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Half-flat structures inducing Einstein metrics on homogeneous spaces

Alberto Raffero

Abstract In this paper, we consider half-flat SU(3)-structures and the subclasses of coupled and double structures. In the general case, we show that the intrinsic torsion form w_1^- is constant in each of the two subclasses. We then consider the problem of finding half-flat structures inducing Einstein metrics on homogeneous spaces. We give an example of a left-invariant half-flat (non-coupled and non-double) structure inducing an Einstein metric on $S^3 \times S^3$ and we show there does not exist any left-invariant coupled structure inducing an $\mathrm{Ad}(S^1)$-invariant Einstein metric on it. Finally, we show that there are no coupled structures inducing the Einstein metric on Einstein solvmanifolds and on homogeneous Einstein manifolds of nonpositive sectional curvature.

Keywords Half-flat structures · Coupled structures · Einstein metrics · Homogeneous manifolds

Mathematics Subject Classification 53C10 · 53C25 · 53C30

1 Introduction

An SU(3)-structure on a six-dimensional smooth manifold N is the data of a Riemannian metric h, an orthogonal almost complex structure J, a 2-form ω related with h and J via the identity $\omega(\cdot, \cdot) = h(J\cdot, \cdot)$ and a (3, 0)-form Ψ of nonzero constant length. Using the results on stable forms [25,32], it can be shown that such a structure actually depends only on ω and $\psi_+ := \Re(\Psi)$.

The intrinsic torsion of an SU(3)-structure lies in a 42-dimensional space whose decomposition into SU(3)-irreducible summands allows to divide the SU(3)-structures in classes, which can be described by the exterior derivatives of ω, ψ_+ and $\psi_- := \Im(\Psi) = J^*\psi_+$

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as shown in [10] or in terms of a characterizing spinor and the spinorial field equations it satisfies as recently shown in [1].

When the intrinsic torsion vanishes, i.e., when ω, ψ_+ and ψ_- are all closed, the manifold has holonomy contained in SU(3) and the metric h is Ricci-flat.

When the forms ψ_+ and $\omega^2 = \omega \wedge \omega$ are both closed, the torsion lies in the 21-dimensional space $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ and the SU(3)-structure is called half-flat. Half-flat structures are the initial values for the Hitchin flow equations and are used as starting point to construct seven-dimensional manifolds with holonomy contained in G_2 (see for example [6,10,12,25]).

There are three classes of SU(3)-structures contained in the half-flat one which have been frequently considered in the literature: the nearly Kähler ($\mathcal{W}_1^-$), the double or co-coupled ($\mathcal{W}_1^- \oplus \mathcal{W}_3$) and the coupled ($\mathcal{W}_1^- \oplus \mathcal{W}_2^-$) (see [33]). It is well known [22,29] that the metric h induced by a nearly Kähler structure is always Einstein, i.e., the Ricci tensor of h satisfies the identity

$$\mathrm{Ric}(h) = \mu h$$

for some real number μ. Moreover, there exist some examples of half-flat structures with torsion class $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ and $\mathcal{W}_1^- \oplus \mathcal{W}_3$ inducing an Einstein metric, but up to now, it seems the coupled case has not been studied.

The almost Hermitian structure (h, J, ω) underlying a coupled SU(3)-structure is quasi-Kähler and depends only on ω. Thus, the properties of quasi-Kähler manifolds, e.g., the curvature identities shown by Gray in [21], are still valid on manifolds endowed with a coupled structure. The class of quasi-Kähler structures contains the almost Kähler one. It was conjectured by Goldberg in [19] that any compact almost Kähler manifold whose associated metric is Einstein is actually Kähler. In [36], Sekigawa showed that this conjecture is true when the scalar curvature is non-negative. Moreover, there exists a noncompact example of an Einstein almost Kähler manifold with negative scalar curvature [3], which is the unique example of six-dimensional Einstein almost Kähler (non-Kähler) solvmanifold by the results contained in [15,24].

In [16], invariant coupled structures on six-dimensional nilpotent Lie groups were classified and it was shown that there is only one case in which the coupled structure induces a Ricci soliton metric. Moreover, this coupled structure was used to construct a locally conformal calibrated G_2-structure inducing an Einstein metric on the rank one solvable extension of the nilpotent Lie algebra.

More in general, it is not difficult to show that on the cylinder and on the cone over a 6-manifold admitting a coupled structure there exists a locally conformal calibrated G_2-structure induced by the coupled one. On the other hand, it is possible to show that a parallel G_2-structure on a seven-dimensional manifold induces a coupled structure (h, J, ω, Ψ) on the oriented hypersurfaces having J-invariant second fundamental form [9].

One of the simplest cases which can be considered when one is looking for examples of special geometric structures with (or without) torsion is the one of left-invariant structures on homogeneous spaces, since in this case the starting analytic problem on the manifold (e.g., the problem of solving the PDEs deriving from the definition of half-flat structure) can often be reduced to an algebraic problem on the tangent space to a point. Following this idea, this paper focuses the attention to the case of left-invariant half-flat structures on six-dimensional homogeneous manifolds.

In [35], Schulte-Hengesbach considered half-flat structures on Lie groups and, in particular, he described left-invariant half-flat structures on $S^3 \times S^3$ and gave two examples of half-flat structures inducing an Einstein metric on it. One of these examples consists of a double structure and the other is the unique (up to homotheties and sign) left-invariant

nearly Kähler structure existing on this manifold, as showed by Butruille in [8]. Moreover, the Einstein metrics are the only two currently known examples of left-invariant Einstein metrics on $S^3 \times S^3$ and can be characterized as the only $\mathrm{Ad}(S^1)$-invariant Einstein metrics existing on it up to isometries and homotheties. It is then natural to ask whether there exist left-invariant half-flat structures (neither coupled nor double) and left-invariant coupled structures on $S^3 \times S^3$ inducing any of these metrics. We show that in the first case the answer is positive by giving an explicit example, while for the coupled case we prove that the answer is negative.

Left-invariant half-flat structures on $S^3 \times S^3$ were also studied by Madsen and Salamon in [28], where they described them using the representation theory of SO(4) and matrix algebra and showed that the moduli space they define is essentially a finite-dimensional symplectic quotient. In that paper they also considered the subclasses of coupled, co-coupled (double) and nearly Kähler structures, in particular they gave some examples of double structures, gave another proof of Butruille's result in their setting and constructed a one-parameter family of double structures which is a solution for the Hitchin flow.

Conversely to the case of the compact manifold $S^3 \times S^3$, in the case of noncompact homogeneous manifolds, it can be shown using the results of Heber [23] and Lauret [27] that every Einstein solvmanifold has a unique left-invariant Einstein metric up to isometries and homotheties. Einstein solvmanifolds constitute the unique example of noncompact homogeneous Einstein manifolds known up to now and it has been conjectured by Alekseevskii that they might be the only case that can occur [5, 7.57].

We then focus on six-dimensional Einstein solvmanifolds, which were classified by Nikitenko and Nikonorov in [30], and we look for left-invariant half-flat structures defined on them inducing the Einstein metric. In particular, the existence of a coupled structure inducing the Einstein metric would provide an example of a solvmanifold endowed with a left-invariant quasi-Kähler Einstein structure. However, we prove that in this case there are no coupled structures satisfying the property we are looking for. Moreover, using another result contained in [30], we are able to conclude that this happens for six-dimensional homogeneous Einstein manifolds of nonpositive sectional curvature too. This gives a constraint to the existence of coupled Einstein structures on six-dimensional homogeneous manifolds and leads to ask whether the non-positivity of the sectional curvature gives a constraint in a more general setting.

This paper is organized as follows: in Sect. 2, we recall the definition of SU(3)-structures and we focus on the half-flat class and its subclasses, in Sect. 3 we consider the case of left-invariant half-flat structures on $S^3 \times S^3$ and in Sect. 4, after recalling some properties of six-dimensional Einstein solvmanifolds, we study the case of left-invariant half-flat structures inducing Einstein metrics on them.

All the algebraic computations in Sects. 3 and 4 have been done with the aid of the software Maple.

2 Preliminaries on SU(3)-structures

Let N be a six-dimensional smooth manifold. It admits an SU(3)-structure if the structure group of its frame bundle can be reduced from $\mathrm{GL}(6, \mathbb{R})$ to SU(3). The existence of an SU(3)-structure is equivalent to the existence of an almost Hermitian structure (h, J, ω) and a $(3, 0)$-form Ψ of nonzero constant length such that the Riemannian metric h, the orthogonal almost complex structure J and the 2-form ω are related via the identity

$$\omega(\cdot, \cdot) = h(J\cdot, \cdot),$$

the forms ω, $\psi_+ := \Re(\Psi)$ and $\psi_- := \Im(\Psi) = J^*\psi_+$ are compatible in the sense that

$$\omega \wedge \psi_\pm = 0$$

and satisfy the normalization condition

$$\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3 = 4\mathrm{d}V_h,$$

where $\mathrm{d}V_h$ is the Riemannian volume form of h.

Using the results on stable forms contained in the works [25,32], it is possible to show that an SU(3)-structure actually depends only on ω and ψ_+ and thus it is possible to give another characterization for SU(3)-structures, which is the one we use in this work. To describe this characterization let us consider a six-dimensional real vector space V, we say that a differential k-form on V is stable if its orbit under the natural action of GL(V) on $\Lambda^k(V^*)$ is open. One can show that a 2-form σ on V is stable if and only if $\sigma^3 \neq 0$, that is, if and only if it is non-degenerate. Moreover, if we denote by $A : \Lambda^5(V^*) \to V \otimes \Lambda^6(V^*)$ the canonical isomorphism given by $A(\gamma) = v \otimes \Omega$, where $i_v\Omega = \gamma$, and define for a fixed $\rho \in \Lambda^3(V^*)$

$$K_\rho : V \to V \otimes \Lambda^6(V^*), \quad K_\rho(v) = A((i_v\rho) \wedge \rho)$$

and

$$\lambda : \Lambda^3(V^*) \to (\Lambda^6(V^*))^{\otimes 2}, \quad \lambda(\rho) = \frac{1}{6}\mathrm{tr}K_\rho^2,$$

we have that a 3-form ρ is stable if and only if $\lambda(\rho) \neq 0$. Whenever this happens, one can choose the orientation of V for which ω^3 is positively oriented and define a volume form by $\sqrt{|\lambda(\rho)|} \in \Lambda^6(V^*)$ and an endomorphism

$$J_\rho = -\frac{1}{\sqrt{|\lambda(\rho)|}}K_\rho,$$

which is an almost complex structure when $\lambda(\rho) < 0$.

The existence of an SU(3)-structure on N is equivalent to the existence of a pair of differential forms $(\omega, \psi_+) \in \Lambda^2(N) \times \Lambda^3(N)$ such that for each $p \in N$ the forms $\omega(p)$ and $\psi_+(p)$ on T_pN are stable with $\lambda(\psi_+(p)) < 0$, compatible, satisfy the normalization condition and define a Riemannian metric $h_p(\cdot, \cdot) = \omega_p(\cdot, J\cdot)$, where $J = J_{\psi_+(p)}$ is the almost complex structure induced by $\psi_+(p)$. The Riemannian metric h can also be described in terms of ω and ψ_+ as

$$h(X, Y)\omega^3 = -3(i_X\omega) \wedge (i_Y\psi_+) \wedge \psi_+,$$

for any $X, Y \in \mathfrak{X}(N)$.

The intrinsic torsion of an SU(3)-structure lies in a 42-dimensional space

$$\mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$$

given by the sum of irreducible SU(3)-modules and completely determined by $\mathrm{d}\omega$, $\mathrm{d}\psi_+$ and $\mathrm{d}\psi_-$ (see [10]).

In this paper, we are mainly interested in SU(3)-structures having both ω^2 and ψ_+ closed, known as half-flat SU(3)-structures in the literature. In this case, the intrinsic torsion lies in the space $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ and in terms of the exterior derivatives of ω, ψ_+ and ψ_- this reads:

$$\begin{aligned} \mathrm{d}\omega &= -\tfrac{3}{2}w_1^-\psi_+ + w_3, \\ \mathrm{d}\psi_+ &= 0, \\ \mathrm{d}\psi_- &= w_1^-\omega^2 - w_2^- \wedge \omega, \end{aligned} \tag{1}$$

where $w_1^- \in C^\infty(N) \cong \mathcal{W}_1^-, w_2^- \in \Lambda_0^{1,1}(N) \cong \mathcal{W}_2^-, w_3 \in \Lambda_0^{2,1}(N) \cong \mathcal{W}_3$ are the (non-vanishing) intrinsic torsion forms.

There are three interesting families of SU(3)-structures with nonzero torsion contained in the family of half-flat ones, we recall here the definitions.

Definition 2.1 A half-flat SU(3)-structure is said to be nearly Kähler if ∇J is skew-symmetric, that is $\nabla_X(J)(X) = 0$ for every $X \in \mathfrak{X}(N)$, coupled if $d\omega = c\psi_+$ for some non-vanishing $c \in C^\infty(N)$ and double (or co-coupled) if $d\psi_- = k\omega^2$ for some non-vanishing $k \in C^\infty(N)$.

It can be shown that the intrinsic torsion of a nearly Kähler structure lies in $\mathcal{W}_1^-$, while it follows easily from the definition that the intrinsic torsion of a coupled lies in $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$ and the intrinsic torsion of a double lies in $\mathcal{W}_1^- \oplus \mathcal{W}_3$. In each case, the expressions of $d\omega$, $d\psi_+$ and $d\psi_-$ can be obtained from (1) having in mind the torsion class to which each family belongs. Looking at these, it is easy to see that coupled and double structures can be thought as a generalization of the nearly Kähler. For each class, it is also possible to write the Ricci tensor and the scalar curvature of h in terms of the non-vanishing intrinsic torsion forms using the results of [7]. One can then recover that the metric induced by a nearly Kähler structure is Einstein and can observe that in the general case, this is not true anymore for the metric induced by a coupled or a double.

It is well known that in the nearly Kähler case the only non-vanishing torsion form w_1^- is constant, using the fact that the coupled and the double structures are in particular half-flat, we can prove that the same is true in these two cases.

Lemma 2.2 *Let N be a six-dimensional connected smooth manifold endowed with an* SU(3)-*structure* (ω, ψ_+). *If* (ω, ψ_+) *is coupled, then there exists a nonzero real constant c such that*

$$d\omega = c\psi_+,$$

if (ω, ψ_+) *is double, then there is a nonzero real constant k such that*

$$d\psi_- = k\omega^2.$$

Proof If (ω, ψ_+) is coupled, using the notations of (1) we know that

$$d\omega = -\frac{3}{2}w_1^-\psi_+,$$

where w_1^- is a smooth nonzero function. Taking the exterior derivative of $d\omega$, we obtain

$$0 = d(w_1^-\psi_+)$$
$$= dw_1^- \wedge \psi_+ + w_1^- d\psi_+,$$

so

$$d\psi_+ = -\frac{1}{w_1^-}dw_1^- \wedge \psi_+.$$

Now, $d(\omega^2) = 2d\omega \wedge \omega = 0$ since ω and ψ_+ are compatible, thus the considered class of SU(3)-structures is contained in the half-flat one if and only if

$$dw_1^- = 0,$$

that is, w_1^- is a nonzero real constant on N connected.

In the double case, we can argue in a similar way: starting from

$$d\psi_- = w_1^- \omega^2,$$

we take the exterior derivative of both sides obtaining

$$0 = dw_1^- \wedge \omega^2 + w_1^- d\omega^2$$

and conclude observing that $d\omega^2 = 2d\omega \wedge \omega = 0$ since $\omega \wedge \psi_+ = 0 = \omega \wedge w_3$ and that wedging 1-forms by ω^2 is an isomorphism.   □

Remark 2.3 The defining conditions of a half-flat structure $d\psi_+ = 0$ and $d\omega^2 = 0$ are obviously satisfied by the subclasses of coupled, double and nearly Kähler, thus to make distinction between these classes is necessary to look at the non-vanishing components of the intrinsic torsion (for example by computing $d\omega$ and $d\psi_-$). Moreover, we could sometimes emphasize the fact that we are considering half-flat structures with torsion in $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ by saying that the structure is half-flat non-coupled and non-double.

Recall that an almost Hermitian structure (h, J, ω) on a $2n$-dimensional manifold M is said to be quasi-Kähler if $\bar{\partial}\omega = (d\omega)^{1,2} = 0$ or, equivalently, if for any $X, Y \in \mathfrak{X}(M)$ it holds $\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$, where ∇ is the Levi Civita connection of h. Properties of quasi-Kähler manifolds have been studied in literature (see for example [20,21,37]) and in dimension six are inherited by manifolds endowed with a coupled structure. Indeed, we have

Proposition 2.4 *Let (ω, ψ_+) be a coupled $SU(3)$-structure on a six-dimensional connected smooth manifold N, then the underlying almost Hermitian structure (h, J, ω) is quasi-Kähler. Moreover, it depends only on ω.*

Proof As we already observed, we have that $d\omega = c\psi_+$ for some $c \in \mathbb{R} - \{0\}$. Thus, $d\omega$ is of type $(3, 0) + (0, 3)$ and $(d\omega)^{1,2} = 0$. Moreover, since J depends only on ψ_+, which is proportional to $d\omega$, the almost Hermitian structure (h, J, ω) depends only on ω.   □

3 Einstein half-flat structures on $S^3 \times S^3$

In this section, we focus on the homogeneous manifold $S^3 \times S^3$ and we consider left-invariant half-flat structures on it inducing Einstein metrics, looking for what happens in the subclasses.

As a Lie group, the manifold is $SU(2) \times SU(2)$ and we can describe the left-invariant tensors only working on the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In particular, a left-invariant metric on $S^3 \times S^3$ can be identified with an inner product on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and a left-invariant $SU(3)$-structure (ω, ψ_+) on $S^3 \times S^3$ can be identified with a 2-form ω and a 3-form ψ_+ defined on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ satisfying the defining properties of such a structure.

Let (e_1, e_2, e_3) denote the standard basis for the first copy of $\mathfrak{su}(2)$, (e_4, e_5, e_6) denote it for the second one and let (e^1, e^2, e^3) and (e^4, e^5, e^6) denote their dual bases. Then, we have the following structure equations for the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$:

$$de^1 = e^{23}, de^2 = e^{31}, de^3 = e^{12},$$
$$de^4 = e^{56}, de^5 = e^{64}, de^6 = e^{45},$$

where we are using the shortening $e^{ijk\cdots}$ for the wedge product $e^i \wedge e^j \wedge e^k \wedge \cdots$.

The problem of classifying left-invariant Einstein metrics on $S^3 \times S^3$ is still open and only two examples are known. These can be characterized as follows:

Theorem 3.1 [31] *Let h be a left-invariant Einstein metric on the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ which is $\mathrm{Ad}(S^1)$-invariant for some embedding $S^1 \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$, then h is isometric up to homotheties either to the standard metric or to Jensen's metric.*

With respect to the basis $(e_1, e_2, e_3, e_4, e_5, e_6)$ and up to scalar multiples, the matrix associated to the standard metric is the identity matrix and the one associated to the Jensen metric is the following

$$
\begin{bmatrix}
\frac{2\sqrt{3}}{3} & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 \\
0 & \frac{2\sqrt{3}}{3} & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 \\
0 & 0 & \frac{2\sqrt{3}}{3} & 0 & 0 & -\frac{\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} & 0 & 0 & \frac{2\sqrt{3}}{3} & 0 & 0 \\
0 & -\frac{\sqrt{3}}{3} & 0 & 0 & \frac{2\sqrt{3}}{3} & 0 \\
0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & \frac{2\sqrt{3}}{3}
\end{bmatrix}. \tag{2}
$$

In [35], Schulte-Hengesbach gave an example of a left-invariant half-flat structure on $S^3 \times S^3$ inducing the standard metric and of one inducing the Jensen metric, we describe them in the following examples.

Example 3.2 [35] The pair of stable forms

$$
\begin{aligned}
\omega &= -e^{14} - e^{25} - e^{36}, \\
\psi_+ &= \frac{1}{\sqrt{2}} \left(e^{123} - e^{156} + e^{246} - e^{345} + e^{126} - e^{135} + e^{234} - e^{456} \right),
\end{aligned}
$$

is compatible, normalized and induces the standard metric, thus it defines an $\mathrm{SU}(3)$-structure on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Moreover, $\mathrm{d}\psi_+ = 0$, $\mathrm{d}\omega^2 = 0$, $\mathrm{d}\psi_- = \frac{1}{\sqrt{2}}\omega^2$ and $\mathrm{d}\omega$ is not proportional to ψ_+, i.e., it is a double structure.

Example 3.3 [35] The following pair of stable, compatible, normalized forms

$$
\begin{aligned}
\omega &= -\frac{\sqrt{3}}{18} \left(e^{14} + e^{25} + e^{36} \right), \\
\psi_+ &= \frac{\sqrt{3}}{54} \left(-e^{234} + e^{156} + e^{135} - e^{246} - e^{126} + e^{345} \right),
\end{aligned}
$$

induces the Jensen metric, therefore, it defines an $\mathrm{SU}(3)$-structure on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ which is nearly Kähler since $\mathrm{d}\omega = 3\psi_+$ and $\mathrm{d}\psi_- = -2\omega^2$.

Remark 3.4 The signs of the differential forms in the previous examples are different from those in the original examples of [35], this is due to the fact that in this paper we are using a convention in the definition of an $\mathrm{SU}(3)$-structure which is slightly different from the one used by the other author in his works.

We can also give a new example of a left-invariant half-flat structure (non-coupled and non-double) inducing the Jensen metric:

Example 3.5 The pair

$$
\begin{aligned}
\omega &= \frac{\sqrt[3]{4}\sqrt[6]{3}}{2} \left(-e^{14} + e^{25} + e^{36} \right), \\
\psi_+ &= e^{123} + e^{135} - e^{246} - e^{126} + e^{345} - e^{456},
\end{aligned}
$$

defines an $\mathrm{SU}(3)$-structure on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and induces a metric which is proportional to the Jensen metric. Moreover, it can be checked that this $\mathrm{SU}(3)$-structure is half-flat since both ψ_+ and ω^2 are closed and it is neither coupled nor double since $\mathrm{d}\omega$ is not proportional to ψ_+ and $\mathrm{d}\psi_-$ is not proportional to ω^2.

Summarizing, on $S^3 \times S^3$ there exist left-invariant half-flat and nearly Kähler structures inducing the Jensen metric and left-invariant double structures inducing the standard metric. We prove now that it is not possible to find left-invariant coupled structures on $S^3 \times S^3$ inducing either the standard or the Jensen metric. In the proof, we use some classical properties of algebraic varieties, the reader can find them for example in [13].

Theorem 3.6 $S^3 \times S^3$ *does not admit left-invariant coupled* SU(3)*-structures* (ω, ψ_+) *inducing an* Ad(S^1)*-invariant Einstein metric.*

Proof Let us consider a left-invariant coupled structure (ω, ψ_+) on $S^3 \times S^3$ which we identify with a 2-form ω and a 3-form ψ_+ defined on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and such that $d\omega = \frac{1}{c}\psi_+$, $c \in \mathbb{R} - \{0\}$. Since ω^2 is closed, it follows that $\omega \in \mathfrak{su}^*(2) \otimes \mathfrak{su}^*(2)$ (cf. [35], Lemma 1.1 p. 81), thus

$$\omega = a_{14}e^{14} + a_{15}e^{15} + a_{16}e^{16} + a_{24}e^{24} + a_{25}e^{25} + a_{26}e^{26} + a_{34}e^{34} + a_{35}e^{35} + a_{36}e^{36},$$

where a_{ij} are real coefficients. Imposing the coupled condition $\psi_+ = c\,d\omega$, $c \neq 0$, we have that

$$\begin{aligned}
\psi_+ = c(&a_{14}e^{234} - a_{14}e^{156} + a_{15}e^{235} + a_{15}e^{146} + a_{16}e^{236} - a_{16}e^{145} - a_{24}e^{134} - a_{24}e^{256} \\
&- a_{25}e^{135} + a_{25}e^{246} - a_{26}e^{136} - a_{26}e^{245} + a_{34}e^{124} - a_{34}e^{356} + a_{35}e^{125} + a_{35}e^{346} \\
&+ a_{36}e^{126} - a_{36}e^{345})
\end{aligned}$$

and from the closedness of ω^2 we know that the compatibility condition $\omega \wedge \psi_+ = 0$ holds. It is now possible to compute $\lambda = \lambda(\psi_+)$, which turns out to be a homogeneous polynomial of degree 4 in the coefficients a_{ij}, the almost complex structure $J = J_{\psi_+}$ and $h(\cdot, \cdot) = \omega(\cdot, J\cdot)$. With respect to the basis $(e_1, \ldots, e_6)$, the matrix H associated to h is symmetric. Moreover, up to a global sign depending on whether the considered basis is positively oriented or not and not affecting the computations afterwards, the nonzero entries are the following:

$$\begin{aligned}
H_{i,i} = &\frac{-2c^2}{\sqrt{-\lambda}}(a_{14}a_{25}a_{36} - a_{14}a_{26}a_{35} - a_{15}a_{24}a_{36} + a_{15}a_{26}a_{34} + a_{16}a_{24}a_{35} - a_{16}a_{25}a_{34}), \\
&i = 1, \ldots, 6
\end{aligned}$$

$$\begin{aligned}
H_{1,4} = &\frac{-c^2}{\sqrt{-\lambda}}(a_{14}^3 + a_{14}a_{15}^2 + a_{14}a_{16}^2 + a_{14}a_{24}^2 - a_{14}a_{25}^2 - a_{14}a_{26}^2 + a_{14}a_{34}^2 - a_{14}a_{35}^2 - a_{14}a_{36}^2 \\
&+ 2a_{15}a_{24}a_{25} + 2a_{15}a_{34}a_{35} + 2a_{16}a_{24}a_{26} + 2a_{16}a_{34}a_{36}),
\end{aligned}$$

$$\begin{aligned}
H_{1,5} = &\frac{-c^2}{\sqrt{-\lambda}}(a_{14}^2a_{15} + 2a_{14}a_{24}a_{25} + 2a_{14}a_{34}a_{35} + a_{15}^3 + a_{15}a_{16}^2 - a_{15}a_{24}^2 + a_{15}a_{25}^2 - a_{15}a_{26}^2 \\
&- a_{15}a_{34}^2 + a_{15}a_{35}^2 - a_{15}a_{36}^2 + 2a_{16}a_{25}a_{26} + 2a_{16}a_{35}a_{36}),
\end{aligned}$$

$$\begin{aligned}
H_{1,6} = &\frac{-c^2}{\sqrt{-\lambda}}(a_{14}^2a_{16} + 2a_{14}a_{24}a_{26} + 2a_{14}a_{34}a_{36} + a_{15}^2a_{16} + 2a_{15}a_{25}a_{26} + 2a_{15}a_{35}a_{36} + a_{16}^3 \\
&- a_{16}a_{24}^2 - a_{16}a_{25}^2 + a_{16}a_{26}^2 - a_{16}a_{34}^2 - a_{16}a_{35}^2 + a_{16}a_{36}^2),
\end{aligned}$$

$$\begin{aligned}
H_{2,4} = &\frac{-c^2}{\sqrt{-\lambda}}(a_{14}^2a_{24} + 2a_{14}a_{15}a_{25} + 2a_{14}a_{16}a_{26} - a_{15}^2a_{24} - a_{16}^2a_{24} + a_{24}^3 + a_{24}a_{25}^2 + a_{24}a_{26}^2 \\
&+ a_{24}a_{34}^2 - a_{24}a_{35}^2 - a_{24}a_{36}^2 + 2a_{25}a_{34}a_{35} + 2a_{26}a_{34}a_{36}),
\end{aligned}$$

$$H_{2,5} = \frac{c^2}{\sqrt{-\lambda}}(a_{14}^2 a_{25} - 2a_{14}a_{15}a_{24} - a_{15}^2 a_{25} - 2a_{15}a_{16}a_{26} + a_{16}^2 a_{25} - a_{24}^2 a_{25} - 2a_{24}a_{34}a_{35}$$
$$- a_{25}^3 - a_{25}a_{26}^2 + a_{25}a_{34}^2 - a_{25}a_{35}^2 + a_{25}a_{36}^2 - 2a_{26}a_{35}a_{36}),$$

$$H_{2,6} = \frac{c^2}{\sqrt{-\lambda}}(a_{14}^2 a_{26} - 2a_{14}a_{16}a_{24} + a_{15}^2 a_{26} - 2a_{15}a_{16}a_{25} - a_{16}^2 a_{26} - a_{24}^2 a_{26} - 2a_{24}a_{34}a_{36}$$
$$- a_{25}^2 a_{26} - 2a_{25}a_{35}a_{36} - a_{26}^3 + a_{26}a_{34}^2 + a_{26}a_{35}^2 - a_{26}a_{36}^2),$$

$$H_{3,4} = \frac{-c^2}{\sqrt{-\lambda}}(a_{14}^2 a_{34} + 2a_{14}a_{15}a_{35} + 2a_{14}a_{16}a_{36} - a_{15}^2 a_{34} - a_{16}^2 a_{34} + a_{24}^2 a_{34} + 2a_{24}a_{25}a_{35}$$
$$+ 2a_{24}a_{26}a_{36} - a_{25}^2 a_{34} - a_{26}^2 a_{34} + a_{34}^3 + a_{34}a_{35}^2 + a_{34}a_{36}^2),$$

$$H_{3,5} = \frac{c^2}{\sqrt{-\lambda}}(a_{14}^2 a_{35} - 2a_{14}a_{15}a_{34} - a_{15}^2 a_{35} - 2a_{15}a_{16}a_{36} + a_{16}^2 a_{35} + a_{24}^2 a_{35} - 2a_{24}a_{25}a_{34}$$
$$- a_{25}^2 a_{35} - 2a_{25}a_{26}a_{36} + a_{26}^2 a_{35} - a_{34}^2 a_{35} - a_{35}^3 - a_{35}a_{36}^2),$$

$$H_{3,6} = \frac{c^2}{\sqrt{-\lambda}}(a_{14}^2 a_{36} - 2a_{14}a_{16}a_{34} + a_{15}^2 a_{36} - 2a_{15}a_{16}a_{35} - a_{16}^2 a_{36} + a_{24}^2 a_{36} - 2a_{24}a_{26}a_{34}$$
$$+ a_{25}^2 a_{36} - 2a_{25}a_{26}a_{35} - a_{26}^2 a_{36} - a_{34}^2 a_{36} - a_{35}^2 a_{36} - a_{36}^3),$$

where $H_{i,j} = h(e_i, e_j)$. Observe that up to multiplication by $\sqrt{-\lambda}$, the nonzero terms are all homogeneous polynomials of third degree in the a_{ij}.

We are looking for coupled structures inducing either the standard metric or the Jensen metric which with respect to the considered basis can be written as the identity matrix and as (2), respectively. Thus, since $\omega \wedge \psi_+ = 0$, $d\psi_+ = 0$ and $d\omega^2 = 0$, we first have to solve the system obtained by imposing that the matrix H is proportional to the identity matrix or to the matrix (2) under the assumption $\lambda < 0$ and then, if we find solutions of this system, we need to impose that the normalization condition is satisfied to obtain what we want.

Case 1: The standard metric

Since rescaling a metric with a positive coefficient does not change the Ricci tensor, we are looking for solutions of the equation

$$H = \alpha I,$$

where α is a positive real number.

Since the entries in the diagonal of H are all equal, we only have to solve the system of equations

$$H_{i,j} = 0, \quad i = 1, 2, 3, \quad j = 4, 5, 6$$

under the assumptions $H_{1,1} \neq 0$ and $\lambda < 0$.

For $i, j = 1, \ldots, 6$, we let

$$\tilde{H}_{i,j} := \sqrt{-\lambda}H_{i,j}.$$

Then, as already observed, the $\tilde{H}_{i,j}$ are homogeneous polynomials of degree 3 in the a_{ij} and under our assumptions $H_{i,j} = 0$ if and only if $\tilde{H}_{i,j} = 0$ for $i = 1, 2, 3$, $j = 4, 5, 6$.

Since we have a system of equations involving homogeneous polynomials of the same degree and we are looking for solutions defined up to a multiplicative constant, let us consider the projective space $\mathbb{CP}^8$ with coordinate ring

$$\mathbb{C}[a_{14}, a_{15}, a_{16}, a_{24}, a_{25}, a_{26}, a_{34}, a_{35}, a_{36}]$$

and the homogeneous ideals

$$P := \langle \tilde{H}_{1,1} \rangle,$$
$$Q := \langle \tilde{H}_{1,4}, \tilde{H}_{2,4}, \tilde{H}_{3,4}, \tilde{H}_{1,5}, \tilde{H}_{2,5}, \tilde{H}_{3,5}, \tilde{H}_{1,6}, \tilde{H}_{2,6}, \tilde{H}_{3,6} \rangle.$$

What we are looking for is the set of points $[a_{14} : \ldots : a_{36}]$ lying in the projective variety $V(Q)$, but not in $V(P)$ and for which $\lambda < 0$. It is known that

$$\overline{V(Q) - V(P)} \subseteq V(Q : P),$$

where $Q : P$ is the ideal quotient of Q by P. In our case, it is possible to show that $Q : P = \langle a_{14}, a_{15}, a_{16}, a_{24}, a_{25}, a_{26}, a_{34}, a_{35}, a_{36} \rangle$, therefore, $V(Q : P) = \emptyset$.

We have then proved that on $S^3 \times S^3$ there are no invariant coupled structures inducing the standard metric.

Case 2: The Jensen metric

Following the same idea of the previous case and looking at the entries of the matrix (2), we have now to consider the ideals P and

$$R := \langle \tilde{H}_{1,5}, \tilde{H}_{1,6}, \tilde{H}_{2,4}, \tilde{H}_{2,6}, \tilde{H}_{3,4}, \tilde{H}_{3,5}, \tilde{H}_{2,5} - \tilde{H}_{3,6}, \tilde{H}_{3,6} - \tilde{H}_{1,4}, \tilde{H}_{1,1} + 2\tilde{H}_{1,4} \rangle$$

and look for those points lying in the projective variety $V(R)$, but not in $V(P)$ and for which $\lambda < 0$. Now

$$R : P = \langle a_{15}, a_{16}, a_{24}, a_{26}, a_{34}, a_{35}, a_{25} - a_{14}, a_{36} - a_{14} \rangle,$$

then

$$V(R : P) = \{[\gamma : 0 : 0 : 0 : \gamma : 0 : 0 : 0 : \gamma] : \gamma \in \mathbb{C} - \{0\}\}$$

is a point in $\mathbb{CP}^8$ and since $\mathbb{C}$ is algebraically closed and R is a radical ideal

$$V(R : P) = \overline{V(R) - V(P)}.$$

Moreover, the requested condition on λ is satisfied, indeed:

$$\lambda = -3c^4\gamma^4 < 0.$$

The coupled structures we are interested in are obtained when γ is a negative real number, in this case we have:

$$\omega = \gamma(e^{14} + e^{25} + e^{36}),$$
$$\psi_+ = c\gamma(e^{234} - e^{156} - e^{135} + e^{246} + e^{126} - e^{345}),$$
$$\psi_- = \frac{c\gamma}{\sqrt{3}}(2e^{123} - e^{126} + e^{135} - e^{156} - e^{234} + e^{246} - e^{345} + 2e^{456}).$$

The forms ω and ψ_+ are stable and the normalization condition implies

$$c = \pm\sqrt{\frac{-2\gamma}{\sqrt{3}}},$$

in both cases we have a nearly Kähler structure.   □

Anyway, it is not difficult to find examples of left-invariant coupled structures on $S^3 \times S^3$:

Example 3.7 The pair

$$\omega = -\sqrt{3}e^{16} - e^{24} - e^{25} - e^{35},$$
$$\psi_+ = \sqrt[4]{3}(-\sqrt{3}e^{236} + \sqrt{3}e^{145} + e^{134} + e^{256} + e^{135} - e^{246} - e^{125} - e^{346}),$$

defines a coupled structure on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ such that $\psi_+ = \sqrt[4]{3}\mathrm{d}\omega$. In particular, we have that (h, J, ω), where $J = J_{\psi_+}$, defines a quasi-Kähler structure on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

4 Einstein half-flat structures on 6-solvmanifolds

In Sect. 3, we saw that on the compact manifold $S^3 \times S^3$ there are examples of left-invariant half-flat, double and nearly Kähler structures inducing an Einstein metric and that there are no left-invariant coupled structures inducing any of the Einstein metrics known up to now. In this section, we turn our attention to the noncompact homogeneous case.

It is well known that noncompact homogeneous Einstein manifolds have nonpositive Ricci curvature. Moreover, if the Ricci curvature is zero, then the Riemannian metric is flat [2] and the manifold is isometric to the product of a flat torus and the Euclidean space. Up to now the only known examples of noncompact homogeneous Einstein manifolds are Einstein solvmanifolds, that is simply connected solvable Lie groups endowed with a left-invariant Einstein metric. It has been conjectured by Alekseevskii that any noncompact homogeneous Einstein manifold might be of this kind (see [5] for the statement of the conjecture and refer to the recent works [4,26] and the references therein for more details and the most recent results on it).

Let (S, h) be a solvmanifold, we can identify the left-invariant metric h on the simply connected solvable Lie group S with the inner product h_0 determined by it on the solvable Lie algebra $\mathfrak{s}$ of S, the pair $(\mathfrak{s}, h_0)$ is said a metric solvable Lie algebra. Two metric Lie algebras $(\mathfrak{s}, h_0)$ and $(\mathfrak{s}', h_0')$ are isomorphic if there exists a Lie algebra isomorphism $F : \mathfrak{s} \to \mathfrak{s}'$ which is also an isometry of Euclidean spaces.

In [27], Lauret showed that any Einstein solvmanifold is standard, i.e., the orthogonal complement $\mathfrak{a}$ to $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is always an abelian subalgebra of $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$. The dimension of $\mathfrak{a}$ is the algebraic rank of $\mathfrak{s}$.

The properties of standard Einstein solvmanifolds were studied by Heber in [23]. In particular, he showed that a standard Einstein metric is unique up to isometries and scaling among left-invariant metrics and that, up to metric Lie algebra isomorphisms, a standard metric Lie algebra with Einstein inner product is an Iwasawa-type algebra.

It was proved by Dotti in [14] that solvmanifolds (S, h) with unimodular solvable Lie group S and Einstein metric h are flat.

The six-dimensional Einstein solvmanifolds were classified by Nikitenko and Nikonorov in [30]. The result is recalled in the next Theorem. Instead of the Lie algebra structure equations given (in the original formulation of the Theorem) by the nontrivial Lie brackets of the basis vectors, we write here the structure equations in terms of the Chevalley–Eilenberg differential of the basis 1-forms, since we will use these in our next computations.

Theorem 4.1 [30] *Let $(\mathfrak{s}, h)$ be a six-dimensional nonunimodular metric solvable Lie algebra with Einstein inner product h such that $\mathrm{Ric}(h) = -r^2 h$, where $r > 0$. Then, $(\mathfrak{s}, h)$ is isomorphic to one of the metric Lie algebras contained in Table 1. For each algebra, $(e_1, \ldots, e_6)$ is an h-orthonormal basis with dual basis $(e^1, \ldots, e^6)$.*

Table 1 Six-dimensional nonunimodular metric solvable Lie algebras with Einstein inner product $h = \sum_{i=1}^{6}(e^i)^2$

$\mathfrak{s}.$	Structure equations (de^1, de^2, de^3, de^4, de^5, de^6)
$\mathfrak{s}_1$	$\left(\frac{r}{2\sqrt{2}}e^{16}, \frac{r}{2\sqrt{2}}e^{26}, \frac{r}{2\sqrt{2}}e^{36}, \frac{r}{2\sqrt{2}}e^{46}, -\frac{r}{\sqrt{2}}e^{12} - \frac{r}{\sqrt{2}}e^{34} + \frac{r}{\sqrt{2}}e^{56}, 0 \right)$
$\mathfrak{s}_2$	$\left(2r\sqrt{\frac{2}{105}}e^{16}, r\sqrt{\frac{3}{70}}e^{26}, -\frac{2r}{\sqrt{7}}e^{12} + r\sqrt{\frac{7}{30}}e^{36}, 2r\sqrt{\frac{3}{70}}e^{46}, -r\sqrt{\frac{2}{7}}e^{14} - \frac{2r}{\sqrt{7}}e^{23} + r\sqrt{\frac{10}{21}}e^{56}, 0 \right)$
$\mathfrak{s}_3$	$\left(\frac{r}{\sqrt{55}}e^{16}, \frac{2r}{\sqrt{55}}e^{26}, -r\sqrt{\frac{6}{11}}e^{12} + \frac{3r}{\sqrt{55}}e^{36}, -r\sqrt{\frac{6}{11}}e^{13} + \frac{4r}{\sqrt{55}}e^{46}, -\frac{2r}{\sqrt{11}}e^{14} - \frac{2r}{\sqrt{11}}e^{23} + \frac{5r}{\sqrt{55}}e^{56}, 0 \right)$
$\mathfrak{s}_4$	$\left(\frac{r\sqrt{6}}{30}e^{16}, \frac{3r\sqrt{6}}{20}e^{26}, -\frac{r}{\sqrt{2}}e^{12} + \frac{11r\sqrt{6}}{60}e^{36}, -r\sqrt{\frac{2}{3}}e^{13} + \frac{13r\sqrt{6}}{60}e^{46}, -\frac{r}{\sqrt{2}}e^{14} + \frac{r\sqrt{6}}{4}e^{56}, 0 \right)$
$\mathfrak{s}_5$	$\left(\frac{r}{3\sqrt{2}}e^{16}, \frac{r}{2\sqrt{2}}e^{26}, \frac{r}{2\sqrt{2}}e^{36}, -\frac{r}{\sqrt{2}}e^{12} + \frac{5r}{6\sqrt{2}}e^{46}, -\frac{r}{\sqrt{2}}e^{13} + \frac{5r}{6\sqrt{2}}e^{56}, 0 \right)$
$\mathfrak{s}_6$	$\left(\frac{r}{2\sqrt{6}}e^{16}, \frac{r}{2\sqrt{6}}e^{26}, -r\sqrt{\frac{2}{3}}e^{12} + \frac{r}{\sqrt{6}}e^{36}, -\frac{r}{\sqrt{2}}e^{13} + r\frac{\sqrt{6}}{4}e^{46}, -\frac{r}{\sqrt{2}}e^{23} + r\frac{\sqrt{6}}{4}e^{56}, 0 \right)$
$\mathfrak{s}_7$	$\left(\frac{r}{\sqrt{39}}e^{16}, \frac{2r}{\sqrt{39}}e^{26}, -r\sqrt{\frac{2}{3}}e^{12} + \frac{3r}{\sqrt{39}}e^{36}, -r\sqrt{\frac{2}{3}}e^{13} + \frac{4r}{\sqrt{39}}e^{46}, \frac{3r}{\sqrt{39}}e^{56}, 0 \right)$
$\mathfrak{s}_8$	$\left(r\sqrt{\frac{2}{21}}e^{16}, r\sqrt{\frac{2}{21}}e^{26}, -r\sqrt{\frac{2}{3}}e^{12} + 2r\sqrt{\frac{2}{21}}e^{36}, r\sqrt{\frac{3}{14}}e^{46}, r\sqrt{\frac{3}{14}}e^{56}, 0 \right)$
$\mathfrak{s}_9$	$\left(\frac{r}{\sqrt{5}}e^{16}, \frac{r}{\sqrt{5}}e^{26}, \frac{r}{\sqrt{5}}e^{36}, \frac{r}{\sqrt{5}}e^{46}, \frac{r}{\sqrt{5}}e^{56}, 0 \right)$
$\mathfrak{s}_{10}$	$\left(\frac{2r}{\sqrt{33}}e^{15} + rte^{16} + r\sqrt{\frac{1}{2} - 11t^2}e^{26}, \frac{2r}{\sqrt{33}}e^{25} + r\sqrt{\frac{1}{2} - 11t^2}e^{16} + rte^{26}, -r\sqrt{\frac{2}{3}}e^{12} + \frac{4r}{\sqrt{33}}e^{35} \right.$ $\left. + 2rte^{36}, \frac{3r}{\sqrt{33}}e^{45} - 4rte^{46}, 0, 0 \right)$
$\mathfrak{s}_{11}$	$\left(\frac{r}{\sqrt{30}}e^{15} + \frac{3r}{\sqrt{30}}e^{16}, \frac{2r}{\sqrt{30}}e^{25} - \frac{4r}{\sqrt{30}}e^{26}, -r\sqrt{\frac{2}{3}}e^{12} + \frac{3r}{\sqrt{30}}e^{35} - \frac{r}{\sqrt{30}}e^{36}, -r\sqrt{\frac{2}{3}}e^{13} + \frac{4r}{\sqrt{30}}e^{45} + \frac{2r}{\sqrt{30}}e^{46}, 0, 0 \right)$
$\mathfrak{s}_{12}$	$\left(\frac{r}{2}e^{15} + r\frac{1+s+t}{2\sqrt{1+t^2+s^2}}e^{16}, \frac{r}{2}e^{25} + r\frac{1-s-t}{2\sqrt{1+t^2+s^2}}e^{26}, \frac{r}{2}e^{35} + r\frac{t-s-1}{2\sqrt{1+t^2+s^2}}e^{36}, \frac{r}{2}e^{45} + r\frac{s-t-1}{2\sqrt{1+t^2+s^2}}e^{46}, 0, 0 \right)$
$\mathfrak{s}_{13}$	$\left(\frac{r}{\sqrt{3}}e^{14} - \frac{2r}{\sqrt{6}}e^{16}, \frac{r}{\sqrt{3}}e^{24} + \frac{r}{\sqrt{2}}e^{25} + \frac{r}{\sqrt{6}}e^{26}, \frac{r}{\sqrt{3}}e^{34} - \frac{r}{\sqrt{2}}e^{35} + \frac{r}{\sqrt{6}}e^{36}, 0, 0, 0 \right)$

The Lie algebra $\mathfrak{s}_{10}$ depends on a parameter $0 \leq t \leq \frac{1}{\sqrt{22}}$ and $\mathfrak{s}_{12}$ depends on two parameters $0 \leq s \leq t \leq 1$

Remark 4.2 All the solvable metric Lie algebras appearing in Table 1 are of Iwasawa type. It is worth emphasizing here that for each of the $\mathfrak{s}_i$, the inner product, with respect to which the basis $(e_1, \ldots, e_6)$ is orthonormal, is the only one having the property of being Einstein. This follows from the result of Heber and the fact that two solvable metric Lie algebras of Iwasawa type are isomorphic if and only if the corresponding solvmanifolds are isometric as Riemannian manifolds.

As a consequence of the previous Theorem, for the six-dimensional homogeneous Einstein manifolds of nonpositive sectional curvature they showed what follows.

Theorem 4.3 [30] *Let (M, h) be a six-dimensional, connected, simply connected, homogeneous Einstein manifold of nonpositive sectional curvature, then it is symmetric or isometric to one of the solvmanifolds of negative sectional curvature generated by the metric Lie algebras $\mathfrak{s}_5$, $\mathfrak{s}_8$. Moreover, in the symmetric case (M, h) is obtained as the solvmanifold corresponding to the metric Lie algebras $\mathfrak{s}_1$, $\mathfrak{s}_9$, $\mathfrak{s}_{10}$ for $t = \frac{1}{\sqrt{22}}$, $\mathfrak{s}_{11}$, $\mathfrak{s}_{12}$ for $(s, t) = (0, 0)$ and $(s, t) = (1, 1)$ and $\mathfrak{s}_{13}$.*

Now, we focus on the problem of finding left-invariant half-flat structures on six-dimensional Einstein solvmanifolds inducing the Einstein (non-Ricci-flat) metric, these are in 1–1 correspondence with half-flat structures inducing the Einstein inner product on six-dimensional nonunimodular solvable metric Lie algebras.

It is worth recalling here an obstruction to the existence of half-flat structures on six-dimensional Lie algebras shown by Freibert and Schulte-Hengesbach in [17]. This result is a refinement of the one obtained by Conti in [11].

Proposition 4.4 [17] *Let $\mathfrak{g}$ be a six-dimensional Lie algebra with volume form $\Omega \in \Lambda^6(\mathfrak{g}^*)$. If there exists a nonzero $\alpha \in \mathfrak{g}^*$ such that*

$$\alpha \wedge \tilde{J}_\rho^* \alpha \wedge \sigma = 0$$

for all closed 3-forms $\rho \in \Lambda^3(\mathfrak{g}^*)$ and closed 4-forms $\sigma \in \Lambda^4(\mathfrak{g}^*)$, where for any $X \in \mathfrak{g}$

$$\tilde{J}_\rho^* \alpha(X)\Omega = \alpha \wedge (i_X \rho) \wedge \rho,$$

then $\mathfrak{g}$ does not admit any half-flat SU(3)-structure.

In [17,18], the authors completely classified the left-invariant half-flat structures on six-dimensional decomposable Lie groups (using also the classification contained in [34]) and on six-dimensional indecomposable Lie groups with five-dimensional nilradical. These classifications will be useful in the proof of the following

Theorem 4.5 *There are no half-flat SU(3)-structures inducing the Einstein metric on the rank 1 solvable metric Lie algebras $\mathfrak{s}_i$, $i = 1 \ldots 9$, and on the rank 2 solvable metric Lie algebra $\mathfrak{s}_{12}$ and there are no coupled SU(3)-structures inducing the Einstein metric on the rank 2 metric Lie algebras $\mathfrak{s}_{10}$, $\mathfrak{s}_{11}$ and on the rank 3 metric Lie algebra $\mathfrak{s}_{13}$.*

Proof We will prove the theorem as follows: in the list of Einstein solvable metric Lie algebras, we first exclude the ones that do not admit a half-flat structure using the results of [17,18], then we will show the result by direct computations in the remaining cases.

The rank 1 Lie algebra $\mathfrak{s}_9$ is indecomposable and has abelian nilradical, therefore, it does not admit any half-flat structure by Proposition 4 of [18]. By Theorem 2 of [18], we have that the Lie algebras $\mathfrak{s}_i$ with $i = 1, 2, 4, 5, 7, 8$ do not admit any half-flat structure since they are isomorphic to the Lie algebras $A_{6,82}^{1,0,0}$, $A_{6,94}^{4/3}$, $A_{6,71}^{9/2}$, $A_{6,54}^{2/5,1}$, $A_{6,39}^{3,2}$, $A_{6,13}^{2/3,2/3,1}$, respectively, whereas the Lie algebras $\mathfrak{s}_3$, $\mathfrak{s}_6$ admit a half-flat structure since the former is isomorphic to $A_{6,99}$ and the latter to $A_{6,76}^1$. By Theorem 1 of [17], we also have that on $\mathfrak{s}_{13}$ there exist half-flat structures since it is isomorphic to $\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$. Finally, applying Proposition 4.4 to $\mathfrak{s}_{12}$ with $\alpha = e^6$ we obtain that this two-parameter family of Lie algebras does not admit any half-flat structure.

We can now start with the second part of the proof. For $k = 3, 6$, let $\omega \in \Lambda^2(\mathfrak{s}_k^*)$ and $\psi_+ \in \Lambda^3(\mathfrak{s}_k^*)$ be generic forms, with respect to the basis $(e^1, \ldots, e^6)$ given in Theorem 4.1 we can write

$$\begin{aligned} \omega = {} & b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{15} + b_5 e^{16} + b_6 e^{23} + b_7 e^{24} + b_8 e^{25} \\ & + b_9 e^{26} + b_{10} e^{34} + b_{11} e^{35} + b_{12} e^{36} + b_{13} e^{45} + b_{14} e^{46} + b_{15} e^{56} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \psi_+ = {} & a_1 e^{123} + a_2 e^{124} + a_3 e^{125} + a_4 e^{126} + a_5 e^{134} + a_6 e^{135} + a_7 e^{136} \\ & + a_8 e^{145} + a_9 e^{146} + a_{10} e^{156} + a_{11} e^{234} + a_{12} e^{235} + a_{13} e^{236} + a_{14} e^{245} \\ & + a_{15} e^{246} + a_{16} e^{256} + a_{17} e^{345} + a_{18} e^{346} + a_{19} e^{356} + a_{20} e^{456}, \end{aligned} \tag{4}$$

where a_i and b_j are real constants. Moreover, we denote by $\beta_{i_1 \ldots i_5}$ and $\gamma_{i_1 \ldots i_5}$ the components of the 5-forms $\omega \wedge \psi_+$ and $d\omega^2$, respectively, so that

$$\begin{aligned} \omega \wedge \psi_+ &= \sum_{1 \leq i_1 < i_2 < \ldots < i_5 \leq 6} \beta_{i_1 \ldots i_5} e^{i_1 \ldots i_5}, \\ d\omega^2 &= \sum_{1 \leq i_1 < i_2 < \ldots < i_5 \leq 6} \gamma_{i_1 \ldots i_5} e^{i_1 \ldots i_5}. \end{aligned}$$

Observe that the non-vanishing β are always homogeneous polynomials of degree 2 in the a_i, b_j, while the non-vanishing γ are always homogeneous polynomials of degree 2 in the b_j.

For each Lie algebra $\mathfrak{s}_k$, we impose the conditions the forms (3) and (4) have to satisfy to be a half-flat SU(3)-structure inducing the Einstein metric. What we have to do is to solve the equations obtained from

$$\begin{cases} \omega \wedge \psi_+ = 0 \\ \mathrm{d}\psi_+ = 0 \\ \mathrm{d}\omega^2 = 0 \end{cases} \tag{5}$$

under the assumptions $\lambda = \lambda(\psi_+) < 0, \omega^3 \neq 0$. Moreover, since we are considering a basis which is orthonormal with respect to the Einstein metric (see Theorem 4.1), we have also to impose that the entries $H_{i,j} = h(e_i, e_j)$ of the matrix H associated to $h(\cdot, \cdot) = \omega(\cdot, J_{\psi_+}\cdot)$ with respect to the basis $(e_1, \ldots, e_6)$ satisfy

$$\begin{aligned} H_{i,j} &= 0, \quad \text{for } 1 \leq i, j \leq 6 \quad \text{and } i \neq j, \\ H_{i,i} - H_{i+1,i+1} &= 0, \quad \text{for } 1 \leq i \leq 5, \end{aligned} \tag{6}$$

with $H_{i,i} > 0$. These conditions give us a system of polynomial equations in 35 unknowns to solve under some constraints on them we will specify case by case. Since the expressions of the unknowns we obtain solving the equations are often too long to be written down, in what follows we will point out only from which equation a certain unknown is obtained, specifying its value only if it is zero.

Let us start with the Lie algebra $\mathfrak{s}_6$, whose structure equations are given in Table 1. We solve all the linear equations in the a_i deriving from $\mathrm{d}\psi_+ = 0$. Then, looking at the expression of λ we deduce that $a_6 \neq 0$. We can then solve all the equations obtained from $\omega \wedge \psi_+ = 0$, $\mathrm{d}\omega^2 = 0$ and $H_{i,j} = 0$ for $i \neq j$ using $a_6 \neq 0$ and comparing case by case each equation with $H_{i,i}$ and ω^3. After doing this, H becomes a diagonal matrix and we have to solve the remaining 5 equations of (6), which do not have any solution under the constraints $H_{i,i} \neq 0$ and $\lambda \neq 0$.

For the Lie algebra $\mathfrak{s}_3$ we can argue in a similar way, but instead of working on it, we can show the result on the Lie algebra $A_{6,99} \cong \mathfrak{s}_3$ since the computations are less involved. The structure equations of $A_{6,99}$ are given for example in [18], with respect to a basis $(f_1, \ldots, f_6)$ with dual basis $(f^1, \ldots, f^6)$ they are

$$\left(5f^{16} + f^{25} + f^{34}, 4f^{26} + f^{35}, 3f^{36} + f^{45}, 2f^{46}, f^{56}, 0 \right).$$

We consider the generic forms ω and ψ_+ as in (3) and (4) with e^i replaced by f^i. Observe that the matrix H associated to the Einstein inner product with respect to the basis $(f_1, \ldots, f_6)$ is not proportional to the identity, but it is still diagonal, thus we still have to solve the equations $H_{i,j} = 0$ for $i \neq j$. First of all we solve the linear equations in the a_i obtained from $\mathrm{d}\psi_+ = 0$. Then, we observe that having $b_1 = 0$ or $a_6 = 0$ leads to a contradiction after solving some equations: if $b_1 = 0$ we can use $H_{1,1}, H_{2,2} \neq 0$ to solve $H_{1,2} = 0$, $H_{1,3} = 0$, $\beta_{12345} = 0$, $\gamma_{12346} = 0$, $\gamma_{12356} = 0$, $\beta_{12346} = 0$, but then γ_{12456} cannot be zero; if $a_6 = 0$ we use $H_{1,1}, H_{2,2} \neq 0$ to solve $H_{1,2} = 0$, $H_{1,3} = 0$, $\beta_{12345} = 0$, $H_{1,5} = 0$, $\beta_{12356} = 0$, $\gamma_{12356} = 0$, $H_{2,3} = 0$, $\beta_{12346} = 0$ and obtain that $\gamma_{12346} = 0$ if and only if $H_{1,1}H_{3,3} = 0$. Thus, we assume $b_1 \neq 0$ and $a_6 \neq 0$. Under these constraints and comparing case by case the polynomial we want to be zero with $H_{i,i}$ and λ, we can get the expression of b_4 from the equation $H_{1,2} = 0$, b_9 from $H_{2,3} = 0$, b_7 from $\beta_{12345} = 0$, b_{10} from $\gamma_{12346} = 0$, b_{11} from $\gamma_{12356} = 0$, a_{18} from β_{12346}, $b_2 = 0$ from β_{12356}, $b_6 = 0$ from $H_{1,3} = 0$, $a_8 = 0$ from

$H_{3,4} = 0$, $a_{14} = 0$ from $H_{3,5} = 0$, $a_{17} = 0$ from $H_{3,6} = 0$, $a_{10} = 0$ from $H_{1,4} = 0$, $a_{19} = 0$ from $H_{1,6} = 0$, $b_{14} = 0$ from $\beta_{13456} = 0$, a_{20} from $H_{1,5} = 0$, b_3 from $H_{2,4} = 0$, b_8 from $H_{2,6} = 0$. Now, $H_{4,6} = 0$ implies $\omega^3 = 0$.

We can now turn our attention to the Lie algebras $\mathfrak{s}_{10}$, $\mathfrak{s}_{11}$ and $\mathfrak{s}_{13}$, we will show that none of these admits a coupled structure inducing the Einstein metric. The way in which we proceed is similar to the one followed for $\mathfrak{s}_6$ and $\mathfrak{s}_3$, but in this case we will consider a generic ω of the form (3) and $\psi_+ = c\,d\omega$ for $c \in \mathbb{R} - \{0\}$. Observe that the second condition of (5) is satisfied since ψ_+ is now an exact 3-form and that the first and the third condition are actually the same. For each Lie algebra, we consider the structure equations given in Table 1.

Consider $\mathfrak{s}_{10}$, this is a one-parameter family of Lie algebras depending on $t \in \left[0, \frac{1}{\sqrt{22}}\right]$. Since $H_{3,3}$ cannot be zero, we have that $b_{10} \neq 0$, $b_2 \neq \pm b_6$ and $t \neq \frac{1}{\sqrt{22}}$. The way in which we solve the equations depends on whether $t = \frac{7}{2\sqrt{330}}$ or not. If $t \neq \frac{7}{2\sqrt{330}}$ we can use $b_{10} \neq 0$ to obtain b_1 from $\gamma_{12345} = 0$, b_{15} from $\gamma_{12456} = 0$, b_5 from $\gamma_{13456} = 0$ and b_9 from $\gamma_{23456} = 0$. Then, b_{12} from $H_{3,4} = 0$, $b_3 = 0$ from $H_{1,3} = 0$, $b_7 = 0$ from $H_{2,3} = 0$, $b_{11} = 0$ from $H_{4,5} = 0$, b_8 from $H_{1,4} = 0$, b_4 from $H_{2,4} = 0$, $b_{13} = 0$ from $H_{3,5} = 0$. Now, $H_{3,6} = 0$ if and only if $\lambda = 0$. If $t = \frac{7}{2\sqrt{330}}$ the computations are the same until we arrive to the equation $H_{2,4} = 0$, which has no solutions since $H_{2,4}$ is proportional to λ.

For $\mathfrak{s}_{11}$ we have that $b_{10} \neq 0$ and $\sqrt{5}b_3 b_7 - b_{10}b_{13} - 3b_{10}b_{14} \neq 0$, otherwise $H_{4,4} = 0$. Using $b_{10} \neq 0$ we can obtain b_1, b_4, b_8, b_{15} from $\gamma_{12345} = 0$, $\gamma_{13456} = 0$, $\gamma_{23456} = 0$, $\gamma_{12456} = 0$, respectively, b_{11} from $H_{3,4} = 0$, b_9 from $H_{2,4} = 0$ and b_5 from $H_{1,4} = 0$. Then, using also the other constraint we get b_{12} from $H_{4,6} = 0$. Now, $H_{4,5} = 0$ if and only if $b_7 = 0$ or $b_{14} = -2b_{13}$. If $b_7 = 0$, from $H_{2,5} = 0$ and $\lambda \neq 0$ we have $b_3 = 0$, but then $H_{1,2} = 0$ only if either $\lambda = 0$ or $H_{1,1} = 0$. Thus, $b_7 \neq 0$ and $b_{14} = -2b_{13}$. Moreover, $b_6 \neq 0$, otherwise λ would be proportional to $H_{2,3}$. Thus, we can solve $H_{2,3} = 0$ to get b_3 and use $\lambda \neq 0$ to solve $H_{1,2} = 0$ and obtain b_6. Now, $H_{3,5}$ is proportional to λ, therefore, it cannot be zero.

In the last case $\mathfrak{s}_{13}$, we can see that $b_1, b_2, b_6 \neq 0$ and $b_3 \neq \sqrt{2}b_5$ from the fact that the entries in the diagonal of H cannot be zero. Solving the equations $\gamma_{12456} = 0$, $\gamma_{12345} = 0$, $\gamma_{12346} = 0$, $\gamma_{13456} = 0$, $\gamma_{23456} = 0$ under the previous constraints we obtain the expressions of $b_{14}, b_8, b_9, b_{13}, b_{15}$, respectively. Then, we get b_{12} from $H_{1,2} = 0$, b_{10} from $H_{1,3} = 0$, b_{11} from $H_{1,5} = 0$, b_5 from $H_{2,5} = 0$ and $b_4 = 0$ from $H_{2,6} = 0$. Now, $H_{2,4} = 0$ if and only if $\lambda = 0$.   □

Remark 4.6 In this case, it is in principle possible to use the properties of algebraic varieties to find solutions as we did in the proof of Theorem 3.6. However, the computations here are more involved since we have more unknowns (35 or 15 instead of 9) and more equations arising from the fact that some defining conditions for an SU(3)-structure that were easily verified in the case of $S^3 \times S^3$ have to be imposed in this case.

From the fact that the class of coupled structures is a subclass of the half-flat one, we can use the result of the previous theorem together with Theorem 4.1 to obtain:

Corollary 4.7 *Let $(\mathfrak{s}, h)$ be a six-dimensional nonunimodular solvable metric Lie algebra with h Einstein. Then, on $\mathfrak{s}$ there are no coupled SU(3)-structures inducing the Einstein inner product.*

Moreover, from the previous theorem and the Theorem 4.3, we obtain a constraint for the existence of coupled structures inducing Einstein metrics on homogeneous spaces:

Corollary 4.8 *Let (M, h) be a six-dimensional, connected, simply connected, homogeneous Einstein manifold of nonpositive sectional curvature. Then, there are no left-invariant coupled SU(3)-structures on M inducing the Einstein metric.*

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