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Regularization via Cheeger deformations

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Abstract We show that Cheeger deformations regularize G-invariant metrics in a very strong sense.

Keywords Cheeger deformation · Riemannian submersion · Normal homogenous · Totally geodesic fibers

Mathematics Subject Classification 53C20

1 Introduction

In the presence of a group of isometries G, Cheeger developed a method for perturbing the metric on a non-negatively curved manifold M [1]. We will show, in the curvature free setting, that this method regularizes the metric in a very strong sense. Before stating our result we recall the definition of a Cheeger deformation.

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Let G be a compact group of isometries of (M, g_M). Let g_{bi} be a bi-invariant metric on G, and consider the one-parameter family $l^2 g_{\mathrm{bi}} + g_M$ of metrics on $G \times M$. G acts on $(G \times M, l^2 g_{\mathrm{bi}} + g_M)$ via

$$g(p, m) = (pg^{-1}, gm),$$

which we will call the Cheeger Action.

Modding out by the Cheeger Action we obtain a one-parameter family g_l of metrics on $M \cong (G \times M)/G$. As $l \to \infty$, (M, g_l) converges to g_M [6].

The quotient map for the Cheeger Action is

$$q\colon (p, m) \mapsto pm.$$

For any point x in the union of the principal orbits, M^{reg}, we define

$$\tilde{g}_l \equiv \frac{1}{l^2} g_l|_{T_x G(x)} + g_l|_{T_x G(x)^\perp},$$

where $T_x G(x)$ is the tangent space to the orbit through x, and $TG(x)^\perp$ is its orthogonal complement.

Theorem A *Let (M, g_M) be a complete, Riemannian G-manifold with G a compact Lie group. For any non-negative integer p and any G-invariant, pre-compact open subset $\mathcal{U} \subset M^{\mathrm{reg}}$, as $l \to 0$ the one-parameter family $\{\tilde{g}_l|_{\mathcal{U}}\}_{l>0}$ converges in the C^p-topology to a G-invariant metric $\tilde{g}$ so that the Riemannian submersion $(\mathcal{U}, \tilde{g}) \longrightarrow \mathcal{U}/G$ has totally geodesic, normal homogeneous fibers.*

The normal homogeneous metrics on the fibers have the following description:

Theorem B *For any $x \in \mathcal{U}$ with isotropy G_x, let $\Phi_x\colon G/G_x \longrightarrow G(x)$ be the G-equivariant diffeomorphism, $\Phi_x(gG_x) = gx$. Let $g_{\mathrm{nh},x}$ be the normal homogeneous metric on G/G_x induced by the submersion $(G, g_{\mathrm{bi}}) \longrightarrow G/G_x$. Then $\Phi_x\colon (G/G_x,\ g_{\mathrm{nh},x}) \longrightarrow (\mathcal{U},\ \tilde{g})$ is a Riemannian embedding whose image is totally geodesic.*

Remark 1 While the embedding $\Phi_x\colon (G/G_x,\ g_{\mathrm{nh},x}) \longrightarrow (\mathcal{U},\ \tilde{g})$ preserves the Riemannian metric and has totally geodesic image, it need not be an isometry in the metric space sense, that is, the intrinsic and extrinsic metrics on the orbits need not coincide. Consider a "Berger" sphere obtained by expanding the constant curvature 1 metric in the Hopf directions, and leaving the metric on the horizontal distribution unchanged. It follows that the Hopf semicircles between pairs of antipodal points have length $>\pi$. Since every geodesic which is horizontal for the Hopf fibration connects antipodal points, the extrinsic distance between any pair of antipodal points is $\leq\pi$, and so the intrinsic and extrinsic metrics on the Hopf fibers are different.

Remark 2 The class, $\mathcal{P}$, of principal G-manifolds with totally geodesic, normal homogeneous orbits is invariant under Cheeger deformation. Theorems A and B say that all principal G-manifolds are attracted to $\mathcal{P}$ by Cheeger deformations.

As Cheeger deformations and G-manifolds have been extensively studied, others may be aware of Theorems A and B. The closest result that we found in the literature, due to Schwachhöfer and Tapp, is Proposition 1.1 in [8], which deals with the case of Cheeger deforming a homogeneous space, $M = G/H$, via G.

We believe there are many potential applications of Theorems A and B. For example, some of the curvature estimates in [7] can be obtained by combining Theorems A and B with the Gray–O'Neill fundamental equations of a submersion [2,5].

The paper is organized as follows. In Sect. 2, we establish our notations and conventions, and in Sect. 3, we prove Theorems A and B.

2 Notations and conventions

Throughout we assume that the compact Lie group, G, with bi-invariant metric g_{bi}, acts isometrically on the complete Riemannian manifold (M, g_M). The orbit through $x \in M$ is called $G(x)$ and the isotropy subgroup at x is G_x. We denote the Lie algebra of G by $\mathfrak{g}$, and the Lie algebra of G_x by $\mathfrak{g}_x$. We call $\mathfrak{m}_x$ the orthogonal complement, with respect to g_{bi}, of $\mathfrak{g}_x$ in $\mathfrak{g}$. For the distribution on M^{reg} given by the tangent spaces to the orbits of G, we write $T(\text{orbits})$.

For an abstract G-manifold, N, let

$$K_N : \mathfrak{g} \times N \longrightarrow TN \tag{2.1}$$

be the bundle map that takes $(k, x) \in \mathfrak{g} \times N$ to the value at x of the Killing field generated by k, and let $K_{N,x} = K_N|_{\mathfrak{g}\times\{x\}}$. Note that the map K_N depends not only on N, but on the particular G-action on N. We adopt the convention that when $N = G$, the G-action is by right multiplication. The corresponding bundle map $K_G : \mathfrak{g} \times G \longrightarrow TG$ is then the trivialization of TG given by the left invariant fields.

For $x \in M^{\mathrm{reg}}$, define $\tilde{\Phi}_x : G \longrightarrow G(x)$ by $\tilde{\Phi}_x(g) = gx$. Let $\pi : G \longrightarrow G/G_x$ be the quotient map, and let $\Phi_x : G/G_x \longrightarrow G(x)$ be the G-equivariant diffeomorphism given by $\Phi_x(gG_x) = gx$. Since $\Phi_x \circ \pi = \tilde{\Phi}_x$, $D\pi_e = K_{G/G_x, eG_x}$ and $(D\tilde{\Phi}_x)_e = K_{M,x}$, the chain rule gives

$$(D\Phi_x)_{eG_x} \circ K_{G/G_x, eG_x} = K_{M,x}.$$

Since $K_{G/G_x, eG_x}\big|_{\mathfrak{m}_x}$ is invertible,

$$(D\Phi_x)_{eG_x} = K_{M,x} \circ K_{G/G_x, eG_x}\big|_{\mathfrak{m}_x}^{-1}. \tag{2.2}$$

Note that the differential of the quotient map

$$q : (p, m) \mapsto pm$$

for the Cheeger Action, $g(p, m) = (pg^{-1}, gm)$ is

$$Dq_{(p,m)}(k, v) = K_{M,x}(k) + v. \tag{2.3}$$

Recall from Chapter 2 of Hirsch [3] that two smooth maps $\Phi, \Psi : M \longrightarrow N$ are close in the weak C^p-topology if all of their values and partials up to order p are close with respect to fixed atlases for M and N. If the atlases are both finite, this leads to a notion of C^p-distance, which depends on the atlases, but will serve our purposes.

For bundle maps and tensors we will need a C^p-norm, which we now define. Recall that a Euclidean metric on a vector bundle E restricts to an inner product on each fiber of E and these inner products vary smoothly. Given vector bundles E_1 and E_2 with Euclidean metrics and a bundle map

$$\varphi : E_1 \longrightarrow E_2,$$

we define the C^p-norm of φ, $|\varphi|_{C^p}$, as follows: Let E_1^1 be the unit sphere bundle of E_1. Define $|\varphi|_{C^p}$ to be the C^p-distance from $\varphi|_{E_1^1}$ to the zero bundle map. The C^p-norm of a tensor is its C^p-distance to the zero section. We note that the C^p-norm of a bundle map or tensor depends on the given Euclidean metrics. With the exception of TM, all of the vector bundles we consider will come with a clear choice of metric. For bundle maps φ that go to or from TM and for tensors ω on M, we adopt the convention that $|\varphi|_{C^p}$ and $|\omega|_{C^p}$ are defined in terms of our initial G-invariant metric g_M.

3 Regular structure theorem

The vertical space for q at $(g, x) \in G \times M$ is

$$\mathcal{V} = \{(-K_G(k), K_M(k)) \mid k \in \mathfrak{g}\}.$$

We recall from [1,6,7] that there is a linear reparametrization of the tangent space, called the *Cheeger reparametrization*. It is denoted by

$$\mathrm{Ch}_l : TM \to TM$$

and defined as

$$\mathrm{Ch}_l(v) = Dq(\hat{v}_l),$$

where $\hat{v}_l \in TG \times TM$ is the horizontal vector for

$$q : (G \times M, l^2 g_{\mathrm{bi}} + g_M) \longrightarrow (M, g_l)$$

that projects to v under the projection $\pi_2 : G \times M \longrightarrow M$.

Although $\hat{v}_l$ is completely determined by v, g_{bi}, g_M, and the G-action, the explicit formula is rather unpleasant [4,9]. Fortunately, we will not need it, as we will use abstract, asymptotic arguments.

Every G-orbit in $G \times M$ has a unique point of the form (e, m). To fix notation, we assume throughout that we are at such a point. When $l = 1$ and $v \in T_x M$, we denote the first factor of $\hat{v}_1$ by $\kappa_x(v)$. Then

$$\hat{v}_1 = (\kappa_x(v), v). \tag{3.1}$$

For any l, we then have

$$\hat{v}_l = \left(\frac{\kappa_x(v)}{l^2}, v\right).$$

For simplicity, we will write $\hat{v}$ for $\hat{v}_l$.

Proposition 3.1 *For $x \in M^{\mathrm{reg}}$ we have the following:*

1. $K_{M,x}|_{\mathfrak{m}_x} : \mathfrak{m}_x \longrightarrow T_x G(x)$ *is an isomorphism that varies smoothly with x.*
2. *The map $\kappa_x : T_x M \longrightarrow \mathfrak{g}_x$, given by $v \mapsto \kappa_x(v)$, takes values in $\mathfrak{m}_x$ and restricts to a linear isomorphism, $T_x G(x) \longrightarrow \mathfrak{m}_x$, that varies smoothly with $x \in M^{\mathrm{reg}}$.*

Proof Part 1 follows from the definition of $K_{M,x}$.

Suppose $(u, v) \in T_{(e,x)}(G \times M)$ with $u \notin \mathfrak{m}_x$. Then there is a $k \in \mathfrak{g}_x$ with $g_{\mathrm{bi}}(k, u) \neq 0$. It follows that

$$\begin{aligned}(l^2 g_{\mathrm{bi}} + g_M)((u, v), (-K_{G,e}(k), K_{M,x}(k))) &= (l^2 g_{\mathrm{bi}} + g_M)((u, v), (-k, 0)) \\ &\neq 0.\end{aligned}$$

So (u, v) is not horizontal. It follows that κ_x takes values in $\mathfrak{m}_x$. κ_x is linear, since $\mathrm{Ch}_l : T_x M \longrightarrow T_x M$ is linear and κ_x is projection to G composed with $\mathrm{Ch}_l|_{T_x M}$.

For $v \in TG(x)$, if $(0, v) \in T(G \times M)$ is horizontal, then $v = 0$, and it follows that κ_x is injective. Since $\dim(\mathfrak{m}_x) = \dim(G(x))$, $\kappa_x : T_x G(x) \longrightarrow \mathfrak{m}_x$ is, in fact, an isomorphism, proving Part 2. □

Before proceeding we define the following vector bundle over M^{reg} :

$$E_{\mathrm{orb}} \equiv \left\{ (x, v) \in M^{\mathrm{reg}} \times \mathfrak{g} \middle| v \in \mathfrak{m}_x \right\}.$$

K and κ are then bundle maps

$$E_{\mathrm{orb}} \xrightarrow{K_M} T(\mathrm{orbits})|_{M^{\mathrm{reg}}},$$
$$T(\mathrm{orbits})|_{M^{\mathrm{reg}}} \xrightarrow{\kappa} E_{\mathrm{orb}}.$$

Proposition 3.2 *Given any compact subset $\mathcal{K} \subset M^{\mathrm{reg}}$ and any $p \geq 0$ there is a constant $C > 0$ so that*

$$\max \left\{ |K|_{C^p}, |\kappa|_{C^p}, \left|K^{-1}\right|_{C^p}, \left|\kappa^{-1}\right|_{C^p} \right\} \leq C.$$

Proof This follows from compactness of the corresponding unit sphere bundles and the fact that K, κ, K^{-1}, and κ^{-1} are C^{∞}. □

The next result shows that along the orbits $\tilde{g}_l$ is approximately $(K_{M,x}|_{\mathfrak{m}_x}^{-1})^*(g_{\mathrm{bi}})$, and the error in this approximation has the form $l^2\tilde{\mathcal{E}}$ for some bounded, symmetric $(0, 2)$-tensor $\tilde{\mathcal{E}}$.

Lemma 3.3 *Given any compact subset $\mathcal{K} \subset M^{\mathrm{reg}}$, there is an $l_0 > 0$ so that for all $l \in (0, l_0)$ there is a symmetric $(0, 2)$-tensor $\tilde{\mathcal{E}}$ and a constant $C > 0$ with the following property:*

$$\tilde{g}_l|_{T(\mathrm{orbits})|_{\mathcal{K}}} = \left(K_M|^{-1}\right)^* (g_{\mathrm{bi}}) + l^2\tilde{\mathcal{E}} \quad \text{and} \tag{3.2}$$
$$\left|\tilde{\mathcal{E}}\right|_{C^p} \leq C.$$

Proof For $x \in \mathcal{K} \subset M^{\mathrm{reg}}$ and $v, w \in T\,(\mathrm{orbits})|_{\mathcal{K}}$, using Eq. 2.3 we find

$$l^2\mathrm{Ch}_l(v) = Dq\left(l^2\left(\frac{\kappa\,(v)}{l^2}, v\right)\right)$$
$$= K_M\,(\kappa\,(v)) + l^2 v. \tag{3.3}$$

The definition of g_l and Ch_l gives

$$\frac{1}{l^2} g_l\left(l^2\mathrm{Ch}_l(v), l^2\mathrm{Ch}_l(w)\right) = l^2\left(l^2 g_{\mathrm{bi}} + g_M\right)\left(\left(\frac{\kappa\,(v)}{l^2}, v\right), \left(\frac{\kappa\,(w)}{l^2}, w\right)\right)$$
$$= g_{\mathrm{bi}}\,(\kappa\,(v)\,, \kappa\,(w)) + l^2 g_M\,(v, w)\,. \tag{3.4}$$

So

$$\frac{1}{l^2}\left(l^2\mathrm{Ch}_l\right)^*\left(g_l|_{T(\mathrm{orbits})}\right) = (\kappa)^*\,(g_{\mathrm{bi}}) + l^2\left(g_M|_{T(\mathrm{orbits})}\right). \tag{3.5}$$

From Eq. 3.3 we have

$$l^2\mathrm{Ch}_l = K_M \circ \kappa + l^2\mathrm{id}.$$

Combining this with Proposition 3.2 we see for small enough l, there is a bundle map

$$E\colon\ T(\mathrm{orbits})|_{M^{\mathrm{reg}}} \longrightarrow T(\mathrm{orbits})|_{M^{\mathrm{reg}}}$$

so that

$$\left(l^2\mathrm{Ch}_l\right)^{-1} = \kappa^{-1} \circ K_M^{-1} + O\left(l^2\right)E, \tag{3.6}$$

and

$$|E|_{C^p} \leq 1. \tag{3.7}$$

Combining Eqs. 3.5 and 3.6 gives

$$\begin{aligned}
\frac{1}{l^2} g_l|_{T(\text{orbits})} &= \left(\left(l^2 \mathrm{Ch}_l\right)^{-1}\right)^* (\kappa)^* (g_{\mathrm{bi}}) + l^2 \left(\left(l^2 \mathrm{Ch}_l\right)^{-1}\right)^* \left(g_M|_{T(\text{orbits})}\right) \\
&= \left(K_M^{-1}\right)^* (g_{\mathrm{bi}}) + O\left(l^2\right)(E)^* (\kappa)^* (g_{\mathrm{bi}}) \\
&\quad + l^2 \left(\kappa^{-1} \circ K_M^{-1}\right)^* \left(g_M|_{T(\text{orbits})}\right) + O\left(l^4\right)(E)^* \left(g_M|_{T(\text{orbits})}\right) \\
&= (K_M^{-1})^* (g_{\mathrm{bi}}) + l^2 \tilde{\mathcal{E}},
\end{aligned}$$

where

$$l^2 \widetilde{\mathcal{E}} = O\left(l^2\right)(E)^* (\kappa)^* (g_{\mathrm{bi}}) + l^2 \left(\kappa^{-1} \circ K_M^{-1}\right)^* \left(g_M|_{T(\text{orbits})}\right) + O\left(l^4\right)(E)^* \left(g_M|_{T(\text{orbits})}\right).$$

Combining this with Proposition 3.2 and Inequality 3.7 it follows that

$$\left|\widetilde{\mathcal{E}}\right|_{C^p} \leq C$$

for some $C > 0$. □

Proposition 3.4 *Given any compact subset $\mathcal{K} \subset M^{\mathrm{reg}}$, there is an $l_0 > 0$ so that for all $l \in (0, l_0)$ there is a $(0,2)$-symmetric tensor $\mathcal{E}$ and a constant $C > 0$ with the following properties. For all $x \in \mathcal{K}$*

$$\begin{aligned}
(\Phi_x)^* (\tilde{g}_l) &= g_{\mathrm{nh},x} + l^2 \mathcal{E} \quad \text{and} \\
|\mathcal{E}|_{C^p} &\leq C.
\end{aligned} \tag{3.8}$$

Proof Since $\Phi_x^* (\tilde{g}_l)$ and $g_{\mathrm{nh},x}$ are G-invariant, it suffices to verify Eq. 3.8 at eG_x. Using Eq. 2.2 and the linearity of $K_{M,x}$ and $K^{-1}_{G/G_x, eG_x}$, we see that applying $(\Phi_x)^*$ to Eq. 3.2 gives

$$\begin{aligned}
(\Phi_x)^* \left(\tilde{g}_l|_{T_x G(x)}\right) &= (\Phi_x)^* \left(K_{M,x}|_{\mathfrak{m}_x}^{-1}\right)^* (g_{\mathrm{bi}}) + l^2 (\Phi_x)^* \left(\tilde{\mathcal{E}}\right) \\
&= \left(K_{M,x} \circ K^{-1}_{G/G_x,\, eG_x}\right)^* \left(K_{M,x}|_{\mathfrak{m}_x}^{-1}\right)^* (g_{\mathrm{bi}}) \\
&\quad + l^2 \left(K_{M,x} \circ K^{-1}_{G/G_x,\, eG_x}\right)^* \left(\tilde{\mathcal{E}}\right) \\
&= \left(K_{M,x}|_{\mathfrak{m}_x}^{-1} \circ K_{M,x} \circ K^{-1}_{G/G_x,\, eG_x}\right)^* (g_{\mathrm{bi}}) \\
&\quad + l^2 \left(K_{M,x} \circ K^{-1}_{G/G_x,\, eG_x}\right)^* \left(\tilde{\mathcal{E}}\right) \\
&= \left(K_{G/G_x,\, eG_x}\Big|_{\mathfrak{m}_x}^{-1}\right)^* (g_{\mathrm{bi}}) + l^2 \left(K_{M,x} \circ K^{-1}_{G/G_x,\, eG_x}\right)^* \left(\tilde{\mathcal{E}}\right) \\
&= g_{\mathrm{nh},x} + l^2 \left(K_{M,x} \circ K^{-1}_{G/G_x,\, eG_x}\right)^* \left(\tilde{\mathcal{E}}\right)
\end{aligned}$$

The result then follows by setting

$$\mathcal{E}_x = \left(K_{M,x} \circ K^{-1}_{G/G_x,\, eG_x}\right)^* \left(\tilde{\mathcal{E}}_x\right)$$

and by appealing to Proposition 3.2 and the fact that $\left|\tilde{\mathcal{E}}\right|_{C^p} \leq C$. □

We are now in a position to begin the proofs of Theorems A and B. First observe that the distribution orthogonal to the orbits

$$x \mapsto TG(x)^{\perp}$$

is the same for g_l, $\tilde{g}_l$, and g_M. Also notice that for $Z \in TG(x)^{\perp}$,

$$g_l(Z, \cdot) = \tilde{g}_l(Z, \cdot) = g_M(Z, \cdot). \tag{3.9}$$

For $x \in \mathcal{K} \subset M^{\mathrm{reg}}$ we set

$$\tilde{g}|_x \equiv g_M|_{TG(x)^{\perp}} + \left(\Phi_x^{-1}\right)^* (g_{\mathrm{nh},x}). \tag{3.10}$$

Our next result shows that $\tilde{g}$ is G-invariant.

Proposition 3.5 *For* $y \in G(x)$, $\left(\Phi_x^{-1}\right)^* \left(g_{\mathrm{nh},x}\right) = \left(\Phi_y^{-1}\right)^* \left(g_{\mathrm{nh},y}\right)$

Proof Let $g_{yx} \in G$ satisfy $g_{yx}x = y$. Then $g_{yx}G_x g_{yx}^{-1} = G_y$ and we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{C_{g_{yx}}} & G \\ \downarrow{\scriptstyle \pi_{G_x}} & & \downarrow{\scriptstyle \pi_{G_y}} \\ G/G_x & \xrightarrow{\bar{C}_{g_{yx}}} & G/G_y \\ \downarrow{\scriptstyle \Phi_x} & & \downarrow{\scriptstyle \Phi_y} \\ G(x) & \xrightarrow{L_{g_{yx}}} & G(y) \end{array}$$

where

$$\begin{aligned} C_{g_{yx}}(a) &= g_{yx} a g_{yx}^{-1}, \\ \bar{C}_{g_{yx}}(aG_x) &= g_{yx} a g_{yx}^{-1} G_y, \\ L_{g_{yx}}(p) &= g_{yx} p, \end{aligned}$$

and π_{G_x} and π_{G_y} are the quotient maps.

It follows that

$$\begin{aligned} \left(\Phi_x^{-1}\right)^* \left(g_{\mathrm{nh},x}\right) &= \left(\left(\bar{C}_{g_{yx}}\right)^{-1} \circ \Phi_y^{-1} \circ L_{g_{yx}}\right)^* \left(g_{\mathrm{nh},x}\right) \\ &= \left(L_{g_{yx}}\right)^* \circ \left(\Phi_y^{-1}\right)^* \circ \left(\bar{C}_{g_{yx}}^{-1}\right)^* \left(g_{\mathrm{nh},x}\right) \\ &= \left(L_{g_{yx}}\right)^* \circ \left(\Phi_y^{-1}\right)^* \left(g_{\mathrm{nh},y}\right) \\ &= \left(\Phi_y^{-1}\right)^* \left(g_{\mathrm{nh},y}\right), \end{aligned}$$

since $L_{g_{yx}}$ is an isometry of $\left(G(y), \left(\Phi_y^{-1}\right)^* \left(g_{\mathrm{nh},y}\right)\right)$. □

Applying $\left(\Phi_x^{-1}\right)^*$ to both sides of Eq. 3.8, we obtain

$$\tilde{g}_l|_{TG(x)} = \left(\Phi_x^{-1}\right)^* \left(g_{\mathrm{nh},x}\right) + l^2 \left(\Phi_x^{-1}\right)^* (\mathcal{E}). \tag{3.11}$$

Combining Eqs. 2.2, 3.9 and 3.11 with the inequality, $|\mathcal{E}|_{C^p} \leq C$, we see that

$$|\tilde{g}_l - \tilde{g}|_{C^p} \leq Cl^2. \tag{3.12}$$

Remark 3 Our proof does not preclude the possibility that the bounds on the higher order derivatives of $\mathcal{E}$ depend on the order p and so does not give convergence in the C^{∞}-topology.

The next result shows that the fibers of $\pi^{\mathrm{reg}} : (\mathcal{U}, \tilde{g}) \longrightarrow \mathcal{U}/G$ are totally geodesic and, combined with Inequality 3.12, completes the proofs of Theorems A and B.

Proposition 3.6 *Let T^{g_M} and $T^{\tilde{g}_l}$ be the T-tensors of the Riemannian submersions*

$$\pi^{\mathrm{reg}}\colon \left(M^{\mathrm{reg}}, g_M\right) \longrightarrow M^{\mathrm{reg}}/G, \quad \text{and}$$
$$\pi^{\mathrm{reg}}\colon \left(M^{\mathrm{reg}}, \tilde{g}_l\right) \longrightarrow M^{\mathrm{reg}}/G,$$

as defined in [5]. *Given any compact subset $\mathcal{K} \subset M^{\mathrm{reg}}$ there is a constant $C > 0$ so that on $\mathcal{K}$*

$$\left|T^{\tilde{g}_l}\right| \leq Cl^2 \left|T^{g_M}\right|. \tag{3.13}$$

Proof Let T^{g_l} be the T-tensor of the Riemannian submersion

$$\pi^{\mathrm{reg}}\colon \left(M^{\mathrm{reg}}, g_l\right) \longrightarrow M^{\mathrm{reg}}/G.$$

The duality between the shape operator and the second fundamental form of the fibers implies that the norm of the T-tensor is determined by its values on just the vertical vectors.

We begin by proving Inequality 3.13 with $T^{\tilde{g}_l}$ replaced by T^{g_l} and then we will show that $\left|T^{\tilde{g}_l}\right| = \left|T^{g_l}\right|$.

For $V, W \in TG(x)$ and $Z \in TG(x)^{\perp}$, we lift $\mathrm{Ch}_l(V)$, $\mathrm{Ch}_l(W)$, and $\mathrm{Ch}_l(Z)$ to $G \times M$ and get

$$\begin{aligned} g_l\left(T_{\mathrm{Ch}_l(V)}\mathrm{Ch}_l(W), \mathrm{Ch}_l(Z)\right) &= \left(l^2 g_{\mathrm{bi}} + g_M\right)\left(\nabla^{l^2 g_{\mathrm{bi}}+g_M}_{\left(\frac{\kappa(V)}{l^2}, V\right)}\left(\frac{\kappa(W)}{l^2}, W\right), (0, Z)\right) \\ &= g_M\left(\nabla^{g_M}_V W, Z\right) \\ &= g_M\left(T^{g_M}_V W, Z\right) \end{aligned}$$

On the other hand if $|V|_{g_M} = |W|_{g_M} = 1$, then

$$|\mathrm{Ch}_l(V)|^2 = \frac{|\kappa(V)|^2_{g_{\mathrm{bi}}}}{l^2} + 1 \quad \text{and} \quad |\mathrm{Ch}_l(W)|^2 = \frac{|\kappa(W)|^2_{g_{\mathrm{bi}}}}{l^2} + 1.$$

Combining the previous two displays with Proposition 3.2 we see that given any compact subset $\mathcal{K} \subset M^{\mathrm{reg}}$ there is a constant $C > 0$ so that

$$\left|T^{g_l}\right| \leq Cl^2 \left|T^{g_M}\right|.$$

To see $\left|T^{\tilde{g}_l}\right| = \left|T^{g_l}\right|$ we use the Koszul formula and find that

$$\begin{aligned} 2\tilde{g}_l\left(T^{\tilde{g}_l}_{lV} lW, Z\right) &= 2l^2 \tilde{g}_l\left(\tilde{\nabla}_V W, Z\right) \\ &= l^2\left(-D_Z \tilde{g}_l(V, W) + \tilde{g}_l([Z, V], W) + \tilde{g}_l([Z, W], V)\right) \\ &= -D_Z g_l(V, W) + g_l([Z, V], W) + g_l([Z, W], V) \\ &= 2g_l\left(T^{g_l}_V W, Z\right). \end{aligned}$$

So $\left|T^{\tilde{g}_l}\right| = \left|T^{g_l}\right|$, and the result follows. □

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