Ann Glob Anal Geom (2015) 47:13–62
DOI 10.1007/s10455-014-9434-0

Existence of CR sections for high power of semi-positive generalized Sasakian CR line bundles over generalized Sasakian CR manifolds

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Received: 22 April 2014 / Accepted: 18 July 2014 / Published online: 4 August 2014

Abstract Let X be a compact generalized Sasakian CR manifold of dimension $2n-1$, $n \geqslant 2$, and let L be a generalized Sasakian CR line bundle over X equipped with a rigid semi-positive Hermitian fiber metric h^L. In this paper, we prove that if h^L is positive at some point of X and conditions $Y(0)$ and $Y(1)$ hold at each point of X, then L is big.

Keywords Szegö kernel asymptotics · Bergman kernel asymptotics · CR manifolds · CR line bundles · Complex variables · CR Grauert–Riemenschneider conjecture

Mathematics Subject Classification 32V30 · 32W10 · 32W25

1 Introduction and statement of the main results

Let X be a compact CR manifold of dimension $2n-1$, $n \geq 2$. When X is strongly pseudoconvex and dimension of X is greater than five, a classical theorem of Boutet de Monvel [3] asserts that X can be globally CR embedded into $\mathbb{C}^N$, for some $N \in \mathbb{N}$. For a strongly pseudoconvex CR manifold of dimension greater than five, the dimension of the kernel of the tangential Cauchy–Riemmann operator $\overline{\partial}_b$ is infinite and we can find many CR functions to embed X into complex space. When the Levi form of X has negative eigenvalues, the dimension of the kernel of $\overline{\partial}_b$ is finite and could be zero and in general, X can not be globally CR embedded into complex space. Inspired by Kodaira, we introduced in [9] (see also [12]) the idea of embedding CR manifolds by means of CR sections of tensor powers L^k of a CR line bundle $L \rightarrow X$. If the dimension of the space $H^0_b(X, L^k)$ of CR sections of L^k is large, when $k \rightarrow \infty$, one should find many CR sections to embed X into projective space. In analogy

C.-Y. Hsiao is partially supported by the DFG funded Project MA 2469/2-1.

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to the Kodaira embedding theorem, it is natural to ask if X can be globally embedded into projective space when it carries a CR line bundle with positive curvature? To understand this question, it is crucial to be able to know if $\dim H^0_b(X, L^k) \sim k^n$, for k large? The following conjecture was implicit in [12, p.47-48]

Conjecture 1.1 *If L is positive and the Levi form of X has at least two negative and two positive eigenvalues, then*

$$\dim H^0_b(X, L^k) \sim k^n,$$

for k large.

The difficulty of this conjecture comes from the presence of positive eigenvalues of the curvature of the line bundle and negative eigenvalues of the Levi form of X and this causes the associated Kohn Laplacian to have no semi-classical spectral gap. This problem is also closely related to the fact that in the global L^2-estimates for the $\overline{\partial}_b$-operator of Kohn–Hörmander, there is a curvature term from the line bundle as well from the boundary and, in general, it is very difficult to control the sign of the total curvature contribution.

In complex geometry, Demailly's holomorphic Morse inequalities [6] handled the corresponding analytical difficulties in a new way. Inspired by Demailly, we established analogs of the holomorphic Morse inequalities of Demailly for CR manifolds (see [9, Theorem 1.8])

Theorem 1.2 (Hsiao–Marinescu, 2009) *We assume that the Levi form of X has at least two negative and two positive eigenvalues. Then, as $k \to \infty$,*

$$\begin{aligned} &-\dim H^0_b(X, L^k) + \dim H^1_b(X, L^k) \\ &\leqslant \frac{k^n}{2(2\pi)^n}\left(-\int_X \int_{\mathbb{R}_{\phi(x),0}} \left|\det(M^\phi_x + s\mathcal{L}_x)\right| ds\, dv_X(x)\right. \\ &\quad \left. + \int_X \int_{\mathbb{R}_{\phi(x),1}} \left|\det(M^\phi_x + s\mathcal{L}_x)\right| ds\, dv_X(x)\right) + o(k^n), \end{aligned} \tag{1.1}$$

where M^ϕ_x is the associated curvature of L at $x \in X$ (see Definition 1.9), $H^1_b(X, L^k)$ denotes the first $\overline{\partial}_b$ cohomology group with values in L^k, $dv_X(x)$ is the volume form on X, $\mathcal{L}_x$ denotes the Levi form of X at $x \in X$, and for $x \in X$, $q = 0, 1$,

$$\begin{aligned} \mathbb{R}_{\phi(x),q} = \{&s \in \mathbb{R};\ M^\phi_x + s\mathcal{L}_x \text{ has exactly } q \text{ negative eigenvalues} \\ &\text{and } n-1-q \text{ positive eigenvalues}\}. \end{aligned} \tag{1.2}$$

From (1.1), we see that if

$$\int_X \int_{\mathbb{R}_{\phi(x),0}} \left|\det(M^\phi_x + s\mathcal{L}_x)\right| ds\, dv_X(x) > \int_X \int_{\mathbb{R}_{\phi(x),1}} \left|\det(M^\phi_x + s\mathcal{L}_x)\right| ds\, dv_X(x) \tag{1.3}$$

then L is big that is $\dim H^0_b(X, L^k) \sim k^n$. This is a very general criterion and it is desirable to refine it in some cases where (1.3) is not easy to verify. The problem still comes from the presence of positive eigenvalues of M^ϕ_x and negative eigenvalues of $\mathcal{L}_x$.

For the better understanding, let's see a simple example. We consider compact analogs of the Heisenberg group H_n. Let $\lambda_1, \ldots, \lambda_{n-1}$ be given non-zero integers. We assume that $\lambda_1 < 0, \ldots, \lambda_{n_-} < 0, \lambda_{n_-+1} > 0, \ldots, \lambda_{n-1} > 0$. Let $\mathscr{C}H_n = (\mathbb{C}^{n-1} \times \mathbb{R})/\sim$, where

$(z,\theta) \sim (\widetilde{z},\widetilde{\theta})$ if

$$\widetilde{z} - z = (\alpha_1,\ldots,\alpha_{n-1}) \in \sqrt{2\pi}\mathbb{Z}^{n-1} + i\sqrt{2\pi}\mathbb{Z}^{n-1},$$
$$\widetilde{\theta} - \theta - i\sum_{j=1}^{n-1}\lambda_j(z_j\overline{\alpha}_j - \overline{z}_j\alpha_j) \in \pi\mathbb{Z}.$$

We can check that $\sim$ is an equivalence relation and $\mathscr{C}H_n$ is a compact manifold of dimension $2n-1$. The equivalence class of $(z,\theta) \in \mathbb{C}^{n-1}\times\mathbb{R}$ is denoted by $[(z,\theta)]$. For a given point $p = [(z,\theta)]$, we define the CR structure $T^{1,0}_p\mathscr{C}H_n$ of $\mathscr{C}H_n$ to be the space spanned by $\left\{\frac{\partial}{\partial z_j} + i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta},\ \ j=1,\ldots,n-1\right\}$. Then, $(\mathscr{C}H_n, T^{1,0}_p\mathscr{C}H_n)$ is a compact CR manifold of dimension $2n-1$. With a suitable choose of a Hermitian metric on the complexified tangent bundle of $\mathscr{C}H_n$, the Levi form of $\mathscr{C}H_n$ at $p\in\mathscr{C}H_n$ is given by $\mathcal{L}_p = \sum_{j=1}^{n-1}\lambda_j dz_j\wedge d\overline{z}_j$. Let $L = (\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C})/_{\equiv}$ where $(z,\theta,\eta)\equiv(\widetilde{z},\widetilde{\theta},\widetilde{\eta})$ if

$$(z,\theta)\sim(\widetilde{z},\widetilde{\theta}),\quad \widetilde{\eta} = \eta\exp\left(\sum_{j,t=1}^{n-1}\mu_{j,t}(z_j\overline{\alpha}_t + \frac{1}{2}\alpha_j\overline{\alpha}_t)\right),\quad \text{for } (\alpha_1,\ldots,\alpha_{n-1}) = \widetilde{z} - z,$$

where $\mu_{j,t} = \mu_{t,j}$, $j,t = 1,\ldots,n-1$, are given integers. We can check that $\equiv$ is an equivalence relation and L is a CR line bundle over $\mathscr{C}H_n$. For $(z,\theta,\eta)\in\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C}$, we denote $[(z,\theta,\eta)]$ its equivalence class. Take the pointwise norm $\left|[(z,\theta,\eta)]\right|^2_{h^L} := |\eta|^2\exp\left(-\sum_{j,t=1}^{n-1}\mu_{j,t}z_j\overline{z}_t\right)$ on L. Then, the associated curvature of L is given by $M^\phi_x = \sum_{j,t=1}^{n-1}\mu_{j,t}dz_j\wedge d\overline{z}_t,\ \ \forall x\in\mathscr{C}H_n$. In this simple example, Conjecture 1.1 becomes

Question 1.3 If $n_-\geq 2$, $n-1-n_-\geq 2$, and the matrix $\left(\mu_{j,t}\right)_{j,t=1}^{n-1}$ is positive definite, then $\dim H^0_b(\mathscr{C}H_n, L^k)\sim k^n$?

If $\mu_{j,t} = \left|\lambda_j\right|\delta_{j,t}$, $j,t = 1,\ldots,n-1$, and $n_-\geq 2$, $n-1-n_-\geq 2$, where $\delta_{j,t} = 1$ if $j=t$, $\delta_{j,t} = 0$ if $j\neq t$, then it is easy to see that $\mathbb{R}_{\phi(x),1} = \emptyset$, where $\mathbb{R}_{\phi(x),1}$ is given by (1.2). Combining this observation with Morse inequalities for CR manifolds [see (1.1)], we get

Theorem 1.4 *If* $n_-\geq 2$, $n-1-n_-\geq 2$, *and* $\mu_{j,t} = \left|\lambda_j\right|\delta_{j,t}$, $j,t = 1,\ldots,n-1$, *then* $\dim H^0_b(\mathscr{C}H_n, L^k)\sim k^n$.

The assumptions in Theorem 1.4 are somehow restrictive. It is clear that we cannot go much further from Morse inequalities.Using Morse inequalities to approach Conjecture 1.1, we always have to impose extra conditions linking the Levi form and the curvature of the line bundle L. Similar problems also appear in the works of Marinescu [12,13] and Berman [2] where they studied the $\overline{\partial}$-Neumann cohomology groups associated to a high power of a given holomorphic line bundle on a compact complex manifold with boundary. To get many holomorphic sections, they also have to assume that, close to the boundary, the curvature of the line bundle is adapted to the Levi form of the boundary. In this work, by carefully studying semi-classical behavior of microlocal Fourier transforms of the extremal functions for the spaces of lower energy forms of the associated Kohn Laplacian, we could solve Conjecture 1.1 under rigidity conditions on X and L without any extra condition linking the Levi form of X and the curvature of L. As an application, we solve Question 1.3 completely. The proof of our main result presents a new way to overcome the analytic difficulty mentioned in the discussion after Conjecture 1.1 under rigidity conditions. Using this new method, it is possible to remove the assumptions linking the curvatures of the line bundle and the boundary in the works of Marinescu [12,13] and Berman [2] under rigidity conditions on the boundary and the line bundle.

The rigidity conditions we used in this work are inspired by the work of Baouendi–Rothschild–Treves [1]. They introduced rigidity condition on CR structure and proved that such a manifold can always be locally CR embedded in complex space as a generic submanifold. From their work, rigidity condition on CR structure seems suitable for our purpose. Initially, it is reasonable to first assume that X can be locally embedded and study global embeddability of X. We can expect that the curvature of the line bundle and its transition functions have to satisfy some rigidity conditions (see Definition 1.7 and Definition 1.12). Moreover, with these geometric conditions, it is possible to establish a micolocal asymptotic expansion of the Szegö kernel and extend Kodaira embedding theorem to this situation.

The geometric objects introduced in this paper form large classes of CR manifolds and CR line bundles. We hope that these geometric objects will be interesting for CR geometers and will be useful in CR geometry.

1.1 Some standard notations

We shall use the following notations: $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R};\ x \geq 0\}$, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. An element $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{N}_0^n$ will be called a multiindex and the length of α is: $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $x = (x_1, \ldots, x_n)$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\partial_{x_j} = \frac{\partial}{\partial x_j}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, $D_x = \frac{1}{i}\partial_x$, $D_{x_j} = \frac{1}{i}\partial_{x_j}$. Let $z = (z_1, \ldots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, be coordinates of $\mathbb{C}^n$. We write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\overline{z}^\alpha = \overline{z}_1^{\alpha_1} \cdots \overline{z}_n^{\alpha_n}$, $\frac{\partial^{|\alpha|}}{\partial z^\alpha} = \partial_z^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$, $\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\right)$, $j = 1, \ldots, n$. $\frac{\partial^{|\alpha|}}{\partial \overline{z}^\alpha} = \partial_{\overline{z}}^\alpha = \partial_{\overline{z}_1}^{\alpha_1} \cdots \partial_{\overline{z}_n}^{\alpha_n}$, $\partial_{\overline{z}_j} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\right)$, $j = 1, \ldots, n$.

Let Ω be a C^∞ paracompact manifold. We let $T\Omega$ and $T^*\Omega$ denote the tangent bundle of Ω and the cotangent bundle of Ω, respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T\Omega$ and $\mathbb{C}T^*\Omega$, respectively. We write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between $T\Omega$ and $T^*\Omega$. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}T\Omega \times \mathbb{C}T^*\Omega$. Let E be a C^∞ vector bundle over Ω. The fiber of E at $x \in \Omega$ will be denoted by E_x. Let F be another vector bundle over Ω. We write $E \boxtimes F$ to denote the vector bundle over $\Omega \times \Omega$ with fiber over $(x, y) \in \Omega \times \Omega$ consisting of the linear maps from E_x to F_y.

1.2 Generalized Sasakian CR manifolds and generalized Sasakian CR line bundles

Let $(X, T^{1,0}X)$ be a CR manifold of dimension $2n-1$, $n \geqslant 2$, where $T^{1,0}X$ is a CR structure of X. That is, $T^{1,0}X$ is a complex $n-1$ dimensional subbundle of the complexified tangent bundle $\mathbb{C}TX$, satisfying $T^{1,0}X \bigcap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$. In this section, we denote $Y := X \times \mathbb{R}$ and we write t to denote the standard coordinate of $\mathbb{R}$. We need

Definition 1.5 We say that $(X, T^{1,0}X)$ is a generalized Sasakian CR manifold if there exists an integrable almost complex structure $J : TY \to TY, \mathbb{C}TY \to \mathbb{C}TY$, such that $Ju = iu$, $\forall u \in T^{1,0}X$.

Let $(X, T^{1,0}X)$ be a CR manifold of dimension $2n-1$, $n \geqslant 2$, and let $J : TY \to TY$, $\mathbb{C}TY \to \mathbb{C}TY$, be an almost complex structure. We say that J is a canonical complex structure on Y if J is integrable and $Ju = iu, \forall u \in T^{1,0}X$. Thus, $(X, T^{1,0}X)$ is a generalized Sasakian CR manifold if and only if there exists a canonical complex structure on Y.

Let $(X, T^{1,0}X)$ be a generalized Sasakian CR manifold and let $J : TY \to TY, \mathbb{C}TY \to \mathbb{C}TY$ be any canonical complex structure on Y. From the Newlander–Nirenberg theorem, J defines a complex structure $T^{1,0}Y \supset T^{1,0}X$. Put $T = J\frac{\partial}{\partial t}$. Then, $T \in C^\infty(X, TX)$, T is a global real vector field on X. Since J is integrable, it is easy to see that

$$\begin{aligned}&\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T;\ \lambda \in \mathbb{C}\},\\ &[T, \mathcal{V}] \subset \mathcal{V}, \quad \mathcal{V} := C^\infty(X, T^{1,0}X).\end{aligned} \tag{1.4}$$

Conversely, let $(X, T^{1,0}X)$ be a CR manifold of dimension $2n-1$, $n \geqslant 2$. We assume that there exists a global real vector field $T \in C^\infty(X, \mathbb{C}TX)$ such that (1.4) hold. Then, one can define a canonical complex structure on Y by the rule:

$$\begin{aligned}&J : TY \to TY, \quad \mathbb{C}TY \to \mathbb{C}TY\\ &Ju = iu, \quad \forall u \in T^{1,0}X, \qquad Jv = -iv, \quad \forall v \in T^{0,1}X, \qquad J\frac{\partial}{\partial t} = T.\end{aligned}$$

Thus, $(X, T^{1,0}X)$ is a generalized Sasakian CR manifold if and only if there exists a global real vector field $T \in C^\infty(X, \mathbb{C}TX)$ such that (1.4) hold. We call T a rigid global real vector field.

Let's see some examples

Example 1.6 (I) Let M be an open subset with C^∞ boundary ∂M of a complex manifold M' of dimension n. If for every $x_0 \in \partial M$, we can find local holomorphic coordinates $(z_1, \ldots, z_n)$ defined in some neighborhood of x_0, such that near x_0, ∂M is given by the equation

$$\operatorname{Im} z_n = f(z_1, \ldots, z_{n-1}), \quad f \in C^\infty \text{ is real valued},$$

then ∂M is a generalized Sasakian CR manifold of dimension $2n-1$.
(II) Let M be a complex manifold and (E, h^E) be a holomorphic Hermitian line bundle on M, where the Hermitian fiber metric on E is denoted by h^E. Let (E^*, h^{E^*}) be the dual bundle of E. We denote

$$G := \left\{v \in L^*;\ |v|_{h^{L^*}} < 1\right\}, \quad \partial G = \left\{v \in L^*;\ |v|_{h^{L^*}} = 1\right\}.$$

The domain G is called *Grauert tube* associated to E. It is easy to see that ∂G is a generalized Sasakian CR manifold.
(III) The hypersurface

$$\left\{(z_1, \ldots, z_n) \in \mathbb{C}^n;\ \sum_{j=1}^n \lambda_j \left|z_j\right|^2 = R\right\}$$

is a generalized Sasakian CR manifolds, where $\lambda_j \in \mathbb{R}$, $j = 0, 1, \ldots, n$, $R \in \mathbb{R}$.
(IV) Heisenberg groups and compact Heisenberg groups (see Sect. 9.1) are generalized Sasakian CR manifolds.

From now on, we assume that $(X, T^{1,0}X)$ is a compact generalized Sasakian CR manifold and we let $\pi : Y \to X$ denote the standard projection.

Definition 1.7 Let L be a complex line bundle over X. (L, J) is a generalized Sasakian CR line bundle over X, where J is a canonical complex structure on Y if the pull back π^*L is a holomorphic line bundle over Y with respect to J.

We need

Definition 1.8 Let $T \in C^\infty(X, TX)$ be a rigid global real vector field on X. Let $U \Subset X$ be an open set. A function $u \in C^\infty(U)$ is said to be a T-rigid CR function on U if $Tu = 0$ and $Zu = 0$ for all $Z \in C^\infty(U, T^{0,1}X)$.

From now on, we let (L, J) be a generalized Sasakian CR line bundle over X and we fix $T = J\frac{\partial}{\partial t}$. T is a rigid global real vector field. Since π^*L is a holomorphic line bundle over Y with respect to the canonical complex structure J on Y, it is easy to see that X can be covered with open sets U_j with trivializing sections s_j, $j = 1, 2, \ldots$, such that the corresponding transition functions are T-rigid CR functions. In this paper, when trivializing sections s are used, we will assume that they are of this special form.

Fix a Hermitian fiber metric h^L on L and we will denote by ϕ the local weights of the Hermitian metric h^L. More precisely, if s is a local trivializing section of L on an open subset $D \subset X$, then the local weight of h^L with respect to s is the function $\phi \in C^\infty(D, \mathbb{R})$ for which

$$|s(x)|^2_{h^L} = \mathrm{e}^{-\phi(x)}, \quad x \in D. \tag{1.5}$$

We write h^{π^*L} to denote the pull back of h^L by the projection π. Then, h^{π^*L} is a Hermitian fiber metric on the holomorphic line bundle π^*L. Let R^{π^*L} be the canonical curvature induced by h^{π^*L}. Let $\overline{\partial}_J$ and ∂_J be the $(0, 1)$ and $(1, 0)$ part of the exterior differential operator d on functions with respect to J. If s is a local trivializing section of L on an open subset $D \subset X$, $|s|^2_{h^L} = \mathrm{e}^{-\phi(x)}$, then

$$R^{\pi^*L}(y) = \partial_J\overline{\partial}_J\phi(\pi(y)) \quad \text{on } D \times \mathbb{R}. \tag{1.6}$$

We need

Definition 1.9 For $p \in X$, we define the Hermitian quadratic form M^ϕ_p on $T^{1,0}_pX$ by

$$M^\phi_p(U, \overline{V}) = \left\langle U \wedge \overline{V}, R^{\pi^*L}(y)\right\rangle, \quad \pi(y) = p,\ U, V \in T^{1,0}_pX. \tag{1.7}$$

Remark 1.10 Let s be a local trivializing section of L on an open subset $D \subset X$ and ϕ the corresponding local weight as in (1.5). Let $\overline{\partial}_b$ denote the tangential Cauchy–Riemann operator on functions (see [4, Chapter 7]). It is not difficult to see that for every $p \in D$, we have

$$M^\phi_p(U, \overline{V}) = \frac{1}{2}\left\langle U \wedge \overline{V}, d\left(\overline{\partial}_b\phi - \partial_b\phi\right)(p)\right\rangle, \quad U, V \in T^{1,0}_pX, \tag{1.8}$$

where d is the usual exterior derivative and $\overline{\partial_b\phi} = \overline{\partial}_b\overline{\phi}$.

For $p \in X$, let $\mathcal{L}_p$ be the *Levi form* (with respect to T) at p (see Definition 1.14, for the precise meaning).

Definition 1.11 We say that h^L is positive at $x_0 \in X$ if the Hermitian quadratic form $M^\phi_{x_0}$ is positive, h^L is semi-positive if there is a positive constant $\delta > 0$ such that for every $x \in X$ and $s \in [-\delta, \delta]$, the Hermitian quadratic form $M^\phi_x + 2s\mathcal{L}_x$ is semi-positive.

Since the transition functions are T-rigid CR functions, we can check that $T\phi$ is a well-defined global smooth function on X.

Definition 1.12 h^L is said to be a T-rigid Hermitian fiber metric on (L, J) if

$$T\phi = C \text{ on } X, \quad \text{for some constant } C, \tag{1.9}$$

where ϕ denotes the corresponding local weight as in (1.5).

Note that the constant C in (1.9) can be non-zero. (See Sect. 9.1).

Definition 1.13 We say that (L, J, h^L) is a rigid generalized Sasakian CR line bundle over X if (L, J) is a generalized Sasakian CR line bundle over X and h^L is a T-rigid Hermitian fiber metric on (L, J), $T = J\frac{\partial}{\partial t}$.

1.3 Hermitian CR geometry and the main results

Fix a smooth Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is pointwise orthogonal to $T^{0,1}X$, T is pointwise orthogonal to $T^{1,0}X \oplus T^{0,1}X$, $\langle T|T\rangle := \|T\|^2 = 1$ and $\langle u|v\rangle$ is real if u, v are real tangent vectors.

Define

$$\begin{aligned} T^{*1,0}X &:= \left\{e \in \mathbb{C}T^*X;\ \langle e, u\rangle = 0, \forall u \in T^{0,1}X \oplus \{\lambda T;\ \lambda \in \mathbb{C}\}\right\}, \\ T^{*0,1}X &:= \left\{f \in \mathbb{C}T^*X;\ \langle f, v\rangle = 0, \forall v \in T^{1,0}X \oplus \{\lambda T;\ \lambda \in \mathbb{C}\}\right\}. \end{aligned}$$

$T^{*1,0}X$ and $T^{*0,1}X$ are subbundles of the complexified cotangent bundle $\mathbb{C}T^*X$. Define the vector bundle of $(0, q)$ forms of X by $\Lambda^{0,q}T^*X := \Lambda^q T^{*0,1}X$. Let $D \subset X$ be an open set. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $\Lambda^{0,q}T^*X$ over D. Similarly, if E is a vector bundle over D, then we let $\Omega^{0,q}(D, E)$ denote the space of smooth sections of $\Lambda^{0,q}T^*X \otimes E$ over D. Let $\Omega_0^{0,q}(D, E)$ be the subspace of $\Omega^{0,q}(D, E)$ whose elements have compact support in D. Let

$$\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X) \tag{1.10}$$

be the tangential Cauchy–Riemann operator (see [4, Chapter 7]).

The Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on $\Lambda^{0,q}T^*X$ the bundle of $(0, q)$ forms of X. We shall also denote all these induced metrics by $\langle\,\cdot\,|\,\cdot\,\rangle$. For $f \in \Omega^{0,q}(X)$, we denote the pointwise norm $|f(x)|^2 := \langle f(x)|f(x)\rangle$. Locally, there is a real 1-form ω_0 of length one which is orthogonal to $T^{*1,0}X \oplus T^{*0,1}X$. The form ω_0 is unique up to the choice of sign. We choose ω_0 so that $\langle T, \omega_0\rangle = -1$. Therefore, ω_0 is uniquely determined. We call ω_0 the uniquely determined global real 1-form. We have the pointwise orthogonal decompositions:

$$\begin{aligned} \mathbb{C}T^*X &= T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0;\ \lambda \in \mathbb{C}\}, \\ \mathbb{C}TX &= T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T;\ \lambda \in \mathbb{C}\}. \end{aligned} \tag{1.11}$$

We recall

Definition 1.14 For $p \in X$, the *Levi form* $\mathcal{L}_p$ is the Hermitian quadratic form on $T_p^{1,0}X$ defined as follows. For any $U,\ V \in T_p^{1,0}X$, pick $\mathcal{U}, \mathcal{V} \in C^\infty(X, T^{1,0}X)$ such that $\mathcal{U}(p) = U$, $\mathcal{V}(p) = V$. Set

$$\mathcal{L}_p(U, \overline{V}) = \frac{1}{2i}\big\langle\big[\mathcal{U}, \overline{\mathcal{V}}\,\big](p), \omega_0(p)\big\rangle, \tag{1.12}$$

where $\big[\mathcal{U}, \overline{\mathcal{V}}\,\big] = \mathcal{U}\,\overline{\mathcal{V}} - \overline{\mathcal{V}}\,\mathcal{U}$ denotes the commutator of $\mathcal{U}$ and $\overline{\mathcal{V}}$. Note that $\mathcal{L}_p$ does not depend on the choices of $\mathcal{U}$ and $\mathcal{V}$.

Since $\mathcal{L}_p$ is a Hermitian form there is a local orthonormal basis $\{\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}\}$ of $T^{1,0}X$ with respect to $\langle\cdot|\cdot\rangle$ such that $\mathcal{L}_p$ is diagonal in this basis, $\mathcal{L}_p(\mathcal{U}_j, \overline{\mathcal{U}}_t) = \delta_{j,t}\lambda_j(p)$, $j, t = 1, \ldots, n-1$, $\delta_{j,t} = 1$ if $j = t$, $\delta_{j,t} = 0$ if $j \neq t$, $\lambda_j(p) \in \mathbb{R}$, $j = 1, \ldots, n-1$. The diagonal entries $\{\lambda_1(p), \ldots, \lambda_{n-1}(p)\}$ are called the *eigenvalues* of the Levi form at $p \in X$ with respect to $\langle\cdot|\cdot\rangle$.

Given $q \in \{0, \ldots, n-1\}$, the Levi form is said to satisfy *condition* $Y(q)$ at $p \in X$, if $\mathcal{L}_p$ has at least either $\max(q+1, n-q)$ eigenvalues of the same sign or $\min(q+1, n-q)$ pairs of eigenvalues with opposite signs. Note that the sign of the eigenvalues does not depend on the choice of the metric $\langle\cdot|\cdot\rangle$.

Let L^k, $k > 0$, be the k-th tensor power of the line bundle L. We write $\overline{\partial}_{b,k}$ to denote the tangential Cauchy–Riemann operator acting on forms with values in L^k, defined locally by:

$$\overline{\partial}_{b,k} : \Omega^{0,q}(X, L^k) \to \Omega^{0,q+1}(X, L^k), \quad \overline{\partial}_{b,k}(s^k u) := s^k \overline{\partial}_b u, \tag{1.13}$$

where s is a local trivialization of L on an open subset $D \subset X$ and $u \in \Omega^{0,q}(D)$. We obtain a $\overline{\partial}_{b,k}$-complex $(\Omega^{0,\bullet}(X, L^k), \overline{\partial}_{b,k})$ with cohomology

$$H_b^\bullet(X, L^k) := \ker \overline{\partial}_{b,k} / \operatorname{Im} \overline{\partial}_{b,k}. \tag{1.14}$$

We assume that X is compact and $Y(0)$ holds. By [11, 7.6-7.8], [7, 5.4.11-12], [4, Props. 8.4.8-9] and [8, Chapter 6], condition $Y(0)$ implies that $\dim H_b^0(X, L^k) < \infty$.

Our main result is the following

Theorem 1.15 *Let $(X, T^{1,0}X)$ be a compact generalized Sasakian CR manifold of dimension $2n-1$, $n \geqslant 2$ and let (L, J, h^L) be a rigid generalized Sasakian CR line bundle over X. Assume that h^L is semi-positive and positive at some point of X. Suppose conditions $Y(0)$ and $Y(1)$ hold at each point of X. Then, for k large, there is a constant $c > 0$ independent of k, such that*

$$\dim H_b^0(X, L^k) \geq ck^n.$$

It should be mentioned that the Levi curvature assumptions in Theorem 1.15 are a bit more general than the ones in Conjecture 1.1.

Remark 1.16 It should be mentioned that Theorem 1.15 implies the famous Grauert–Riemenschneider conjecture in complex geometry. Let M be a compact complex manifold of complex dimension n and let $E \to M$ be a holomorphic line bundle with a Hermitian fiber metric h^E. Let R^E denotes the canonical curvature on E induced by h^E. We assume that R^E is semi-positive and positive at some point of M. Then, Grauert–Riemenschneider conjecture claims that L is big, that is, $\dim H^0(M, E^k) \sim k^n$, where $H^0(M, E^k)$ denotes the space of global holomorphic sections of E^k the k-th power of E. This conjecture was first solved by Siu [14]. Let's see how to obtain this conjecture from Theorem 1.15. With the notations used above, let $(\widetilde{X}, T^{1,0}\widetilde{X})$ be a compact generalized Sasakian CR manifold of dimension $2m-1$, $m \geqslant 2$, such that the Levi form of $\widetilde{X}$ has at least two negative and two positive eigenvalues and let $(\widetilde{L}, \widetilde{J}, h^{\widetilde{L}})$ be a rigid generalized Sasakian CR line bundle over $\widetilde{X}$ with $h^{\widetilde{L}}$ is positive at every point of $\widetilde{X}$. We can find such $(\widetilde{X}, T^{1,0}\widetilde{X})$ and $(\widetilde{L}, \widetilde{J}, h^{\widetilde{L}})$ (see Sect. 9). Consider $X = M \oplus \widetilde{X}$, $T^{1,0}X := T^{1,0}M \oplus T^{1,0}\widetilde{X}$, where $T^{1,0}M$ denotes the holomorphic tangent bundle of M. Then, $(X, T^{1,0}X)$ is a compact generalized Sasakian CR manifold of dimension $2(m+n)-1$ and the Levi form of X has at least two negative and two positive eigenvalues. Thus, conditions $Y(0)$ and $Y(1)$ hold at each point of X. Put $L := E \otimes \widetilde{L}$. Then, L is a complex line bundle over X. Let J be the canonical complex structure on $X \times \mathbb{R}$ induced by $\widetilde{J}$ and the complex structure on M. It is obviously that (L, J) is a generalized Sasakian CR line bundle over X. Put $h^L = h^E \otimes h^{\widetilde{L}}$. Then, h^L is a Hermitian fiber metric on L and (L, J, h^L) is a rigid generalized Sasakian CR line bundle over X. Moreover, it is easy to check that h^L is semi-positive and positive at some point of X. From Theorem 1.15, we conclude that for k large, there is a constant $C_0 > 0$ such that

$$\dim H_b^0(X, L^k) \geq C_0 k^{n+m}. \tag{1.15}$$

We notice that $\dim H_b^0(X, L^k) = \dim H^0(M, E^k) \times \dim H_b^0(\widetilde{X}, \widetilde{L}^k)$ and it is well known that there is a constant $C_1 > 0$ such that $\dim H_b^0(\widetilde{X}, \widetilde{L}^k) \leq C_1 k^m$ (see [9, Theorem 1.5]). Combining this observation and (1.15), we conclude that there is a constant $c > 0$ such that $\dim H^0(M, E^k) \geq ck^n$.

We investigate Theorem 1.15 on generalized torus CR manifolds. Let $\Phi^t(x)$ be the T-flow. That is, $\Phi^t(x)$ is a differentiable mapping:

$$t \to \Phi^t(x) \in X : I \to X,$$

I is an open interval in $\mathbb{R}$, $0 \in I$, such that $\Phi^0(x) = x$, $\forall x \in X$, and $\frac{\mathrm{d}\Phi^t(x)}{\mathrm{d}t} = T(\Phi^t(x))$. We need

Definition 1.17 We say that $(X, T^{1,0}X)$ is a generalized torus CR manifold if there is a constant $\gamma_0 > 0$ such that $\Phi^t(x)$ is well defined, $\forall\, |t| \leq \gamma_0$, $\forall x \in X$, and $\Phi^{\gamma_0}(x) = x$ for every $x \in X$.

Definition 1.18 We say that (L, J) is an admissible generalized Sasakian CR line bundle over a compact generalized torus CR manifold X if we can find an open covering $\{U_j\}_{j=1}^N$ of X such that L is trivial on U_j, for each j, and

$$\left\{\Phi^t(x);\ x \in U_j, |t| \leq \gamma_0\right\} = U_j,$$

for each j, where $\gamma_0 > 0$ is as in Definition 1.17.

Let (L, J) be an admissible generalized Sasakian CR line bundle over a compact generalized torus CR manifold $(X, T^{1,0}X)$. Take any Hermitian fiber metric h^L on L and let ϕ denotes the corresponding local weight as in (1.5). Let h_1^L be the Hermitian fiber metric on L locally given by $|s|^2_{h_1^L} = \mathrm{e}^{-\phi_1}$, where $\phi_1 = \frac{1}{\gamma_0}\int_0^{\gamma_0} \phi(\Phi^t(x))\mathrm{d}t$, $\gamma_0 > 0$ is as in Definition 1.17, s is a local trivializing section of L with the special form in Definition 1.18. It is easy to check that h_1^L is well defined and $T\phi_1 = 0$. Thus, (L, J, h_1^L) is a rigid generalized Sasakian CR line bundle over $(X, T^{1,0}X)$. Moreover, we can show that if M_x^{ϕ} is positive on X then $M_x^{\phi_1}$ is positive on X (see Proposition 3.3, for the proof). Combining this with Theorem 1.15, we obtain

Theorem 1.19 *Let $(X, T^{1,0}X)$ be a compact generalized torus CR manifold of dimension $2n-1$, $n \geqslant 2$ and let (L, J) be an admissible generalized Sasakian CR line bundle over X with a Hermitian fiber metric h^L. We assume that h^L is positive on X and conditions $Y(0)$ and $Y(1)$ hold at each point of X. Then, for k large, there is a constant $c > 0$ independent of k, such that*

$$\dim H_b^0(X, L^k) \geq ck^n.$$

1.4 The outline of the proof of Theorem 1.15

Let $\Box^{(q)}_{b,k}$ denote the Kohn Laplacian with values in L^k (see Sect. 2). Fix $q = 0, 1, \ldots, n-1$. We assume that $Y(q)$ holds. It is well known that $\Box^{(q)}_{b,k}$ has a discrete spectrum, each eigenvalues occurs with finite multiplicity and all eigenforms are smooth and $\operatorname{Ker}\Box^{(q)}_{b,k} := \mathscr{H}^q_b(X, L^k) \cong H^q_b(X, L^k)$. For $\lambda \geq 0$, let $\mathscr{H}^q_{b,\leq\lambda}(X, L^k)$ denote the space spanned by the eigenforms of $\Box^{(q)}_{b,k}$ whose eigenvalues are bounded by λ. Now, we assume that $Y(0)$ and $Y(1)$ hold and (L, h^L) is semi-positive and positive at some point of X. Take $\delta_0 > 0$ be a small constant so that $M^\phi_x + 2s\mathcal{L}_x \geq 0$, $\forall |s| \leq \delta_0$, $\forall x \in X$. Take $\psi(\eta) \in C^\infty_0(]-\delta_0, \delta_0[, \overline{\mathbb{R}}_+)$ so that $\psi(\eta) = 1$ if $-\frac{\delta_0}{2} \leq \eta \leq \frac{\delta_0}{2}$. Take $\chi(t) \in C^\infty_0(]-2, 2[, \overline{\mathbb{R}}_+)$ so that $0 \leq \chi(t) \leq 1$ and $\chi(t) = 1$ if $-1 \leq t \leq 1$ and $\chi(-t) = \chi(t)$ for all $t \in \mathbb{R}$. Fix $M > 0$. Under the rigidity assumptions in Theorem 1.15, we can construct global continuous operators $Q^{(0)}_{M,k} : C^\infty(X, L^k) \to C^\infty(X, L^k)$ and $Q^{(1)}_{M,k} : \Omega^{0,1}(X, L^k) \to \Omega^{0,1}(X, L^k)$ such that

$$\overline{\partial}_{b,k} Q^{(0)}_{M,k} = Q^{(1)}_{M,k} \overline{\partial}_{b,k} \quad \text{on } C^\infty(X, L^k) \tag{1.16}$$

and $Q^{(0)}_{M,k}$, $Q^{(1)}_{M,k}$ are formally given by the following. Let s be a local section of L on $D \subset X$, $|s|^2_{h^L} = \mathrm{e}^{-\phi}$, and let $\Phi^t(x)$ be the T-flow. Then,

$$\begin{aligned}\left(Q^{(0)}_{M,k} f\right)(x) &= s^k \mathrm{e}^{\frac{k}{2}\phi(x)} \int \mathrm{e}^{-it\eta}\psi(\eta)\chi\left(\frac{t}{M}\right) \mathrm{e}^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(x))} \widetilde{f}(\Phi^{\frac{t}{k}}(x))\mathrm{d}t\mathrm{d}\eta \quad \text{on } D,\\ \left(Q^{(1)}_{M,k} g\right)(x) &= s^k \mathrm{e}^{\frac{k}{2}\phi(x)} \int \mathrm{e}^{-it\eta}\psi(\eta)\chi\left(\frac{t}{M}\right) \mathrm{e}^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(x))} \widetilde{g}(\Phi^{\frac{t}{k}}(x))\mathrm{d}t\mathrm{d}\eta \quad \text{on } D,\end{aligned} \tag{1.17}$$

where $f = s^k \widetilde{f} \in C^\infty_0(D, L^k)$, $g = s^k \widetilde{g} \in \Omega^{0,1}_0(D, L^k)$. (See Sect. 5, for the precise definitions of the operators $Q^{(0)}_{M,k}$, $Q^{(1)}_{M,k}$.) Let $\langle \cdot | \cdot \rangle_{h^{L^k}}$ denote the Hermitian metric on $\Lambda^{0,q}T^*X \otimes L^k$ induced by h^L and $\langle \cdot | \cdot \rangle$. Let $\mathrm{d}v_X = \mathrm{d}v_X(x)$ be the volume form on X induced by $\langle \cdot | \cdot \rangle$ and let $(\cdot | \cdot)_{h^{L^k}}$ be the L^2 inner product on $\Omega^{0,q}(X, L^k)$ induced by $\langle \cdot | \cdot \rangle_{h^{L^k}}$ and $\mathrm{d}v_X$. For $\lambda \geq 0$, define

$$\begin{aligned}\left(Q^{(0)}_{M,k} \Pi^{(0)}_{k,\leq\lambda}\right)(x) &:= \sum_{j=1}^{m_k} \langle Q^{(0)}_{M,k} f_j(x) | f_j(x) \rangle_{h^{L^k}},\\ \left(Q^{(1)}_{M,k} \Pi^{(1)}_{k,\leq\lambda} \overline{Q^{(1)}_{M,k}})(x\right) &:= \sum_{j=1}^{p_k} \langle Q^{(1)}_{M,k} g_j(x) | Q^{(1)}_{M,k} g_j(x) \rangle_{h^{L^k}},\end{aligned} \tag{1.18}$$

where $f_j(x) \in C^\infty(X, L^k)$, $j = 1, \ldots, m_k$, is an orthonormal frame for the space $\mathscr{H}^0_{b,\leq\lambda}(X, L^k)$ with respect to $(\cdot|\cdot)_{h^{L^k}}$, $g_j(x) \in \Omega^{0,1}(X, L^k)$, $j = 1, \ldots, p_k$, is an orthonormal frame for $\mathscr{H}^1_{b,\leq\lambda}(X, L^k)$ with respect to $(\cdot|\cdot)_{h^{L^k}}$. It is straightforward to see that the definitions (1.18) are independent of the choices of orthonormal frames. The point of our proof is that there exists a sequence $\nu_k > 0$ with $\nu_k \to 0$ as $k \to \infty$, such that

$$\text{For each } x \in X,\ \lim_{k\to\infty} k^{-n}(Q^{(0)}_{M,k} \Pi^{(0)}_{k,\leq k\nu_k})(x) \text{ exists and is real valued,} \tag{1.19}$$

$$\lim_{k\to\infty} k^{-n}(Q^{(0)}_{M,k} \Pi^{(0)}_{k,\leq k\nu_k})(x)$$

$$\geq (2\pi)^{1-n}\int \psi(\xi)\det(M_x^\phi+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)d\xi-\frac{C_1}{M^2},\quad \forall x\in X, \tag{1.20}$$

$$\limsup_{k\to\infty} k^{-n}(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x)\leq \frac{C_1}{M^2},\quad \forall x\in X, \tag{1.21}$$

and

$$\sup\left\{k^{-n}\left|(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)\right|;\ k>0,x\in X\right\}<\infty,$$
$$\sup\left\{k^{-n}(Q^{(1)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x);\ k>0,x\in X\right\}<\infty, \tag{1.22}$$

where $\mathbb{R}_{x,0}$ is given by (2.15) and $\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)=1$ if $\xi\in\mathbb{R}_{x,0}$, $\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)=0$ if $\xi\notin\mathbb{R}_{x,0}$ and $C_1>0$ is a constant independent of k and M.

From (1.22), we can apply Lebesgue dominate theorem and Fatou's lemma and we get using (1.20) and (1.21),

$$\left|\int_X (Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)dv_X(x)\right|$$
$$\geq k^n\Big((2\pi)^{1-n}\int_X\Big(\int\psi(\xi)\det(M_x^\phi+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)d\xi\Big)dv_X(x)-\frac{C_2}{M^2}\Big)+o(k^n), \tag{1.23}$$

$$\int_X(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x)dv_X(x)\leq k^n\frac{C_2}{M^2}+o(k^n), \tag{1.24}$$

where $C_2>0$ is a constant independent of M and k.

Let $f_{1,k},f_{2,k},\ldots,f_{d_k,k}$ be an orthonormal basis for $\mathscr{H}^0_b(X,L^k)$, where $d_k=\dim\mathscr{H}^0_b(X,L^k)$. Let $\widetilde{f}_{1,k},\widetilde{f}_{2,k},\ldots,\widetilde{f}_{n_k,k}$ be an orthonormal basis for the space $\mathscr{H}^0_{b,0<\mu\leq k\nu_k}(X,L^k)$. From (1.23) and (1.18), we see that if M is large enough, then

$$\sum_{j=1}^{d_k}\left|\int_X\langle Q^{(0)}_{M,k}f_{j,k}|f_{j,k}\rangle_{h^{L^k}}(x)dv_X(x)\right|+\sum_{j=1}^{n_k}\left|\int_X\langle Q^{(0)}_{M,k}\widetilde{f}_{j,k}|\widetilde{f}_{j,k}\rangle_{h^{L^k}}(x)dv_X(x)\right|$$
$$\geq\frac{k^n}{2}(2\pi)^{1-n}\int_X\Big(\int\psi(\xi)\det(M_x^\phi+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)d\xi\Big)dv_X(x) \tag{1.25}$$

for k large. From (1.16) and (1.18), it is not difficult to check that

$$\sum_{j=1}^{n_k}\left|\int_X\langle Q^{(0)}_{M,k}\widetilde{f}_{j,k}|\widetilde{f}_{j,k}\rangle_{h^{L^k}}(x)dv_X(x)\right|$$
$$\leq\Big(\int_X(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x)dv_X(x)\Big)^{\frac12}\Big(\sum_{j=1}^{n_k}\int_X\langle\widetilde{f}_{j,k}|\widetilde{f}_{j,k}\rangle_{h^{L^k}}(x)dv_X(x)\Big)^{\frac12}. \tag{1.26}$$

It is well known (see [9, Theorem 1.4]) that

$$\sup\left\{k^{-n}\sum_{j=1}^{n_k}\langle\widetilde{f}_{j,k}|\widetilde{f}_{j,k}\rangle_{h^{L^k}}(x);\ k>0,x\in X\right\}<\infty. \tag{1.27}$$

From (1.27), (1.24), (1.26) and (1.25), it is straightforward to see that if M is large enough, then

$$\sum_{j=1}^{d_k}\left|\int_X \langle Q^{(0)}_{M,k}f_{j,k}|f_{j,k}\rangle_{h^{L^k}}(x)dv_X(x)\right| \geq \frac{k^n}{4}(2\pi)^{1-n}\int_X\left(\int \psi(\xi)\det(M^\phi_x+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)d\xi\right)dv_X(x) \tag{1.28}$$

for k large. Moreover, it is straightforward to see that there is a constant $C_M > 0$ independent of k such that $\left|\int_X \langle Q^{(0)}_{M,k}u|u\rangle_{h^{L^k}}(x)dv_X(x)\right| \leq C_M\int_X\langle u|u\rangle_{h^{L^k}}(x)dv_X(x)$, for all $u\in C^\infty(X,L^k)$. Combining this with (1.28), we have

$$C_Md_k = C_M\sum_{j=1}^{d_k}\int_X\langle f_{j,k}|f_{j,k}\rangle_{h^{L^k}}(x)dv_X(x) \geq \sum_{j=1}^{d_k}\left|\int_X\langle Q^{(0)}_{M,k}f_{j,k}|f_{j,k}\rangle_{h^{L^k}}(x)dv_X(x)\right| \geq \frac{k^n}{4}(2\pi)^{1-n}\int_X\left(\int \psi(\xi)\det(M^\phi_x+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)d\xi\right)dv_X(x).$$

Theorem 1.15 follows.

The paper is organized as follows. In Sect. 2, we review the results in [9] about the asymptotic behavior of the Szegö kernel for lower energy forms to prove (1.19), (1.20) and (1.22). We introduce the extremal function for the space of lower energy forms with respect to a given continuous operator and relate it to the function $Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq\lambda}\overline{Q^{(1)}_{M,k}}$ (see Lemma 2.2). This result will be used in the proof of (1.21). In Sect. 3, we introduce canonical coordinates on generalized Sasakian CR manifolds and prove that locally we can always find canonical coordinates and local section such that the corresponding local weight has a simple form (see Proposition 3.2). Canonical coordinates will be used in the constructions of the operators $Q^{(0)}_{M,k}$ and $Q^{(1)}_{M,k}$ and Proposition 3.2 will be used in Sect. 4 and the proofs of (1.19), (1.20) and (1.21). In Sect. 4, we modify the scaling technique developed in [9] and [10] to establish the semi-classical Kohn estimates (see Propositions 4.2) and a result about the asymptotic behavior of a sequence of forms with small energy (see Proposition 4.3). These results play important roles in the proofs of (1.19), (1.20) and (1.21). In Sect. 5, we construct the operators $Q^{(0)}_{M,k}$ and $Q^{(1)}_{M,k}$. In Sect. 6, we prove (1.22), (1.19), (1.20) and (1.23). In Sect. 7, we prove (1.21) and (1.24). In Sect. 8, we first prove the inequality (1.26) and then we complete the proof of Theorem 1.15. In Sect. 9, we exemplify our main result in two concrete examples, one of a quotient of the Heisenberg group and the other of a Grauert tube over the torus.

2 Szegö kernels for lower energy forms

We will use the same notations as Sect. 1. From now on, we assume that (L, J, h^L) is a rigid generalized Sasakian CR line bundle over X.

The Hermitian fiber metric on L induces a Hermitian fiber metric on L^k that we shall denote by h^{L^k}. If s is a local trivializing section of L then s^k is a local trivializing section of L^k. The Hermitian metrics $\langle\cdot|\cdot\rangle$ on $\Lambda^{0,q}T^*X$ and h^{L^k} induce Hermitian metrics on $\Lambda^{0,q}T^*X\otimes L^k$. We shall denote these induced metrics by $\langle\cdot|\cdot\rangle_{h^{L^k}}$. For $f\in\Omega^{0,q}(X,L^k)$, we denote the pointwise norm $|f(x)|^2_{h^{L^k}} := \langle f(x)|f(x)\rangle_{h^{L^k}}$. As (1.13), let

$$\overline{\partial}_{b,k} : \Omega^{0,q}(X, L^k) \to \Omega^{0,q+1}(X, L^k) \tag{2.1}$$

denote the tangential Cauchy–Riemann operator acting on forms with values in L^k. We denote by $dv_X = dv_X(x)$ the volume form on X induced by the fixed Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$. Then, we get natural global L^2 inner products $(\,|\,)_{h^{L^k}}$, $(\,|\,)$ on $\Omega^{0,q}(X, L^k)$ and $\Omega^{0,q}(X)$, respectively. We denote by $L^2_{(0,q)}(X, L^k)$ the completion of $\Omega^{0,q}(X, L^k)$ with respect to $(\,|\,)_{h^{L^k}}$. For $f \in \Omega^{0,q}(X, L^k)$, we denote $\|f\|^2_{h^{L^k}} := (f \mid f)_{h^{L^k}}$. Similarly, for $f \in \Omega^{0,q}(X)$, we denote $\|f\|^2 := (f \mid f)$. Let

$$\overline{\partial}^*_{b,k} : \Omega^{0,q+1}(X, L^k) \to \Omega^{0,q}(X, L^k) \tag{2.2}$$

be the formal adjoint of $\overline{\partial}_{b,k}$ with respect to $(\,|\,)_{h^{L^k}}$. The *Kohn Laplacian* with values in L^k is given by

$$\Box^{(q)}_{b,k} = \overline{\partial}^*_{b,k}\overline{\partial}_{b,k} + \overline{\partial}_{b,k}\overline{\partial}^*_{b,k} : \Omega^{0,q}(X, L^k) \to \Omega^{0,q}(X, L^k). \tag{2.3}$$

We extend $\overline{\partial}_{b,k}$ to $L^2_{(0,r)}(X, L^k)$, $r = 0, 1, \ldots, n-1$, by

$$\overline{\partial}_{b,k} : \operatorname{Dom}\overline{\partial}_{b,k} \subset L^2_{(0,r)}(X, L^k) \to L^2_{(0,r+1)}(X, L^k), \tag{2.4}$$

where $\operatorname{Dom}\overline{\partial}_{b,k} := \{u \in L^2_{(0,r)}(X, L^k);\ \overline{\partial}_{b,k}u \in L^2_{(0,r+1)}(X, L^k)\}$, where for any $u \in L^2_{(0,r)}(X, L^k)$, $\overline{\partial}_{b,k}u$ is defined in the sense of distribution. We also write

$$\overline{\partial}^*_{b,k} : \operatorname{Dom}\overline{\partial}^*_{b,k} \subset L^2_{(0,r+1)}(X, L^k) \to L^2_{(0,r)}(X, L^k) \tag{2.5}$$

to denote the Hilbert space adjoint of $\overline{\partial}_{b,k}$ in the L^2 space with respect to $(\,|\,)_{h^{L^k}}$. Let $\Box^{(q)}_{b,k}$ also denote the Gaffney extension of the Kohn Laplacian given by

$$\begin{aligned}\operatorname{Dom}\Box^{(q)}_{b,k} = \{s \in L^2_{(0,q)}(X, L^k);\ s \in \operatorname{Dom}\overline{\partial}_{b,k} \cap \operatorname{Dom}\overline{\partial}^*_{b,k},\\ \overline{\partial}_{b,k}u \in \operatorname{Dom}\overline{\partial}^*_{b,k},\ \overline{\partial}^*_{b,k}u \in \operatorname{Dom}\overline{\partial}_{b,k}\},\end{aligned} \tag{2.6}$$

and $\Box^{(q)}_{b,k}s = \overline{\partial}_{b,k}\overline{\partial}^*_{b,k}s + \overline{\partial}^*_{b,k}\overline{\partial}_{b,k}s$ for $s \in \operatorname{Dom}\Box^{(q)}_{b,k}$. We notice that $\Box^{(q)}_{b,k}$ is a positive self-adjoint operator. For a Borel set $B \subset \mathbb{R}$, we denote by $E(B)$ the spectral projection of $\Box^{(q)}_{b,k}$ corresponding to the set B, where E is the spectral measure of $\Box^{(q)}_{b,k}$ (see [5, section 2], for the precise meanings of spectral projection and spectral measure). We notice that the spectrum of $\Box^{(q)}_{b,k}$ is contained in $\overline{\mathbb{R}}_+$. For $\lambda \geq 0$, we set

$$\begin{aligned}\mathscr{H}^q_{b,\leq\lambda}(X, L^k) &:= \operatorname{Range} E\big((-\infty, \lambda]\big) \subset L^2_{(0,q)}(X, L^k),\\ \mathscr{H}^q_{b,>\lambda}(X, L^k) &:= \operatorname{Range} E\big((\lambda, \infty)\big) \subset L^2_{(0,q)}(X, L^k).\end{aligned} \tag{2.7}$$

It is well known (see [5, section 2]) that for all $\lambda > 0$,

$$L^2_{(0,q)}(X, L^k) = \mathscr{H}^q_{b,\leq\lambda}(X, L^k) \oplus \mathscr{H}^q_{b,>\lambda}(X, L^k) \tag{2.8}$$

and

$$\|u\|^2_{h^{L^k}} \leq \frac{1}{\lambda}\left(\Box^{(q)}_{b,k}u \mid u\right)_{h^{L^k}}, \quad \forall u \in \mathscr{H}^q_{b,>\lambda}(X, L^k) \bigcap \operatorname{Dom}\Box^{(q)}_{b,k}. \tag{2.9}$$

For $\lambda = 0$, we denote

$$\mathscr{H}^q_b(X, L^k) := \mathscr{H}^q_{b,\leq 0}(X, L^k) = \operatorname{Ker}\Box^{(q)}_{b,k}. \tag{2.10}$$

Now, fix $q \in \{0, 1, \ldots, n-1\}$ and until further notice we assume that $Y(q)$ holds. By [11, 7.6-7.8], [7, 5.4.11-12], [4, Props. 8.4.8-9] and [8, Chapter 6], we know that $\Box^{(q)}_{b,k}$ is hypoelliptic, has compact resolvent, the strong Hodge decomposition holds and $\Box^{(q)}_{b,k}$ has a discrete spectrum, each eigenvalues occurs with finite multiplicity and all eigenforms are smooth. Hence, for any $\lambda \geq 0$,

$$\dim \mathscr{H}^q_{b,\leq\lambda}(X, L^k) < \infty, \quad \mathscr{H}^q_{b,\leq\lambda}(X, L^k) \subset \Omega^{0,q}(X, L^k), \quad \mathscr{H}^q_b(X, L^k) \cong H^q_b(X, L^k). \tag{2.11}$$

Let $g_j(x) \in \Omega^{0,q}(X, L^k)$, $j = 1, \ldots, d_k$, $d_k = \dim \mathscr{H}^q_{b,\leqslant\lambda}(X, L^k)$, be any orthonormal frame for the space $\mathscr{H}^q_{b,\leq\lambda}(X, L^k)$ with respect to $(\cdot\,|\,\cdot)_{h^{L^k}}$. The Szegö kernel function $\Pi^{(q)}_{k,\leq\lambda}(x)$ of the space $\mathscr{H}^q_{b,\leqslant\lambda}(X, L^k)$ is given by

$$\Pi^{(q)}_{k,\leqslant\lambda}(x) := \sum_{j=1}^{d_k} \left|g_j(x)\right|^2_{h^{L^k}}. \tag{2.12}$$

Let

$$A : \Omega^{0,q}(X, L^k) \to \Omega^{0,q}(X, L^k)$$

be a continuous operator. We define

$$\left(A\Pi^{(q)}_{k,\leqslant\lambda}\right)(x) := \sum_{j=1}^{d_k} \langle Ag_j(x)|g_j(x)\rangle_{h^{L^k}}, \tag{2.13}$$

$$\left(A\Pi^{(q)}_{k,\leqslant\lambda}\overline{A}\right)(x) := \sum_{j=1}^{d_k} \left|Ag_j(x)\right|^2_{h^{L^k}}. \tag{2.14}$$

It is straightforward to see that the definitions (2.12), (2.13) and (2.14) are independent of the choices of orthonormal frame g_j, $j = 1, \ldots, d_k$.

For $q = 0, 1, \ldots, n-1$ and $x \in X$, set

$$\begin{aligned}\mathbb{R}_{x,q} = \big\{s \in \mathbb{R};\ & M^\phi_x + 2s\mathcal{L}_x \text{ has exactly } q \text{ negative eigenvalues} \\ & \text{and } n-1-q \text{ positive eigenvalues}\big\},\end{aligned} \tag{2.15}$$

where M^ϕ_x is given by (1.8) and the eigenvalues of the Hermitian quadratic form $M^\phi_x + 2s\mathcal{L}_x$, $s \in \mathbb{R}$, are calculated with respect to the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$. It is not difficult to see that if $Y(q)$ holds at each point of X then there is a constant $C > 0$ such that

$$\mathbb{R}_{x,q} \subset [-C, C] \quad \text{for all } x \in X. \tag{2.16}$$

Denote by $\det(M^\phi_x + 2s\mathcal{L}_x)$ the product of all the eigenvalues of $M^\phi_x + 2s\mathcal{L}_x$. Assuming (2.16) holds, the function

$$X \longrightarrow \mathbb{R}, \quad x \longmapsto \int_{\mathbb{R}_{x,q}} \left|\det(M^\phi_x + 2s\mathcal{L}_x)\right| \mathrm{d}s \tag{2.17}$$

is well defined. Since M^ϕ_x and $\mathcal{L}_x$ are continuous functions of $x \in X$, we conclude that the function (2.17) is continuous.

The following is well known (see [9, Theorem 1.6])

Theorem 2.1 *Assume that condition $Y(q)$ holds at each point of X. Then, for any sequence $\nu_k > 0$ with $\nu_k \to 0$ as $k \to \infty$, there is a constant $C_0 > 0$ independent of k, such that*

$$k^{-n}\Pi^{(q)}_{k,\leqslant k\nu_k}(x) \leqslant C_0 \tag{2.18}$$

for all $x \in X$. Moreover, there is a sequence $\mu_k > 0$, $\mu_k \to 0$ as $k \to \infty$, such that for any sequence $\nu_k > 0$ with $\lim_{k\to\infty}\frac{\mu_k}{\nu_k} = 0$ and $\nu_k \to 0$ as $k \to \infty$, we have

$$\lim_{k\to\infty} k^{-n}\Pi^{(q)}_{k,\leqslant k\nu_k}(x) = (2\pi)^{-n}\int_{\mathbb{R}_{x,q}} \left|\det(M^\phi_x + 2s\mathcal{L}_x)\right| \mathrm{d}s, \tag{2.19}$$

for all $x \in X$.

We introduce some notations. For $p \in X$, we can choose a smooth orthonormal frame $e_1, \ldots, e_{n-1}$ of $T^{*0,1}X$ over a neighborhood U of p. We say that a multiindex $J = (j_1, \ldots, j_q) \in \{1, \ldots, n-1\}^q$ has length q and write $|J| = q$. We say that J is strictly increasing if $1 \leqslant j_1 < j_2 < \cdots < j_q \leqslant n-1$. For $J = (j_1, \ldots, j_q)$ we define $e_J := e_{j_1} \wedge \cdots \wedge e_{j_q}$. Then, $\{e_J;\ |J| = q,\ J \text{ strictly increasing}\}$ is an orthonormal frame for $\Lambda^{0,q}T^*X$ over U.

For $f \in \Omega^{0,q}(X, L^k)$, we may write

$$f|_U = \sum_{|J|=q}{}' f_J e_J, \quad \text{with } f_J = \langle f|e_J\rangle \in C^\infty(U, L^k),$$

where $\sum'$ means that the summation is performed only over strictly increasing multiindices. We call f_J the component of f along e_J. It will be clear from the context what frame is being used. For $q > 0$, the *extremal function* $S^{(q)}_{k,\leq\lambda,J}$ for the space $\mathscr{H}^q_{b,\leq\lambda}(X, L^k)$ along the direction e_J is defined by

$$S^{(q)}_{k,\leq\lambda,J}(y) = \sup_{\alpha\in\mathscr{H}^q_{b,\leq\lambda}(X,L^k),\ \|\alpha\|_{h^{L^k}}=1} |\alpha_J(y)|^2_{h^{L^k}}, \tag{2.20}$$

where α_J denotes the component of α along e_J. Let

$$A : \Omega^{0,q}(X, L^k) \to \Omega^{0,q}(X, L^k)$$

be a continuous operator. For $|J| = q$, J is strictly increasing, we define

$$(AS^{(q)}_{k,\leqslant\lambda,J}\overline{A})(y) := \sup_{\alpha\in\mathscr{H}^q_{b,\leq\lambda}(X,L^k),\ \|\alpha\|_{h^{L^k}}=1} |(A\alpha)_J(y)|^2_{h^{L^k}}, \tag{2.21}$$

where $(A\alpha)_J$ denotes the component of $A\alpha$ along e_J. Similarly, when $q = 0$, we define

$$S^{(0)}_{k,\leq\lambda}(y) = \sup_{\alpha\in\mathscr{H}^0_{b,\leq\lambda}(X,L^k),\ \|\alpha\|_{h^{L^k}}=1} |\alpha(y)|^2_{h^{L^k}},$$
$$(AS^{(0)}_{k,\leqslant\lambda}\overline{A})(y) := \sup_{\alpha\in\mathscr{H}^0_{b,\leq\lambda}(X,L^k),\ \|\alpha\|_{h^{L^k}}=1} |(A\alpha)(y)|^2_{h^{L^k}}. \tag{2.22}$$

We need the following

Lemma 2.2 *Fix $\lambda \geq 0$. Let $A : \Omega^{0,q}(X, L^k) \to \Omega^{0,q}(X, L^k)$ be a continuous operator. For every local orthonormal frame*

$$\{e_J(y);\ |J| = q,\ J \text{ strictly increasing}\}$$

of $\Lambda^{0,q}T^*X$ over an open set $U \subset X$, we have when $q > 0$,

$$\left(A\Pi^{(q)}_{k,\leq\lambda}\overline{A}\right)(y)=\sum_{|J|=q}{}^{'}\left(AS^{(q)}_{k,\leq\lambda,J}\overline{A}\right)(y),\tag{2.23}$$

for every $y\in U$.

Similarly, when $q=0$, *we have*

$$\left(A\Pi^{(0)}_{k,\leq\lambda}\overline{A}\right)(y)=\left(AS^{(0)}_{k,\leq\lambda}\overline{A}\right)(y),\tag{2.24}$$

for every $y\in U$.

We remind that $A\Pi^{(q)}_{k,\leq\lambda}\overline{A}$ *is given by* (2.14).

Proof Let $(f_j)_{j=1,\ldots,d_k}$ be an orthonormal frame for the space $\mathscr{H}^q_{b,\leq\lambda}(X,L^k)$. Let s be a local section of L on U, $|s|^2_{h^L}=\mathrm{e}^{-\phi}$. On U, we write

$$\begin{aligned}&Af_j=s^k\widetilde{g}_j,\quad \widetilde{g}_j\in\Omega^{0,q}(U),\quad j=1,\ldots,d_k,\\&\widetilde{g}_j=\sum_{|J|=q}{}^{'}\widetilde{g}_{j,J}e_J,\quad j=1,\ldots,d_k.\end{aligned}$$

On U, we write

$$\left(A\Pi^{(q)}_{k,\leq\lambda}\overline{A}\right)(y)=\sum_{|J|=q}{}^{'}\left(A\Pi^{(q)}_{k,\leq\lambda,J}\overline{A}\right)(y),\tag{2.25}$$

where

$$(A\Pi^{(q)}_{k,\leq\lambda,J}\overline{A})(y):=\mathrm{e}^{-\phi(y)}\sum_j\left|\widetilde{g}_{j,J}(y)\right|^2.$$

It is easy to see that $(A\Pi^{(q)}_{k,\leq\lambda,J}\overline{A})(y)$ is independent of the choice of the orthonormal frame $(f_j)_{j=1,\ldots,d_k}$. Take $\alpha\in\mathscr{H}^q_{b,\leq\lambda}(X,L^k)$ of unit norm. Since α is contained in an orthonormal base, obviously $|(A\alpha)_J(y)|^2_{h^{L^k}}\leqslant(A\Pi^{(q)}_{k,\leq\lambda,J}\overline{A})(y)$, where $(A\alpha)_J$ denotes the component of $A\alpha$ along e_J. Thus,

$$(AS^{(q)}_{k,\leq\lambda,J}\overline{A})(y)\leqslant(A\Pi^{(q)}_{k,\leq\lambda,J}\overline{A})(y),\quad\text{for all strictly increasing } J, |J|=q.\tag{2.26}$$

Fix a point $p\in U$ and a strictly increasing multiindex J with $|J|=q$. We may assume that $\sum_{j=1}^{d_k}\left|\widetilde{g}_{j,J}(p)\right|^2\neq 0$. Put

$$u(y)=\left(\sum_{j=1}^{d_k}\left|\widetilde{g}_{j,J}(p)\right|^2\right)^{-1/2}\cdot\sum_{j=1}^N\overline{\widetilde{g}_{j,J}(p)}f_j(y).$$

We can easily check that $u\in\mathscr{H}^q_{b,\leq\lambda}(X,L^k)$ and $\|u\|_{h^{L^k}}=1$. Hence,

$$|(Au)_J(p)|^2_{h^{L^k}}\leqslant(AS^{(q)}_{k,\leq\lambda,J}\overline{A})(p),$$

therefore,

$$\left(A\Pi^{(q)}_{k,\leq\lambda,J}\overline{A}\right)(p)=\sum_{j=1}^{d_k}\mathrm{e}^{-\phi(p)}\left|\widetilde{g}_{j,J}(p)\right|^2=|(Au)_J(p)|^2_{h^{L^k}}\leqslant(AS^{(q)}_{k,\leq\lambda,J}\overline{A})(p).$$

From this and (2.26), we conclude that $A\Pi^{(q)}_{k,\leq\lambda,J}\overline{A}=AS^{(q)}_{k,\leq\lambda,J}\overline{A}$ for all strictly increasing multiindices J with $|J|=q$. Combining this with (2.25), (2.23) follows.

The proof of (2.24) is the same. The lemma follows. □

3 Canonical coordinates of generalized Sasakian CR manifolds

In this work, we need the following beautiful result due to Baouendi–Rothschild–Treves [1, section1]

Theorem 3.1 *We recall that we work with the assumption that* $(X, T^{1,0}X)$ *is a generalized Sasakian CR manifold and we fix a rigid global real vector field* $T = J\frac{\partial}{\partial t}$. *For every point* $x_0 \in X$, *there exists local coordinates* $x = (x_1, \ldots, x_{2n-1}) = (z, \theta) = (z_1, \ldots, z_{n-1}, \theta)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, $\theta = x_{2n-1}$, *defined in some small neighborhood* U *of* x_0 *such that*

$$T = \frac{\partial}{\partial \theta},$$
$$Z_j = \frac{\partial}{\partial z_j} + i\frac{\partial \varphi}{\partial z_j}(z)\frac{\partial}{\partial \theta}, \quad j = 1, \ldots, n-1, \tag{3.1}$$

where $Z_j(x)$, $j = 1, \ldots, n-1$, *form a basis of* $T^{1,0}_x X$, *for each* $x \in U$, *and* $\varphi(z) \in C^\infty(U, \mathbb{R})$ *independent of* θ.

Let $x = (x_1, \ldots, x_{2n-1})$ be local coordinates of X defined in some open set in X. In this paper, when we write $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$ we mean that $z = (z_1, \ldots, z_{n-1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, $\theta = x_{2n-1}$. We call x canonical coordinates if x satisfies (3.1).

We also need

Proposition 3.2 *For a given point* $p \in X$, *we can find canonical coordinates* $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$ *and local section* s, $|s|^2_{h^L} = e^{-\phi}$, *defined in some small neighborhood* D *of* p *such that*

$$\begin{aligned}
&x(p) = 0,\\
&Z_j = \frac{\partial}{\partial z_j} + i\lambda_j \overline{z}_j \frac{\partial}{\partial \theta} + O(|z|^2)\frac{\partial}{\partial \theta}, \quad j = 1, \ldots, n-1,\\
&\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n-1}} \text{ is an orthonormal frame for } T^{1,0}_p X,\\
&\phi(z, \theta) = \beta\theta + \sum_{j,t=1}^{n-1} \mu_{j,t}\overline{z}_j z_t + O(|z|\,|\theta|) + O(|\theta|^2) + O(|(z, \theta)|^3),
\end{aligned} \tag{3.2}$$

where $Z_1(x), \ldots, Z_{n-1}(x)$ *form a basis of* $T^{1,0}_x X$ *varying smoothly with* x *in a neighborhood of* p, $\lambda_1, \ldots, \lambda_{n-1}$ *are the eigenvalues of* $\mathcal{L}_p$ *with respect to* $\langle \cdot | \cdot \rangle$, $\beta \in \mathbb{R}$, $\mu_{j,t} \in \mathbb{C}$, $\mu_{j,t} = \overline{\mu_{t,j}}$, $j, t = 1, \ldots, n-1$.

Proof Fix $p \in X$. Let $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_{2n-1}) = (\widetilde{z}, \widetilde{\theta}) = (\widetilde{z}_1, \ldots, \widetilde{z}_{n-1}, \widetilde{\theta})$, $\widetilde{z}_j = \widetilde{x}_{2j-1} + i\widetilde{x}_{2j}$, $j = 1, \ldots, n-1$, $\widetilde{\theta} = \widetilde{x}_{2n-1}$ be canonical coordinates of X defined in some small neighborhood D of p. We have

$$T = \frac{\partial}{\partial \widetilde{\theta}},$$
$$Z_j = \frac{\partial}{\partial \widetilde{z}_j} + i\frac{\partial \widetilde{\varphi}}{\partial \widetilde{z}_j}(\widetilde{z})\frac{\partial}{\partial \widetilde{\theta}}, \quad j = 1, \ldots, n-1, \tag{3.3}$$

where $Z_j(\widetilde{x})$, $j = 1, \ldots, n-1$, form a basis of $T^{1,0}_{\widetilde{x}} X$, for each $\widetilde{x} \in D$, and $\widetilde{\varphi}(\widetilde{z}) \in C^\infty(D, \mathbb{R})$ independent of $\widetilde{\theta}$. It is easy to see that we can take $\widetilde{x}$ so that $\widetilde{x}(p) = 0$. Near p, we write

$$\widetilde{\varphi}(\widetilde{z}) = a + \sum_{j=1}^{n-1}(\alpha_j \widetilde{z}_j + \overline{\alpha}_j \overline{\widetilde{z}_j}) + O(|\widetilde{z}|^2), \tag{3.4}$$

where $a \in \mathbb{C}$, $\alpha_j \in \mathbb{C}$, $j = 1, \ldots, n-1$. Let $\hat{z} = \widetilde{z}$, $\hat{\theta} = \widetilde{\theta} - \sum_{j=1}^{n-1}(i\alpha_j \widetilde{z}_j - i\overline{\alpha}_j \overline{\widetilde{z}_j})$. Then, $(\hat{z}, \hat{\theta})$ form canonical coordinates of X near p and we can check that

$$\begin{aligned} \frac{\partial}{\partial \hat{\theta}} &= \frac{\partial}{\partial \widetilde{\theta}}, \\ \frac{\partial}{\partial \hat{z}_j} &= \frac{\partial}{\partial \widetilde{z}_j} + i\alpha_j \frac{\partial}{\partial \widetilde{\theta}}, \quad j = 1, \ldots, n-1. \end{aligned} \tag{3.5}$$

From (3.5), (3.4) and (3.3), we see that

$$\begin{aligned} T &= \frac{\partial}{\partial \hat{\theta}}, \\ Z_j &= \frac{\partial}{\partial \hat{z}_j} + i\frac{\partial \hat{\varphi}}{\partial \hat{z}_j}(\hat{z})\frac{\partial}{\partial \hat{\theta}}, \quad j = 1, \ldots, n-1, \end{aligned} \tag{3.6}$$

where

$$\hat{\varphi}(\hat{z}) = \widetilde{\varphi}(\hat{z}) - \sum_{j=1}^{n-1}\left(\alpha_j \hat{z}_j + \overline{\alpha}_j \overline{\hat{z}_j}\right) = a + O\left(\left|\hat{z}\right|^2\right).$$

Thus, $\frac{\partial}{\partial \hat{z}_1}, \ldots, \frac{\partial}{\partial \hat{z}_{n-1}}$ is a basis of $T_p^{1,0}X$. By taking some linear transformation, we can take $\hat{z}$ so that $\frac{\partial}{\partial \hat{z}_j}$, $j = 1, \ldots, n-1$, is an orthonormal frame for $T_p^{1,0}X$ and the Levi form is diagonal at p with respect to $\frac{\partial}{\partial \hat{z}_j}$, $j = 1, \ldots, n-1$. We write

$$\hat{\varphi}(\hat{z}) = \alpha + \sum_{j,t=1}^{n-1}(\beta_{j,t}\hat{z}_j\hat{z}_t + \overline{\beta}_{j,t}\overline{\hat{z}_j}\,\overline{\hat{z}_t}) + \sum_{j,t=1}^{n-1}\gamma_{j,t}\overline{\hat{z}_j}\hat{z}_t + O(\left|\hat{z}\right|^3), \tag{3.7}$$

where $\beta_{j,t} \in \mathbb{C}$, $\gamma_{j,t} \in \mathbb{C}$, $\gamma_{j,t} = \overline{\gamma_{t,j}}$, $j, t = 1, \ldots, n-1$. Since the Levi form is diagonal at p with respect to $\frac{\partial}{\partial \hat{z}_j}$, $j = 1, \ldots, n-1$, we can check that

$$\gamma_{j,t} = \lambda_j \delta_{j,t}, \quad j, t = 1, \ldots, n-1, \tag{3.8}$$

where $\lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $\mathcal{L}_p$ with respect to $\langle \cdot | \cdot \rangle$. Let $z = \hat{z}$, $\theta = \hat{\theta} - \sum_{j,t=1}^{n-1} i(\beta_{j,t}\hat{z}_j\hat{z}_t - \overline{\beta}_{j,t}\overline{\hat{z}_j}\,\overline{\hat{z}_t})$. Then, (z, θ) form canonical coordinates of X near p and we can check that

$$\begin{aligned} \frac{\partial}{\partial \theta} &= \frac{\partial}{\partial \hat{\theta}}, \\ \frac{\partial}{\partial z_j} &= \frac{\partial}{\partial \hat{z}_j} + i\sum_{t=1}^{n-1}\beta_{j,t}\hat{z}_t\frac{\partial}{\partial \hat{\theta}}, \quad j = 1, \ldots, n-1. \end{aligned} \tag{3.9}$$

From (3.6), (3.7), (3.8) and (3.9), we can check that

$$\begin{aligned} T &= \frac{\partial}{\partial \theta}, \\ Z_j &= \frac{\partial}{\partial z_j} + i\lambda_j \overline{z}_j \frac{\partial}{\partial \theta} + O(|z|^2)\frac{\partial}{\partial \theta}, \quad j = 1, \ldots, n-1. \end{aligned}$$

Since $\frac{\partial}{\partial z_j}$, $j = 1, \ldots, n-1$, is an orthonormal frame of $T_p^{1,0}X$, we conclude that $x = (z, \theta)$ satisfies the first three properties in (3.2).

Let $\hat{s}$ be a local section defined in some neighborhood of p, $\left|\hat{s}\right|^2_{h^L} = e^{-\hat{\phi}}$. Near p, we write

$$\hat{\phi}(z,\theta) = c + \beta\theta + \sum_{j=1}^{n-1}(a_j z_j + \overline{a}_j \overline{z}_j) + O(|(z,\theta)|^2), \tag{3.10}$$

where $c \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $a_j \in \mathbb{C}$, $j = 1, \ldots, n-1$. Let

$$g(z) = e^{\frac{c}{2}}\left(1 + \sum_{j=1}^{n-1} a_j z_j\right). \tag{3.11}$$

Then, $g(z)$ is a rigid CR function. We may replace $\hat{s}$ by $g\hat{s} := \widetilde{s}$. We have

$$|\widetilde{s}|^2_{h^L} = e^{-\widetilde{\phi}} = |g|^2 e^{-\hat{\phi}} = e^{2\log|g| - \hat{\phi}}. \tag{3.12}$$

From (3.11), we can check that

$$2\log|g| = c + \sum_{j=1}^{n-1}(a_j z_j + \overline{a}_j \overline{z}_j) + O(|z|^2).$$

Combining this with (3.12) and (3.10), we conclude that

$$\widetilde{\phi}(z,\theta) = \beta\theta + O(|(z,\theta)|^2).$$

Near p, we write

$$\begin{aligned}\widetilde{\phi}(z,\theta) = \beta\theta &+ \sum_{j,t=1}^{n-1}(c_{j,t} z_j z_t + \overline{c_{j,t}}\,\overline{z}_j \overline{z}_t) + \sum_{j,t=1}^{n} \mu_{j,t} \overline{z}_j z_t \\ &+ O(|z|\,|\theta|) + O(|\theta|^2) + O(|(z,\theta)|^3),\end{aligned} \tag{3.13}$$

where $c_{j,t} \in \mathbb{C}$, $\mu_{j,t} \in \mathbb{C}$, $\mu_{j,t} = \overline{\mu_{t,j}}$, $j, t = 1, \ldots, n-1$. Let

$$g_1(z) = 1 + \sum_{j,t=1}^{n-1} c_{j,t} z_j z_t. \tag{3.14}$$

Then, $g_1(z)$ is a rigid CR function. We may replace $\widetilde{s}$ by $g_1\widetilde{s} := s$. We have

$$|s|^2_{h^L} = e^{-\phi} = |g_1|^2 e^{-\widetilde{\phi}} = e^{2\log|g_1| - \widetilde{\phi}}. \tag{3.15}$$

From (3.14), we can check that

$$2\log|g_1| = \sum_{j,t=1}^{n-1}(c_{j,t} z_j z_t + \overline{c_{j,t}}\,\overline{z}_j \overline{z}_t) + O(|z|^3).$$

Combining this with (3.15) and (3.13), we conclude that

$$\phi(z,\theta) = \beta\theta + \sum_{j,t=1}^{n-1} \mu_{j,t}\overline{z}_j z_t + O(|z|\,|\theta|) + O(|\theta|^2) + O(|(z,\theta)|^3).$$

The proposition follows. □

Proposition 3.3 *We assume that X is a generalized torus CR manifold and (L, J) is an admissible generalized Sasakian CR line bundle over X (see Definition 1.17 and Definition 1.18). Let ϕ and ϕ_1 be as in the discussion after Definition 1.18. If M_x^{ϕ} is positive on X, then $M_x^{\phi_1}$ is positive on X.*

Proof Let $\left\{W_1 \subset W_1', \ldots, W_N \subset W_N'\right\}$ be open sets of X such that $X = \bigcup_{j=1}^{N} W_j$ and there exist canonical coordinates on W_j', for each j and there is a constant $\epsilon_0 > 0$ such that for each $x \in X$, $\Phi^t(x)$ is well defined, $\forall\, |t| \leq \epsilon_0$, and

$$\left\{\Phi^t(x);\ x \in W_j, |t| \leq \epsilon_0\right\} \subset W_j',$$

for each j, where $\Phi^t(x)$ is the T-flow. Fix $t_0 \in [-\epsilon_0, \epsilon_0]$. Put $\widetilde{\phi}(x) = \phi(\Phi^{t_0} x)$. It is obviously that $\widetilde{\phi}(x)$ also define a Hermitian fiber metric on L. Using canonical coordinates (3.1), we can check that

$$d(\overline{\partial}_b \widetilde{\phi} - \partial_b \widetilde{\phi})(x) = d(\overline{\partial}_b \phi - \partial \phi)(\Phi^{t_0} x), \quad \forall x \in X.$$

Thus,

$$M_x^{\widetilde{\phi}} = M_{\Phi^{t_0}(x)}^{\phi}, \quad \forall x \in X.$$

Similarly, fix $t_1 \in [-\epsilon_0, \epsilon_0]$ and put $\hat{\phi}(x) = \widetilde{\phi}(\Phi^{t_1} x) = \phi(\Phi^{t_0+t_1} x)$. We have

$$M_x^{\hat{\phi}} = M_{\Phi^{t_1}(x)}^{\widetilde{\phi}} = M_{\Phi^{t_0+t_1}(x)}^{\phi}, \quad \forall x \in X.$$

Continuing in this way, we obtain for any $t \in [0, \gamma_0]$, $M_x^{\phi(\Phi^t(x))} = M_{\Phi^t(x)}^{\phi}$, $\forall x \in X$, where γ_0 is as in Definition 1.17. Thus,

$$M_x^{\phi_1} = \frac{1}{\gamma_0} \int_0^{\gamma_0} M_{\Phi^t(x)}^{\phi} \mathrm{d}t, \quad \forall x \in X.$$

The proposition follows. □

4 The scaling technique

In this section, we modify the scaling technique developed in [9] and [10] to prove (1.19), (1.20) and (1.21).

Fix a point $p \in X$. Let $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$ be canonical coordinates of X defined in some small neighborhood D of p and let s be a local section of L on D, $|s|_{h^L}^2 = \mathrm{e}^{-\phi}$. We take x and s so that

$$\begin{aligned}
& x(p) = 0, \\
& Z_j = \frac{\partial}{\partial z_j} + i\lambda_j \overline{z}_j \frac{\partial}{\partial \theta} + O(|z|^2)\frac{\partial}{\partial \theta}, \quad j = 1, \ldots, n-1, \\
& \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n-1}} \text{ is an orthonormal frame for } T_p^{1,0}X, \\
& \phi(z, \theta) = \beta\theta + \sum_{j,t=1}^{n-1} \mu_{j,t} \overline{z}_j z_t + O(|z|\,|\theta|) + O(|\theta|^2) + O(|(z, \theta)|^3),
\end{aligned} \tag{4.1}$$

where $Z_1(x), \ldots, Z_{n-1}(x)$ form a basis of $T_x^{1,0}X$ varying smoothly with x in a neighborhood of p, $\lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $\mathcal{L}_p$ with respect to $\langle \cdot | \cdot \rangle$, $\beta \in \mathbb{R}$, $\mu_{j,k} \in \mathbb{C}$,

$\mu_{j,t} = \overline{\mu_{t,j}}$, $j, t = 1, \ldots, n-1$. By Proposition 3.2, this is always possible. Fix $q \in \{0, 1, \ldots, n-1\}$. In this section, we work on $(0, q)$ forms and we work with this local coordinates $x = (z, \theta)$.

Let $(\,|\,)_{k\phi}$ be the inner product on the space $\Omega_0^{0,q}(D)$ defined as follows:

$$(f \mid g)_{k\phi} = \int_D \langle f|g\rangle e^{-k\phi} dv_X,$$

where $f, g \in \Omega_0^{0,q}(D)$. Let $\overline{\partial}_b^{*,k\phi} : \Omega^{0,q+1}(D) \to \Omega^{0,q}(D)$ be the formal adjoint of $\overline{\partial}_b$ with respect to $(\,|\,)_{k\phi}$. Put

$$\Box_{b,k\phi}^{(q)} = \overline{\partial}_b \overline{\partial}_b^{*,k\phi} + \overline{\partial}_b^{*,k\phi} \overline{\partial}_b : \Omega^{0,q}(D) \to \Omega^{0,q}(D).$$

Let $u \in \Omega^{0,q}(D, L^k)$. Then, there exists $\widetilde{u} \in \Omega^{0,q}(D)$ such that $u = s^k \widetilde{u}$ and we have

$$\Box_{b,k}^{(q)} u = s^k \Box_{b,k\phi}^{(q)} \widetilde{u}. \tag{4.2}$$

Let $U_1(z,\theta), \ldots, U_{n-1}(z,\theta)$ be an orthonormal frame of $T_{(z,\theta)}^{1,0}X$ varying smoothly with (z,θ) in a neighborhood of p. We take $U_1, \ldots, U_{n-1}$ so that $U_j(0,0) = \frac{\partial}{\partial z_j}$, $j = 1, \ldots, n-1$. Put

$$U_j(z,\theta) = \sum_{t=1}^{n-1} a_{j,t}(z,\theta) Z_t, \quad j = 1, \ldots, n-1, \tag{4.3}$$

where $a_{j,t} \in C^\infty$, $j, t = 1, \ldots, n-1$, $Z_1, \ldots, Z_{n-1}$ are as in (4.1). Then, we have

$$a_{j,t}(z,\theta) = \delta_{j,t} + O(|(z,\theta)|), \quad j, t = 1, \ldots, n-1. \tag{4.4}$$

Let $(e_j(z,\theta))_{j=1,\ldots,n-1}$ denote the basis of $T_{(z,\theta)}^{*0,1}X$, dual to $(\overline{U}_j(z,\theta))_{j=1,\ldots,n-1}$. If $w \in T_z^{*0,1}X$, let $(w\wedge)^* : \Lambda^{0,q+1}T_z^*X \to \Lambda^{0,q}T_z^*X$, $q \geqslant 0$, be the adjoint of the left exterior multiplication $w\wedge : \Lambda^{0,q}T_z^*X \to \Lambda^{0,q+1}T_z^*X$, $u \mapsto w \wedge u$:

$$\langle w \wedge u|v\rangle = \langle u|(w\wedge)^* v\rangle, \tag{4.5}$$

for all $u \in \Lambda^{0,q}T_z^*X$, $v \in \Lambda^{0,q+1}T_z^*X$. Notice that $(w\wedge)^*$ depends $\mathbb{C}$-anti-linearly on w. It is easy to see that

$$\overline{\partial}_b = \sum_{j=1}^{n-1} e_j \wedge \overline{U}_j + \sum_{j=1}^{n-1} (\overline{\partial}_b e_j) \wedge (e_j\wedge)^* \tag{4.6}$$

and correspondingly

$$\overline{\partial}_b^{*,k\phi} = \sum_{j=1}^{n-1} (e_j\wedge)^* \overline{U}_j^{*,k\phi} + \sum_{j=1}^{n-1} e_j \wedge (\overline{\partial}_b e_j\wedge)^*, \tag{4.7}$$

where $\overline{U}_j^{*,k\phi}$ is the formal adjoint of $\overline{U}_j$ with respect to $(\,|\,)_{k\phi}$, $j = 1, \ldots, n-1$. We can check that for $j = 1, \ldots, n-1$,

$$\overline{U}_j^{*,k\phi} = -U_j + k(U_j\phi) + s_j(z,\theta), \tag{4.8}$$

where $s_j \in C^\infty(D)$, s_j is independent of k, $j = 1, \ldots, n-1$.

For $r>0$, let $D_r=\{x=(z,\theta)\in\mathbb{R}^{2n-1};\ |x_j|<r,\ j=1,\ldots,2n-1\}$. Let F_k be the scaling map: $F_k(z,\theta)=(\frac{z}{\sqrt{k}},\frac{\theta}{k})$. From now on, we assume that k is large enough so that $F_k(D_{\log k})\subset D$. We define the scaled bundle $F_k^*\Lambda^{0,q}T^*X$ on $D_{\log k}$ to be the bundle whose fiber at $(z,\theta)\in D_{\log k}$ is

$$F_k^*\Lambda^{0,q}T^*_{(z,\theta)}X:=\Big\{\sum\nolimits'_{|J|=q}a_Je_J\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big);\ a_J\in\mathbb{C},|J|=q\Big\}.$$

We take the Hermitian metric $\langle\cdot|\cdot\rangle_{F_k^*}$ on $F_k^*\Lambda^{0,q}T^*X$ so that at each point $(z,\theta)\in D_{\log k}$,

$$\Big\{e_J\big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\big);\ |J|=q,\ J\text{ strictly increasing}\Big\},$$

is an orthonormal basis for $F_k^*\Lambda^{0,q}T^*_{(z,\theta)}X$. For $r>0$, let $F_k^*\Omega^{0,q}(D_r)$ denote the space of smooth sections of $F_k^*\Lambda^{0,q}T^*X$ over D_r. Let $F_k^*\Omega_0^{0,q}(D_r)$ be the subspace of $F_k^*\Omega^{0,q}(D_r)$ whose elements have compact support in D_r. Given $f\in\Omega^{0,q}(F_k(D_{\log k}))$, we write $f=\sum'_{|J|=q}f_Je_J$. We define the scaled form $F_k^*f\in F_k^*\Omega^{0,q}(D_{\log k})$ by:

$$F_k^*f=\sum_{|J|=q}{}'f_J\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)e_J\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big).\tag{4.9}$$

Let P be a partial differential operator of order one on $F_k(D_{\log k})$ with C^∞ coefficients. We write $P=a(z,\theta)\frac{\partial}{\partial\theta}+\sum_{j=1}^{2n-2}a_j(z,\theta)\frac{\partial}{\partial x_j}$, $a,a_j\in C^\infty(F_k(D_{\log k}))$, $j=1,\ldots,2n-2$. The partial differential operator $P_{(k)}$ on $D_{\log k}$ is given by

$$P_{(k)}=\sqrt{k}F_k^*a\frac{\partial}{\partial\theta}+\sum_{j=1}^{2n-2}F_k^*a_j\frac{\partial}{\partial x_j}=\sqrt{k}a\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\frac{\partial}{\partial\theta}+\sum_{j=1}^{2n-2}a_j\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\frac{\partial}{\partial x_j}.\tag{4.10}$$

Let $f\in C^\infty(F_k(D_{\log k}))$. We can check that

$$P_{(k)}(F_k^*f)=\frac{1}{\sqrt{k}}F_k^*(Pf).\tag{4.11}$$

The scaled differential operator $\overline{\partial}_{b,(k)}:F_k^*\Omega^{0,q}(D_{\log k})\to F_k^*\Omega^{0,q+1}(D_{\log k})$ is given by (compare to the formula (4.6) for $\overline{\partial}_b$):

$$\begin{aligned}\overline{\partial}_{b,(k)}=&\sum_{j=1}^{n-1}e_j\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\wedge\overline{U}_{j(k)}\\&+\sum_{j=1}^{n-1}\frac{1}{\sqrt{k}}(\overline{\partial}_be_j)\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\wedge\Big(e_j\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\wedge\Big)^*.\end{aligned}\tag{4.12}$$

From (4.6) and (4.11), we can check that if $f\in\Omega^{0,q}(F_k(D_{\log k}))$, then

$$\overline{\partial}_{b,(k)}F_k^*f=\frac{1}{\sqrt{k}}F_k^*(\overline{\partial}_bf).\tag{4.13}$$

Let $(\ |\ )_{kF_k^*\phi}$ be the inner product on the space $F_k^*\Omega_0^{0,q}(D_{\log k})$ defined as follows:

$$(f\mid g)_{kF_k^*\phi}=\int_{D_{\log k}}\langle f|g\rangle_{F_k^*}e^{-kF_k^*\phi}(F_k^*m)(z,\theta)dv(z)d\theta,$$

where $dv_X = m dv(z)d\theta$ is the volume form, $dv(z) = 2^{n-1}dx_1 \cdots dx_{2n-2}$. Note that $m(0,0) = 1$. Let $\overline{\partial}^{*,k\phi}_{b,(k)} : F^*_k\Omega^{0,q+1}(D_{\log k}) \to F^*_k\Omega^{0,q}(D_{\log k})$ be the formal adjoint of $\overline{\partial}_{b,(k)}$ with respect to $(\,|\,)_{kF^*_k\phi}$. Then, we can check that [compare the formula for $\overline{\partial}^{*,k\phi}_b$, see (4.7) and (4.8)]

$$\begin{aligned}\overline{\partial}^{*,k\phi}_{b,(k)} &= \sum_{j=1}^{n-1}\Big(e_j\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\wedge\Big)^*\Big(-U_{j(k)}+\sqrt{k}F^*_k(U_j\phi)+\frac{1}{\sqrt{k}}F^*_k s_j\Big)\\ &\quad+\sum_{j=1}^{n-1}\frac{1}{\sqrt{k}}\,e_j\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\wedge\Big((\overline{\partial}_b e_j)\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\wedge\Big)^*,\end{aligned} \tag{4.14}$$

where $s_j \in C^\infty(D_{\log k})$, $j = 1, \ldots, n-1$, are independent of k. We also have

$$\overline{\partial}^{*,k\phi}_{b,(k)} F^*_k f = \frac{1}{\sqrt{k}} F^*_k(\overline{\partial}^{*,k\phi}_b f), \quad f \in \Omega^{0,q+1}(F_k(D_{\log k})). \tag{4.15}$$

We define now the *scaled Kohn Laplacian*:

$$\Box^{(q)}_{b,k\phi,(k)} := \overline{\partial}^{*,k\phi}_{b,(k)}\overline{\partial}_{b,(k)} + \overline{\partial}_{b,(k)}\overline{\partial}^{*,k\phi}_{b,(k)} : F^*_k\Omega^{0,q}(D_{\log k}) \to F^*_k\Omega^{0,q}(D_{\log k}). \tag{4.16}$$

From (4.13) and (4.15), we see that if $f \in \Omega^{0,q}(F_k(D_{\log k}))$, then

$$\Big(\Box^{(q)}_{b,k\phi,(k)}\Big) F^*_k f = \frac{1}{k} F^*_k(\Box^{(q)}_{b,k\phi} f). \tag{4.17}$$

From (4.3), (4.4) and (4.1), we can check that

$$\overline{U}_{j(k)} = \frac{\partial}{\partial \overline{z}_j} - i\lambda_j z_j \frac{\partial}{\partial\theta} + \epsilon_k Z_{j,k}, \quad j = 1, \ldots, n-1, \tag{4.18}$$

on $D_{\log k}$, where ϵ_k is a sequence tending to zero with $k \to \infty$ and $Z_{j,k}$ is a first order differential operator and all the derivatives of the coefficients of $Z_{j,k}$ are uniformly bounded in k on $D_{\log k}$, $j = 1, \ldots, n-1$. Similarly, from (4.3), (4.4) and (4.1), it is straightforward to see that

$$\begin{aligned}&-U_{t(k)} + \sqrt{k}F^*_k(U_t\phi) + \frac{1}{\sqrt{k}}F^*_k s_t\\ &= -\frac{\partial}{\partial z_t} - i\lambda_t\overline{z}_t\frac{\partial}{\partial\theta} + i\lambda_t\overline{z}_t\beta + \sum_{j=1}^{n-1}\mu_{j,t}\,\overline{z}_j + \delta_k V_{t,k}, \quad t = 1, \ldots, n-1,\end{aligned} \tag{4.19}$$

on $D_{\log k}$, where δ_k is a sequence tending to zero with $k \to \infty$ and $V_{t,k}$ is a first order differential operator and all the derivatives of the coefficients of $V_{t,k}$ are uniformly bounded in k on $D_{\log k}$, $t = 1, \ldots, n-1$. From (4.19), (4.18) and (4.16), (4.14), (4.12), it is straightforward to obtain the following

Proposition 4.1 *We have that*

$$\begin{aligned}\Box^{(q)}_{b,k\phi,(k)} &= \sum_{j=1}^{n-1}\Big[\Big(-\frac{\partial}{\partial z_j} - i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta} + i\lambda_j\overline{z}_j\beta + \sum_{t=1}^{n-1}\mu_{t,j}\,\overline{z}_t\Big)\Big(\frac{\partial}{\partial\overline{z}_j} - i\lambda_j z_j\frac{\partial}{\partial\theta}\Big)\Big]\\ &\quad+\sum_{j,t=1}^{n-1} e_j\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\wedge\Big(e_t\Big(\frac{z}{\sqrt{k}},\frac{\theta}{k}\Big)\wedge\Big)^*\Big(\Big(\mu_{j,t}+i\lambda_j\delta_{j,t}\beta\Big)-2i\lambda_j\delta_{j,t}\frac{\partial}{\partial\theta}\Big)\\ &\quad+\varepsilon_k P_k,\end{aligned}$$

on $D_{\log k}$*, where* ε_k *is a sequence tending to zero with* $k \to \infty$*,* P_k *is a second order differential operator and all the derivatives of the coefficients of* P_k *are uniformly bounded in* k *on* $D_{\log k}$*.*

Let $D \subset D_{\log k}$ be an open set and let $W^s_{kF_k^*\phi}(D, F_k^*\Lambda^{0,q}T^*X)$, $s \in \mathbb{N}_0$, denote the Sobolev space of order s of sections of $F_k^*\Lambda^{0,q}T^*X$ over D with respect to the weight $\mathrm{e}^{-kF_k^*\phi}$. The Sobolev norm on this space is given by

$$\|u\|^2_{kF_k^*\phi,s,D} = \sum'_{\substack{\alpha\in\mathbb{N}_0^{2n-1},\ |\alpha|\leqslant s\\ |J|=q}} \int_D \left|\partial_x^\alpha u_J\right|^2 \mathrm{e}^{-kF_k^*\phi}(F_k^*m)(z,\theta)\mathrm{d}v(z)\mathrm{d}\theta, \tag{4.20}$$

where $u = \sum'_{|J|=q} u_J e_J\big(\frac{z}{\sqrt{k}}, \frac{\theta}{k}\big) \in W^s_{kF_k^*\phi}(D, F_k^*\Lambda^{0,q}T^*X)$ and m is the volume form. If $s = 0$, we write $\|\cdot\|_{kF_k^*\phi,D}$ to denote $\|\cdot\|_{kF_k^*\phi,0,D}$. The following is well known (see [9, Proposition 2.4 and Lemma 2.5])

Proposition 4.2 *Assume that* $Y(q)$ *holds at each point of* X*. For every* $r > 0$ *with* $D_{2r} \subset D_{\log k}$ *and* $s \in \mathbb{N}_0$*, there are constants* $C_{r,s} > 0$, $C_r > 0$, $C_{r,s}$ *and* C_r *are independent of* k*, such that*

$$\|u\|^2_{kF_k^*\phi,s+1,D_r} \leqslant C_{r,s}\Big(\|u\|^2_{kF_k^*\phi,D_{2r}} + \big\|\Box^{(q)}_{b,k\phi,(k)}u\big\|^2_{kF_k^*\phi,s,D_{2r}}\Big),\quad u \in F_k^*\Omega^{0,q}(D_{\log k}) \tag{4.21}$$

and

$$\sup_{x\in D_r}|u(x)|^2 \leqslant C_r\Big(\|u\|^2_{kF_k^*\phi,D_{2r}} + \sum_{m=1}^n \big\|(\Box^{(q)}_{b,k\phi,(k)})^m u\big\|^2_{kF_k^*\phi,D_{2r}}\Big),\quad u \in F_k^*\Omega^{0,q}(D_{\log k}). \tag{4.22}$$

We pause and introduce some notations. We identify $\mathbb{R}^{2n-1}$ with the Heisenberg group $H_n := \mathbb{C}^{n-1}\times\mathbb{R}$. We also write (z,θ) to denote the coordinates of H_n, $z = (z_1,\ldots,z_{n-1}) \in \mathbb{C}^{n-1}$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1,\ldots,n-1$, and $\theta \in \mathbb{R}$. Then,

$$\Big\{U_{j,H_n} = \frac{\partial}{\partial z_j} + i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta};\ j = 1,\ldots,n-1\Big\},$$
$$\Big\{U_{j,H_n},\ \overline{U}_{j,H_n},\ T = \frac{\partial}{\partial\theta};\ j = 1,\ldots,n-1\Big\}$$

are orthonormal bases for the bundles $T^{1,0}H_n$ and $\mathbb{C}TH_n$, respectively. Then,

$$\Big\{\mathrm{d}z_j,\ \mathrm{d}\overline{z}_j,\ \omega_0 = -\mathrm{d}\theta + \sum_{j=1}^{n-1}(i\lambda_j\overline{z}_j\mathrm{d}z_j - i\lambda_j z_j\mathrm{d}\overline{z}_j);\ j = 1,\ldots,n-1\Big\}$$

is the basis of $\mathbb{C}T^*H_n$ which is dual to $\{U_{j,H_n}, \overline{U}_{j,H_n}, -T;\ j = 1,\ldots,n-1\}$. We take the Hermitian metric $\langle\cdot|\cdot\rangle$ on $\Lambda^{0,q}T^*H_n$ such that

$$\{\mathrm{d}\overline{z}_J;\ |J| = q,\ J \text{ strictly increasing}\}$$

is an orthonormal basis of $\Lambda^{0,q}T^*H_n$. The Cauchy–Riemann operator $\overline{\partial}_{b,H_n}$ on H_n is given by

$$\overline{\partial}_{b,H_n} = \sum_{j=1}^{n-1}\mathrm{d}\overline{z}_j\wedge\overline{U}_{j,H_n} : \Omega^{0,q}(H_n) \to \Omega^{0,q+1}(H_n). \tag{4.23}$$

Put $\phi_0(z,\theta)=\beta\theta+\sum_{j,t=1}^{n-1}\mu_{j,t}\overline{z}_jz_t\in C^\infty(H_n,\mathbb{R})$, where β and $\mu_{j,t}$, $j,t=1,\ldots,n-1$, are as in (4.1). Note that

$$\sup_{(z,\theta)\in D_{\log k}}\left|kF_k^*\phi-\phi_0\right|\to 0,\quad \text{as } k\to\infty. \tag{4.24}$$

Let $(\,|\,)_{\phi_0}$ be the inner product on $\Omega_0^{0,q}(H_n)$ defined as follows:

$$(f\mid g)_{\phi_0}=\int_{H_n}\langle f|g\rangle e^{-\phi_0}dv(z)d\theta,\quad f,g\in\Omega_0^{0,q}(H_n),$$

where $dv(z)=2^{n-1}dx_1dx_2\cdots dx_{2n-2}$. Let $\overline{\partial}^{*,\phi_0}_{b,H_n}:\Omega^{0,q+1}(H_n)\to\Omega^{0,q}(H_n)$ be the formal adjoint of $\overline{\partial}_{b,H_n}$ with respect to $(\,|\,)_{\phi_0}$. We have

$$\overline{\partial}^{*,\phi_0}_{b,H_n}=\sum_{t=1}^{n-1}(d\overline{z}_t\wedge)^*\,\overline{U}^{*,\phi_0}_{t,H_n}:\Omega^{0,q+1}(H_n)\to\Omega^{0,q}(H_n), \tag{4.25}$$

where

$$\overline{U}^{*,\phi_0}_{t,H_n}=-U_{t,H_n}+U_{t,H_n}\phi_0=-U_{t,H_n}+\sum_{j=1}^{n-1}\mu_{j,t}\overline{z}_j+i\lambda_t\overline{z}_t\beta. \tag{4.26}$$

The Kohn Laplacian on H_n is given by

$$\Box^{(q)}_{b,H_n}=\overline{\partial}_{b,H_n}\overline{\partial}^{*,\phi_0}_{b,H_n}+\overline{\partial}^{*,\phi_0}_{b,H_n}\overline{\partial}_{b,H_n}:\Omega^{0,q}(H_n)\to\Omega^{0,q}(H_n). \tag{4.27}$$

From (4.23), (4.25) and (4.26), we can check that

$$\begin{aligned}\Box^{(q)}_{b,H_n}&=\sum_{j=1}^{n-1}\overline{U}^{*,\phi_0}_{j,H_n}\overline{U}_{j,H_n}+\sum_{j,t=1}^{n-1}d\overline{z}_j\wedge(d\overline{z}_t\wedge)^*\Big[\Big(\mu_{j,t}+i\lambda_j\delta_{j,t}\beta\Big)-2i\lambda_j\delta_{j,t}\frac{\partial}{\partial\theta}\Big]\\&=\sum_{j=1}^{n-1}\Big[\Big(-\frac{\partial}{\partial z_j}-i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta}+i\lambda_j\overline{z}_j\beta+\sum_{t=1}^{n-1}\mu_{t,j}\overline{z}_t\Big)\Big(\frac{\partial}{\partial\overline{z}_j}-i\lambda_jz_j\frac{\partial}{\partial\theta}\Big)\Big]\\&\quad+\sum_{j,t=1}^{n-1}d\overline{z}_j\wedge(d\overline{z}_t\wedge)^*\Big[\Big(\mu_{j,t}+i\lambda_j\delta_{j,t}\beta\Big)-2i\lambda_j\delta_{j,t}\frac{\partial}{\partial\theta}\Big].\end{aligned} \tag{4.28}$$

Now, we can prove

Proposition 4.3 *Assume that $Y(q)$ holds at each point of X. For each k, let $\alpha_k\in F_k^*\Omega^{0,q}(D_{\log k})$. We assume that $\|\alpha_k\|_{kF_k^*\phi,D_{\log k}}\leqslant 1$ for each , k and there is a sequence $\nu_k>0$, $\nu_k\to 0$ as $k\to\infty$, such that for each k,*

$$\left\|(\Box^{(q)}_{b,k\phi,(k)})^m\alpha_k\right\|_{kF_k^*\phi,D_{\log k}}\leq\nu_k^m,\quad\forall m\in\mathbb{N}.$$

Identify α_k with a form on H_n by extending it with zero and write

$$\alpha_k=\sum_{|J|=q}{}'\alpha_{k,J}e_J\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right).$$

Then, there is a subsequence $\{\alpha_{k_j}\}$ of $\{\alpha_k\}$ such that for each strictly increasing multiindex J, $|J|=q$, $\alpha_{k_j,J}$ converges uniformly with all its derivatives on any compact subset of H_n to a smooth function α_J. Furthermore, if we put $\alpha=\sum'_{|J|=q}\alpha_Jd\overline{z}_J$, then $\Box^{(q)}_{b,H_n}\alpha=0$.

Proof From (4.21) and using induction, we get for any $r > 0$ and for every $s \in \mathbb{N}_0$, there is a constant $C_{r,s} > 0$ independent of k, such that

$$\begin{aligned}\|\alpha_k\|^2_{kF_k^*\phi,s+1,D_r} &\leqslant C_{r,s}\Big(\|\alpha_k\|^2_{kF_k^*\phi,D_{2r}} + \sum_{m=1}^{s+1}\Big\|(\Box^{(q)}_{b,k\phi,(k)})^m\alpha_k\Big\|^2_{kF_k^*\phi,D_{2r}}\Big)\\ &\leq C_{r,s}\Big(1+\sum_{m=1}^{\infty}\nu_k^m\Big) \leq \widetilde{C}_{r,s}\end{aligned} \tag{4.29}$$

for k large, where $\widetilde{C}_{r,s} > 0$ is independent of k. Fix a strictly increasing multiindex J, $|J| = q$, and $r > 0$. Combining (4.29) with Rellich's compactness theorem [15, p. 281], we conclude that there is a subsequence of $\{\alpha_{k,J}\}$, which converges in all Sobolev spaces $W^s(D_r)$ for $s > 0$. From the Sobolev embedding theorem [15, p. 170], we see that the sequence converges in all $C^l(D_r)$, $l \geqslant 0$, $l \in \mathbb{Z}$, locally uniformly. Choosing a diagonal sequence, with respect to a sequence of D_r exhausting H_n, we get a subsequence $\{\alpha_{k_j,J}\}$ of $\{\alpha_{k,J}\}$ such that $\alpha_{k_j,J}$ converges uniformly with all derivatives on any compact subset of H_n to a smooth function α_J.

Let J' be another strictly increasing multiindex, $|J'| = q$. We can repeat the procedure above and get a subsequence $\{\alpha_{k_{j_s},J'}\}$ of $\{\alpha_{k_j,J'}\}$ such that $\alpha_{k_{j_s},J'}$ converges uniformly with all derivatives on any compact subset of H_n to a smooth function $\alpha_{J'}$. Continuing in this way, we get the first statement of the proposition.

Now, we prove the second statement of the proposition. Let $P = (p_1, \ldots, p_q)$, $R = (r_1, \ldots, r_q)$ be multiindices, $|P| = |R| = q$. Define

$$\varepsilon^P_R = \begin{cases} 0, & \text{if } \{p_1,\ldots,p_q\} \neq \{r_1,\ldots,r_q\},\\ \text{the sign of permutation } \begin{pmatrix}P\\R\end{pmatrix}, & \text{if } \{p_1,\ldots,p_q\} = \{r_1,\ldots,r_q\}.\end{cases}$$

For $j, t = 1, \ldots, n-1$, define

$$\sigma^{jtP}_R = \begin{cases} 0, & \text{if } d\overline{z}_j \wedge (d\overline{z}_t\wedge)^*(d\overline{z}^P) = 0,\\ \varepsilon^Q_R, & \text{if } d\overline{z}_j \wedge (d\overline{z}_t\wedge)^*(d\overline{z}^P) = d\overline{z}^Q,\ |Q| = q.\end{cases}$$

We may assume that $\alpha_{k,J}$ converges uniformly with all derivatives on any compact subset of H_n to a smooth function α_J, for all strictly increasing J, $|J| = q$. As (4.29), we have for any $r > 0$ and for every $s \in \mathbb{N}_0$, there is a constant $C_{r,s} > 0$ independent of k, such that

$$\begin{aligned}&\Big\|\Box^{(q)}_{k,k\phi,(k)}\alpha_k\Big\|^2_{kF_k^*\phi,s+1,D_r}\\ &\leqslant C_{r,s}\Big(\Big\|\Box^{(q)}_{k,k\phi,(k)}\alpha_k\Big\|^2_{kF_k^*\phi,D_{2r}} + \sum_{m=1}^{s+1}\Big\|(\Box^{(q)}_{b,k\phi,(k)})^{m+1}\alpha_k\Big\|^2_{kF_k^*\phi,D_{2r}}\Big)\\ &\leq C_{r,s}\sum_{m=1}^{\infty}\nu_k^m \to 0 \quad \text{as } k\to\infty.\end{aligned} \tag{4.30}$$

Put

$$\beta_k := \Box^{(q)}_{k,k\phi,(k)}\alpha_k = \sum_{|J|=q}{}^{'}\beta_{k,J}e_J\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right) \in F_k^*\Omega^{0,q}(D_{\log k}).$$

Combining (4.30) with Sobolev embedding theorem [15, p. 170], we conclude that

$$\begin{aligned}&\beta_{k,J}\text{ converges uniformly with all derivatives on any compact subset of } H_n\\ &\quad\text{to zero, for all strictly increasing } J, |J|=q.\end{aligned}\tag{4.31}$$

From the explicit formula of $\Box^{(q)}_{b,k\phi,(k)}$ (see Proposition 4.1), it is not difficult to see that for all strictly increasing J, $|J|=q$, we have

$$\begin{aligned}\sum_{j=1}^{n-1}\overline{U}^{*,\phi_0}_{j,H_n}\overline{U}_{j,H_n}\alpha_{k,J} = -\sideset{}{'}\sum_{\substack{|P|=q,\\ 1\leqslant j,t\leqslant n-1}}\sigma^{jtP}_J\Big[\Big(\mu_{j,t}+i\lambda_j\delta_{j,t}\beta\Big)-2i\lambda_j\delta_{j,t}\frac{\partial}{\partial\theta}\Big]\alpha_{k,P}\\ +\epsilon_k P_{k,J}\alpha_k+\beta_{k,J}\end{aligned}\tag{4.32}$$

on $D_{\log k}$, where ϵ_k is a sequence tending to zero with $k\to\infty$, $P_{k,J}$ is a second order differential operator and all the derivatives of the coefficients of $P_{k,J}$ are uniformly bounded in k on $D_{\log k}$ and $\beta_{k,J}$ is as in (4.31). By letting $k\to\infty$ in (4.32) we get

$$\sum_{j=1}^{n-1}\overline{U}^{*,\phi_0}_{j,H_n}\overline{U}_{j,H_n}\alpha_J = -\sideset{}{'}\sum_{\substack{|P|=q,\\ 1\leqslant j,t\leqslant n-1}}\sigma^{jtP}_J\Big[\Big(\mu_{j,t}+i\lambda_j\delta_{j,t}\beta\Big)-2i\lambda_j\delta_{j,t}\frac{\partial}{\partial\theta}\Big]\alpha_P\tag{4.33}$$

on H_n, for all strictly increasing J, $|J|=q$. From this and the explicit formula of $\Box^{(q)}_{b,H_n}$ [see (4.28)], we conclude that $\Box^{(q)}_{b,H_n}\alpha=0$. The proposition follows. □

5 The operators $Q^{(0)}_{M,k}$ and $Q^{(1)}_{M,k}$

From now on, we assume that h^L is semi-positive on X and positive at some point of X and conditions $Y(0)$ and $Y(1)$ hold at each point of X.

Take $\delta_0>0$ be a small constant so that

$$M^\phi_x+2s\mathcal{L}_x\geq 0,\quad \forall|s|\leq\delta_0,\ \forall x\in X.\tag{5.1}$$

Take $\psi(\eta)\in C^\infty_0(]-\delta_0,\delta_0[,\overline{\mathbb{R}}_+)$ so that $\psi(\eta)=1$ if $-\frac{\delta_0}{2}\leq\eta\leq\frac{\delta_0}{2}$. Let $\hat{\psi}(t)=\int e^{-it\eta}\psi(\eta)d\eta$ be the Fourier transform of ψ. Put

$$C_0:=\sup_{t\in\mathbb{R}}t^2\left|\hat{\psi}(t)\right|.\tag{5.2}$$

Let $E>0$ be a small constant so that

$$\begin{aligned}&\sqrt{\int_X E^2 dv_X(x)}\sqrt{2(2\pi)^{-n}\int_X\Big(\int\det(M^\phi_x+2\xi\mathcal{L}_x)1_{\mathbb{R}_{x,0}}(\xi)d\xi\Big)dv_X(x)}\\ &\quad\leq\frac{(2\pi)^{1-n}}{4}\int_X\Big(\int\psi(\xi)\det(M^\phi_x+2\xi\mathcal{L}_x)1_{\mathbb{R}_{x,0}}(\xi)d\xi\Big)dv_X(x).\end{aligned}\tag{5.3}$$

Fix $M>0$ be a large constant so that

$$\begin{aligned}&\frac{2C_0}{M}(2\pi)^{-n}\int_X\Big(\int\det(M^\phi_x+2\xi\mathcal{L}_x)1_{\mathbb{R}_{x,0}}(\xi)d\xi\Big)dv_X(x)\\ &\quad<\frac{(2\pi)^{1-n}}{2}\int_X\Big(\int\psi(\xi)\det(M^\phi_x+2\xi\mathcal{L}_x)1_{\mathbb{R}_{x,0}}(\xi)d\xi\Big)dv_X(x)\end{aligned}\tag{5.4}$$

and

$$\frac{2C_0}{M}\left((2\pi)^{-n}\int\left|\det(M_x^\phi+2\xi\mathcal{L}_x)\right|\mathbb{1}_{\mathbb{R}_{x,1}}(\xi)d\xi\right)^{\frac{1}{2}}<\frac{E}{\sqrt{n-1}},\quad\forall x\in X,\tag{5.5}$$

where $\mathbb{1}_{\mathbb{R}_{x,1}}(\xi)=1$ if $\xi\in\mathbb{R}_{x,1}$, $\mathbb{1}_{\mathbb{R}_{x,1}}(\xi)=0$ if $\xi\notin\mathbb{R}_{x,1}$. Take $\chi(t)\in C_0^\infty(]-2,2[,\overline{\mathbb{R}}_+)$ so that $0\leq\chi(t)\leq 1$ and $\chi(t)=1$ if $-1\leq t\leq 1$ and $\chi(-t)=\chi(t)$, for all $t\in\mathbb{R}$. Put

$$\chi_M(t):=\chi\left(\frac{t}{M}\right).\tag{5.6}$$

As before, let $\Phi^t(x)$ be the T-flow. The operator $Q^{(0)}_{M,k}$ is a continuous operator $C^\infty(X,L^k)\to C^\infty(X,L^k)$ defined as follows. Let $u\in C^\infty(X,L^k)$. Let $D\Subset D'\Subset X$ be open sets of X and let s be a local section of L on D', $|s|^2_{h^L}=e^{-\phi}$. On D', we write $u=s^k\widetilde{u}$, $\widetilde{u}\in C^\infty(D')$. Then,

$$\left(Q^{(0)}_{M,k}u\right)(x):=s^k e^{\frac{k}{2}\phi(x)}\int e^{-it\eta}\psi(\eta)\chi_M(t)e^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(x))}\widetilde{u}(\Phi^{\frac{t}{k}}(x))dtd\eta\quad\text{on } D.\tag{5.7}$$

We first notice that for k large, $\Phi^{\frac{t}{k}}(x)$ is well defined for all $t\in\operatorname{Supp}\chi_M$, every $x\in X$ and $\Phi^{\frac{t}{k}}(x)\in D'$ for all $t\in\operatorname{Supp}\chi_M$, every $x\in D$. We may assume that $\Phi^{\frac{t}{k}}(x)$ is well defined for all $t\in\operatorname{Supp}\chi_M$, every $x\in X$ and $\Phi^{\frac{t}{k}}(x)\in D'$ for all $t\in\operatorname{Supp}\chi_M$, every $x\in D$. Now, we check that the definition above is independent of the choice of local sections. Let $\hat{s}$ be another local section of L on D', $\left|\hat{s}\right|^2_{h^L}=e^{-\hat{\phi}}$. Then, we have $\hat{s}=gs$ for some non-zero rigid CR function g. We can check that

$$\begin{aligned}&\hat{\phi}=\phi-2\log|g|,\\&e^{-\frac{k}{2}\hat{\phi}}=e^{-\frac{k}{2}\phi}|g|^k.\end{aligned}\tag{5.8}$$

Let $u\in C^\infty(X,L^k)$. On D, we write $u=s^k\widetilde{u}=\hat{s}^k\hat{u}$. We have

$$\hat{u}=g^{-k}\widetilde{u}.\tag{5.9}$$

From (5.8) and (5.9), we can check that

$$e^{-\frac{k}{2}\hat{\phi}}\hat{u}=e^{-\frac{k}{2}\phi}|g|^k g^{-k}\widetilde{u}.\tag{5.10}$$

Since $Tg=0$, we have $(|g|^k g^{-k})(\Phi^{\frac{t}{k}}x)=(|g|^k g^{-k})(x)$ for all $t\in\operatorname{Supp}\chi_M$, $x\in D$. From this observation and (5.10), it is easy to see that

$$\begin{aligned}&\int e^{-it\eta}\psi(\eta)\chi_M(t)e^{-\frac{k}{2}\hat{\phi}(\Phi^{\frac{t}{k}}(x))}\hat{u}(\Phi^{\frac{t}{k}}(x))dtd\eta\\&=\left(|g|^k g^{-k}\right)(x)\int e^{-it\eta}\psi(\eta)\chi_M(t)e^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(x))}\widetilde{u}(\Phi^{\frac{t}{k}}(x))dtd\eta\quad\text{on } D.\end{aligned}\tag{5.11}$$

Furthermore, we can check that

$$\hat{s}^k e^{\frac{k}{2}\hat{\phi}}=|g|^{-k}g^k s^k e^{\frac{k}{2}\phi}.$$

Combining this with (5.11), we obtain

$$\begin{aligned}&\hat{s}^k e^{\frac{k}{2}\hat{\phi}(x)}\int e^{-it\eta}\psi(\eta)\chi_M(t)e^{-\frac{k}{2}\hat{\phi}(\Phi^{\frac{t}{k}}(x))}\hat{u}(\Phi^{\frac{t}{k}}(x))dtd\eta\\&=s^k e^{\frac{k}{2}\phi(x)}\int e^{-it\eta}\psi(\eta)\chi_M(t)e^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(x))}\widetilde{u}(\Phi^{\frac{t}{k}}(x))dtd\eta\quad\text{on } D.\end{aligned}$$

Thus, the definition of $Q^{(0)}_{M,k}$ is well defined.

We consider (0, 1) forms. The operator $Q^{(1)}_{M,k}$ is a continuous operator

$$Q^{(1)}_{M,k} : \Omega^{0,1}(X, L^k) \to \Omega^{0,1}(X, L^k)$$

defined as follows. Let D be an open set of X. We assume that there exist canonical coordinates x defined in some neighborhood W of $\overline{D}$ and L is trivial on W. Let $\psi(\eta)$ and χ_M be as in (5.7). For k large, we have

$$\left\{\Phi^{\frac{t}{k}}(x) \in W;\ \forall x \in D, t \in \operatorname{Supp}\chi_M\right\}.$$

Let s be a local section of L on W, $|s|^2_{h^L} = \mathrm{e}^{-\phi}$. Let $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$ be canonical coordinates on W. Then,

$$\begin{aligned} T &= \frac{\partial}{\partial\theta}, \\ Z_j &= \frac{\partial}{\partial z_j} + i\frac{\partial\varphi}{\partial z_j}(z)\frac{\partial}{\partial\theta}, \quad j = 1, \ldots, n-1, \end{aligned} \tag{5.12}$$

where $Z_j(x)$, $j = 1, \ldots, n-1$, form a basis of $T^{1,0}_x X$, for each $x \in D$, and $\varphi(z) \in C^\infty(D, \mathbb{R})$ independent of θ. We can check that $\mathrm{d}\overline{z}_j$, $j = 1, \ldots, n-1$, is the basis of $T^{*0,1}X$, dual to $\overline{Z}_j$, $j = 1, \ldots, n-1$. Let $u \in \Omega^{0,1}(X, L^k)$. On W, we write

$$u = s^k \sum_{j=1}^{n-1} \widetilde{u}_j(x)\mathrm{d}\overline{z}_j, \quad \widetilde{u}_j \in C^\infty(D),\ j = 1, \ldots, n-1.$$

Then,

$$(Q^{(1)}_{M,k}u)(x) := s^k e^{\frac{k}{2}\phi(x)} \sum_{j=1}^{n-1} \left(\int \mathrm{e}^{-it\eta}\psi(\eta)\chi_M(t)\mathrm{e}^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(x))}\widetilde{u}_j(\Phi^{\frac{t}{k}}(x))\mathrm{d}t\mathrm{d}\eta\right)\mathrm{d}\overline{z}_j \text{ on } D. \tag{5.13}$$

As before, we can show that the definition (5.13) is independent of the choices of local sections. Now, we check that the definition (5.13) is independent of the choice of canonical coordinates. Let $y = (y_1, \ldots, y_{2n-1}) = (w, \gamma)$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, n-1$, $\gamma = y_{2n-1}$, be another canonical coordinates on W. Then,

$$\begin{aligned} T &= \frac{\partial}{\partial\gamma}, \\ \widetilde{Z}_j &= \frac{\partial}{\partial w_j} + i\frac{\partial\widetilde{\varphi}}{\partial w_j}(w)\frac{\partial}{\partial\gamma}, \quad j = 1, \ldots, n-1, \end{aligned} \tag{5.14}$$

where $\widetilde{Z}_j(y)$, $j = 1, \ldots, n-1$, form a basis of $T^{1,0}_y X$, for each $y \in D$, and $\widetilde{\varphi}(w) \in C^\infty(D, \mathbb{R})$ independent of γ. From (5.14) and (5.12), it is not difficult to see that on W, we have

$$\begin{aligned} &w = (w_1, \ldots, w_{n-1}) = (H_1(z), \ldots, H_{n-1}(z)) = H(z), \quad H_j(z) \in C^\infty,\ \forall j, \\ &\gamma = \theta + G(z), \quad G(z) \in C^\infty, \end{aligned} \tag{5.15}$$

where for each $j = 1, \ldots, n-1$, $H_j(z)$ is holomorphic. From (5.15), we can check that

$$d\overline{w}_j = \sum_{l=1}^{n-1} \overline{\left(\frac{\partial H_j}{\partial z_l}\right)} d\overline{z}_l, \quad j = 1, \ldots, n-1. \tag{5.16}$$

From this observation, we have for $u \in \Omega^{0,1}(X, L^k)$,

$$\begin{aligned} u &= s^k \sum_{j=1}^{n-1} \widetilde{u}_j(x) d\overline{z}_j = s^k \sum_{j=1}^{n-1} \hat{u}_j(y) d\overline{w}_j \quad \text{on } W, \\ \widetilde{u}_l(x) &= \sum_{j=1}^{n-1} \hat{u}_j(H(z), \theta + G(z)) \overline{\frac{\partial H_j}{\partial z_l}(z)}, \quad l = 1, \ldots, n-1. \end{aligned} \tag{5.17}$$

On D, we have $\Phi^{\frac{t}{k}}(x) = (z, \frac{t}{k} + \theta)$, $\Phi^{\frac{t}{k}}(y) = (w, \frac{t}{k} + \gamma)$ and $\frac{\partial H_j}{\partial z_l}(\Phi^{\frac{t}{k}}(z)) = \frac{\partial H_j}{\partial z_l}(z)$, $j, l = 1, \ldots, n-1$, $t \in \operatorname{Supp} \chi_M$. From this observation and (5.17), (5.16), it is straightforward to see that

$$\begin{aligned} &s^k e^{\frac{k}{2}\phi(x)} \sum_{l=1}^{n-1} \left(\int e^{-it\eta} \psi(\eta) \chi_M(t) e^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(x))} \widetilde{u}_l(\Phi^{\frac{t}{k}}(x)) dt d\eta \right) d\overline{z}_l \\ &= s^k e^{\frac{k}{2}\phi(y)} \sum_{j=1}^{n-1} \left(\int e^{-it\eta} \psi(\eta) \chi_M(t) e^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(y))} \hat{u}_j(\Phi^{\frac{t}{k}}(y)) dt d\eta \right) d\overline{w}_j. \end{aligned}$$

Thus, the definition (5.13) is independent of the choice of canonical coordinates. The operator $Q^{(1)}_{M,k}$ is well defined.

Now, we claim that

$$Q^{(1)}_{M,k} \overline{\partial}_{b,k} u = \overline{\partial}_{b,k} Q^{(0)}_{M,k} u, \quad \forall u \in C^\infty(X, L^k). \tag{5.18}$$

We work with canonical coordinates $x = (z, \theta)$ as (5.12). For $u \in C^\infty(X, L^k)$, we can check that

$$\overline{\partial}_{b,k} u = s^k \sum_{j=1}^{n} (\overline{Z}_j \widetilde{u}) d\overline{z}_j = s^k \sum_{j=1}^{n-1} \left(\frac{\partial \widetilde{u}}{\partial \overline{z}_j} - i \frac{\partial \varphi}{\partial \overline{z}_j}(z) \frac{\partial \widetilde{u}}{\partial \theta} \right) d\overline{z}_j \tag{5.19}$$

on W, where $u = s^k \widetilde{u}$ on W. Combining (5.19) with (5.13), (5.7) and notice that $\frac{\partial \widetilde{u}}{\partial \theta}(\Phi^t(x)) = \frac{\partial}{\partial \theta}\big(\widetilde{u}(\Phi^t(x))\big)$, it is easy to see that

$$\begin{aligned} &Q^{(1)}_{M,k} \overline{\partial}_{b,k} u - \overline{\partial}_{b,k} Q^{(0)}_{M,k} u \\ &\quad = -\frac{k}{2} s^k e^{\frac{k}{2}\phi(x)} \\ &\qquad \times \sum_{j=1}^{n-1} \left(\int e^{-it\eta} \psi(\eta) \chi_M(t) e^{-\frac{k}{2}\phi(\Phi^{\frac{t}{k}}(x))} \overline{Z}_j \big(\phi(x) - \phi(\Phi^{\frac{t}{k}}(x))\big) \widetilde{u}(\Phi^{\frac{t}{k}}(x)) dt d\eta \right) d\overline{z}_j. \end{aligned} \tag{5.20}$$

Since $\overline{\partial}_b T\phi = 0$, we have $\overline{Z}_j \phi(x) = \overline{Z}_j \phi(\Phi^{\frac{t}{k}}(x))$, $j = 1, \ldots, n-1$, $t \in \operatorname{Supp} \chi_M$. From this and (5.20), (5.18) follows.

6 The asymptotic behavior of $(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)$

We will use the same notations as before. We recall that we work with the assumption that $Y(0)$ and $Y(1)$ hold at each point of X. We first need

Theorem 6.1 *For any sequence $\nu_k > 0$ with $\nu_k \to 0$ as $k \to \infty$, there is a constant $C > 0$ independent of k, such that*

$$\left|\left(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leqslant k\nu_k}\overline{Q^{(0)}_{M,k}}\right)(x)\right| \leqslant Ck^n \tag{6.1}$$

and

$$\left|\left(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leqslant k\nu_k}\right)(x)\right| \leqslant Ck^n, \tag{6.2}$$

for all $x \in X$, $k > 0$. Recall that $(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)$ and $(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k}\overline{Q^{(0)}_{M,k}})(x)$ are given by (2.13) and (2.14), respectively.

Proof Let $\nu_k > 0$ be any sequence with $\nu_k \to 0$ as $k \to \infty$. Let $f_j \in C^\infty(X, L^k)$, $j = 1, \ldots, d_k$, be an orthonormal frame for $\mathscr{H}^0_{b,\leq k\nu_k}(X, L^k)$. From (2.24), we see that for each $x \in X$,

$$\begin{aligned}(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k}\overline{Q^{(0)}_{M,k}})(x) &= \sum_{j=1}^{d_k}\left|(Q^{(0)}_{M,k}f_j)(x)\right|^2_{h^{L^k}} \\ &= \sup_{\alpha\in\mathscr{H}^0_{b,\leq k\nu_k}(X,L^k),\|\alpha\|_{h^{L^k}}=1}\left|(Q^{(0)}_{M,k}\alpha)(x)\right|^2_{h^{L^k}}.\end{aligned} \tag{6.3}$$

In view of (2.18), we see that there is a constant $C > 0$ independent of k such that

$$\Pi^{(0)}_{k,\leq k\nu_k}(x) = \sum_{j=1}^{d_k}\left|f_j(x)\right|^2_{h^{L^k}} \leq Ck^n, \quad \forall x \in X. \tag{6.4}$$

For $\alpha \in \mathscr{H}^0_{b,\leq k\nu_k}(X, L^k)$, $\|\alpha\|_{h^{L^k}} = 1$, we have

$$|\alpha(x)|^2_{h^{L^k}} \leq \Pi^{(0)}_{k,\leq k\nu_k}(x) \leq Ck^n, \quad \forall x \in X, \tag{6.5}$$

where $C > 0$ is a constant independent of k and α. From (6.5) and (5.7), it is easy to see that there is a constant $C_1 > 0$ independent of k such that

$$\left|(Q^{(0)}_{M,k}\alpha)(x)\right|^2_{h^{L^k}} \leq C_1k^n, \quad \forall x \in X,\ \forall\alpha \in \mathscr{H}^0_{b,\leq k\nu_k}(X, L^k),\ \|\alpha\|_{h^{L^k}} = 1. \tag{6.6}$$

From (6.6) and (6.3), (6.1) follows.

We have

$$\begin{aligned}\left|(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)\right| &= \left|\sum_{j=1}^{d_k}\langle(Q^{(0)}_{M,k}f_j)(x)|f_j(x)\rangle_{h^{L^k}}\right| \\ &\leq\left(\sum_{j=1}^{d_k}\left|(Q^{(0)}_{M,k}f_j)(x)\right|^2_{h^{L^k}}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{d_k}\left|f_j(x)\right|^2_{h^{L^k}}\right)^{\frac{1}{2}}.\end{aligned} \tag{6.7}$$

From (6.7), (6.4), (6.3) and (6.1), (6.2) follows. □

Fix a point $p \in X$. Let $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$ be canonical coordinates of X defined in some small neighborhood D of p and let s be a local section of L on D, $|s|^2_{h^L} = \mathrm{e}^{-\phi}$. We take x and s so that (4.1) holds. Until further notice, we work with the local coordinates x and the local section s and we will use the same notations as Sect. 4. We identify D with some open set in $\mathbb{C}^{n-1} \times \mathbb{R}$. Put

$$u(z,\theta) = (2\pi)^{-\frac{n}{2}}\left(\int_{\mathbb{R}_{p,0}} \det(M^\phi_p + 2s\mathcal{L}_p)\mathrm{d}s\right)^{-\frac{1}{2}} \times \int \mathrm{e}^{i\theta\xi + \frac{\beta\theta}{2} + (-\xi + \frac{i}{2}\beta)\sum_{j=1}^{n-1}\lambda_j|z_j|^2} \det(M^\phi_p + 2\xi\mathcal{L}_p)\mathbb{1}_{\mathbb{R}_{p,0}}(\xi)\mathrm{d}\xi. \quad (6.8)$$

$u(z,\theta) \in C^\infty(\mathbb{C}^{n-1} \times \mathbb{R})$. We remind that $\mathbb{R}_{p,0}$ is given by (2.15). Set

$$\alpha_k = k^{\frac{n}{2}} s^k \chi_1\left(\frac{\sqrt{k}}{\log k}z, \frac{\sqrt{k}}{\log k}\theta\right)u(\sqrt{k}z, k\theta) \in C^\infty_0(D, L^k), \quad (6.9)$$

where $\chi_1 \in C^\infty$, $0 \leq \chi_1 \leq 1$,

$$\mathrm{Supp}\,\chi_1 \subset \left\{(z,\theta) \in \mathbb{C}^{n-1} \times \mathbb{R};\ |z| \leq 1, |\theta| \leq 1\right\},$$

$\chi_1(z,\theta) = 1$ if $|z| \leq \frac{1}{2}$, $|\theta| \leq \frac{1}{2}$. We notice that

$$\mathrm{Supp}\,\alpha_k \subset \left\{(z,\theta) \in \mathbb{C}^{n-1} \times \mathbb{R};\ |z| \leq \frac{\log k}{\sqrt{k}}, |\theta| \leq \frac{\log k}{\sqrt{k}}\right\}.$$

Thus, for k large, $\mathrm{Supp}\,\alpha_k \Subset D$ and α_k is well defined. The following is well known (see [9, section 5]).

Proposition 6.2 *With the notations used above, we have*

$$\lim_{k\to\infty} k^{-n}\,|\alpha_k(0)|^2_{h^{L^k}} = (2\pi)^{-n}\int_{\mathbb{R}_{p,0}} \det(M^\phi_p + 2s\mathcal{L}_p)\mathrm{d}s, \quad (6.10)$$

$$\lim_{k\to\infty} \|\alpha_k\|_{h^{L^k}} = 1, \quad (6.11)$$

$$\lim_{k\to\infty} \left\|\left(\frac{1}{k}\Box^{(0)}_{b,k}\alpha_k\right)^m\right\|_{h^{L^k}} = 0, \quad \forall m \in \mathbb{N}, \quad (6.12)$$

and there is a sequence $\gamma_k > 0$, independent of the point p and tending to zero as $k \to \infty$, such that

$$\left(\frac{1}{k}\Box^{(0)}_{b,k}\alpha_k \mid \alpha_k\right)_{h^{L^k}} \leq \gamma_k, \quad \forall k > 0. \quad (6.13)$$

We have the following

Proposition 6.3 *Let $\nu_k > 0$ be any sequence with $\lim_{k\to\infty}\frac{\gamma_k}{\nu_k} = 0$ and $\nu_k \to 0$ as $k \to \infty$, where γ_k is as in (6.13). Let α_k be as in (6.9). Let*

$$\alpha_k = \alpha^1_k + \alpha^2_k,$$
$$\alpha^1_k \in \mathscr{H}^0_{b,\leq k\nu_k}(X, L^k), \quad \alpha^2_k \in \mathscr{H}^0_{b,>k\nu_k}(X, L^k). \quad (6.14)$$

Then,

$$\lim_{k\to\infty} \|\alpha^1_k\|_{h^{L^k}} = 1 \quad (6.15)$$

and

$$\lim_{k\to\infty} k^{-n}\left|\alpha^1_k(0)\right|^2_{h^{L^k}} = (2\pi)^{-n}\int_{\mathbb{R}_{p,0}} \det(M^\phi_p + 2s\mathcal{L}_p)\mathrm{d}s. \tag{6.16}$$

Moreover, on D, we put

$$\alpha^2_k = k^{\frac{n}{2}} s^k \beta^2_k,\quad \beta^2_k \in C^\infty(D). \tag{6.17}$$

Fix $r > 0$. Then, for every $\varepsilon > 0$, there is a $k_0 > 0$ such that for all $k \geq k_0$, we have $F_k(D_{2r}) \subset D$ and

$$\left|\beta^2_k\left(\frac{z}{\sqrt{k}}, \frac{\theta}{k}\right)\right| \leq \varepsilon,\quad \forall (z,\theta)\in D_r. \tag{6.18}$$

In particular,

$$\lim_{k\to\infty}\left|\beta^2_k\left(\frac{z}{\sqrt{k}}, \frac{\theta}{k}\right)\right| = 0,\quad \forall (z,\theta)\in D. \tag{6.19}$$

Proof From (2.9), we have

$$\left\|\alpha^2_k\right\|^2_{h^{L^k}} \leqslant \frac{1}{k\nu_k}\left(\Box^{(0)}_{b,k}\alpha^2_k \mid \alpha^2_k\right)_{h^{L^k}} \leqslant \frac{1}{k\nu_k}\left(\Box^{(0)}_{b,k}\alpha_k \mid \alpha_k\right)_{h^{L^k}} \leqslant \frac{\gamma_k}{\nu_k} \to 0,$$

as $k\to\infty$. Thus, $\lim_{k\to\infty}\left\|\alpha^2_k\right\|_{h^{L^k}} = 0$. Since $\|\alpha_k\|_{h^{L^k}} \to 1$ as $k\to\infty$, (6.15) follows.

Now, we prove (6.18). As (6.17), on D, we write $\alpha^2_k = s^k k^{\frac{n}{2}}\beta^2_k$, $\beta^2_k \in C^\infty(D)$. From (4.22), we know that

$$\begin{aligned}\sup_{(z,\theta)\in D_r}\left|F^*_k\beta^2_k(z,\theta)\right|^2 &= \sup_{(z,\theta)\in D_r}\left|\beta^2_k\left(\frac{z}{\sqrt{k}}, \frac{\theta}{k}\right)\right|^2\\ &\leqslant C_r\left(\left\|F^*_k\beta^2_k\right\|^2_{kF^*_k\phi, D_{2r}} + \sum_{m=1}^{n}\left\|(\Box^{(q)}_{k,k\phi,(k)})^m F^*_k\beta^2_k\right\|^2_{kF^*_k\phi, D_{2r}}\right),\end{aligned} \tag{6.20}$$

where $C_r > 0$ is independent of k. Now, we have

$$\left\|F^*_k\beta^2_k\right\|^2_{kF^*_k\phi, D_{2r}} \leqslant \left\|\alpha^2_k\right\|^2_{h^{L^k}} \to 0,\quad \text{as } k\to\infty. \tag{6.21}$$

Moreover, from (4.17), it is easy to can check that for all $m\in\mathbb{N}$,

$$\begin{aligned}\left\|\left(\Box^{(q)}_{b,k\phi,(k)}\right)^m F^*_k\beta^2_k\right\|^2_{kF^*_k\phi, D_{2r}} &\leqslant \left\|\left(\tfrac{1}{k}\Box^{(q)}_{b,k}\right)^m\alpha^2_k\right\|^2_{h^{L^k}}\\ &\leqslant \left\|\left(\tfrac{1}{k}\Box^{(q)}_{b,k}\right)^m\alpha_k\right\|^2_{h^{L^k}} \to 0 \quad \text{as } k\to\infty.\end{aligned} \tag{6.22}$$

Here, we used (6.12). Combining (6.20) with (6.21) and (6.22), (6.18) follows.

From (6.18), we deduce

$$\lim_{k\to\infty}\left|F^*_k\beta^2_k(0)\right|^2 = \lim_{k\to\infty}\left|\beta^2_k(0)\right|^2 = \lim_{k\to\infty} k^{-n}\left|\alpha^2_k(0)\right|^2_{h^{L^k}} = 0. \tag{6.23}$$

From this and (6.10), (6.16) follows. □

Now, we can prove

Theorem 6.4 *Let $\delta_k = \min\{\mu_k, \gamma_k\}$, where μ_k is as in Theorem 2.1 and γ_k is as in (6.13). Let $\nu_k > 0$ be any sequence with $\lim_{k\to\infty} \frac{\delta_k}{\nu_k} = 0$ and $\nu_k \to 0$ as $k \to \infty$. Then,*

$$\begin{aligned}&\lim_{k\to\infty} k^{-n}(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leqslant k\nu_k})(x)\\&\quad= (2\pi)^{-n}\int e^{it\xi}\hat{\psi}(t)\chi_M(t)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)dtd\xi\end{aligned} \tag{6.24}$$

for all $x \in X$, where $\psi(\eta)$ is as in the discussion after (5.1) and $\chi_M(t)$ is given by (5.6), $\hat{\psi}(t) = \int e^{-it\eta}\psi(\eta)d\eta$. We remind that Supp $\psi \bigcap \mathbb{R}_{x,1} = \emptyset$, *for every $x \in X$.*

Proof Let $\nu_k > 0$ be any sequence with $\lim_{k\to\infty} \frac{\delta_k}{\nu_k} = 0$ and $\nu_k \to 0$ as $k \to \infty$. Fix a point $p \in X$. Let $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$ be canonical coordinates of X defined in some small neighborhood D of p and let s be a local section of L on D, $|s|^2_{h^L} = e^{-\phi}$. As before we take x and s so that (4.1) hold and let $\alpha^1_k \in \mathscr{H}^0_{b,\leq k\nu_k}(X, L^k)$ be as in (6.14). We take

$$f^1_k := \frac{\alpha^1_k}{\|\alpha^1_k\|_{h^{L^k}}}, f^2_k, \ldots, f^{d_k}_k$$

to be an orthonormal frame for $\mathscr{H}^0_{b,\leq k\nu_k}(X, L^k)$. From (6.15), (6.16) and (2.19), we conclude that

$$\lim_{k\to\infty} k^{-n}\left|f^1_k(0)\right|^2_{h^{L^k}} = \lim_{k\to\infty} k^{-n}\Pi^{(0)}_{k,\leq k\nu_k}(0) = (2\pi)^{-n}\int_{\mathbb{R}_{p,0}} \det(M^\phi_p + 2s\mathcal{L}_p)ds. \tag{6.25}$$

Thus,

$$\lim_{k\to\infty} k^{-n}\sum_{j=2}^{d_k}\left|f^j_k(0)\right|^2_{h^{L^k}} = 0. \tag{6.26}$$

Now,

$$(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(0) = \langle(Q^{(0)}_{M,k}f^1_k)(0)|f^1_k(0)\rangle_{h^{L^k}} + \sum_{j=2}^{d_k}\langle(Q^{(0)}_{M,k}f^j_k)(0)|f^j_k(0)\rangle_{h^{L^k}}. \tag{6.27}$$

From (6.1) and (6.26), we have

$$\begin{aligned}&\lim_{k\to\infty} k^{-n}\left|\sum_{j=2}^{d_k}\langle(Q^{(0)}_{M,k}f^j_k)(0)|f^j_k(0)\rangle_{h^{L^k}}\right|\\&\leq \lim_{k\to\infty} k^{-n}\sqrt{\sum_{j=2}^{d_k}\left|(Q^{(0)}_{M,k}f^j_k)(0)\right|^2_{h^{L^k}}}\sqrt{\sum_{j=2}^{d_k}\left|f^j_k(0)\right|^2_{h^{L^k}}} \to 0, \quad \text{as } k\to\infty.\end{aligned} \tag{6.28}$$

Combining (6.28) with (6.27), we conclude that

$$\lim_{k\to\infty} k^{-n}(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq \nu_k})(0) = \lim_{k\to\infty} k^{-n}\langle(Q^{(0)}_{M,k}f^1_k)(0)|f^1_k(0)\rangle_{h^{L^k}}. \tag{6.29}$$

Let α^2_k be as in (6.14). From (6.18) and the definition of $Q^{(0)}_{M,k}$ [see (5.7)], it is not difficult to see that

$$\lim_{k\to\infty} k^{-n}\langle(Q^{(0)}_{M,k}\alpha^2_k)(0)|\alpha^2_k(0)\rangle_{h^{L^k}} = 0. \tag{6.30}$$

Combining (6.30) with (6.29) and (6.15), we deduce

$$\lim_{k\to\infty} k^{-n}(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq\nu_k})(0)=\lim_{k\to\infty}k^{-n}\langle(Q^{(0)}_{M,k}\alpha_k)(0)|\alpha_k(0)\rangle_{h^{L^k}}, \tag{6.31}$$

where α_k is as in (6.9). On D, we put

$$Q^{(0)}_{M,k}\alpha_k=s^kq_k,\quad q_k\in C^\infty(D). \tag{6.32}$$

By the definitions of $Q^{(0)}_{M,k}$ and α_k [see (5.7) and (6.9)], we can check that

$$\begin{aligned}q_k(0)&=k^{\frac{n}{2}}(2\pi)^{-\frac{n}{2}}\left(\int_{\mathbb{R}_{p,0}}\det(M^\phi_p+2s\mathcal{L}_p)\mathrm{d}s\right)^{-\frac{1}{2}}\\&\times\int \mathrm{e}^{-it\eta}\psi(\eta)\chi_M(t)\mathrm{e}^{-\frac{k}{2}\phi(0,\frac{t}{k})}\chi_1\left(0,\frac{t}{\sqrt{k}\log k}\right)\mathrm{e}^{it\xi+\frac{\beta}{2}t}\mathbb{1}_{\mathbb{R}_{p,0}}(\xi)\det(M^\phi_p+2\xi\mathcal{L}_p)\mathrm{d}\xi\mathrm{d}t\mathrm{d}\eta.\end{aligned} \tag{6.33}$$

We notice that $\frac{k}{2}\phi(0,\frac{t}{k})=\frac{\beta}{2}t+\epsilon_k(t)$, where $\epsilon_k(t)\to 0$ as $k\to\infty$, uniformly on Supp χ_M and $\chi_1(0,\frac{t}{\sqrt{k}\log k})\to 1$ as $k\to\infty$, uniformly on Supp χ_M. Combining this observation with (6.33), (6.9) and (6.8), we can check that

$$\begin{aligned}&\lim_{k\to\infty}k^{-n}\langle(Q^{(0)}_{M,k}\alpha_k)(0)|\alpha_k(0)\rangle_{h^{L^k}}\\&\quad=\lim_{k\to\infty}k^{-\frac{n}{2}}q_k(0)\overline{u(0,0)}\mathrm{e}^{-k\phi(0)}\\&\quad=(2\pi)^{-n}\int \mathrm{e}^{-it\eta+it\xi}\psi(\eta)\chi_M(t)\mathbb{1}_{\mathbb{R}_{p,0}}(\xi)\det(M^\phi_p+2\xi\mathcal{L}_p)\mathrm{d}\xi\mathrm{d}t\mathrm{d}\eta\\&\quad=(2\pi)^{-n}\int \mathrm{e}^{it\xi}\hat{\psi}(t)\chi_M(t)\mathbb{1}_{\mathbb{R}_{p,0}}(\xi)\det(M^\phi_p+2\xi\mathcal{L}_p)\mathrm{d}\xi\mathrm{d}t,\end{aligned} \tag{6.34}$$

where $\hat{\psi}(t):=\int \mathrm{e}^{-it\eta}\psi(\eta)\mathrm{d}\eta$, u is as in (6.8). From (6.34) and (6.31), (6.24) follows. We get Theorem 6.4. □

We need

Theorem 6.5 *Let $\delta_k>0$, $\delta_k\to 0$, as $k\to\infty$, be as in Theorem 6.4 and let $\nu_k>0$ be any sequence with $\lim_{k\to\infty}\frac{\delta_k}{\nu_k}=0$ and $\nu_k\to 0$ as $k\to\infty$. Then, there is a $k_0>0$ such that for all $k\geq k_0$,*

$$\begin{aligned}&\left|\int_X(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)\mathrm{d}v_X(x)\right|\\&\quad\geq\frac{k^n}{2}(2\pi)^{1-n}\int_X\left(\int\psi(\xi)\det(M^\phi_x+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\right)\mathrm{d}v_X(x).\end{aligned} \tag{6.35}$$

Proof For each $x\in X$, put

$$C(x):=(2\pi)^{-n}\int \mathrm{e}^{it\xi}\hat{\psi}(t)\chi_M(t)\det(M^\phi_x+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t. \tag{6.36}$$

From (6.2), (6.24) and the Lebesgue dominated Theorem, we conclude that

$$\int_X(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)\mathrm{d}v_X(x)=k^n\int_X C(x)\mathrm{d}v_X(x)+o(k^n)$$

and hence

$$\left|\int_X (Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)\mathrm{d}v_X(x)\right| \geq k^n \left|\int_X C(x)\mathrm{d}v_X(x)\right| + o(k^n). \tag{6.37}$$

We first claim that for each $x \in X$, $C(x)$ is real. We notice that $\overline{\hat{\psi}(t)} = \hat{\psi}(-t)$ and $\chi_M(t) = \chi_M(-t)$. From this observation, we can check that

$$\begin{aligned}\overline{C(x)} &= (2\pi)^{-n}\int \mathrm{e}^{-it\xi}\overline{\hat{\psi}(t)}\chi_M(t)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t\\ &= (2\pi)^{-n}\int \mathrm{e}^{-it\xi}\hat{\psi}(-t)\chi_M(-t)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t\\ &= (2\pi)^{-n}\int \mathrm{e}^{it\xi}\hat{\psi}(t)\chi_M(t)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t = C(x).\end{aligned}$$

Thus, $C(x)$ is real.

Now, we claim that $\int_X C(x)\mathrm{d}v_X(x)$ is positive and

$$\int_X C(x)\mathrm{d}v_X(x) > \frac{1}{2}(2\pi)^{1-n}\int_X\Big(\int \psi(\xi)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\Big)\mathrm{d}v_X(x). \tag{6.38}$$

We have

$$\begin{aligned}C(x) &= (2\pi)^{-n}\int \mathrm{e}^{it\xi}\hat{\psi}(t)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t\\ &\quad +(2\pi)^{-n}\int \mathrm{e}^{it\xi}\hat{\psi}(t)(\chi_M(t)-1)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t\\ &= (2\pi)^{1-n}\int \psi(\xi)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\\ &\quad +(2\pi)^{-n}\int \mathrm{e}^{it\xi}\hat{\psi}(t)(\chi_M(t)-1)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t.\end{aligned} \tag{6.39}$$

Here, we used Fourier's inversion formula. Since $0 \leq \chi_M \leq 1$ and $\chi_M = 1$ if $-M \leq t \leq M$, we have

$$\begin{aligned}&\left|\int \mathrm{e}^{it\xi}\hat{\psi}(t)(\chi_M(t)-1)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t\right|\\ &\quad = \left|\int_{|t|\geq M} \mathrm{e}^{it\xi}\hat{\psi}(t)(\chi_M(t)-1)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\mathrm{d}t\right|\\ &\quad \leq \int_{|t|\geq M}\left|\hat{\psi}(t)\right|\mathrm{d}t\left|\int \det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\right|\\ &\quad \leq \frac{2C_0}{M}\int \det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi,\end{aligned} \tag{6.40}$$

where $C_0 = \sup_{t\in\mathbb{R}} t^2\left|\hat{\psi}(t)\right|$. Combining (6.40) with (6.39), we get

$$\begin{aligned}C(x) \geq\ &(2\pi)^{1-n}\int \psi(\xi)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\\ &-\frac{2C_0}{M}(2\pi)^{-n}\int \det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi.\end{aligned}$$

Combining this with (5.4), (6.38) follows.

From (6.38) and (6.37), we obtain (6.35). □

7 The asymptotic behavior of $\left(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}}\right)(x)$

We will use the same notations as before. Fix $p\in X$. Let $x=(x_1,\ldots,x_{2n-1})=(z,\theta)$ be canonical coordinates of X defined in some small neighborhood D of p and let s be a local section of L on D, $|s|^2_{h^L}=\mathrm{e}^{-\phi}$. We take x and s so that (4.1) hold. Until further notice, we work with the local coordinates x and the local section s. We also write t to denote the coordinate θ. We identify D with some open set in $H_n=\mathbb{C}^{n-1}\times\mathbb{R}$. Let $\nu_k>0$ be any sequence with $\nu_k\to 0$ as $k\to\infty$. We are going to estimate $\limsup_{k\to\infty}k^{-n}(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(p)$. For the convenience of the reader we recall some notations we used before. Let $e_j(z,\theta)$, $j=1,\ldots,n-1$, denote the basis of $T^{*(0,1)}X$, dual to $\overline{U}_j(z,\theta)$, $j=1,\ldots,n-1$, where U_j, $j=1,\ldots,n-1$, are as in (4.3). For $f\in\Omega^{0,1}(X,L^k)$, we write $f=\sum_{j=1}^{n-1}f_je_j$, $f_j\in C^\infty(X,L^k)$, $j=1,\ldots,n-1$. We call f_j the component of f along e_j. As (2.21), for $j=1,\ldots,n-1$, we define

$$(Q^{(1)}_{M,k}S^{(1)}_{k,\leqslant k\nu_k,j}\overline{Q^{(1)}_{M,k}})(y):=\sup_{\alpha\in\mathscr{H}^1_{b,\leq k\nu_k}(X,L^k),\ \|\alpha\|_{h^{L^k}}=1}\left|(Q^{(1)}_{M,k}\alpha)_j(y)\right|^2_{h^{L^k}},\tag{7.1}$$

where $(Q^{(1)}_{M,k}\alpha)_j$ denotes the component of $Q^{(1)}_{M,k}\alpha$ along e_j. From (2.23), we know that

$$(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leqslant k\nu_k}\overline{Q^{(1)}_{M,k}})(y)=\sum_{j=1}^{n-1}(Q^{(1)}_{M,k}S^{(1)}_{k,\leqslant k\nu_k,j}\overline{Q^{(1)}_{M,k}})(y),\quad\forall y\in D.\tag{7.2}$$

We consider H_n. Let $\psi(\eta)$ be as in the discussion after (5.1) and let $\chi_M(t)$ be as in (5.6). The operator $Q^{(1)}_{M,H_n}$ is a continuous operator $\Omega^{0,1}(H_n)\to\Omega^{0,1}(H_n)$ defined as follows. Let $u\in\Omega^{0,1}(H_n)$. We write $u=\sum_{j=1}^{n-1}u_jd\overline{z}_j$, $u_j\in C^\infty(H_n)$, $j=1,\ldots,n-1$. Then,

$$\begin{aligned}(Q^{(1)}_{M,H_n}u)(z,\theta)&=\sum_{j=1}^{n-1}\left(\int\mathrm{e}^{-it\eta}\psi(\eta)\chi_M(t)\mathrm{e}^{-\frac{\beta}{2}(t+\theta)}u_j(z,t+\theta)dtd\eta\right)d\overline{z}_j\\&:=\int\mathrm{e}^{-it\eta}\psi(\eta)\chi_M(t)\mathrm{e}^{-\frac{\beta}{2}(t+\theta)}u(z,t+\theta)dtd\eta.\end{aligned}\tag{7.3}$$

We remind that β is as in (4.1). For $j=1,\ldots,n-1$, put [compare (7.1)]

$$\begin{aligned}&(Q^{(1)}_{M,H_n}S^{(1)}_{j,H_n}\overline{Q^{(1)}_{M,H_n}})(0)\\&\quad=\sup\left\{\left|(Q^{(1)}_{M,H_n}\alpha)_j(0)\right|^2;\ \alpha\in\Omega^{0,1}(H_n),\Box^{(1)}_{b,H_n}\alpha=0,\|\alpha\|_{\phi_0}=1\right\},\end{aligned}\tag{7.4}$$

where

$$(Q^{(1)}_{M,H_n}\alpha)(x)=\sum_{j=1}^{n-1}(Q^{(1)}_{M,H_n}\alpha)_j(x)d\overline{z}_j,\quad(Q^{(1)}_{M,H_n}\alpha)_j\in C^\infty(H_n),\qquad j=1,\ldots,n-1,$$

and

$$\|\alpha\|^2_{\phi_0}=\int|\alpha(z,\theta)|^2\,\mathrm{e}^{-\phi_0(z,\theta)}dv(z)d\theta,\quad dv(z)=2^{n-1}dx_1dx_2\ldots dx_{2n-1}.$$

We recall that ϕ_0 is as in the discussion after (4.23). We first need

Theorem 7.1 *We have*

$$\limsup_{k\to\infty} k^{-n}\left(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leqslant k\nu_k}\overline{Q^{(1)}_{M,k}}\right)(0)\leqslant\sum_{j=1}^{n-1}\left(Q^{(1)}_{M,H_n}S^{(1)}_{j,H_n}\overline{Q^{(1)}_{M,H_n}}\right)(0).$$

Proof Fix $j\in\{1,2,\ldots,n-1\}$. We claim that

$$\limsup_{k\to\infty} k^{-n}\left(Q^{(1)}_{M,k}S^{(1)}_{k,\leqslant k\nu_k,j}\overline{Q^{(1)}_{M,k}}\right)(0)\leqslant\left(Q^{(1)}_{M,H_n}S^{(1)}_{j,H_n}\overline{Q^{(1)}_{M,H_n}}\right)(0). \tag{7.5}$$

The definition (7.1) of $(Q^{(1)}_{M,k}S^{(1)}_{k,\leqslant k\nu_k,j}\overline{Q^{(1)}_{M,k}})(0)$ yields a sequence

$$\alpha_{k_s}\in\mathscr{H}^1_{b,\leqslant k_s\nu_{k_s}}(X,L^{k_s}),\quad k_1<k_2<\cdots,$$

such that $\left\|\alpha_{k_s}\right\|_{h^{L^{k_s}}}=1$ and

$$\lim_{s\to\infty}k_s^{-n}\left|(Q^{(1)}_{M,k_s}\alpha_{k_s})_j(0)\right|^2_{h^{L^{k_s}}}=\limsup_{k\to\infty}k^{-n}(Q^{(1)}_{M,k}S^{(1)}_{k,\leqslant k\nu_k,j}\overline{Q^{(1)}_{M,k}})(0), \tag{7.6}$$

where $(Q^{(1)}_{M,k_s}\alpha_{k_s})_j$ is the component of $Q^{(1)}_{M,k_s}\alpha_{k_s}$ along e_j. On D, we write

$$\alpha_{k_s}=s^{k_s}\widetilde{\alpha}_{k_s},\quad\widetilde{\alpha}_{k_s}\in\Omega^{0,1}(D),$$

and on $D_{\log k_s}$, put

$$\gamma_{k_s}=k_s^{-\frac{n}{2}}F^*_{k_s}\widetilde{\alpha}_{k_s}\in F^*_{k_s}\Omega^{0,1}(D_{\log k_s}).$$

We recall that $F^*_{k_s}$ is the scaling map given by (4.9). It is not difficult to see that

$$\left\|\gamma_{k_s}\right\|_{k_sF^*_{k_s}\phi,D_{\log k_s}}\leqslant 1.$$

Moreover, from (4.17) and (4.2), it is straightforward to see that

$$\left\|(\Box^{(1)}_{b,k_s\phi,(k_s)})^m\gamma_{k_s}\right\|_{k_sF^*_{k_s}\phi,D_{\log k_s}}\leq\frac{1}{k_s^m}\left\|(\Box^{(1)}_{b,k_s})^m\alpha_{k_s}\right\|_{h^{L^{k_s}}}\leq\nu^m_{k_s},\quad\forall m\in\mathbb{N}.$$

Proposition 4.3 yields a subsequence $\{\gamma_{k_{s_u}}\}$ of $\{\gamma_{k_s}\}$ such that for each t in the set $\{1,2,\ldots,n-1\}$, $\gamma_{k_{s_u},t}$ converges uniformly with all derivatives on any compact subset of H_n to a smooth function γ_t, where $\gamma_{k_{s_u},t}$ denotes the component of $\gamma_{k_{s_u}}$ along $e_t(\frac{z}{\sqrt{k}},\frac{\theta}{k})$. Set $\gamma=\sum_{t=1}^{n-1}\gamma_t d\overline{z}_t$. Then, we have $\Box^{(1)}_{b,H_n}\gamma=0$ and, by (4.24), $\|\gamma\|_{\phi_0}\leqslant 1$. Thus,

$$\left|(Q^{(1)}_{M,H_n}\gamma)_j(0)\right|^2\leqslant\frac{\left|(Q^{(1)}_{M,H_n}\gamma)_j(0)\right|^2}{\|\gamma\|^2_{\phi_0}}\leqslant\left(Q^{(1)}_{M,H_n}S^{(1)}_{j,H_n}\overline{Q^{(1)}_{M,H_n}}\right)(0), \tag{7.7}$$

where

$$Q^{(1)}_{M,H_n}\gamma=\sum_{t=1}^{n-1}(Q^{(1)}_{M,H_n}\gamma)_t d\overline{z}_t,\qquad(Q^{(1)}_{M,H_n}\gamma)_t\in C^\infty(H_n),\quad t=1,\ldots,n-1.$$

We claim that

$$\lim_{u\to\infty}k_{s_u}^{-n}\left|\left(Q^{(1)}_{M,k_{s_u}}\alpha_{k_{s_u}}\right)_j(0)\right|^2=\left|\left(Q^{(1)}_{M,H_n}\gamma\right)_j(0)\right|^2. \tag{7.8}$$

We write

$$\widetilde{\alpha}_{k_{s_u}} = \sum_{j=1}^{n-1} \widetilde{\alpha}_{k_{s_u},j} e_j = \sum_{j=1}^{n-1} \hat{\alpha}_{k_{s_u},j} \mathrm{d}\overline{z}_j.$$

Since $e_t = \mathrm{d}\overline{z}_t + O(|(z,\theta)|, t = 1, \ldots, n-1$, we conclude that for all $t = 1, \ldots, n-1$,

$$\lim_{u\to\infty} k_{s_u}^{-\frac{n}{2}} F^*_{k_{s_u}} \widetilde{\alpha}_{k_{s_u},t} = \lim_{u\to\infty} k_{s_u}^{-\frac{n}{2}} F^*_{k_{s_u}} \hat{\alpha}_{k_{s_u},t} = \gamma_t. \tag{7.9}$$

Moreover, from the definition of $Q^{(1)}_{M,k_{s_u}}$ [see (5.13)], it is easy to see that

$$\left|(Q^{(1)}_{M,k_{s_u}} \alpha_{k_{s_u}})_j(0)\right|_{h^{L^k}} = \left|\int \mathrm{e}^{-it\eta} \psi(\eta) \chi_M(t) \mathrm{e}^{-\frac{k_{s_u}}{2}(F^*_{k_{s_u}}\phi)(0,t)} F^*_{k_{s_u}} \hat{\alpha}_{k_{s_u},j}(0,t)\mathrm{d}t\right|. \tag{7.10}$$

Combining (7.10) with (7.9), (7.3) and notice that $-\frac{k}{2}(F^*_k\phi)(0,t) \to -\frac{\beta}{2}t$, as $k \to \infty$, uniformly on Supp χ_M, (7.8) follows. The claim (7.5) follows from (7.6), (7.7) and (7.8). Finally, (7.5) and (7.2) imply the conclusion of the theorem. □

To estimate $\sum_{j=1}^{n-1}(Q^{(1)}_{M,H_n} S^{(1)}_{j,H_n} \overline{Q^{(1)}_{M,H_n}})(0)$, we need the some preparation. Put

$$\Phi_0 = \sum_{j,t=1}^{n-1} \mu_{j,t}\overline{z}_j z_t, \tag{7.11}$$

where $\mu_{j,t}$, $j, t = 1, \ldots, n-1$, are as in (4.1). Note that

$$\phi_0(z,\theta) = \Phi_0(z) + \beta\theta.$$

For $q = 0, 1, \ldots, n-1$, we denote by $L^2_{(0,q)}(H_n, \Phi_0)$ the completion of $\Omega^{0,q}_0(H_n)$ with respect to the norm $\|\cdot\|_{\Phi_0}$, where

$$\|u\|^2_{\Phi_0} = \int_{H_n} |u|^2 \mathrm{e}^{-\Phi_0} \mathrm{d}v(z)\mathrm{d}\theta, \quad u \in \Omega^{(0,q)}_0(H_n).$$

Let $u(z,\theta) \in \Omega^{0,1}(H_n)$ with $\|u\|_{\phi_0} = 1$, $\Box^{(1)}_{b,H_n} u = 0$. Put $v(z,\theta) = u(z,\theta)\mathrm{e}^{-\frac{\beta}{2}\theta}$. We have

$$\int_{H_n} |v(z,\theta)|^2 \mathrm{e}^{-\Phi_0(z)} \mathrm{d}v(z)\mathrm{d}\theta = 1.$$

Choose $\chi(\theta) \in C^\infty_0(\mathbb{R})$ so that $\chi(\theta) = 1$ when $|\theta| < 1$ and $\chi(\theta) = 0$ when $|\theta| > 2$ and set $\chi_j(\theta) = \chi(\theta/j)$, $j \in \mathbb{N}$. Let

$$\hat{v}_j(z,\eta) = \int_{\mathbb{R}} v(z,\theta)\chi_j(\theta)\mathrm{e}^{-i\theta\eta}\mathrm{d}\theta \in \Omega^{0,1}(H_n), \quad j = 1, 2, \ldots. \tag{7.12}$$

From Parseval's formula, we have

$$\begin{aligned} &\int_{H_n} \left|\hat{v}_j(z,\eta) - \hat{v}_t(z,\eta)\right|^2 \mathrm{e}^{-\Phi_0(z)} \mathrm{d}\eta \mathrm{d}v(z) \\ &= 2\pi \int_{H_n} |v(z,\theta)|^2 \left|\chi_j(\theta) - \chi_t(\theta)\right|^2 \mathrm{e}^{-\Phi_0(z)} \mathrm{d}\theta \mathrm{d}v(z) \to 0, \quad j, t \to \infty. \end{aligned}$$

Thus, there is $\hat{v}(z,\eta)\in L^2_{(0,1)}(H_n,\Phi_0)$ such that $\hat{v}_j(z,\eta)\to\hat{v}(z,\eta)$ in $L^2_{(0,1)}(H_n,\Phi_0)$. We have

$$\int\left|\hat{v}(z,\eta)\right|^2 \mathrm{e}^{-\Phi_0(z)}\mathrm{d}v(z)\mathrm{d}\eta=2\pi. \tag{7.13}$$

We call $\hat{v}(z,\eta)$ the Fourier transform of $v(z,\theta)$ with respect to θ. Formally,

$$\hat{v}(z,\eta)=\int_{\mathbb{R}}\mathrm{e}^{-i\theta\eta}v(z,\theta)\mathrm{d}\theta. \tag{7.14}$$

The following theorem is one of the main technical results in [9] (see [9, section 3], for the proof).

Theorem 7.2 *With the notations used above. Let $u(z,\theta)\in\Omega^{0,1}(H_n)$ with $\|u\|_{\phi_0}=1$, $\Box^{(1)}_{b,H_n}u=0$ and let $\hat{v}(z,\eta)\in L^2_{(0,1)}(H_n,\Phi_0)$ be the Fourier transform of the function $u(z,\theta)\mathrm{e}^{-\frac{\beta}{2}\theta}$ with respect to θ (see the discussion before (7.14)). Then, for almost all $\eta\in\mathbb{R}$, we have $\hat{v}(z,\eta)$ is smooth with respect to z and*

$$\int_{\mathbb{C}^{n-1}}\left|\hat{v}(z,\eta)\right|^2\mathrm{e}^{-\Phi_0(z)}\mathrm{d}v(z)<\infty$$

and

$$\left|\hat{v}(z,\eta)\right|^2\leq(2\pi)^{-n+1}\mathrm{e}^{\Phi_0(z)}\mathbb{1}_{\mathbb{R}_{p,1}}(\eta)\left|\det(M^\phi_p+2\eta\mathcal{L}_p)\right|\int_{\mathbb{C}^{n-1}}\left|\hat{v}(w,\eta)\right|^2\mathrm{e}^{-\Phi_0(w)}\mathrm{d}v(w) \tag{7.15}$$

for all $z\in\mathbb{C}^{n-1}$.

Now, we can prove

Proposition 7.3 *Let $u(z,\theta)\in\Omega^{0,1}(H_n)$ with $\|u\|_{\phi_0}=1$, $\Box^{(1)}_{b,H_n}u=0$. We have*

$$\left|(Q^{(1)}_{M,H_n}u)(0)\right|^2\leq\frac{E^2}{n-1}, \tag{7.16}$$

where E is as in (5.3).

Proof Let $\varphi\in C^\infty_0(\mathbb{C}^{n-1},\mathbb{R})$ such that $\int_{\mathbb{C}^{n-1}}\varphi(z)\mathrm{d}v(z)=1$, $\varphi\geqslant 0$, $\varphi(z)=0$ if $|z|>1$. Put $g_m(z)=m^{2n-2}\varphi(mz)\mathrm{e}^{\Phi_0(z)}$, $m=1,2,\ldots$. Then, $\int_{\mathbb{C}^{n-1}}g_m(z)\mathrm{e}^{-\Phi_0(z)}\mathrm{d}v(z)=1$ and

$$\begin{aligned}(Q^{(1)}_{M,H_n}u)(0)&=\lim_{m\to\infty}\int\mathrm{e}^{-it\eta}\psi(\eta)\chi_M(t)\mathrm{e}^{-\frac{\beta}{2}t}\mathrm{e}^{-\Phi_0(z)}g_m(z)u(z,t)\mathrm{d}t\mathrm{d}v(z)\\&=\lim_{m\to\infty}\int\hat{\psi}(t)\chi_M(t)\mathrm{e}^{-\frac{\beta}{2}t}\mathrm{e}^{-\Phi_0(z)}g_m(z)u(z,t)\mathrm{d}t\mathrm{d}v(z).\end{aligned} \tag{7.17}$$

Choose $\chi(t)\in C^\infty_0(\mathbb{R})$ so that $\chi(t)=1$ when $|t|<1$ and $\chi(t)=0$ when $|t|>2$ and set $\chi_j(t)=\chi(t/j)$, $j\in\mathbb{N}$. For each m, we have

$$\begin{aligned}&\int\hat{\psi}(t)\chi_M(t)\mathrm{e}^{-\frac{\beta}{2}t}\mathrm{e}^{-\Phi_0(z)}g_m(z)u(z,t)\mathrm{d}t\mathrm{d}v(z)\\&\quad=\lim_{j\to\infty}\int\hat{\psi}(t)\chi_M(t)\mathrm{e}^{-\frac{\beta}{2}t}\mathrm{e}^{-\Phi_0(z)}g_m(z)u(z,t)\chi_j(t)\mathrm{d}t\mathrm{d}v(z).\end{aligned} \tag{7.18}$$

From Parseval's formula, we can check that for each j,

$$\int \hat{\psi}(t)\chi_M(t)e^{-\frac{\beta}{2}t}e^{-\Phi_0(z)}g_m(z)u(z,t)\chi_j(t)dtdv(z)$$
$$= \frac{1}{2\pi}\int \alpha(\eta)\hat{v}_j(z,\eta)g_m(z)e^{-\Phi_0(z)}d\eta dv(z), \tag{7.19}$$

where $\hat{v}_j(z,\eta)$ is as in (7.12) and

$$\alpha(\eta) = \int e^{-it\eta}\hat{\psi}(\eta)\chi_M(t)dt. \tag{7.20}$$

From (7.19) and (7.18), we obtain for each m,

$$\int \hat{\psi}(\eta)\chi_M(t)e^{-\frac{\beta}{2}t}e^{-\Phi_0(z)}g_m(z)u(z,t)dv(z)dt$$
$$= \frac{1}{2\pi}\int \hat{v}(z,\eta)g_m(z)\alpha(\eta)e^{-\Phi_0(z)}dv(z)d\eta, \tag{7.21}$$

where $\hat{v}(z,\eta)$ is as in (7.14). Now,

$$\begin{aligned}\alpha(\eta) &= \int e^{-it\eta}\hat{\psi}(t)\chi_M(t)dt\\ &= \int e^{-it\eta}\hat{\psi}(t)dt + \int e^{-it\eta}\hat{\psi}(t)(\chi_M(t)-1)dt\\ &= (2\pi)\psi(\eta)+\alpha_1(\eta),\end{aligned}$$

where

$$\alpha_1(\eta) = \int e^{-it\eta}\hat{\psi}(t)(\chi_M(t)-1)dt.$$

Combining this with (7.21), we have

$$\int \hat{\psi}(t)\chi_M(t)e^{-\frac{\beta}{2}t}g_m(z)u(z,t)e^{-\Phi_0(z)}dv(z)dt$$
$$= \int \hat{v}(z,\eta)g_m(z)\psi(\eta)e^{-\Phi_0(z)}dv(z)d\eta + \frac{1}{2\pi}\int \hat{v}(z,\eta)g_m(z)\alpha_1(\eta)e^{-\Phi_0(z)}dv(z)d\eta. \tag{7.22}$$

Since $\hat{v}(z,\eta) \in L^2_{(0,1)}(H_n,\Phi_0)$, it is easy to see that

$$\int |\psi(\eta)|\left|\hat{v}(z,\eta)\right||g_m(z)|e^{-\Phi_0(z)}d\eta dv(z) < \infty, \quad \forall m > 0. \tag{7.23}$$

From (7.15), we see that $\hat{v}(z,\eta) = 0$ almost everywhere on $\mathbb{R}\setminus\mathbb{R}_{p,1}$, for every $z \in \mathbb{C}^{n-1}$. Since $\operatorname{Supp}\psi \bigcap \mathbb{R}_{p,1} = \emptyset$ [see the discussion after (5.1)], we conclude that for each $m > 0$,

$$z \to \int \psi(\eta)\hat{v}(z,\eta)g_m(z)e^{-\Phi_0(z)}d\eta = 0. \tag{7.24}$$

From (7.23), (7.24) and Fubini's theorem, we obtain

$$\int \hat{v}(z,\eta)g_m(z)\psi(\eta)e^{-\Phi_0(z)}d\eta dv(z) = 0 \tag{7.25}$$

for every $m > 0$. From (7.25) and (7.22), we get for each m,

$$\int \hat{\psi}(t)\chi_M(t)\mathrm{e}^{-\frac{\beta}{2}t} g_m(z)u(z,t)\mathrm{e}^{-\Phi_0(z)}\mathrm{d}v(z)\mathrm{d}t = \frac{1}{2\pi}\int \hat{v}(z,\eta)g_m(z)\alpha_1(\eta)\mathrm{e}^{-\Phi_0(z)}\mathrm{d}v(z)\mathrm{d}\eta. \tag{7.26}$$

Since $0 \le \chi_M \le 1$ and $\chi_M = 1$ if $-M \le t \le M$, we have

$$|\alpha_1(\eta)| = \left|\int_{|t|\ge M} \mathrm{e}^{-it\eta}\hat{\psi}(t)(\chi_M(t)-1)\mathrm{d}t\right| \le \int_{|t|\ge M}\left|\hat{\psi}(t)\right|\mathrm{d}t \le \frac{2C_0}{M}, \quad \forall \eta \in \mathbb{R}, \tag{7.27}$$

where $C_0 = \sup_{t\in\mathbb{R}} t^2 \left|\hat{\psi}(t)\right|$. Put

$$f(\eta) := \int_{\mathbb{C}^{n-1}} \left|\hat{v}(z,\eta)\right|^2 \mathrm{e}^{-\Phi_0(z)}\mathrm{d}v(z).$$

From (7.27) and (7.26), we have for each m,

$$\begin{aligned}
&\left|\int \hat{\psi}(\eta)\chi_M(t)\mathrm{e}^{-\frac{\beta}{2}t} g_m(z)u(z,t)\mathrm{e}^{-\Phi_0(z)}\mathrm{d}v(z)\mathrm{d}t\right| \\
&\le \frac{2C_0}{M}\frac{1}{2\pi}\int \left|\hat{v}(z,\eta)\right| g_m(z)\mathrm{e}^{-\Phi_0(z)}\mathrm{d}v(z)\mathrm{d}\eta = \frac{2C_0}{M}\frac{1}{2\pi}\int_{|z|\le 1}\left|\hat{v}\left(\frac{z}{m},\eta\right)\right|\varphi(z)\mathrm{d}v(z)\mathrm{d}\eta \\
&\overset{\text{by (7.15)}}{\le} \frac{2C_0}{M}(2\pi)^{-\frac{n+1}{2}}\int_{|z|\le 1} \mathrm{e}^{\Phi(\frac{z}{m})}\mathbb{1}_{\mathbb{R}_{p,1}}(\eta)\left|\det(M_p^\phi + 2\eta\mathcal{L}_p)\right|^{\frac{1}{2}}\sqrt{f(\eta)}\varphi(z)\mathrm{d}v(z)\mathrm{d}\eta \\
&\le \frac{2C_0}{M}(2\pi)^{-\frac{n+1}{2}}\sup\{\mathrm{e}^{\Phi_0(\frac{z}{m})};\ |z|\le 1\}\left(\int_{\mathbb{R}_{p,1}}\left|\det(M_p^\phi + 2\eta\mathcal{L}_p)\right|\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int f(\eta)\mathrm{d}\eta\right)^{\frac{1}{2}} \\
&\overset{\text{by (7.13)}}{=} \frac{2C_0}{M}(2\pi)^{-\frac{n}{2}}\sup\{\mathrm{e}^{\Phi_0(\frac{z}{m})};\ |z|\le 1\}\left(\int_{\mathbb{R}_{p,1}}\left|\det(M_p^\phi + 2\eta\mathcal{L}_p)\right|\mathrm{d}\eta\right)^{\frac{1}{2}}.
\end{aligned} \tag{7.28}$$

Combining (7.28) with (7.17) and (5.5), we get

$$\left|(Q^{(1)}_{M,H_n}u)(0)\right| \le \frac{2C_0}{M}\left((2\pi)^{-n}\int_{\mathbb{R}_{p,1}}\left|\det(M_p^\phi + 2\eta\mathcal{L}_p)\right|\mathrm{d}\eta\right)^{\frac{1}{2}} < \frac{E}{\sqrt{n-1}},$$

where E is as in (5.3). (7.16) follows. □

In view of Proposition 7.3, we have proved that for all $u(z,\theta) \in \Omega^{0,1}(H_n)$ with $\|u\|_{\phi_0} = 1$, $\Box^{(1)}_{b,H_n}u = 0$, we have

$$\left|(Q^{(1)}_{M,H_n}u)(0)_j\right|^2 \le \left|(Q^{(1)}_{M,H_n}u)(0)\right|^2 < \frac{E^2}{n-1},$$

for all $j = 1,\ldots,n-1$, where $(Q^{(1)}_{M,H_n}u)(0) = \sum_{j=1}^{n-1}(Q^{(1)}_{M,H_n}u)_j(0)\mathrm{d}\overline{z}_j$ and E is as in (5.3). Thus, for every $j = 1,\ldots,n-1$, we have

$$\left(Q^{(1)}_{M,H_n}S^{(1)}_{j,H_n}\overline{Q^{(1)}_{M,H_n}}\right)(0) < \frac{E^2}{n-1}$$

and

$$\sum_{j=1}^{n-1}(Q^{(1)}_{M,H_n}S^{(1)}_{j,H_n}\overline{Q^{(1)}_{M,H_n}})(0) < E^2. \tag{7.29}$$

From (7.29) and Theorem 7.1, we obtain the main result of this section

Theorem 7.4 *Let $\nu_k > 0$ be any sequence with $\nu_k \to 0$ as $k \to \infty$. For each $x \in X$, we have*

$$\limsup_{k\to\infty} k^{-n}(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x) < E^2, \tag{7.30}$$

where E is as in (5.3).

The proof of the following theorem is essentially the same as the proof of (6.1). We omit the proof.

Theorem 7.5 *For any sequence $\nu_k > 0$ with $\nu_k \to 0$ as $k \to \infty$, there is a constant $C > 0$ independent of k, such that*

$$\left|(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leqslant k\nu_k}\overline{Q^{(1)}_{M,k}})(x)\right| \leqslant Ck^n, \quad \forall x \in X. \tag{7.31}$$

Now, we can prove

Theorem 7.6 *Let $\nu_k > 0$ be any sequence with $\nu_k \to 0$ as $k \to \infty$. Then, there is a $k_0 > 0$ such that for all $k \geq k_0$,*

$$\int_X (Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x)dv_X(x) \leq k^n \int_X E^2 dv_X(x), \tag{7.32}$$

where E is as in (5.3).

Proof In view of Theorem 7.5, $\sup_k k^{-n}Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(\cdot)$ is integrable on X. Thus, we can apply Fatou's lemma and we get using Theorem 7.4:

$$\begin{aligned}
&\limsup_{k\to\infty} k^{-n}\int_X (Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x)dv_X(x)\\
&\leq \int_X \limsup_{k\to\infty} k^{-n}(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x)dv_X(x)\\
&< \int_X E^2 dv_X(x).
\end{aligned}$$

The theorem follows. □

8 The proof of Theorem 1.15

Let $\delta_k > 0$, $\delta_k \to \infty$ as $k \to \infty$, be as in Theorem 6.4 and let $\nu_k > 0$ be any sequence with $\lim_{k\to\infty}\frac{\delta_k}{\nu_k} = 0$ and $\nu_k \to 0$ as $k \to \infty$. Let $\gamma_{1,k} < \gamma_{2,k} < \cdots < \gamma_{m_k,k}$ be all the distinct non-zero eigenvalues of $\Box^{(0)}_{b,k}$ between 0 and $k\nu_k$. Thus, $\gamma_{1,k} > 0$ and $\gamma_{m_k,k} \leq k\nu_k$. We notice that $\gamma_{j,k}$, $j = 1, \ldots, m_k$, are also eigenvalues of $\Box^{(1)}_{b,k}$. For $\mu \in \mathbb{R}$, let $\mathscr{H}^q_{b,\mu}(X, L^k)$

denote the space spanned by the eigenforms of $\Box^{(q)}_{b,k}$ whose eigenvalues are λ. For each $j \in \{1, \ldots, m_k\}$, let $f^1_{j,k}, f^2_{j,k}, \ldots, f^{d_{j,k}}_{j,k}$ be an orthonormal basis for $\mathscr{H}^0_{b,\gamma_{j,k}}(X, L^k)$, where $d_{j,k} = \dim \mathscr{H}^0_{b,\gamma_{j,k}}(X, L^k)$. Let $f^1_{0,k}, f^2_{0,k}, \ldots, f^{d_{0,k}}_{0,k}$ be an orthonormal basis for $\mathscr{H}^0_b(X, L^k)$, where $d_{0,k} = \dim \mathscr{H}^0_b(X, L^k)$.

Let $Q^{(0)}_{M,k}$ and $Q^{(1)}_{M,k}$ be as in (5.7) and (5.13), respectively. By the definition of $(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x)$ [see (2.13)], we have

$$(Q^{(0)}_{M,k}\Pi^{(0)}_{k,\leq k\nu_k})(x) = \sum_{t=1}^{d_{0,k}} \langle (Q^{(0)}_{M,k} f^t_{0,k})(x) | f^t_{0,k}(x)\rangle_{h^{L^k}} + \sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}} \langle (Q^{(0)}_{M,k} f^t_{j,k})(x) | f^t_{j,k}(x)\rangle_{h^{L^k}}. \tag{8.1}$$

From (8.1) and (6.35), we conclude that

$$\sum_{t=1}^{d_{0,k}} \left|(Q^{(0)}_{M,k} f^t_{0,k} \mid f^t_{0,k})_{h^{L^k}}\right| + \sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}} \left|(Q^{(0)}_{M,k} f^t_{j,k} \mid f^t_{j,k})_{h^{L^k}}\right| \geq \frac{k^n}{2}(2\pi)^{1-n}\int_X \Big(\int \psi(\xi)\det(M^\phi_x + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)d\xi\Big)dv_X(x), \tag{8.2}$$

for k large. For $j = 1, \ldots, m_k$, we put

$$g^t_{j,k} = \frac{1}{\left\|\overline{\partial}_{b,k} f^t_{j,k}\right\|_{h^{L^k}}}\overline{\partial}_{b,k} f^t_{j,k} = \frac{1}{\sqrt{\gamma_{j,k}}}\overline{\partial}_{b,k} f^t_{j,k} \in \mathscr{H}^1_{b,\gamma_{j,k}}(X, L^k), \quad t = 1, \ldots, d_{j,k}.$$

For each $j = 1, \ldots, m_k$,

$$\begin{aligned}(Q^{(1)}_{M,k} g^t_{j,k} \mid g^t_{j,k})_{h^{L^k}} &= \frac{1}{\gamma_{j,k}}(Q^{(1)}_{M,k}\overline{\partial}_{b,k} f^t_{j,k} \mid \overline{\partial}_{b,k} f^t_{j,k})_{h^{L^k}} \\ &= \frac{1}{\gamma_{j,k}}(\overline{\partial}_{b,k} Q^{(0)}_{M,k} f^t_{j,k} \mid \overline{\partial}_{b,k} f^t_{j,k})_{h^{L^k}} \text{ here we used (5.18)} \\ &= \frac{1}{\gamma_{j,k}}(Q^{(0)}_{M,k} f^t_{j,k} \mid \Box^{(0)}_{b,k} f^t_{j,k})_{h^{L^k}} \\ &= (Q^{(0)}_{M,k} f^t_{j,k} \mid f^t_{j,k})_{h^{L^k}}, \quad t = 1, \ldots, d_{j,k}.\end{aligned} \tag{8.3}$$

Hence,

$$\begin{aligned}&\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}} \left|(Q^{(0)}_{M,k} f^t_{j,k} \mid f^t_{j,k})_{h^{L^k}}\right| \\ &= \sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}} \left|(Q^{(1)}_{M,k} g^t_{j,k} \mid g^t_{j,k})_{h^{L^k}}\right| \\ &\leq \sqrt{\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}} \left\|Q^{(1)}_{M,k} g^t_{j,k}\right\|^2_{h^{L^k}}}\sqrt{\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}} \left\|g^t_{j,k}\right\|^2_{h^{L^k}}}.\end{aligned} \tag{8.4}$$

Since $\left\|g^t_{j,k}\right\|^2_{h^{L^k}} = \left\|f^t_{j,k}\right\|^2_{h^{L^k}} = 1$, for every j and t, it is obviously that

$$\sqrt{\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left\|g^t_{j,k}\right\|^2_{h^{L^k}}} = \sqrt{\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left\|f^t_{j,k}\right\|^2_{h^{L^k}}}.$$

Combining this with (8.4) and (8.2), we get

$$\begin{aligned}&\sum_{t=1}^{d_{0,k}}\left|(Q^{(0)}_{M,k}f^t_{0,k}\mid f^t_{0,k})_{h^{L^k}}\right| + \sqrt{\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left\|Q^{(1)}_{M,k}g^t_{j,k}\right\|^2_{h^{L^k}}}\sqrt{\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left\|f^t_{j,k}\right\|^2_{h^{L^k}}}\\ &\quad\geq \frac{k^n}{2}(2\pi)^{1-n}\int_X\Big(\int\psi(\xi)\det(M^\phi_x+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)d\xi\Big)dv_X(x),\end{aligned} \tag{8.5}$$

for k large.

We can check that for each $j=1,\ldots,m_k$, $g^t_{j,k}$, $t=1,\ldots,d_{j,k}$ is an orthonormal basis of the space $\overline{\partial}_{b,k}\mathscr{H}^0_{b,\gamma_{j,k}}(X,L^k)\subset\mathscr{H}^1_{b,\gamma_{j,k}}(X,L^k)$. From this observation and the definition of $(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x)$ [see (2.14)], we conclude that

$$\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left|Q^{(1)}_{M,k}g^t_{j,k}\right|^2_{h^{L^k}}(x)\leq(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x). \tag{8.6}$$

Thus,

$$\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left\|Q^{(1)}_{M,k}g^t_{j,k}\right\|^2_{h^{L^k}}\leq\int_X(Q^{(1)}_{M,k}\Pi^{(1)}_{k,\leq k\nu_k}\overline{Q^{(1)}_{M,k}})(x)dv(x). \tag{8.7}$$

Combining (8.7) with (7.32), we get

$$\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left\|Q^{(1)}_{M,k}g^t_{j,k}\right\|^2_{h^{L^k}}\leq k^n\int_X E^2dv_X(x) \tag{8.8}$$

for k large, where E is as in (5.3).

From (2.19) and (2.18), we conclude that

$$\int_X\Pi^{(0)}_{k,\leqslant k\nu_k}(x)dv_X(x)=k^n(2\pi)^{-n}\int_X\Big(\int_{\mathbb{R}_{x,0}}\left|\det(M^\phi_x+2s\mathcal{L}_x)\right|ds\Big)dv_X(x)+o(k^n), \tag{8.9}$$

for k large. It is obviously the case that

$$\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left\|f^t_{j,k}\right\|^2_{h^{L^k}}\leq\int_X\Pi^{(0)}_{k,\leqslant k\nu_k}(x)dv_X(x).$$

Combining this with (8.9), we get

$$\sum_{j=1}^{m_k}\sum_{t=1}^{d_{j,k}}\left\|f^t_{j,k}\right\|^2_{h^{L^k}} \leq k^n 2(2\pi)^{-n}\int_X\Big(\int \det(M^\phi_x+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\Big)\mathrm{d}v_X(x) \tag{8.10}$$

for k large. From (8.8), (8.10), (8.5) and (5.3), we obtain

Theorem 8.1 *Let $f^1_{0,k}, f^2_{0,k}, \ldots, f^{d_{0,k}}_{0,k}$ be an orthonormal basis for $\mathscr{H}^0_b(X, L^k)$, where $d_{0,k} = \dim \mathscr{H}^0_b(X, L^k)$. Then, for k large, we have*

$$\sum_{t=1}^{d_{0,k}}\left|(Q^{(0)}_{M,k}f^t_{0,k}\mid f^t_{0,k})_{h^{L^k}}\right| \geq \frac{k^n}{4}(2\pi)^{1-n}\int_X\Big(\int \psi(\xi)\det(M^\phi_x+2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\Big)\mathrm{d}v_X(x). \tag{8.11}$$

The following is straightforward

Lemma 8.2 *For k large, there is a constant $C > 0$ independent of k, such that*

$$\left\|Q^{(0)}_{M,k}u\right\|^2_{h^{L^k}} \leq C\left\|u\right\|^2_{h^{L^k}}, \quad \forall u \in C^\infty(X, L^k).$$

Proof Let $D \Subset D' \Subset D'' \Subset X$ be open sets of X and let s be a local section of L on D''. We assume that there exist canonical coordinates $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$ on D''. Let χ_M be as in (5.6). For k large, we have

$$\left\{\Phi^{\frac{t}{k}}(x) \in D';\ \forall x \in D, t \in \operatorname{Supp}\chi_M\right\}$$

and $\operatorname{Supp} f(\Phi^{\frac{t}{k}}x) \subset D'$, $\forall t \in \operatorname{Supp}\chi_M$, $\forall f \in C^\infty_0(D, L^k)$. In canonical coordinates $x = (z, \theta)$, we have $\Phi^{\frac{t}{k}}(x) = (z, \frac{t}{k} + \theta)$. Let $m(z, \theta)\mathrm{d}v(z)\mathrm{d}\theta$ be the volume form on D'', where $\mathrm{d}v(z) = 2^{n-1}\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_{2n-2}$. Since $m(z, \theta)$ is strictly positive, for k large, there is a constant $C_1 > 0$ independent of k, such that

$$m(z,\theta) \leq C_1 m\left(z, \theta+\frac{t}{k}\right), \quad \forall (z,\theta) \in D',\ t \in \operatorname{Supp}\chi_M. \tag{8.12}$$

Let $u \in C_0^\infty(D, L^k)$. On D'', we write $u = s^k \widetilde{u}$, $\widetilde{u} \in C_0^\infty(D)$. From the definition of $Q_{M,k}^{(0)}$ [see (5.7)], we can check that for k large,

$$\begin{aligned}
&\int \left|(Q_{M,k}^{(0)}u)(x)\right|^2_{h^{L^k}} \mathrm{d}v_X(x) \\
&= \int_{D'} \left| \int \mathrm{e}^{-it\eta} \psi(\eta)\chi_M(t)\mathrm{e}^{-\frac{k}{2}\phi(z,\theta+\frac{t}{k})}\widetilde{u}\left(z, \theta + \frac{t}{k}\right) \mathrm{d}t\mathrm{d}\eta\right|^2 m(z,\theta)\mathrm{d}v(z)\mathrm{d}\theta \\
&\leq \widetilde{C} \int_{(z,\theta)\in D'} \chi_M(t)\mathrm{e}^{-k\phi(z,\theta+\frac{t}{k})} \left|\widetilde{u}\left(z,\theta+\frac{t}{k}\right)\right|^2 m(z,\theta)\mathrm{d}td\theta\mathrm{d}v(z) \\
&\leq \widetilde{C}C_1 \int_{(z,\theta)\in D'} \chi_M(t)\mathrm{e}^{-k\phi(z,\theta+\frac{t}{k})} \left|\widetilde{u}\left(z,\theta+\frac{t}{k}\right)\right|^2 m\left(z,\theta+\frac{t}{k}\right)\mathrm{d}td\theta\mathrm{d}v(z) \\
&= \widetilde{C}C_1 \int_{(z,\lambda-\frac{t}{k})\in D'} \chi_M(t)\mathrm{e}^{-k\phi(z,\lambda)} \left|\widetilde{u}(z,\lambda)\right|^2 m(z,\lambda)\mathrm{d}td\lambda\mathrm{d}v(z) \\
&\leq C \int \mathrm{e}^{-k\phi(z,\theta)} \left|\widetilde{u}(z,\theta)\right|^2 m(z,\theta)\mathrm{d}v(z)\mathrm{d}\theta = C\left\|u\right\|^2_{h^{L^k}}.
\end{aligned} \tag{8.13}$$

where $\widetilde{C} > 0$, $C > 0$ are independent of k and u and C_1 is as in (8.12). From (8.13) and using partition of unity, the lemma follows. □

Proof of Theorem 1.15 From Lemma 8.2 and (8.11), we see that for k large,

$$\begin{aligned}
\sqrt{C}d_{0,k} &= \sqrt{C}\sum_{t=1}^{d_{0,k}} \left\|f_{0,k}^t\right\|^2_{h^{L^k}} \geq \sum_{t=1}^{d_{0,k}} \left|(Q_{M,k}^{(0)} f_{0,k}^t \mid f_{0,k}^t)_{h^{L^k}}\right| \\
&\geq \frac{k^n}{4}(2\pi)^{1-n} \int_X \left(\int \psi(\xi)\det(M_x^\phi + 2\xi\mathcal{L}_x)\mathbb{1}_{\mathbb{R}_{x,0}}(\xi)\mathrm{d}\xi\right)\mathrm{d}v_X(x),
\end{aligned}$$

where $C > 0$ is the constant as in Lemma 8.2 and $d_{0,k} = \dim \mathscr{H}_b^0(X, L^k)$. Theorem 1.15 follows. □

9 Examples

In this section, some examples are collected. The aim is to illustrate the main results in some simple situations.

9.1 Compact Heisenberg groups

Let $\lambda_1, \ldots, \lambda_{n-1}$ be given non-zero integers. Let $\mathscr{C}H_n = (\mathbb{C}^{n-1} \times \mathbb{R})/\sim$, where $(z,\theta) \sim (\widetilde{z}, \widetilde{\theta})$ if

$$\begin{aligned}
&\widetilde{z} - z = (\alpha_1, \ldots, \alpha_{n-1}) \in \sqrt{2\pi}\mathbb{Z}^{n-1} + i\sqrt{2\pi}\mathbb{Z}^{n-1}, \\
&\widetilde{\theta} - \theta - i\sum_{j=1}^{n-1}\lambda_j(z_j\overline{\alpha}_j - \overline{z}_j\alpha_j) \in \pi\mathbb{Z}.
\end{aligned}$$

We can check that $\sim$ is an equivalence relation and $\mathscr{C}H_n$ is a compact manifold of dimension $2n-1$. The equivalence class of $(z,\theta) \in \mathbb{C}^{n-1} \times \mathbb{R}$ is denoted by $[(z,\theta)]$. For a given point $p = [(z,\theta)]$, we define $T_p^{1,0}\mathscr{C}H_n$ to be the space spanned by

$$\left\{\tfrac{\partial}{\partial z_j} + i\lambda_j\overline{z}_j\tfrac{\partial}{\partial\theta},\ j = 1, \ldots, n-1\right\}.$$

It is easy to see that the definition above is independent of the choice of a representative (z,θ) for $[(z,\theta)]$. Moreover, we can check that $T^{1,0}\mathscr{C}H_n$ is a CR structure and $T := \frac{\partial}{\partial\theta}$ is a rigid global real vector field. Thus, $(\mathscr{C}H_n, T^{1,0}\mathscr{C}H_n)$ is a compact generalized Sasakian CR manifold of dimension $2n-1$. Let J denote the canonical complex structure on $X\times\mathbb{R}$ given by $J\frac{\partial}{\partial t}=T$, where t denotes the coordinate of $\mathbb{R}$. We take a Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on the complexified tangent bundle $\mathbb{C}T\mathscr{C}H_n$ such that

$$\left\{\frac{\partial}{\partial z_j}+i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta},\ \frac{\partial}{\partial\overline{z}_j}-i\lambda_j z_j\frac{\partial}{\partial\theta},\ -\frac{\partial}{\partial\theta};\ j=1,\ldots,n-1\right\}$$

is an orthonormal basis. The dual basis of the complexified cotangent bundle is

$$\left\{\mathrm{d}z_j,\ \mathrm{d}\overline{z}_j,\ \omega_0 := -\mathrm{d}\theta+\textstyle\sum_{j=1}^{n-1}(i\lambda_j\overline{z}_j\mathrm{d}z_j-i\lambda_j z_j\mathrm{d}\overline{z}_j);\ j=1,\ldots,n-1\right\}.$$

The Levi form $\mathcal{L}_p$ of $\mathscr{C}H_n$ at $p\in\mathscr{C}H_n$ is given by $\mathcal{L}_p=\sum_{j=1}^{n-1}\lambda_j\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j$.

Now, we construct a generalized Sasakian CR line bundle (L,J) over $\mathscr{C}H_n$. Let $L=(\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C})/_{\equiv}$ where $(z,\theta,\eta)\equiv(\widetilde{z},\widetilde{\theta},\widetilde{\eta})$ if

$$\begin{aligned}&(z,\theta)\sim(\widetilde{z},\widetilde{\theta}),\\ &\widetilde{\eta}\exp\left(\widetilde{\theta}+i\sum_{j=1}^{n-1}\lambda_j\left|\widetilde{z}_j\right|^2\right)=\eta\exp\left(\theta+i\sum_{j=1}^{n-1}\lambda_j\left|z_j\right|^2\right)\exp\left(\sum_{j,t=1}^{n-1}\mu_{j,t}\left(z_j\overline{\alpha}_t+\frac{1}{2}\alpha_j\overline{\alpha}_t\right)\right),\end{aligned}$$

where $\alpha=(\alpha_1,\ldots,\alpha_{n-1})=\widetilde{z}-z$, $\mu_{j,t}=\mu_{t,j}$, $j,t=1,\ldots,n-1$, are given integers. We can check that $\equiv$ is an equivalence relation and (L,J) is a generalized Sasakian CR line bundle over $\mathscr{C}H_n$. For $(z,\theta,\eta)\in\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C}$, we denote $[(z,\theta,\eta)]$ its equivalence class. It is straightforward to see that the pointwise norm

$$\left|[(z,\theta,\eta)]\right|^2_{h^L} := |\eta|^2\exp\left(2\theta-\textstyle\sum_{j,t=1}^{n-1}\mu_{j,t}z_j\overline{z}_t\right)$$

is well defined. In local coordinates (z,θ,η), the weight function of this metric is

$$\phi=-2\theta+\sum_{j,t=1}^{n-1}\mu_{j,t}z_j\overline{z}_t.$$

We can check that $T\phi=-2$. Thus, (L,J,h^L) is a rigid generalized Sasakian CR line bundle over $\mathscr{C}H_n$. Note that

$$\overline{\partial}_b=\textstyle\sum_{j=1}^{n-1}\mathrm{d}\overline{z}_j\wedge\left(\frac{\partial}{\partial\overline{z}_j}-i\lambda_j z_j\frac{\partial}{\partial\theta}\right),\quad \partial_b=\sum_{j=1}^{n-1}\mathrm{d}z_j\wedge\left(\frac{\partial}{\partial z_j}+i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta}\right).$$

Thus, $d(\overline{\partial}_b\phi-\partial_b\phi)=2\sum_{j,t=1}^{n-1}\mu_{j,t}\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_t$ and for any $p\in\mathscr{C}H_n$,

$$M^\phi_p=\sum_{j,t=1}^{n-1}\mu_{j,t}\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_t.$$

From this and Theorem 1.15, we obtain

Theorem 9.1 *If the matrix $\left(\mu_{j,t}\right)_{j,t=1}^{n-1}$ is positive definite and $Y(0)$, $Y(1)$ hold on $\mathscr{C}H_n$, then for k large, there is a constant $c>0$ independent of k, such that*

$$\dim H^0_b(\mathscr{C}H_n,L^k)\geq ck^n.$$

9.2 Holomorphic line bundles over a complex torus

Let

$$T_n := \mathbb{C}^n/(\sqrt{2\pi}\mathbb{Z}^n + i\sqrt{2\pi}\mathbb{Z}^n)$$

be the flat torus. Let $\lambda = \left(\lambda_{j,t}\right)_{j,t=1}^n$, where $\lambda_{j,t} = \lambda_{t,j}$, $j, t = 1, \ldots, n$, are given integers. Let L_λ be the holomorphic line bundle over T_n with curvature the $(1, 1)$-form $\Theta_\lambda = \sum_{j,t=1}^n \lambda_{j,t} dz_j \wedge d\overline{z}_t$. More precisely, $L_\lambda := (\mathbb{C}^n \times \mathbb{C})/\sim$, where $(z, \theta) \sim (\widetilde{z}, \widetilde{\theta})$ if

$$\widetilde{z} - z = (\alpha_1, \ldots, \alpha_n) \in \sqrt{2\pi}\mathbb{Z}^n + i\sqrt{2\pi}\mathbb{Z}^n, \quad \widetilde{\theta} = \exp\left(\sum_{j,t=1}^n \lambda_{j,t}(z_j\overline{\alpha}_t + \tfrac{1}{2}\alpha_j\overline{\alpha}_t)\right)\theta.$$

We can check that $\sim$ is an equivalence relation and L_λ is a holomorphic line bundle over T_n. For $[(z, \theta)] \in L_\lambda$, we define the Hermitian metric by

$$\left|[(z, \theta)]\right|^2 := |\theta|^2 \exp(-\sum_{j,t=1}^n \lambda_{j,t} z_j \overline{z}_t)$$

and it is easy to see that this definition is independent of the choice of a representative (z, θ) of $[(z, \theta)]$. We denote by $\phi_\lambda(z)$ the weight of this Hermitian fiber metric. Note that $\partial\overline{\partial}\phi_\lambda = \Theta_\lambda$.

Let L^*_λ be the dual bundle of L_λ and let $\|\cdot\|_{L^*_\lambda}$ be the norm of L^*_λ induced by the Hermitian fiber metric on L_λ. Consider the compact CR manifold of dimension $2n + 1$: $X = \{v \in L^*_\lambda;\ \|v\|_{L^*_\lambda} = 1\}$; this is the boundary of the Grauert tube associated to L^*_λ. The manifold X is equipped with a natural S^1-action. Locally, X can be represented in local holomorphic coordinates (z, η), where η is the fiber coordinate, as the set of all (z, η) such that $|\eta|^2 e^{\phi_\lambda(z)} = 1$. The S^1-action on X is given by $e^{i\theta} \circ (z, \eta) = (z, e^{i\theta}\eta)$, $e^{i\theta} \in S^1$, $(z, \eta) \in X$. Let T be the global real vector field on X determined by $Tu(x) = \frac{\partial}{\partial\theta}u(e^{i\theta} \circ x)\big|_{\theta=0}$, for all $u \in C^\infty(X)$. We can check that T is a rigid global real vector field on X. Thus, X is a compact generalized Sasakian CR manifold of dimension $2n + 1$. Let J denote the canonical complex structure on $X \times \mathbb{R}$ given by $J\frac{\partial}{\partial t} = T$, where t denotes the coordinate of $\mathbb{R}$.

Let $\pi : L^*_\lambda \to T_n$ be the natural projection from L^*_λ onto T_n. Let $\mu = \left(\mu_{j,t}\right)_{j,t=1}^n$, where $\mu_{j,t} = \mu_{t,j}$, $j, t = 1, \ldots, n$, are given integers. Let L_μ be another holomorphic line bundle over T_n determined by the constant curvature form $\Theta_\mu = \sum_{j,t=1}^n \mu_{j,t} dz_j \wedge d\overline{z}_t$ as above. The pullback line bundle π^*L_μ is a holomorphic line bundle over L^*_λ. If we restrict π^*L_μ on X, then we can check that (π^*L_μ, J) is a generalized Sasakian CR line bundle over X.

The Hermitian fiber metric on L_μ induced by ϕ_μ induces a Hermitian fiber metric on π^*L_μ that we shall denote by $h^{\pi^*L_\mu}$. We let ψ to denote the weight of $h^{\pi^*L_\mu}$. The part of X that lies over a fundamental domain of T_n can be represented in local holomorphic coordinates (z, ξ), where ξ is the fiber coordinate, as the set of all (z, ξ) such that $r(z, \xi) := |\xi|^2 \exp(\sum_{j,t=1}^n \lambda_{j,t} z_j \overline{z}_t) - 1 = 0$ and the weight ψ may be written as $\psi(z, \xi) = \sum_{j,t=1}^n \mu_{j,t} z_j \overline{z}_t$. From this we see that $(\pi^*L_\mu, J, h^{\pi^*L_\mu})$ is a rigid generalized Sasakian CR line bundle over X. It is straightforward to check that for any $p \in X$, we have $M^\psi_p = \frac{1}{2}d(\overline{\partial}_b\psi - \partial_b\psi)(p)|_{T^{1,0}X} = \sum_{j,t=1}^n \mu_{j,t} dz_j \wedge d\overline{z}_t$. From this observation and Theorem 1.15, we obtain

Theorem 9.2 *If the matrix* $\left(\mu_{j,t}\right)_{j,t=1}^{n-1}$ *is positive definite and* $Y(0)$, $Y(1)$ *hold on* X, *then for* k *large, there is a constant* $c > 0$ *independent of* k, *such that*

$$\dim H^0_b(X, (\pi^*L_\mu)^k) \geq ck^{n+1}.$$

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