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Conjugate points on the symplectomorphism group

James Benn[1]

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Abstract Let $\mathcal{D}^s_\omega(M)$ denote the group of symplectic diffeomorphisms of a closed symplectic manifold M, which are of Sobolev class H^s for sufficiently high s. When equipped with the L^2 metric on vector fields, $\mathcal{D}^s_\omega$ becomes an infinite-dimensional Hilbert manifold whose tangent space at a point η consists of H^s sections X of the pull-back bundle η^*TM for which the corresponding vector field $u = X \circ \eta^{-1}$ on M satisfies $\mathcal{L}_u\omega = 0$. Geodesics of the L^2 metric are globally defined, so that the L^2 metric admits an exponential mapping defined on the whole tangent space. It was shown that this exponential mapping is a non-linear Fredholm map of index zero. Singularities of the exponential map are known as conjugate points and in this paper we construct explicit examples of them on $\mathcal{D}^s_\omega(\mathbb{C}P^n)$. We then solve the Jacobi equation explicitly along a geodesic in $\mathcal{D}^s_\omega$, generated by a Killing vector field, and characterize all conjugate points along such a geodesic.

Keywords Diffeomorphism group · Hydrodynamics · Plasma dynamics · Geodesic · Conjugate point · Monoconjugate · Epiconjugate · Fredholm map · Euler equations · Hilbert manifold

Mathematics Subject Classification 35Q05 · 35Q35 · 47H99 · 58C40 · 58C99 · 76E99 · 37K65

1 Introduction

Let M be a closed symplectic manifold with symplectic form ω and Riemannian metric g. We assume that ω and g are compatible, in the sense that there exists an almost complex structure $J : TM \to TM$ satisfying $J^2 = -I$, $g(Jv, Jw) = g(v, w)$, and $g(v, Jw) = \omega(v, w)$, for any fields v, w. Let $\mathcal{D}^s_\omega(M)$ denote the group of all diffeomorphisms of Sobolev class H^s

✉ James Benn
jbenn2@nd.edu

1 Department of Mathematics, University of Notre Dame, 255 Hurley Hall,
Notre Dame, IN 46556, USA

preserving the symplectic form ω on M. If $s > \frac{\dim M}{2} + 1$ then $\mathcal{D}^s_\omega(M)$ becomes an infinite-dimensional Hilbert manifold whose tangent space at a point η consists of H^s sections X of the pull-back bundle $\eta^* TM$ for which the corresponding vector field $u = X \circ \eta^{-1}$ on M satisfies $\mathcal{L}_u \omega = 0$, where $\mathcal{L}$ is the usual Lie derivative. Using right translations, the L^2 inner product on vector fields,

$$(u, v)_{L^2} = \int_M g\,(u, v)\,\mathrm{d}\mu, \quad u, v \in T_e\mathcal{D}^s_\omega, \tag{1}$$

defines a right-invariant metric on the group. The L^2 metric induces a smooth invariant Levi–Civita connection on $\mathcal{D}^s_\omega$ whose curvature tensor R is also invariant with respect to right multiplication by $\mathcal{D}^s_\omega(M)$. Our main references for basic facts regarding diffeomorphism groups are [2,4,6].

It is useful to consider $\mathcal{D}^s_\omega(M)$ as a Riemannian submanifold of the group $\mathcal{D}^s(M)$ with all H^s diffeomorphisms within the same L^2 metric. The action of $\mathcal{D}^s_\omega(M)$ on $\mathcal{D}^s(M)$ by composition on the right is an isometry of (1) and combined with the Hodge decomposition gives an L^2 orthogonal splitting of each tangent space (see [4])

$$T_\eta \mathcal{D}^s = T_\eta \mathcal{D}^s_\omega \oplus \omega^\sharp \left(\delta \mathrm{d} H^s(T^*M)\right) \circ \eta \tag{2}$$

where $\omega^\flat : TM \to T^*M$ is an isomorphism defined by $v \mapsto i_v\omega$ with inverse $\omega^\sharp : T^*M \to TM$ given by contracting a 1-form with the inverse components of the symplectic form. The projections onto the first and second summands of (2) will be denoted by P_η and Q_η, respectively, or simply by P and Q if $\eta = e$ the identity. We also mention that the almost complex structure J may be written in the form $J = g^\sharp \omega^\flat$, where $g^\sharp$ is defined analogously to $\omega^\sharp$.

A strong motivation to study the geometry of diffeomorphism groups comes from hydrodynamics. In his celebrated paper, Arnold [1] related motions of a perfect fluid in M to geodesics in $D_\mu(M) = \cap_s \mathcal{D}^s_\mu(M)$, the group of smooth diffeomorphisms preserving the volume of M. He observed that a curve $\eta(t)$ is a geodesic of the L^2 metric (1) starting from the identity e in the direction v_o if and only if the time-dependent vector field $v = \dot\eta \circ \eta^{-1}$ on M solves the Euler equations of hydrodynamics[1]. Soon after, Ebin and Marsden [6] discovered that there is a technical advantage in rewriting the Euler equations in this way. Their result was that the corresponding geodesic equation on the group $\mathcal{D}^s_\mu$ is in fact a smooth ODE and can therefore be solved uniquely for small values of t using a Picard iteration argument. Furthermore, since the solutions depend smoothly on initial data, it follows that the L^2 metric has a smooth exponential map.

The L^2 geometry of $\mathcal{D}^s_\mu$ provides the Lagrangian description of incompressible fluids. In two dimensions, solutions to the Euler equations of hydrodynamics exist globally in time [21] and the exponential map is defined on the whole tangent space. In three dimensions, existence of global solutions to the Euler equations is a celebrated open problem. The exponential map corresponds to the solution operator, in Lagrangian coordinates, of the Euler equations so it is of interest to study its singularities, i.e. conjugate points. In contrast with finite-dimensional geometry, two types of singularities of the exponential map exist in infinite dimensions. Let $\eta(t)$ be a geodesic in an infinite-dimensional Hilbert manifold. Following Grosman [8], $\eta(t_0)$ is monoconjugate to $\eta(0)$ if $d\exp_{\eta(0)}(t_0\dot\eta(0))$ fails to be injective and epiconjugate if $d\exp_{\eta(0)}(t_0\dot\eta(0))$ fails to be surjective. The first examples of conjugate points in $\mathcal{D}^s_\mu$ were given in [13,14] and later in [20]. Singularities of the exponential map have been studied in [7] where it is shown that in two dimensions conjugate points are isolated, of finite multiplicity

[1] A $\dot{}$ denotes a derivative with respect to t.

and the two types of conjugacies mentioned above coincide. In particular, the exponential map is a non-linear Fredholm map of index zero, i.e. its differential has finite-dimensional Kernel and Cokernel, and has closed range. In three dimensions, the geometry of $\mathcal{D}^s_\mu$ is drastically different: in [7] it is shown, by example, that monoconjugate points accumulate and converge to an epiconjugate point. In particular, the exponential map is no longer a non-linear Fredholm map. Moreover, the accumulation of monoconjugate points on epiconjugate points in three-dimensional incompressible fluids is a typical pathology [17].

Here, we study the L^2 geometry of the group $\mathcal{D}^s_\omega$. The subgroup of Hamiltonian diffeomorphisms plays a role in plasma dynamics analogous to the role played by $\mathcal{D}^s_\mu$ in incompressible hydrodynamics [2,9–11,15,18]. A curve $\eta(t)$ is a geodesic of the L^2 metric (1) starting from the identity e in the direction v_o if and only if the time-dependent vector field $v = \dot{\eta} \circ \eta^{-1}$ on M solves the symplectic Euler equations:

$$\begin{aligned} \partial_t v + P(\nabla_v v) &= 0 \\ \mathcal{L}_v \omega &= 0, \\ v(0) &= v_o, \end{aligned} \tag{3}$$

see [4]. The initial value problem for the geodesic equation on $\mathcal{D}^s_\omega$ of the L^2 metric (1), which is the Lagrangian formulation of the symplectic Euler equations (3), has the following form

$$\partial_t^2 \eta(t) = F(\eta(t), \quad \dot{\eta}(t)) = \omega^\sharp \delta \triangle^{-1} \mathrm{d}\omega^b \nabla_{\dot{\eta}(t)\circ\eta^{-1}(t)} \dot{\eta}(t) \circ \eta^{-1}(t) \tag{4}$$

subject to the initial conditions

$$\partial_t \eta(0) = v_o \quad \eta(0) = e.$$

This is a smooth ODE which can be solved for small values of t (cf. [4] or [6]). Furthermore, since the solutions depend smoothly on initial data, it follows that the L^2 metric has a smooth exponential map

$$\exp_e : T_e\mathcal{D}^s_\omega \to \mathcal{D}^s_\omega(M) \tag{5}$$

defined, for small t, by

$$\exp_e(tv_o) = \eta(t),$$

where η is the unique geodesic from the identity with initial velocity $v_o \in T_e\mathcal{D}^s_\omega$. Furthermore, the manifold $\mathcal{D}^s_\omega(M)$ is geodesically complete for any closed symplectic manifold, see [5,10], and consequently the map (5) is defined on the whole tangent space.

If M is a closed symplectic manifold of any dimension, then the L^2 exponential map is a non-linear Fredholm map of index zero on the group of symplectomorphisms $\mathcal{D}^s_\omega$ (see [3]). As a consequence, the two types of conjugacies mentioned above coincide and conjugate points are of finite multiplicity along any finite geodesic segment.

In this paper, we show that conjugate points always exist in the symplectomorphism group. In particular, geodesics of the L^2 metric which lie in the isometry subgroup always have conjugate points. We solve the Jacobi equation, along such a geodesic, explicitly and show that the multiplicity of every conjugate is always even. In part 2, we show that the existence of conjugate points along stationary geodesics is characterized by the existence of a fixed point of a product of operators.

The paper is structured as follows. In Sect. 2 we give examples of conjugate points in $\mathcal{D}^s_\omega$, where the underlying symplectic manifold is of arbitrary dimension. Next, in Sect. 3, we show that every geodesic which is generated by a Killing vector field on a closed symplectic manifold M contains conjugate points. In Sect. 4 we solve the Jacobi equation, along a geodesic generated by a Killing field on M, explicitly and determine the multiplicity of conjugate points.

2 Conjugate points on $\mathcal{D}^s_\omega(\mathbb{CP}^n)$

The isometry group of a symplectic manifold (M, ω) with compatible Riemannian metric g, denoted by $\mathrm{Iso}(M)$, consists of those diffeomorphisms η satisfying

$$\eta^*\omega = \omega \quad \eta^* g = g.$$

If M is symplectic, then the isometry group is contained in the group of symplectomorphisms as a finite-dimensional Lie subgroup with Lie algebra $T_e\mathrm{Iso}(M) = \{v \in T_e\mathcal{D}^s_\omega : \mathcal{L}_v g = 0\}$. Elements of $T_e\mathrm{Iso}(M)$ are called Killing fields.

Proposition 2.1 *Let $v \in T_e\mathcal{D}^s_\omega$ be a Killing vector field. Then, v generates a stationary solution to the symplectic Euler equation and the corresponding geodesic in $\mathcal{D}^s_\omega$ consists of isometries for all t.*

Proof Let $v \in T_e\mathcal{D}^s_\omega$ be a Killing vector field. Let w be any other C^∞ vector field on M. We compute

$$g(\nabla_v v, w) = -g(v, \nabla_w v) = -\frac{1}{2} w \cdot g(v, v) = g\left(-\frac{1}{2}\nabla |v|^2, w\right)$$

and since this holds for any w

$$\nabla_v v = -\frac{1}{2}\nabla |v|^2 .$$

Since a gradient vector field is orthogonal to $T_e\mathcal{D}^s_\omega$ in the L^2 metric by (2)

$$P(\nabla_v v) = P\left(-\frac{1}{2}\nabla |v|^2\right) = 0$$

and v generates a stationary solution to the symplectic Euler equations.

If $\eta(t)$ is the unique geodesic in $\mathcal{D}^s_\omega$ with initial velocity v, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)^* g = \eta(t)^*\mathcal{L}_v g = 0$$

so that $\eta(t)$ consists of isometries for each t. □

Theorem 2.2 *Conjugate points exist on $\mathcal{D}^s_\omega(\mathbb{C}P^n)$ for all $n \geq 2$.*

Proof The complex projective space $\mathbb{CP}^n$ is a Kähler manifold with the Fubini–Study metric, which is given in components as

$$h_{i\bar{j}} = h(\partial_i, \bar{\partial}_j) = \frac{(1 + |z|^2)\delta_{i\bar{j}} - \bar{z}_i z_j}{(1 + |z|^2)^2}$$

where $z = (z_1, \ldots, z_n)$ is a point in $\mathbb{CP}^n$, $|z|^2 = z_1^2 + \cdots + z_n^2$. The isometry group of $\mathbb{CP}^n$ is given by $PU(n+1)$, the projective unitary group. $PU(n+1)$ is given by the quotient of the unitary group, $U(n+1)$, by its center, $U(1)$, embedded as scalars. Thus, in terms of matrices, $U(n+1)$ consists of complex $n+1 \times n+1$ matrices whose center consists of elements of the form $\mathrm{e}^{i\theta}I$. Elements of $PU(n+1)$ correspond to equivalence classes of unitary matrices, where two matrices A and B are equivalent if $A = \mathrm{e}^{i\theta}I \times B$ and we write $A \equiv B$.

If n is even, consider the following 2-parameter variation of isometries

$$\gamma(s,t) = A(s)B(t)A^{-1}(s)$$

where

$$A(s) = \begin{bmatrix} i & & & & \\ & A'(s) & & & \\ & & A'(s) & & \\ & & & \ddots & \\ & & & & A'(s) \end{bmatrix} \quad B(t) = \begin{bmatrix} B'(s) & & & & \\ & B'(s) & & 0 & \\ & & \ddots & & \\ & 0 & & B'(s) & \\ & & & & i \end{bmatrix}$$

where $A'(s) = \begin{bmatrix} i\cos s & \sin s \\ \sin s & i\cos s \end{bmatrix}$ and $B'(s) = \begin{bmatrix} i\cos t & \sin t \\ \sin t & i\cos t \end{bmatrix}$ are 2×2 block matrices. Observe that $\gamma(s,0) = iI \equiv I$ for all s. We shall show that for each s, $\gamma(s,t)$ is a family of geodesics in $\mathcal{D}^s_\omega(\mathbb{C}P^n)$ and compute its variation field, $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\gamma(s,t)$, along $\gamma(0,t)$.

We first find that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(s,t)\right) \circ \gamma^{-1}(s,t) = \begin{bmatrix} 0 & -i\cos s & \sin s & & & & & & \\ -i\cos s & 0 & 0 & C_1(s) & & & & & \\ -\sin s & 0 & 0 & & & & & & \\ & C_2(s) & & 0 & 0 & C_1(s) & & & \\ & & & 0 & 0 & & & & \\ & & & C_2(s) & & 0 & 0 & \ddots & \\ & & & & & 0 & 0 & & \\ & & & & & \ddots & & \ddots & \end{bmatrix}$$

with

$$C_1(s) = \begin{bmatrix} -\sin s\cos s & -i\sin^2 s \\ -i\cos^2 s & \sin s\cos s \end{bmatrix} \quad C_2(s) = \begin{bmatrix} \sin s\cos s & -i\cos^2 s \\ -i\sin^2 s & -\sin s\cos s \end{bmatrix}$$

where $\gamma^{-1}(s,t) = (A(s)B(t)A^{-1}(s))^{-1} = A(s)B^{-1}(t)A^{-1}(s)$ and composition is given by the usual matrix multiplication.

The matrix associated with the vector field $v(s,t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(s,t)\right) \circ \gamma^{-1}(s,t)$ is clearly a skew-hermitian matrix and therefore lies in the Lie algebra to $PU(n+1)$. Thus, $v(s,t)$ is a Killing vector field so that v generates a stationary solution to the symplectic Euler equation by Proposition 2.1. Hence, $\gamma(s,t)$ satisfies the geodesic equation (4) in $\mathcal{D}^s_\omega(\mathbb{C}P^n)$. The variation field of this family of geodesics is given by

$$J(t) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\gamma(s,t) = \begin{bmatrix} \mathbf{0} & D_1 & & & & & \\ -D_1 & \mathbf{0} & D_1 & & & & \\ & -D_1 & \mathbf{0} & \ddots & & & \\ & & \ddots & \ddots & D_1 & & \\ & & & D_1 & \mathbf{0} & \ddots & \\ & & & & \ddots & \ddots & D_2 \\ & & & & & -D_2 & \mathbf{0} \end{bmatrix},$$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D_1 = \begin{bmatrix} -\sin t & 0 \\ 0 & \sin t \end{bmatrix}, \quad D_2 = \begin{bmatrix} -\sin t & 0 \\ 0 & i(1-\cos t) \end{bmatrix}$$

which clearly vanishes at $t = 0$ and $t = 2\pi$. Therefore, the point $\gamma(2\pi) = B(2\pi)$ is conjugate to the identity $\gamma(0) = e$.

The proof for odd n is exactly the same except we take

$$\gamma(s,t) = A(s)B(t)A^{-1}(s)$$

with

$$A(s) = \begin{bmatrix} i & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_1 & & & & \\ 0 & & A_1 & & & \\ 0 & & & \ddots & & \\ \vdots & & & & A_1 & \\ 0 & & & & & i \end{bmatrix} \quad B(t) = \begin{bmatrix} B_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & B_1 & & & \\ \mathbf{0} & & \ddots & & \\ \vdots & & & B_1 & \\ \mathbf{0} & & & & iI \end{bmatrix}$$

as our two-parameter variation of isometries, where A_1 and B_1 are as in the n odd case. □

Remark 2.3 In the particular case when $n = 2$, the geodesic

$$\gamma(t) = \begin{bmatrix} i\cos t & \sin t & 0 \\ \sin t & i\cos t & 0 \\ 0 & 0 & i \end{bmatrix}$$

has a point, conjugate to the identity, at $\gamma(2\pi)$.

3 Geodesics in the isometry subgroup

In standard Lie group notation, the group adjoint operator on $T_e\mathcal{D}^s_\omega$ is $\mathrm{Ad}_\eta = dR_{\eta^{-1}}dL_\eta : T_e\mathcal{D}^s_\omega \to T_e\mathcal{D}^s_\omega$, where $\eta \in \mathcal{D}^s_\omega$ and R_η and L_η are the right and left translations on $\mathcal{D}^s_\omega$ given by the composition with H^s diffeomorphisms on the right, respectively, the left. Consequently

$$\mathrm{Ad}_\eta(w) = D\eta \cdot w \circ \eta^{-1}. \tag{6}$$

and for the algebra adjoint action

$$\mathrm{ad}_u : T_e\mathcal{D}^s_\omega \to T_e\mathcal{D}^s_\omega$$

$$\mathrm{ad}_u v = -\mathcal{L}_u v \tag{7}$$

See [2] for details on these formulas.

The group coadjoint $\mathrm{Ad}^*_\eta : T_e\mathcal{D}^s_\omega \to T_e\mathcal{D}^s_\omega$ is defined so that

$$(\mathrm{Ad}^*_\eta v, w)_{L^2} = (v, \mathrm{Ad}_\eta w)_{L^2} \;\; w \in T_e\mathcal{D}^s_\omega \tag{8}$$

and the Lie algebra coadjoint $\mathrm{ad}^*_v : T_e\mathcal{D}^s_\omega \to T_e\mathcal{D}^s_\omega$ is defined so that

$$(\mathrm{ad}^*_u v, w)_{L^2} = (v, \mathrm{ad}_u w)_{L^2} \;\; u, w \in T_e\mathcal{D}^s_\omega. \tag{9}$$

Lemma 3.1 *The restriction of the L^2 metric (1) to* $\mathrm{Iso}(M) \subset \mathcal{D}^s_\omega(M)$ *is bi-invariant. Consequently, for a killing field v, which generates a stationary geodesic $\eta(t)$ in $\mathcal{D}^s_\omega$,*

$$\mathrm{Ad}^*_{\eta(t)} w = \mathrm{Ad}_{\eta^{-1}(t)} w \tag{10}$$
$$\mathrm{ad}^*_v w = -\mathrm{ad}_v w \tag{11}$$

for any vector field $w \in T_e\mathcal{D}^s_\omega$.

Proof Since $\mathcal{D}^s_\omega(M)$ is a group and right translation is an isometry of the L^2 metric (1), it suffices to do the necessary computations at the identity. We will show that the restriction of (1) to $\mathrm{Iso}(M)$ is invariant under the adjoint action restricted to $\mathrm{Iso}(M)$. Let η_t be a geodesic of the metric (1) in $\mathrm{Iso}(M)$ with initial velocity v_o and $u, w \in T_e\mathcal{D}^s_\omega(M)$. Then

$$\begin{aligned}(\mathrm{Ad}_{\eta_t} u, \mathrm{Ad}_{\eta_t} w)_{L^2} &= \int_M g(D\eta_t \cdot u \circ \eta_t^{-1}, D\eta_t \cdot w \circ \eta_t^{-1}) \mathrm{d}\mu \\ &= \int_M (\eta_t^* g)(u, w) \mathrm{d}\mu = (u, w)_{L^2}\end{aligned}$$

since $\eta_t \in \mathrm{Iso}(M)$.

To prove the formula (11), we differentiate both sides of the identity

$$(\mathrm{Ad}_{\eta_t} u, \mathrm{Ad}_{\eta_t} w)_{L^2} = (u, w)_{L^2}$$

in t and set $t = 0$ with u and w as above. We obtain

$$(\mathrm{ad}_v u, w)_{L^2} + (u, \mathrm{ad}_v w)_{L^2} = 0$$

and the formula is proved. □

Let ∇ be the covariant derivative on M. The L^2 metric (1) induces a smooth right invariant Levi–Civita connection $\nabla^\omega = P \circ \nabla$ on $\mathcal{D}^s_\omega(M)$ whose curvature tensor R is also right invariant with respect to right multiplication by $\mathcal{D}^s_\omega(M)$, cf. [6]. The curvature tensor is a smooth, bounded, multi-linear operator in the strong Sobolev H^s topology, [13]. Arnold first computed sectional curvatures of $\mathcal{D}^s_\omega$ for the two-dimensional torus [1]. He found that in most directions the curvature is non-positive, although in some directions it is positive. In finite dimensions, the curvature along a geodesic tells us something about the stability of small perturbations along it. If the curvature is strictly negative, then small perturbations grow exponentially; if it is strictly positive then small perturbations are bounded (at least up to the first conjugate point). An analogous interpretation of curvature holds for diffeomorphism groups, see [2,13,16], and positivity of curvatures is necessary for the existence of conjugate points along geodesics in diffeomorphism groups.

Lemma 3.2 1. *Let v be a Killing field and $u \in T_e\mathcal{D}^s_\omega$. Then, the covariant derivative of the L^2-metric (1) on $\mathcal{D}^s_\omega$ reduces to*

$$\nabla^\omega_v u = -\frac{1}{2}\mathrm{ad}^*_u v$$
$$\nabla^\omega_u v = -\mathrm{ad}_v u + \frac{1}{2}\mathrm{ad}^*_u v.$$

If u is also a Killing field then

$$\nabla^\omega_v u = -\frac{1}{2}\mathrm{ad}_v u.$$

2. *For Killing vector fields $u, v, w \in T_e\mathrm{Iso}(M)$*

$$R^{\omega}(u, v)\, w = \frac{1}{4}\,[w, [u, v]]\,. \tag{12}$$

Proof This follows directly from the bi-invariance of the L^2 metric when restricted to $\mathrm{Iso}(M)$. However, to see this another way let v be a Killing field and $u \in T_e\mathcal{D}^s_\omega$. We have

$$\nabla^{\omega}_v u = P(\nabla_v u) = \frac{1}{2} P(\mathrm{ad}_v u + \mathrm{ad}^*_v u + \mathrm{ad}^*_u v).$$

By Lemma 3.1, $\mathrm{ad}_v u = -\mathrm{ad}^*_v u$ so that

$$\begin{aligned}\nabla^{\omega}_v u = P(\nabla_v u) &= \frac{1}{2} P(\mathrm{ad}_v u - \mathrm{ad}_v u + \mathrm{ad}^*_u v)\\ &= \frac{1}{2} P(\mathrm{ad}^*_u v).\end{aligned}$$

We also have

$$\begin{aligned}\nabla^{\omega}_u v = P(\nabla_u v) &= \frac{1}{2} P(\mathrm{ad}^*_v u + \mathrm{ad}^*_u v - \mathrm{ad}_v u)\\ &= \frac{1}{2} P(-\mathrm{ad}_v u + \mathrm{ad}^*_u v - \mathrm{ad}_v u)\\ &= -\mathrm{ad}_v u + \frac{1}{2}\mathrm{ad}^*_u v.\end{aligned}$$

The expression for the curvature tensor follows from the expression for the covariant derivative. □

Proposition 3.3 *Let M be a closed Symplectic manifold with compatible Riemannian metric. Then, $(\mathrm{Iso}(M), (\cdot,\cdot)_{L^2})$ is an isometrically embedded, totally geodesic submanifold of $(\mathcal{D}^s_\omega(M), (\cdot,\cdot)_{L^2})$, each with the L^2 metric.*

Proof Endow the compact Lie group $\mathrm{Iso}(M)$ with the L^2 metric (1), which is bi-invariant, and denote the associated covariant derivative by ∇^I. Since every isometry of M is contained in the group of symplectic diffeomorphisms, the identity inclusion map

$$i : \mathrm{Iso}(M) \hookrightarrow \mathcal{D}^s_\omega(M)$$

gives an isometric embedding of $(\mathrm{Iso}(M), (\cdot,\cdot)_{L^2})$ in $(\mathcal{D}^s_\omega(M), (\cdot,\cdot)_{L^2})$ with the L^2 metric (1).

Since the L^2 metric is bi-invariant on $\mathrm{Iso}(M)$, the associated covariant derivative is given by the Lie bracket of vector fields:

$$\nabla^I_u v = \frac{1}{2}\mathcal{L}_u v$$

for right invariant Killing vector fields u and v. On the other hand, Lemma 3.2 says that

$$\nabla^{\omega}_u v = -\frac{1}{2}\mathrm{ad}_u v = \frac{1}{2}\mathcal{L}_u v$$

for Killing vector fields $u, v \in T_e\mathcal{D}^s_\omega$. The difference of these two covariant derivatives is zero so that the second fundamental tensor of $\mathrm{Iso}(M)$ vanishes identically. It follows that $\mathrm{Iso}(M)$ is totally geodesic in $\mathcal{D}^s_\omega$. □

With this Proposition in hand, the main result of this section follows from pushing forward finite-dimensional Jacobi fields on the compact manifold $(\mathrm{Iso}(M), (\cdot,\cdot)_{L^2})$ to $(\mathcal{D}^s_\omega(M), (\cdot,\cdot)_{L^2})$. Compare this with the constructions in Sect. 2.

Theorem 3.4 *Every geodesic of the L^2 metric (1) which is generated by a Killing field and which is of length greater than πr (for some positive constant r) has conjugate points.*

Proof It is known that every geodesic in a compact Lie group with bi-invariant metric, which is of length greater than πr (for some positive constant r), has conjugate points, cf. Milnor [12]. Since $\mathrm{Iso}(M)$ is isometrically embedded in $\mathcal{D}^s_\omega$ and is totally geodesic, any Jacobi field along a geodesic $\eta(t)$ in the manifold $\mathrm{Iso}(M)$ is also a Jacobi field along the same geodesic $i(\eta(t)) = \eta(t)$ in $\mathcal{D}^s_\omega$. Consequently, every geodesic in $\mathcal{D}^s_\omega$ which is generated by a Killing vector field, and which is of length greater than πr (for some positive constant r), has conjugate points. □

4 The Jacobi equation along geodesics of isometries

In Sect. 3, we restricted the L^2 metric to the finite-dimensional, compact subgroup of isometries and considered variations of geodesics within this subgroup. Here, we consider variations of a geodesic of isometries in the full group of symplectomorphisms, solve the Jacobi equation explicitly and use this solution to describe conjugate points explicitly.

Lemma 4.1 *For v and w in $T_e\mathcal{D}^s_\omega$,*

$$\mathrm{ad}^*_v w = P((\mathrm{div} Jw)\cdot Jv) \tag{13}$$

Proof Let u be any vector in $T_e\mathcal{D}^s_\omega$. Using (9)

$$\begin{aligned}(\mathrm{ad}^*_v w, u)_{L^2} &= (w, \mathrm{ad}_v u)_{L^2}\\ &= (w, J\nabla\omega(v,u))_{L^2} = -\int_M g(Jw, \nabla\omega(v,u))\,\mathrm{d}\mu\\ &= \int_M \omega(u,v)\cdot(\mathrm{div} Jw)\,\mathrm{d}\mu = \int_M g(u, Jv)\cdot(\mathrm{div} Jw)\,\mathrm{d}\mu\\ &= ((\mathrm{div} Jw)Jv, u)_{L^2},\end{aligned}$$

and consequently

$$\mathrm{ad}^*_v w = P\left((\mathrm{div} Jw)\cdot Jv\right).$$

□

For $v \in T_e\mathcal{D}^s_\omega$, define an operator

$$\begin{aligned}K_v : T_e\mathcal{D}^s_\omega &\to T_e\mathcal{D}^s_\omega\\ w &\mapsto \mathrm{ad}^*_w v\end{aligned} \tag{14}$$

using Lemma 4.1. The operator (14) is skew self-adjoint in the metric (1).

Let $T_e\mathcal{D}_\omega = \cap_{s\geq 0} T_e\mathcal{D}^s_\omega$ be the subspace consisting of smooth vector fields on M and observe that $T_e\mathcal{D}_\omega$ is dense in $T_e\mathcal{D}^s_\omega$. Given $v \in T_e\mathcal{D}^s_\omega$, let v_k be a sequence of smooth vector fields in $T_e\mathcal{D}_\omega$ approximating v in the H^s norm. Consequently, v_k is a smooth sequence of H^s vector fields approximating v in the H^s norm.

Lemma 4.2 *Let $s > \frac{\dim M}{2} + 1$, $s \geq \sigma + 1$ and let v and $\{v_k\}_{k\in\mathbb{N}}$ be as above. Then, $K_{v_k} \to K_v$ in the H^σ norm.*

Proof Let $w \in T_e\mathcal{D}^\sigma_\omega$. We estimate

$$\begin{aligned}\left\|K_v w - K_{v_k} w\right\|_{H^\sigma} &= \|P\left((\mathrm{div} Jv)\cdot Jw - (\mathrm{div} Jv_k)\cdot Jw\right)\|_{H^\sigma}\\ &\lesssim \|\mathrm{div} J\left(v - v_k\right)\cdot Jw\|_{H^\sigma}\\ &\lesssim \|\mathrm{div} J\left(v - v_k\right)\|_{H^\sigma}\cdot\|w\|_{H^\sigma}\\ &\lesssim \|v - v_k\|_{H^{\sigma+1}}\cdot\|w\|_{H^\sigma}\\ &\lesssim \|v - v_k\|_{H^s}\cdot\|w\|_{H^\sigma}\end{aligned}$$

and the Lemma follows. □

Lemma 4.3 *Let $s > \frac{\dim M}{2} + 1$, $s \geq \sigma + 1$. For any vector field $v \in T_e\mathcal{D}^s_\omega$ the operator K_v defined by (14) is compact on $T_e\mathcal{D}^\sigma_\omega$.*

Proof By Lemma 4.2 we can approximate v in the H^s norm by a sequence of smooth vector fields v_k such that $K_{v_k} \to K_v$ in the H^σ operator norm. Since a limit of compact operators is compact it suffices to show that K_v is compact when v is smooth.

By Lemma 4.1, the operator K_v may be written as

$$K_v(w) = P\left((\mathrm{div} Jv)\cdot Jw\right).$$

Any $w \in T_e\mathcal{D}^s_\omega$ can be written as $w = J\nabla H + h$, where H is an H^{s+1} function with zero mean on M and h is a harmonic vector field. Therefore, for any $w \in T_e\mathcal{D}^s_\omega$, the operator K_v can be written as

$$K_v = K_v \circ \pi_{rg} + K_v \circ \pi_h$$

where π_{rg} denotes projection onto the space of rotated gradients (i.e., vector fields of the form $J\nabla H$, for a function H on M) and π_h denotes the projection onto the space of harmonic vector fields. The projections π_{rg} and π_h are both continuous in the H^σ topology. Since the space of harmonic vector fields is finite dimensional, the operator $K_v \circ \pi_h$ has finite rank and is therefore compact. We have

$$K_v \circ \pi_{rg}(w) = P\left((\mathrm{div} Jv)\cdot Jw\right) = -P\left((\mathrm{div} Jv)\cdot \nabla H\right) = P(H\cdot\nabla\left(\mathrm{div} Jv\right))$$

since the projection of a gradient vector field vanishes. Then

$$\begin{aligned}\left\|K_v \circ \pi_{rg}(w)\right\|_{H^{\sigma+1}} &= \|P(H\cdot\nabla\left(\mathrm{div} Jv\right))\|_{H^{\sigma+1}}\\ &\lesssim \|H\cdot\nabla\left(\mathrm{div} Jv\right)\|_{H^{\sigma+1}} \lesssim \|H\|_{H^{\sigma+1}}\\ &\lesssim \|w\|_{H^\sigma}.\end{aligned}$$

Therefore, the map $w \mapsto K_v \circ \pi_{rg}(w)$, as a map from H^σ vector fields to $H^{\sigma+1}$ vector fields, is compact by the Rellich embedding Theorem. Consequently, the operator K_v is compact. □

Theorem 4.4 *Let $\eta(t) = \exp(tv_o)$ be a geodesic of the L^2 metric (1) generated by a Killing vector v_o. Let $J(t)$ be a Jacobi field along $\eta(t)$, with initial conditions $J(0) = 0$, $J'(0) = w_o$. Then*

$$J(t) = D\eta(t)\cdot\frac{I - \mathrm{e}^{-tK_{v_o}}}{K_{v_o}}\,w_o,$$

where $K_{v_o}(\cdot) = \mathrm{ad}^*_{(\cdot)} v_o$*, and we have the spectral representations* $\mathrm{e}^{-tK_{v_o}} = \int_{\mathbb{R}} \mathrm{e}^{it\lambda}\, \mathrm{d}E(\lambda)$ *and* $\frac{\mathrm{e}^{tK_{v_o}} - I}{K_{v_o}} = \int_{\mathbb{R}} \frac{\mathrm{e}^{it\lambda}-1}{i\lambda}\, \mathrm{d}E(\lambda)$*, where* E *is the unique spectral measure determined by* K_{v_o}*.*

Proof Let v_o be any vector in $T_e\mathcal{D}^s_\omega$ and η be the geodesic of (1) starting from the identity with initial velocity v_o. The Cauchy problem for the symplectic Euler equation is globally well posed and it follows that the corresponding geodesic in $\mathcal{D}^s_\omega$ can be extended indefinitely [4]. The Jacobi equation along $\eta(t) = \exp_e(tv_o)$ is given by

$$\nabla^\omega_{\dot\eta}\nabla^\omega_{\dot\eta} J + R^\omega(J, \dot\eta)\dot\eta = 0 \tag{15}$$

with initial conditions

$$J(0) = 0, \quad \dot J(0) = w_o. \tag{16}$$

The fact that R is a bounded multi-linear operator implies the Jacobi fields exist, and are unique and global in time [13].

It can be shown that the Jacobi equation (15) is equivalent to the linearization of the symplectic Euler equations and the flow equation

$$\partial_t v + P(\nabla_v v) = 0 \tag{17}$$

$$\dot\eta(t) = v(t) \circ \eta(t), \tag{18}$$

where $\eta(t)$ is a geodesic of the L^2 metric in $\mathcal{D}^s_\omega$ and $v(t)$ solves the symplectic Euler equations. Linearizing equations (17) and (18) yields

$$\partial_t z(t) + P(\nabla_{v(t)} z(t) + \nabla_{z(t)} v(t)) = 0 \tag{19}$$

$$\partial_t Y(t) + [v(t), Y(t)] = z(t) \tag{20}$$

where $Y(t) = J(t) \circ \eta(t)^{-1}$ and $J(t)$ a solution to the Jacobi equation. Since we have that

$$P(\nabla_{v(t)} z(t) + \nabla_{z(t)} v(t)) = \mathrm{ad}^*_v z + \mathrm{ad}^*_z v, \tag{21}$$

Equations (19) and (20) become

$$\partial_t z + \mathrm{ad}^*_v z + \mathrm{ad}^*_z v = 0$$
$$\partial_t Y - \mathrm{ad}_v Y = z.$$

Using (14), the Jacobi equation (15) can be rewritten in the following form

$$(\partial_t + \mathrm{ad}^*_v + K_v)(\partial_t - \mathrm{ad}_v)Y = 0. \tag{22}$$

Equation (22) is equivalent to

$$(\partial_t + \mathrm{ad}^*_v)(\partial_t - \mathrm{ad}_v + K_v)Y = 0 \tag{23}$$

which follows from self-adjointness of the Jacobi equation and skew-self-adjointness of the operator K_v.

Using (23), we rewrite (15) as a system of equations

$$(\partial_t + \mathrm{ad}^*_v)w = 0$$
$$(\partial_t - \mathrm{ad}_v + K_v)Y = w.$$

We can rewrite the operators ad_v and ad_v^* in terms of the push-forward Ad_η and its adjoint Ad_η^*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\eta^{-1}} = -\mathrm{Ad}_{\eta^{-1}}\mathrm{ad}_v \tag{24}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\eta^{-1}}^* = -\mathrm{ad}_v^*\mathrm{Ad}_{\eta^{-1}}^*. \tag{25}$$

Using these equations, the factored, right-translated Jacobi equation can be written as the pair of equations

$$\mathrm{Ad}^*_{\eta(t)^{-1}}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}^*_{\eta(t)}w(t)) = 0 \tag{26}$$
$$\mathrm{Ad}_{\eta(t)}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}_{\eta(t)^{-1}}Y(t)) + K_{v(t)}(Y(t)) = w(t). \tag{27}$$

The solution of (27) is obviously $w(t) = \mathrm{Ad}^*_{\eta(t)^{-1}}w_o$, and from this we rewrite (27) as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}_{\eta(t)^{-1}}Y(t)) + \mathrm{Ad}_{\eta(t)^{-1}}K_{v(t)}\,(Y(t)) = \mathrm{Ad}_{\eta(t)^{-1}}\mathrm{Ad}^*_{\eta(t)^{-1}}w_o. \tag{28}$$

For any geodesic $\eta(t)$ of the L^2 metric, generated by a solution $v(t)$ of the symplectic Euler equations with initial condition v_0,

$$K_{v(t)} = \mathrm{Ad}^*_{\eta^{-1}(t)}K_{v_o}\mathrm{Ad}_{\eta^{-1}(t)}. \tag{29}$$

Indeed, comparing Eqs. (22) and (23) one obtains the expression $\partial_t K_{v(t)} = -K_{v(t)}\mathrm{ad}_{v(t)} - \mathrm{ad}^*_{v(t)}K_{v(t)}$. Rewriting this equation as

$$\begin{aligned}\partial_t(\mathrm{Ad}^*_{\eta^{-1}(t)}\mathrm{Ad}^*_{\eta(t)}K_{v(t)}\mathrm{Ad}_{\eta(t)}\mathrm{Ad}_{\eta^{-1}(t)}) &= -\mathrm{Ad}^*_{\eta^{-1}(t)}\mathrm{Ad}^*_{\eta(t)}K_{v(t)}\mathrm{Ad}_{\eta(t)}\mathrm{Ad}_{\eta^{-1}(t)}\mathrm{ad}_{v(t)}\\ &\quad -\mathrm{ad}^*_{v(t)}\mathrm{Ad}^*_{\eta^{-1}(t)}\mathrm{Ad}^*_{\eta(t)}K_{v(t)}\mathrm{Ad}_{\eta(t)}\mathrm{Ad}_{\eta^{-1}(t)},\end{aligned}$$

letting $X(t) = \mathrm{Ad}^*_{\eta(t)}K_{v(t)}\mathrm{Ad}_{\eta(t)}$, and using (24) and (25), we obtain

$$\partial_t X(t) = 0$$

and (29) follows.

Using Eq. (29) and letting $u(t) = \mathrm{Ad}_{\eta(t)^{-1}}Y(t)$, Eq. (28) becomes

$$\partial_t u(t) = \mathrm{Ad}_{\eta(t)^{-1}}\mathrm{Ad}^*_{\eta(t)^{-1}}w_o - \mathrm{Ad}_{\eta(t)^{-1}}\mathrm{Ad}^*_{\eta(t)^{-1}}K_{v_o}u(t). \tag{30}$$

Now suppose that $\eta(t) = \exp_e(tv_o)$ is a geodesic generated by a Killing field v_o. Then $\mathrm{Ad}^*_{\eta(t)^{-1}} = \mathrm{Ad}_{\eta(t)}$, by Lemma 3.1, and Eq. (30) reduces to

$$\partial_t u(t) = w_o - K_{v_o}u(t).$$

The solution to the homogeneous part is given by Stone's Theorem:

$$u(t) = \mathrm{e}^{-tK_{v_o}}u(0),$$

where $\mathrm{e}^{-tK_{v_o}}$ has the spectral representation

$$\mathrm{e}^{-tK_{v_o}} = \int_{\mathbb{R}} \mathrm{e}^{-it\lambda}\,\mathrm{d}E(\lambda)$$

and E is the unique spectral measure associated to the operator K_{v_o}. Therefore, the solution to the inhomogeneous equation is given by Duhamel's principle:

$$u(t) = \mathrm{e}^{-tK_{v_o}}u(0) + \int_0^t \mathrm{e}^{-(t-s)K_{v_o}}w_o\,\mathrm{d}s = \int_0^t \mathrm{e}^{-(t-s)K_{v_o}}w_o\,\mathrm{d}s$$

since $u(0) = \mathrm{Ad}_{\eta^{-1}(0)} Y(0) = Y(0) = 0$. Explicitly,

$$u(t) = \mathrm{e}^{-tK_{v_o}} \int_0^t \int_{\mathbb{R}} \mathrm{e}^{is\lambda}\, \mathrm{d}E(\lambda)\, \mathrm{d}s\, w_o = \mathrm{e}^{-tK_{v_o}} S(t) w_o.$$

For any $x \in T_e\mathcal{D}^0_\omega$, we have

$$\int_0^t \int_{\mathbb{R}} \left|\mathrm{e}^{is\lambda}\right| \mathrm{d}(E(\lambda)x, x)_{L^2}\, \mathrm{d}s = \int_0^t \int_{\mathbb{R}} \left|\mathrm{e}^{is\lambda}\right| \mathrm{d}E_x(\lambda) < \infty,$$

where $E_x(\lambda) = (E(\lambda)x, x)_{L^2}$ is a real, finite scalar measure. For any $y \in T_e\mathcal{D}^0_\omega$, there is a complex measure $E_{x,y}$ on $\mathfrak{B}(\mathbb{R})$: $E_{x,y}(\lambda) = (E(\lambda)x, y)_{L^2}$ which is a linear combination of four positive finite measures, by the polarization formula, each of which yields a finite integral as above. We deduce that the measure space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), E_{x,y})$ is finite and

$$\int_0^1 \int_{\mathbb{R}} |\mathrm{e}^{is\lambda}|\, \mathrm{d}\left\langle E(\lambda)x, y\right\rangle\, \mathrm{d}s = \int_0^1 \int_{\mathbb{R}} \left|\mathrm{e}^{is\lambda}\right| \mathrm{d}E_{x,y}(\lambda) < \infty.$$

Making use of the Fubini–Tonelli theorem, we have

$$\begin{aligned}(S(t)w_o, x)_{L^2} &= \int_0^1 \int_{\mathbb{R}} \mathrm{e}^{is\lambda}\, \mathrm{d}\left(E(\lambda)w_o, x\right)_{L^2}\, \mathrm{d}s = \int_{\mathbb{R}} \int_0^1 \mathrm{e}^{is\lambda}\, \mathrm{d}s\, \mathrm{d}\left(E(\lambda)w_o, x\right)_{L^2}\\ &= \int_{\mathbb{R}} \frac{\mathrm{e}^{i\lambda} - 1}{i\lambda}\, \mathrm{d}\left(E(\lambda)w_o, x\right)_{L^2} = \left(\int_{\mathbb{R}} \frac{\mathrm{e}^{i\lambda} - 1}{i\lambda}\, \mathrm{d}E(\lambda)w_o, x\right)_{L^2}.\end{aligned}$$

Since this relation holds for any $x \in T_e\mathcal{D}^0_\omega$ we deduce that

$$S(t)w_o = \int_{\mathbb{R}} \frac{\mathrm{e}^{i\lambda} - 1}{i\lambda}\, \mathrm{d}E(\lambda)w_o = \frac{\mathrm{e}^{tK_{v_o}} - I}{K_{v_o}}\, w_o.$$

Consequently, the Jacobi fields along $\eta(t)$ are given explicitly as

$$J(t) = D\eta(t) \cdot \frac{I - \mathrm{e}^{-tK_{v_o}}}{K_{v_o}}\, w_o.$$

□

Corollary 4.5 *Let $\eta(t)$ be a geodesic of the L^2 metric generated by a Killing field v_o on M. Then $\eta(t^*)$, $t^* > 0$, is conjugate to the identity if and only if $\frac{2\pi ik}{t^*}$ is an eigenvalue of K_{v_o} for some non-zero integer k. Consequently, the multiplicity of every conjugate point along $\eta(t)$ is even.*

Proof Let $\eta(t)$ be a geodesic of the L^2 metric, generated by a Killing field v on M. By the above construction, Jacobi fields along $\eta(t)$ have the form

$$J(t) = D\eta(t) \cdot \frac{I - \mathrm{e}^{-tK_{v_o}}}{K_{v_o}}\, w_o$$

where

$$J(0) = 0 \quad J'(0) = w_o.$$

A point $\eta(t^*)$ is conjugate to the identity if and only if there exists a Jacobi field $J(t)$, along $\eta(t)$, such that $J(t^*) = 0$. Since $D\eta(t)$ is an invertible linear operator, conjugate points are determined by the operator

$$\frac{I - e^{-t^* K_{v_o}}}{K_{v_o}} = \int_{\mathbb{R}} \frac{1 - e^{-it^*\lambda}}{i\lambda} \, dE(\lambda).$$

The spectrum of the operator $\frac{I-e^{-t^* K_{v_o}}}{K_{v_o}}$ is determined by the essential range of the function $f(\lambda) = \frac{1-e^{-it^*\lambda}}{i\lambda}$ [19]. In particular, 0 is an eigenvalue of $\frac{I-e^{-t^* K_{v_o}}}{K_{v_o}}$ if and only if $E\left(\left\{\lambda \in \mathbb{R} : \frac{1-e^{-it^*\lambda}}{i\lambda} = 0\right\}\right) \neq 0$. From the Taylor series of the function $f(\lambda) = \frac{1-e^{-it^*\lambda}}{i\lambda}$ we see that $f(0) = 1$. Therefore,

$$\left\{\lambda \in \mathbb{R} : \frac{1 - e^{-it^*\lambda}}{i\lambda} = 0\right\} = \{\lambda \in \mathbb{R} \setminus \{0\} : e^{-it^*\lambda} = 1\}.$$

This shows that 0 is an eigenvalue of $\frac{I-e^{-t^* K_{v_o}}}{K_{v_o}}$ if and only if 1 is an eigenvalue of $e^{-t^* K_{v_o}}$. Since the function $f(\lambda) = e^{-t\lambda}$ is an analytic function, the semigroup $e^{-tK_{v_o}}$ is an analytic semigroup. Applying the spectral mapping theorem (see [5]) to the operator $e^{-t^* K_{v_o}}$, we have that 1 is an eigenvalue of $e^{-t^* K_{v_o}}$ if and only if $\frac{2\pi i k}{t^*}$ is an eigenvalue of K_{v_o} for some non-zero integer k. We have the following chain of equalities

$$\begin{aligned} E\left(\left\{\lambda \in \mathbb{R} : \tfrac{1-e^{-it^*\lambda}}{i\lambda} = 0\right\}\right) &= E(\{\lambda \in \mathbb{R} \setminus \{0\} : e^{-it^*\lambda} = 1\}) = \sigma(e^{-t^* K_{v_o}}) \\ &= e^{-t^*\sigma(K_{v_o})} = E\left(\left\{\tfrac{2\pi i k}{t^*} : k \in \mathbb{Z} \setminus \{0\}\right\}\right) \end{aligned}$$

which shows that $\eta(t^*)$ is conjugate to the identity if and only if $\frac{2\pi i k}{t^*}$ is an eigenvalue of K_{v_o} for some non-zero integer k. By Lemma 4.3, the operator K_{v_o} is compact and skew self-adjoint in the L^2 metric and therefore has a complete set of orthonormal eigenvectors spanning $T_e\mathcal{D}^0_\omega$. As K_{v_o} is a map from H^σ vector fields to $H^{\sigma+1}$ vector fields, $s \geq \sigma + 1$, each eigenvector must be at least H^s and therefore in $T_e\mathcal{D}^s_\omega$. Since complex eigenvalues always occur in conjugate pairs, whose associated eigenvectors are orthonormal, the multiplicity of every conjugate point along $\eta(t)$ is even. □

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