operator $(D^*D)^k$ is essentially self-adjoint for all $k \in \mathbb{Z}_+$. Furthermore, in Example 3.1 of [5] it was shown that for the considered operator D the condition (2.4) is not satisfied. Thus, the result stated in Remark 2.2 does not apply.

3 Roelcke-type inequality

Let M, $\mathrm{d}\mu$, D, and $\sigma(D)$ be as in Sect. 2.1. Set $\widehat{D} := -i\sigma(D)$, where $i = \sqrt{-1}$. Then for any Lipschitz function $\psi\colon M \to \mathbb{R}$ and $u \in W^{1,2}_{\mathrm{loc}}(E)$ we have

$$D(\psi u) \;=\; \widehat{D}(d\psi)u + \psi Du, \tag{3.1}$$

where we have suppressed x for simplicity. We also note that $\widehat{D^*}(\xi) = -(\widehat{D}(\xi))^*$, for all $\xi \in T^*_xM$.

For a compact set $K \subset M$, and $u,\ v \in W^{1,2}_{\mathrm{loc}}(E)$, we define

$$(u, v)_K := \int_K \langle u(x), v(x)\rangle \,\mathrm{d}\mu(x), \quad (Du, Dv)_K := \int_K \langle Du(x), Dv(x)\rangle \,\mathrm{d}\mu(x). \tag{3.2}$$

In order to prove Theorem 1 we need the following important lemma, which is an extension of Lemma IV.2.1 in [12] to operator (2.3). In the context of the scalar Laplacian on a Riemannian manifold, this kind of result is originally due to Roelcke [29].

Lemma 3.1 *Let M, g^{TM}, and $\mathrm{d}\mu$ be as in Sect. 2.1. Let E and F be Hermitian vector bundles over M, and let $D\colon C^\infty_c(E) \to C^\infty_c(F)$ be a first order differential operator satisfying the Assumption (A0). Let $\rho\colon M \to [0, \infty)$ be a function satisfying the following properties*:

(i) *$\rho(x)$ is Lipschitz continuous with respect to the distance induced by the metric g^{TM};*
(ii) *$\rho(x_0) = 0$, for some fixed $x_0 \in M$;*
(iii) *the set $B_T := \{x \in M\colon \rho(x) \le T\}$ is compact, for some $T > 0$.*

Then the following inequality holds for all $u \in W^{2,2}_{\mathrm{loc}}(E)$ and $v \in W^{2,2}_{\mathrm{loc}}(E)$:

$$\int_0^T |(Du, Dv)_{B_t} - (D^*Du, v)_{B_t}|\,\mathrm{d}t \le \lambda_0 \int_{B_T} |\mathrm{d}\rho(x)||Du(x)||v(x)|\,\mathrm{d}\mu(x), \tag{3.3}$$

where B_t is as in (iii) (*with t instead of T*)*, the constant λ_0 is as in* (2.2)*, and $|\mathrm{d}\rho(x)|$ is the length of $\mathrm{d}\rho(x) \in T^*_xM$ induced by g^{TM}.*

Proof For $\varepsilon > 0$ and $t \in (0, T)$, we define a continuous piecewise linear function $F_{\varepsilon,t}$ as follows:

$$F_{\varepsilon,t}(s) = \begin{cases} 1 & \text{for } s < t - \varepsilon \\ (t-s)/\varepsilon & \text{for } t - \varepsilon \le s < t \\ 0 & \text{for } s \ge t. \end{cases}$$

The function $f_{\varepsilon,t}(x) := F_{\varepsilon,t}(\rho(x))$, is Lipschitz continuous with respect to the distance induced by the metric g^{TM}, and $df_{\varepsilon,t}(x) = (F'_{\varepsilon,t}(\rho(x)))\mathrm{d}\rho(x)$. Moreover we have $f_{\varepsilon,t}v \in W^{1,2}_{\mathrm{loc}}(E)$ for all $v \in W^{1,2}_{\mathrm{loc}}(E)$, since

$$D(f_{\varepsilon,t}v) = \widehat{D}(df_{\varepsilon,t})v + f_{\varepsilon,t}Dv.$$

It follows from the compactness of B_T that B_t is compact for all $t \in (0, T)$. Using integration by parts (see Lemma 8.8 in [5]), for all $u \in W^{2,2}_{\text{loc}}(E)$ and $v \in W^{2,2}_{\text{loc}}(E)$ we have

$$(D^*Du, vf_{\varepsilon,t})_{B_t} = (Du, D(vf_{\varepsilon,t}))_{B_t} = (Du, f_{\varepsilon,t}Dv)_{B_t} + (Du, \widehat{D}(df_{\varepsilon,t})v)_{B_t},$$

which, together with (2.2), gives

$$\begin{aligned} |(Du, f_{\varepsilon,t}Dv)_{B_t} - (D^*Du, vf_{\varepsilon,t})_{B_t}| &= |(Du, \widehat{D}(df_{\varepsilon,t})v)_{B_t}| \\ &\leq \int_{B_t} |Du(x)||\widehat{D}(df_{\varepsilon,t}(x))v(x)|\,\mathrm{d}\mu(x) \leq \lambda_0 \int_{B_t} |Du(x)||df_{\varepsilon,t}(x)||v(x)|\,\mathrm{d}\mu(x) \\ &= \lambda_0 \int_{B_t} |Du(x)||F'_{\varepsilon,t}(\rho(x))||\mathrm{d}\rho(x)||v(x)|\,\mathrm{d}\mu(x) \\ &\leq \lambda_0 \int_{B_T} |Du(x)||F'_{\varepsilon,t}(\rho(x))||\mathrm{d}\rho(x)||v(x)|\,\mathrm{d}\mu(x), \end{aligned} \tag{3.4}$$

where $|df_{\varepsilon,t}(x)|$ and $|\mathrm{d}\rho(x)|$ are the norms of $df_{\varepsilon,t}(x) \in T^*_xM$ and $\mathrm{d}\rho(x) \in T^*_xM$ induced by g^{TM}.

Fixing $\varepsilon > 0$, integrating the leftmost and the rightmost side of (3.4) from $t = 0$ to $t = T$, and noting that $F'_{\varepsilon,t}(\rho(x))$ is the only term on the rightmost side depending on t, we obtain

$$\begin{aligned} &\int_0^T |(Du, f_{\varepsilon,t}Dv)_{B_t} - (D^*Du, vf_{\varepsilon,t})_{B_t}|\,\mathrm{d}t \\ &\quad \leq \lambda_0 \int_{B_T} |Du(x)||\mathrm{d}\rho(x)||v(x)|I_\varepsilon(x)\,\mathrm{d}\mu(x), \end{aligned} \tag{3.5}$$

where

$$I_\varepsilon(x) := \int_0^T |F'_{\varepsilon,t}(\rho(x))|\,\mathrm{d}t.$$

We now let $\varepsilon \to 0+$ in (3.5). On the left-hand side of (3.5), as $\varepsilon \to 0+$, we have $f_{\varepsilon,t}(x) \to \chi_{B_t}(x)$ almost everywhere, where $\chi_{B_t}(x)$ is the characteristic function of the set B_t. Additionally, $|f_{\varepsilon,t}(x)| \leq 1$ for all $x \in B_t$ and all $t \in (0, T)$; thus, by dominated convergence theorem, as $\varepsilon \to 0+$ the left-hand side of (3.5) converges to the left-hand side of (3.3). On the right-hand side of (3.5) an easy calculation shows that $I_\varepsilon(x) \to 1$, as $\varepsilon \to 0+$. Additionally, we have $|I_\varepsilon(x)| \leq 1$, a.e. on B_T; hence, by the dominated convergence theorem, as $\varepsilon \to 0+$ the right-hand side of (3.5) converges to the right-hand side of (3.3). This establishes the inequality (3.3). □

4 Proof of Theorem 1

We first give the definitions of minimal and maximal operators associated with the expression H in (2.3).

4.1 Minimal and maximal operators

We define $H_{\min}u := Hu$, with $\text{Dom}(H_{\min}) := C_c^\infty(E)$, and $H_{\max} := (H_{\min})^*$, where T^* denotes the adjoint of operator T. Denoting $\mathscr{D}_{\max} := \{u \in L^2(E) : Hu \in L^2(E)\}$, we recall the following well-known property: $\text{Dom}(H_{\max}) = \mathscr{D}_{\max}$ and $H_{\max}u = Hu$ for all $u \in \mathscr{D}_{\max}$.

From now on, throughout this section, we assume that the hypotheses of Theorem 1 are satisfied. Let $x_0 \in M$, and define $\rho(x) := d_{g^{\text{TM}}}(x_0, x)$, where $d_{g^{\text{TM}}}$ is the distance function corresponding to the metric g^{TM}. By the definition of $\rho(x)$ and the geodesic completeness of (M, g^{TM}), it follows that $\rho(x)$ satisfies all hypotheses of Lemma 3.1. Using Lemma 3.1 and Proposition 4.1 below, we are able to apply the method of Cordes [11,12] to our context. As we will see, Cordes's technique reduces our problem to a system of ordinary differential inequalities of the same type as in Section IV.3 of [12].

Proposition 4.1 *Let A be a densely defined operator with domain $\mathscr{D}$ in a Hilbert space $\mathscr{H}$. Assume that A is semi-bounded from below, that $A\mathscr{D} \subseteq \mathscr{D}$, and that there exists $c_0 \in \mathbb{R}$ such that the following two properties hold*:

(i) $((A + c_0 I)u, u)_{\mathscr{H}} \geq \|u\|^2_{\mathscr{H}}$, *for all $u \in \mathscr{D}$, where I denotes the identity operator in $\mathscr{H}$*;
(ii) $(A + c_0 I)^k$ *is essentially self-adjoint on $\mathscr{D}$, for some $k \in \mathbb{Z}_+$.*

Then, $(A + cI)^j$ is essentially self-adjoint on $\mathscr{D}$, for all $j = 1, 2, \ldots, k$ and all $c \in \mathbb{R}$.

Remark 4.2 To prove Proposition 4.1, one may mimick the proof of Proposition IV.1.4 in [12], which was carried out for the operator P defined in (1.2) with $\mathscr{D} = C_c^\infty(M)$, since only abstract functional analysis facts and the property $P\mathscr{D} \subseteq \mathscr{D}$ were used.

We start the proof of Theorem 1 by noticing that the operator $H_{\min}$ is essentially self-adjoint on $C_c^\infty(E)$; see Corollary 2.9 in [5]. Thanks to Proposition 4.1, whithout any loss of generality we can change $V(x)$ to $V(x) + C\,\mathrm{Id}(x)$, where C is a sufficiently large constant in order to have

$$V(x) \geq (\lambda_0^2 + 1)\mathrm{Id}(x), \quad \text{for all } x \in M, \tag{4.1}$$

where λ_0 is as in (2.2) and $\mathrm{Id}(x)$ is the identity endomorphism of E_x. Using non-negativity of D^*D and (4.1) we have

$$(H_{\min}u, u) \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E), \tag{4.2}$$

which leads to

$$\|u\|^2 \leq (Hu, u) \leq \|Hu\|\|u\|, \quad \text{for all } u \in C_c^\infty(E),$$

and, hence, $\|Hu\| \geq \|u\|$, for all $u \in C_c^\infty(E)$. Therefore,

$$(H^2u, u) = (Hu, Hu) = \|Hu\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E), \tag{4.3}$$

and

$$(H^3u, u) = (HHu, Hu) \geq \|Hu\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E).$$

By (4.3) we have

$$\|u\|^2 \leq (H^2u, u) \leq \|H^2u\|\|u\|, \quad \text{for all } u \in C_c^\infty(E),$$

and, hence, $\|H^2u\| \geq \|u\|$, for all $u \in C_c^\infty(E)$. This, in turn, leads to

$$(H^4u, u) = (H^2u, H^2u) = \|H^2u\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E).$$

Continuing like this, we obtain $(H^ku, u) \geq \|u\|^2$, for all $u \in C_c^\infty(E)$ and all $k \in \mathbb{Z}_+$. In this case, by an abstract fact (see Theorem X.26 in [28]), the essential self-adjointness of H^k on $C_c^\infty(E)$ is equivalent to the following statement: if $u \in L^2(E)$ satisfies $H^k u = 0$, then $u = 0$.

Let $u \in L^2(E)$ satisfy $H^k u = 0$. Since $V \in C^\infty(E)$, by local elliptic regularity it follows that $u \in C^\infty(E) \cap L^2(E)$. Define

$$f_j := H^{k-j} u, \quad j = 0, \pm 1, \pm 2, \dots \tag{4.4}$$

Here, in the case $k - j < 0$, the definition (4.4) is interpreted as $((H_{\max})^{-1})^{j-k}$. We already noted that $H_{\min}$ is essentially self-adjoint and positive. Furthermore, it is well known that the self-adjoint closure of $H_{\min}$ coincides with $H_{\max}$. Therefore $H_{\max}$ is a positive self-adjoint operator, and $(H_{\max})^{-1} \colon L^2(E) \to L^2(E)$ is bounded. This, together with $f_k = u \in L^2(E)$ explains the following property: $f_j \in L^2(E)$, for all $j \geq k$. Additionally, observe that $f_j = 0$ for all $j \leq 0$ because $f_0 = 0$. Furthermore, we note that $f_j \in C^\infty(E)$, for all $j \in \mathbb{Z}$. The last assertion is obvious for $j \leq k$, and for $j > k$ it can be seen by showing that $H^j f_j = 0$ in distributional sense and using $f_j \in L^2(E)$ together with local elliptic regularity. To see this, let $v \in C_c^\infty(E)$ be arbitrary, and note that

$$(f_j, H^j v) = (H^{k-j} u, H^j v) = (u, H^k v) = (H^k u, v) = 0.$$

Finally, observe that

$$H^l f_j = f_{j-l}, \quad \text{for all } j \in \mathbb{Z} \text{ and } l \in \mathbb{Z}_+ \cup \{0\}. \tag{4.5}$$

With f_j as in (4.4), define the functions α_j and β_j on the interval $0 \leq T < \infty$ by the formulas

$$\alpha_j(T) := \lambda_0^2 \int_0^T (f_j, f_j)_{B_t} \, \mathrm{d}t, \quad \beta_j(T) := \int_0^T (Df_j, Df_j)_{B_t} \, \mathrm{d}t, \tag{4.6}$$

where λ_0 is as in (4.1) and $(\cdot, \cdot)_{B_t}$ is as in (3.2).

In the sequel, to simplify the notations, the functions $\alpha_j(T)$ and $\beta_j(T)$, the inner products $(\cdot, \cdot)_{B_t}$, and the corresponding norms $\| \cdot \|_{B_t}$ appearing in (4.6) will be denoted by α_j, β_j, $(\cdot, \cdot)_t$, and $\| \cdot \|_t$, respectively.

Note that α_j and β_j are absolutely continuous on $[0, \infty)$. Furthermore, α_j and β_j have a left first derivative and a right first derivative at each point. Additionally, α_j and β_j are differentiable, except at (at most) countably many points. In the sequel, to simplify notations, we shall denote the right first derivatives of α_j and β_j by α'_j and β'_j. Note that α_j, β_j, α'_j and β'_j are non-decreasing and non-negative functions. Note also that α_j and β_j are convex functions. Furthermore, since $f_j = 0$ for all $j \leq 0$, it follows that $\alpha_j \equiv 0$ and $\beta_j \equiv 0$ for all $j \leq 0$. Finally, using (4.1) and the property $f_j \in L^2(E) \cap C^\infty(E)$ for all $j \geq k$, observe that

$$\lambda_0^2 (f_j, f_j) + (Df_j, Df_j) \leq (Vf_j, f_j) + (Df_j, Df_j) = (f_j, Hf_j) = (f_j, f_{j-1}) < \infty,$$

for all $j > k$. Here, "integration by parts" in the first equality is justified because $H_{\min}$ is essentially self-adjoint (i.e. $C_c^\infty(E)$ is an operator core of $H_{\max}$). Hence, α'_j and β'_j are bounded for all $j > k$. It turns out that α_j and β_j satisfy a system of differential inequalities, as seen in the next proposition.

Proposition 4.3 *Let α_j and β_j be as in (4.6). Then, for all $j \geq 1$ and all $T \geq 0$ we have*

$$\alpha_j + \beta_j \leq \sqrt{\alpha'_j \beta'_j} + \sum_{l=0}^{\infty} \left(\sqrt{\alpha'_{j+l+1} \beta'_{j-l-1}} + \sqrt{\alpha'_{j-l-1} \beta'_{j+l+1}} \right) \tag{4.7}$$

and

$$\alpha_j \leq \lambda_0^2 \left(\sum_{l=0}^{\infty} \left(\sqrt{\alpha'_{j+l+1}\beta'_{j-l}} + \sqrt{\alpha'_{j-l}\beta'_{j+l+1}} \right) \right), \tag{4.8}$$

where λ_0 *is as in* (4.1) *and* α'_i, β'_i *denote the right-hand derivatives.*

Remark 4.4 Note that the sums in (4.7) and (4.8) are finite since $\alpha_i \equiv 0$ and $\beta_i \equiv 0$ for $i \leq 0$. As our goal is to show that $f_k = u = 0$, we will only use the first k inequalities in (4.7) and the first k inequalities in (4.8).

Proof of Proposition 4.3 From (4.6) and (4.1) it follows that

$$\alpha_j + \beta_j \leq \int_0^T \big((f_j, Vf_j)_t + (Df_j, Df_j)_t \big)\, \mathrm{d}t. \tag{4.9}$$

We start from (4.9), use (3.3), Cauchy–Schwarz inequality, and (4.5) to obtain

$$\begin{aligned} \alpha_j + \beta_j &\leq \int_0^T ((f_j, Vf_j)_t + (Df_j, Df_j)_t)\, \mathrm{d}t \\ &= \int_0^T |(f_j, Hf_j)_t - (f_j, D^*Df_j)_t + (Df_j, Df_j)_t|\, \mathrm{d}t \\ &\leq \lambda_0 \int_{B_T} |Df_j(x)||f_j(x)|\, \mathrm{d}\mu(x) + \int_0^T |(f_j, Hf_j)_t|\, \mathrm{d}t \\ &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(Hf_{j+1}, f_{j-1})_t|\, \mathrm{d}t. \end{aligned}$$

We continue the process as follows:

$$\begin{aligned} \alpha_j + \beta_j &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(Hf_{j+1}, f_{j-1})_t|\, \mathrm{d}t \\ &= \sqrt{\alpha'_j \beta'_j} + \int_0^T |(D^*Df_{j+1}, f_{j-1})_t + (f_{j+1}, Vf_{j-1})_t|\, \mathrm{d}t \\ &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(D^*Df_{j+1}, f_{j-1})_t - (Df_{j+1}, Df_{j-1})_t|\, \mathrm{d}t \\ &\quad + \int_0^T |(Df_{j+1}, Df_{j-1})_t - (f_{j+1}, D^*Df_{j-1})_t|\, \mathrm{d}t + \int_0^T |(f_{j+1}, Hf_{j-1})_t|\, \mathrm{d}t \\ &\leq \sqrt{\alpha'_j \beta'_j} + \sqrt{\alpha'_{j+1}\beta'_{j-1}} + \sqrt{\alpha'_{j-1}\beta'_{j+1}} + \int_0^T |(Hf_{j+2}, f_{j-2})_t|\, \mathrm{d}t, \end{aligned}$$

where we used triangle inequality, (3.3), Cauchy–Schwarz inequality, and (4.5). We continue like this until the last term reaches the subscript $j - l \leq 0$, which makes the last term equal zero by properties of f_i discussed above. This establishes (4.7).

To show (4.8), we start from the definition of α_j, use (3.3), Cauchy–Schwarz inequality, and (4.5) to obtain

$$\begin{aligned}\alpha_j &= \lambda_0^2 \int_0^T (f_j, f_j)_t \,\mathrm{d}t = \lambda_0^2 \int_0^T |(f_j, Hf_{j+1})_t| \,\mathrm{d}t \\ &= \lambda_0^2 \int_0^T |(f_j, D^*Df_{j+1})_t + (Vf_j, f_{j+1})_t| \,\mathrm{d}t \\ &\le \lambda_0^2 \int_0^T |(f_j, D^*Df_{j+1})_t - (Df_j, Df_{j+1})_t| \,\mathrm{d}t \\ &\quad + \lambda_0^2 \int_0^T |(Df_j, Df_{j+1})_t - (D^*Df_j, f_{j+1})_t| \,\mathrm{d}t + \lambda_0^2 \int_0^T |(Hf_j, f_{j+1})_t| \,\mathrm{d}t \\ &\le \lambda_0^2 \left(\sqrt{\alpha'_{j+1}\beta'_j} + \sqrt{\alpha'_j \beta'_{j+1}}\right) + \lambda_0^2 \int_0^T |(f_{j-1}, f_{j+1})_t| \,\mathrm{d}t.\end{aligned}$$

We continue like this until the last term reaches the subscript $j - l \le 0$, which makes the last term equal zero by properties of f_i discussed above. This establishes (4.8). □

End of the proof of Theorem 1 We will now transform the system (4.7) and (4.8) by introducing new variables:

$$\omega_j(T) := \alpha_j(T) + \beta_j(T), \quad \theta_j(T) := \alpha_j(T) - \beta_j(T) \quad T \in [0, \infty). \tag{4.10}$$

To carry out the transformation, observe that Cauchy–Schwarz inequality applied to vectors $\left\langle \sqrt{\alpha'_i}, \sqrt{\beta'_i} \right\rangle$ and $\left\langle \sqrt{\beta'_p}, \sqrt{\alpha'_p} \right\rangle$ in $\mathbb{R}^2$ gives

$$\sqrt{\alpha'_i \beta'_p} + \sqrt{\alpha'_p \beta'_i} \le \sqrt{\omega'_i \omega'_p},$$

which, together with (4.7) and (4.8) leads to

$$\omega_j \le \frac{1}{2}\sqrt{(\omega'_j)^2 - (\theta'_j)^2} + \sum_{l=0}^{\infty} \sqrt{\omega'_{j+l+1}\omega'_{j-l-1}} \tag{4.11}$$

and

$$\frac{1}{2}(\omega_j + \theta_j) \le \lambda_0^2 \left(\sum_{l=0}^{\infty} \sqrt{\omega'_{j+l+1}\omega'_{j-l}} \right), \tag{4.12}$$

where λ_0 is as in (4.1) and ω'_i, θ'_i denote the right-hand derivatives.

The functions ω_j and θ_j satisfy the following properties: (i) ω_j and θ_j are absolutely continuous on $[0, \infty)$, and the right-hand derivatives ω'_j and θ'_j exist everywhere; (ii) ω_j and ω'_j are non-negative and non-increasing; (iii) ω_j is convex; (iv) ω'_j is bounded for all $j \ge k$; (v) $\omega_j(0) = \theta_j(0) = 0$; and (vi) $|\theta_j(T)| \le \omega_j(T)$ and $|\theta'_j(T)| \le \omega'_j(T)$ for all $T \in [0, \infty)$.

In Section IV.3 of [12] it was shown that if ω_j and θ_j are functions satisfying the above described properties (i)–(vi) and the system (4.11) and (4.12), then $\omega_j \equiv 0$ for all $j = 1, 2, \ldots, k$. In particular, we have $\omega_k(T) = 0$, for all $T \in [0, \infty)$, and hence $f_k = 0$. Going back to (4.4), we get $u = 0$, and this concludes the proof of essential self-adjointness of H^k on $C_c^\infty(E)$. The essential self-adjointness of $H^2, H^3, \ldots$, and H^{k-1} on $C_c^\infty(E)$ follows by Proposition 4.1. □

5 Proof of Theorem 2

We adapt the proof of Theorem 1.1 in [13] to our type of operator. By assumption (2.6) it follows that

$$((\Delta_{M,\mu} + q - C + 1)u, u) \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(M). \tag{5.1}$$

Since (5.1) is satisfied and since M is non-compact and g^{TM} is geodesically complete, a result of Agmon [1] (see also Proposition III.6.2 in [12]) guarantees the existence of a function $\gamma \in C^\infty(M)$ such that $\gamma(x) > 0$ for all $x \in M$, and

$$(\Delta_{M,\mu} + q - C + 1)\gamma = \gamma. \tag{5.2}$$

We now use the function γ to transform the operator $H = \nabla^*\nabla + V$. Let $L^2_{\mu_1}(E)$ be the space of square integrable sections of E with inner product $(\cdot,\cdot)_{\mu_1}$ as in (2.1), where $\mathrm{d}\mu$ is replaced by $\mathrm{d}\mu_1 := \gamma^2 \mathrm{d}\mu$. For clarity, we denote $L^2(E)$ from Sect. 2.1 by $L^2_\mu(E)$. In what follows, the formal adjoints of ∇ with respect to inner products $(\cdot,\cdot)_\mu$ and $(\cdot,\cdot)_{\mu_1}$ will be denoted by $\nabla^{*,\mu}$ and ∇^{*,μ_1}, respectively. It is easy to check that the map $T_\gamma\colon L^2_\mu(E) \to L^2_{\mu_1}(E)$ defined by $Tu := \gamma^{-1}u$ is unitary. Furthermore, under the change of variables $u \mapsto \gamma^{-1}u$, the differential expression $H = \nabla^{*,\mu}\nabla + V$ gets transformed into $H_1 := \gamma^{-1}H\gamma$. Since T is unitary, the essential self-adjointness of $H^k|_{C_c^\infty(E)}$ in $L^2_\mu(E)$ is equivalent to essential self-adjointness of $(H_1)^k|_{C_c^\infty(E)}$ in $L^2_{\mu_1}(E)$.

In the sequel, we will show that H_1 has the following form:

$$H_1 = \nabla^{*,\mu_1}\nabla + \widetilde{V}, \tag{5.3}$$

with

$$\widetilde{V}(x) := \frac{\Delta_{M,\mu}\gamma}{\gamma}\,\mathrm{Id}(x) + V(x).$$

To see this, let $w,\ z \in C_c^\infty(E)$ and consider

$$\begin{aligned}(H_1 w, z)_{\mu_1} &= \int_M \langle \gamma^{-1}H(\gamma w), z\rangle\, \gamma^2 \mathrm{d}\mu = \int_M \langle H(\gamma w), \gamma z\rangle\, \mathrm{d}\mu = (H(\gamma w), \gamma z)_\mu \\ &= (\nabla(\gamma w), \nabla(\gamma z))_\mu + (V\gamma w, \gamma z)_\mu = (\gamma^2\nabla w, \nabla z)_\mu + (\mathrm{d}\gamma\otimes w, \mathrm{d}\gamma\otimes z)_{L^2_\mu(T^*M\otimes E)} \\ &\quad + (\gamma\nabla w, \mathrm{d}\gamma\otimes z)_{L^2_\mu(T^*M\otimes E)} + (\mathrm{d}\gamma\otimes w, \gamma\nabla z)_{L^2_\mu(T^*M\otimes E)} + (V\gamma w, \gamma z)_\mu.\end{aligned} \tag{5.4}$$

Setting $\xi := d(\gamma^2/2) \in T^*M$ and using equation (1.34) in Appendix C of [32] we have

$$(\gamma\nabla w, \mathrm{d}\gamma\otimes z)_{L^2_\mu(T^*M\otimes E)} = (\nabla w, \xi\otimes z)_{L^2_\mu(T^*M\otimes E)} = (\nabla_X w, z)_\mu, \tag{5.5}$$

where X is the vector field associated with $\xi \in T^*M$ via the metric g^{TM}.

Furthermore, by equation (1.35) in Appendix C of [32] we have

$$\begin{aligned}(\mathrm{d}\gamma\otimes w, \gamma\nabla z)_{L^2_\mu(T^*M\otimes E)} &= (\xi\otimes w, \nabla z)_{L^2_\mu(T^*M\otimes E)} = (\nabla^{*,\mu}(\xi\otimes w), z)_\mu \\ &= -(\mathrm{div}_\mu(X)w, z)_\mu - (\nabla_X w, z)_\mu,\end{aligned} \tag{5.6}$$

where, in local coordinates $x^1,\ x^2, \ldots, x^n$, for $X = X^j \frac{\partial}{\partial x^j}$, with Einstein summation convention,

$$\mathrm{div}_\mu(X) := \frac{1}{\kappa}\left(\frac{\partial}{\partial x^j}\left(\kappa X^j\right)\right).$$

[Recall that $d\mu = \kappa(x)\, dx^1 dx^2 \ldots dx^n$, where $\kappa(x)$ is a positive C^∞-density.] Since $X^j = (g^{TM})^{jl}\left(\gamma \frac{\partial\gamma}{\partial x^l}\right)$, we have

$$\operatorname{div}_\mu(X) = |d\gamma|^2 - \gamma(\Delta_{M,\mu}\gamma), \tag{5.7}$$

where $|d\gamma(x)|$ is the norm of $d\gamma(x) \in T_x^*M$ induced by g^{TM}, and $\Delta_{M,\mu}$ is as in (1.1) with metric g^{TM}. Combining (5.4)–(5.7) and noting that

$$(d\gamma \otimes w, d\gamma \otimes z)_{L^2_\mu(T^*M\otimes E)} = \int_M |d\gamma|^2 \langle w, z\rangle\, d\mu,$$

we obtain

$$\begin{aligned}(H_1 w, z)_{\mu_1} &= \int_M \langle \nabla w, \nabla z\rangle \gamma^2\, d\mu + \int_M \langle Vw, z\rangle\gamma^2\, d\mu + \int_M \gamma(\Delta_{M,\mu}\gamma)\langle w, z\rangle\, d\mu \\ &= (\nabla w, \nabla z)_{L^2_{\mu_1}(T^*M\otimes E)} + (Vw, z)_{\mu_1} + (\gamma^{-1}(\Delta_{M,\mu}\gamma)w, z)_{\mu_1} \\ &= (\nabla^{*,\mu_1}\nabla w, z)_{\mu_1} + (Vw, z)_{\mu_1} + (\gamma^{-1}(\Delta_{M,\mu}\gamma)w, z)_{\mu_1},\end{aligned} \tag{5.8}$$

which shows (5.3).

By (2.5) and (5.2) it follows that

$$\widetilde{V}(x) = \frac{\Delta_{M,\mu}\gamma}{\gamma}\operatorname{Id}(x) + V(x) \geq (C-1)\operatorname{Id}(x), \quad \text{for all } x \in M,$$

where C is as in (2.6). Thus, by Theorem 1 the operator $(H_1)^k|_{C_c^\infty(E)}$ is essentially self-adjoint in $L^2_{\mu_1}(E)$ for all $k \in \mathbb{Z}_+$. □

6 Proof of Theorem 3

Throughout the section, we assume that the hypotheses of Theorem 3 are satisfied. In subsequent discussion, the notation $\widehat{D}$ is as in (3.1) and the operators $H_{\min}$ and $H_{\max}$ are as in Sect. 4.1. We begin with the following lemma, whose proof is a direct consequence of the definition of $H_{\max}$ and local elliptic regularity.

Lemma 6.1 *Under the assumption $V \in L^\infty_{loc}(\operatorname{End} E)$, we have the following inclusion:* $\operatorname{Dom}(H_{\max}) \subset W^{2,2}_{loc}(E)$.

The proof of the next lemma is given in Lemma 8.10 of [5].

Lemma 6.2 *For any $u \in \operatorname{Dom}(H_{\max})$ and any Lipschitz function with compact support $\psi\colon M \to \mathbb{R}$, we have:*

$$(D(\psi u), D(\psi u)) + (V\psi u, \psi u) = \operatorname{Re}(\psi Hu, \psi u) + \|\widehat{D}(d\psi)u\|^2. \tag{6.1}$$

Corollary 6.3 *Let H be as in (2.3), let $u \in L^2(E)$ be a weak solution of $Hu = 0$, and let $\psi\colon M \to \mathbb{R}$ be a Lipschitz function with compact support. Then*

$$(\psi u,\ H(\psi u)) = \|\widehat{D}(d\psi)u\|^2, \tag{6.2}$$

where $(\cdot,\cdot)$ on the left-hand side denotes the duality between $W^{1,2}_{loc}(E)$ and $W^{-1,2}_{comp}(E)$.

Proof Since $u \in L^2(E)$ and $Hu = 0$, we have $u \in \mathrm{Dom}(H_{\max}) \subset W^{2,2}_{\mathrm{loc}}(E) \subset W^{1,2}_{\mathrm{loc}}(E)$, where the first inclusion follows by Lemma 6.1. Since ψ is a Lipschitz compactly supported function, we get $\psi u \in W^{1,2}_{\mathrm{comp}}(E)$ and, hence, $H(\psi u) \in W^{-1,2}_{\mathrm{comp}}(E)$. Now the equality (6.2) follows from (6.1), the assumption $Hu = 0$, and

$$(\psi u,\ H(\psi u)) = (\psi u,\ D^*D(\psi u)) + (V\psi u, \psi u) = (D(\psi u), D(\psi u)) + (V\psi u, \psi u),$$

where in the second equality we used integration by parts; see Lemma 8.8 in [5]. Here, the two leftmost symbols $(\cdot, \cdot)$ denote the duality between $W^{1,2}_{\mathrm{comp}}(E)$ and $W^{-1,2}_{\mathrm{loc}}(E)$, while the remaining ones stand for L^2-inner products. □

The key ingredient in the proof of Theorem 3 is the Agmon-type estimate given in the next lemma, whose proof, inspired by an idea of [24], is based on the technique developed in [10] for magnetic Laplacians on an open set with compact boundary in $\mathbb{R}^n$.

Lemma 6.4 *Let $\lambda \in \mathbb{R}$ and let $v \in L^2(E)$ be a weak solution of $(H - \lambda)v = 0$. Assume that that there exists a constant $c_1 > 0$ such that, for all $u \in W^{1,2}_{\mathrm{comp}}(E)$,*

$$(u,\ (H-\lambda)u) \geq \lambda_0^2 \int_M \max\left(\frac{1}{r(x)^2}, 1\right) |u(x)|^2 \,\mathrm{d}\mu(x) + c_1\|u\|^2, \tag{6.3}$$

where $r(x)$ is as in (2.7), λ_0 is as in (2.2), the symbol $(\cdot, \cdot)$ on the left-hand side denotes the duality between $W^{1,2}_{\mathrm{comp}}(E)$ and $W^{-1,2}_{\mathrm{loc}}(E)$, and $|\cdot|$ is the norm in the fiber E_x.

Then, the following equality holds: $v = 0$.

Proof Let ρ and R be numbers satisfying $0 < \rho < 1/2$ and $1 < R < +\infty$. For any $\varepsilon > 0$, we define the function $f_\varepsilon \colon M \to \mathbb{R}$ by $f_\varepsilon(x) = F_\varepsilon(r(x))$, where $r(x)$ is as in (2.7) and $F_\varepsilon \colon [0, \infty) \to \mathbb{R}$ is the continuous piecewise affine function defined by

$$F_\varepsilon(s) = \begin{cases} 0 & \text{for } s \leq \varepsilon \\ \rho(s-\varepsilon)/(\rho-\varepsilon) & \text{for } \varepsilon \leq s \leq \rho \\ s & \text{for } \rho \leq s \leq 1 \\ 1 & \text{for } 1 \leq s \leq R \\ R+1-s & \text{for } R \leq s \leq R+1 \\ 0 & \text{for } s \geq R+1. \end{cases}$$

Let us fix $x_0 \in M$. For any $\alpha > 0$, we define the function $p_\alpha \colon M \to \mathbb{R}$ by

$$p_\alpha(x) = P_\alpha(d_{g^{\mathrm{TM}}}(x_0, x)),$$

where $P_\alpha \colon [0, \infty) \to \mathbb{R}$ is the continuous piecewise affine function defined by

$$P_\alpha(s) = \begin{cases} 1 & \text{for } s \leq 1/\alpha \\ -\alpha s + 2 & \text{for } 1/\alpha \leq s \leq 2/\alpha \\ 0 & \text{for } s \geq 2/\alpha. \end{cases}$$

Since $\widehat{d}_{g^{\mathrm{TM}}}(x_0, x) \leq d_{g^{\mathrm{TM}}}(x_0, x)$, it follows that the support of $f_\varepsilon p_\alpha$ is contained in the set $B_\alpha := \{x \in M \colon \widehat{d}_{g^{\mathrm{TM}}}(x_0, x) \leq 2/\alpha\}$. By Assumption (A1) we know that $\widehat{M}$ is a geodesically complete Riemannian manifold. Hence, by Hopf–Rinow Theorem the set B_α is compact. Therefore, the support of $f_\varepsilon p_\alpha$ is compact. Additionally, note that $f_\varepsilon p_\alpha$ is a β-Lipschitz function (with respect to the distance corresponding to the metric g^{TM}) with $\beta = \frac{\rho}{\rho-\varepsilon} + \alpha$.

Since $v \in L^2(E)$ and $(H - \lambda)v = 0$, we have $v \in \mathrm{Dom}(H_{\max}) \subset W^{2,2}_{\mathrm{loc}}(E) \subset W^{1,2}_{\mathrm{loc}}(E)$, where the first inclusion follows by Lemma 6.1. Since $f_\varepsilon p_\alpha$ is a Lipschitz compactly supported function, we get $f_\varepsilon p_\alpha v \in W^{1,2}_{\mathrm{comp}}(E)$ and, hence, $((H - \lambda)(f_\varepsilon p_\alpha v)) \in W^{-1,2}_{\mathrm{comp}}(E)$.

Using (2.2) we have

$$\|\widehat{D}(\mathrm{d}(f_\varepsilon p_\alpha))v\|^2 \leq \lambda_0^2 \int_M |\mathrm{d}(f_\varepsilon p_\alpha)(x)|^2 |v(x)|^2 \, \mathrm{d}\mu(x), \tag{6.4}$$

where $|\mathrm{d}(f_\varepsilon p_\alpha)(x)|$ is the norm of $\mathrm{d}(f_\varepsilon p_\alpha)(x) \in T_x^*M$ induced by g^{TM}.

By Corollary 6.3 with $H - \lambda$ in place of H and the inequality (6.4), we get

$$(f_\varepsilon p_\alpha v,\ (H-\lambda)(f_\varepsilon p_\alpha v)) \leq \lambda_0^2 \left(\frac{\rho}{\rho - \varepsilon} + \alpha\right)^2 \|v\|^2. \tag{6.5}$$

On the other hand, using the definitions of f_ε and p_α and the assumption (6.3) we have

$$(f_\varepsilon p_\alpha v,\ (H-\lambda)(f_\varepsilon p_\alpha v)) \geq \lambda_0^2 \int_{S_{\rho,R,\alpha}} |v(x)|^2 \, \mathrm{d}\mu(x) + c_1 \|f_\varepsilon p_\alpha v\|^2, \tag{6.6}$$

where

$$S_{\rho,R,\alpha} := \{x \in M \colon \rho \leq r(x) \leq R \text{ and } d_{g^{\mathrm{TM}}}(x_0, x) \leq 1/\alpha\}.$$

In (6.6) and (6.5), the symbol $(\cdot,\cdot)$ stands for the duality between $W^{1,2}_{\mathrm{comp}}(E)$ and $W^{-1,2}_{\mathrm{loc}}(E)$. We now combine (6.6) and (6.5) to get

$$\lambda_0^2 \int_{S_{\rho,R,\alpha}} |v(x)|^2 \, \mathrm{d}\mu(x) \; + \; c_1 \|f_\varepsilon p_\alpha v\|^2 \leq \lambda_0^2 \left(\frac{\rho}{\rho - \varepsilon} + \alpha\right)^2 \|v\|^2.$$

We fix ρ, R, and ε, and let $\alpha \to 0+$. After that we let $\varepsilon \to 0+$. The last step is to do $\rho \to 0+$ and $R \to +\infty$. As a result, we get $v = 0$. □

End of the proof of Theorem 3 Using integration by parts (see Lemma 8.8 in [5]), we have

$$\begin{aligned}(u,\ Hu) \;=\; (u, D^*Du) \;+\; (Vu, u) \;=\; (Du, Du) \\ +\,(Vu, u) \;\geq\; (Vu, u), \quad \text{for all } u \in W^{1,2}_{\mathrm{comp}}(E),\end{aligned}$$

where the two leftmost symbols $(\cdot,\cdot)$ denote the duality between $W^{1,2}_{\mathrm{comp}}(E)$ and $W^{-1,2}_{\mathrm{loc}}(E)$, while the remaining ones stand for L^2-inner products. Hence, by assumption (2.8) we get:

$$\begin{aligned}(u,\ (H-\lambda)u) &\geq \lambda_0^2 \int_M \frac{1}{r(x)^2} |u(x)|^2 \, \mathrm{d}\mu(x) - (\lambda + C)\|u\|^2 \\ &\geq \lambda_0^2 \int_M \max\left(\frac{1}{r(x)^2}, 1\right) |u(x)|^2 \, \mathrm{d}\mu(x) - (\lambda + C + 1)\|u\|^2. \end{aligned} \tag{6.7}$$

Choosing, for instance, $\lambda = -C - 2$ in (6.7) we get the inequality (6.3) with $c_1 = 1$.

Thus, $H_{\min} - \lambda$ with $\lambda = -C - 2$ is a symmetric operator satisfying $(u,\ (H_{\min} - \lambda)u) \geq \|u\|^2$, for all $u \in C_c^\infty(E)$. In this case, it is known (see Theorem X.26 in [28]) that the essential self-adjointness of $H_{\min} - \lambda$ is equivalent to the following statement: if $v \in L^2(E)$ satisfies $(H - \lambda)v = 0$, then $v = 0$. Thus, by Lemma 6.4, the operator $(H_{\min} - \lambda)$ is essentially self-adjoint. Hence, $H_{\min}$ is essentially self-adjoint. □

Acknowledgments We are grateful to the referee for useful suggestions and comments.

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