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Finiteness of non-parabolic ends on submanifolds in spheres

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Abstract We study a complete noncompact submanifold M^n in a sphere $\mathbb{S}^{n+p}$. We prove that the dimension of the space of L^2 harmonic 1-forms on M is finite and there are finitely many non-parabolic ends on M if the total curvature of M is finite and $n \geq 3$. This result is an improvement of Fu–Xu theorem on submanifolds in spheres and a generalized version of Cavalcante, Mirandola and Vitorio's result on submanifolds in Hadamard manifolds.

Keywords Non-parabolic ends, Total curvature, L^2 harmonic forms

Mathematics Subject Classification (2000) 53C20 · 53C40

1 Introduction

Suppose that $x : M^n \to N^{n+p}$ is an isometric immersion of an n-dimensional manifold M in an $(n+p)$-dimensional Riemannian manifold N. Let A denote the second fundamental form and H the mean curvature vector of the immersion x. Let

$$\Phi(X, Y) = A(X, Y) - H\langle X, Y\rangle,$$

for all vector fields X and Y, where $\langle , \rangle$ is the induced metric of M. We say the immersion x has finite total curvature if

$$\|\Phi\|_{L^n(M)} < +\infty.$$

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If M^n $(n \geq 3)$ is a complete minimal hypersurface in $\mathbb{R}^{n+1}$ with finite index, Li and Wang [9] proved that M has finitely many ends. More generally, Zhu [12] showed that: suppose that N^{n+1} $(n \geq 3)$ is a complete simply connected manifold with non-positive sectional curvature and M^n is a complete minimal hypersurface in N with finite index. If the bi-Ricci curvature satisfies

$$b - \overline{\mathrm{Ric}}(X, Y) + \frac{1}{n}|A|^2 \geq 0,$$

for all orthonormal tangent vectors X, Y in T_pN for $p \in M$, then M must has finitely many ends. Cavalcante et al. [1] considered a complete noncompact submanifold M^n $(n \geq 3)$ isometric immersed in a Hadamard manifold N^{n+p} with sectional curvature satisfying $-k^2 \leq K_N \leq 0$ for some constant k and obtained that if the total curvature is finite and the first eigenvalue of the Laplacian operator of M is bounded from below by a suitable constant, then the dimension of the space of the L^2 harmonic 1-forms on M is finite and M has finitely many non-parabolic ends. Fu and Xu [3] considered a complete submanifold M^n in a sphere $\mathbb{S}^{n+p}$ with finite total curvature and bounded mean curvature and showed that the dimension of $H^1(L^2(M))$ is finite and there are finitely many non-parabolic ends on M .

In this paper, we discuss a complete noncompact submanifold M^n in a sphere $\mathbb{S}^{n+p}$ with finite total curvature and no restriction of mean curvature. We recall some relevant definitions. The Hodge operator $* : \wedge^k(M) \to \wedge^{n-k}(M)$ is defined by

$$*e^{i_1} \wedge \ldots \wedge e^{i_p} = \mathrm{sgn}\sigma(i_1, i_2, \ldots, i_n) e^{i_{p+1}} \wedge \ldots \wedge e^{i_n},$$

where $\sigma(i_1, i_2, \ldots, i_n)$ denotes a permutation of the set $(i_1, i_2, \ldots, i_n)$ and $\mathrm{sgn}\sigma$ is the sign of σ. The operator $d^* : \wedge^k(M) \to \wedge^{k-1}(M)$ is given by

$$d^*\omega = (-1)^{(nk+k+1)} * d * \omega.$$

The Laplacian operator is defined by

$$\triangle\omega = -dd^*\omega - d^*d\omega.$$

A k-form ω is called L^2-harmonic if $\triangle\omega = 0$ and

$$\int_M \omega \wedge *\omega < +\infty.$$

We denote $H^1(L^2(M))$ by the space of all L^2 harmonic 1-forms on M. We obtain finiteness of non-parabolic ends for the submanifold in a sphere with finite total curvature:

Theorem 1.1 *Let M^n $(n \geq 3)$ be an n-dimensional complete noncompact oriented manifold isometrically immersed in an $(n+p)$-dimensional sphere $\mathbb{S}^{n+p}$. If the total curvature is finite, then the dimension of $H^1(L^2(M))$ is finite and there are finitely many non-parabolic ends on M.*

Remark 1.2 Theorem 1.1 generalizes Theorem 1.4 in [3] without the restriction of the mean curvature vector and is also an extension of finiteness of non-parabolic ends on submanifolds in Hadamard manifolds in [1].

2 Proof of main results

We initially introduce several results which will be used to prove Theorem 1.1.

Proposition 2.1 [8,9] *If M is a complete Riemannian manifold, then the number of non-parabolic ends of M is bounded from above by* $\dim H^1(L^2(M)) + 1$.

Proposition 2.2 [4,13] *Let M^n be a complete noncompact oriented manifold isometrically immersed in a sphere $\mathbb{S}^{n+p}$. Then*

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_0\left(\int_M |\nabla f|^2 + n^2\int_M (|H|^2+1)f^2\right)$$

for each $f \in C_0^1(M)$, where C_0 depends only on n and H is the mean curvature vector of M in $\mathbb{S}^{n+p}$.

Proof of Theorem 1.1 Suppose that $\omega \in H^1(L^2(M))$. Then we have

$$\triangle|\omega|^2 = 2|\nabla|\omega||^2 + 2|\omega|\triangle|\omega|. \tag{2.1}$$

Note that the following Bochner's formula holds [6]:

$$\begin{aligned}\triangle|\omega|^2 &= 2\langle\Delta\omega, \omega\rangle + 2|\nabla\omega|^2 + 2\text{Ric}(\omega^\sharp, \omega^\sharp)\\ &= 2|\nabla\omega|^2 + 2\text{Ric}(\omega^\sharp, \omega^\sharp).\end{aligned} \tag{2.2}$$

Equalities (2.1) and (2.2) imply that

$$|\omega|\triangle|\omega| = |\nabla\omega|^2 - |\nabla|\omega|| + \text{Ric}(\omega^\sharp, \omega^\sharp). \tag{2.3}$$

There exists the Kato inequality [2,11]:

$$|\nabla|\omega||^2 \leq \frac{n-1}{n}|\nabla\omega|^2. \tag{2.4}$$

Combining (2.3) and (2.4), we get that

$$|\omega|\triangle|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 + \text{Ric}(\omega^\sharp, \omega^\sharp). \tag{2.5}$$

Take $h = |\omega|$. There is an estimate for the Ricci curvature of the submanifold M in [5,10]:

$$\begin{aligned}\text{Ric}(\omega^\sharp, \omega^\sharp) &\geq (n-1)(|H|^2+1)h^2\\ &\quad - \frac{n-1}{n}|\Phi|^2h^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi|h^2.\end{aligned}$$

By (2.5), we obtain that

$$\begin{aligned}h\triangle h &\geq \frac{1}{n-1}|\nabla h|^2 + (n-1)(|H|^2+1)h^2\\ &\quad - \frac{n-1}{n}|\Phi|^2h^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi|h^2.\end{aligned} \tag{2.6}$$

Suppose that η is a compactly supported piecewise smooth function on M. Then

$$\begin{aligned}\text{div}(\eta^2h\nabla h) &= \eta^2h\triangle h + \langle\nabla(\eta^2h), \nabla h\rangle\\ &= \eta^2h\triangle h + \eta^2|\nabla h|^2 + 2\eta h\langle\nabla\eta, \nabla h\rangle.\end{aligned}$$

Integrating by parts on M, we obtain that

$$\int_M \eta^2 h \triangle h + \int_M \eta^2 |\nabla h|^2 + 2\int_M \eta h \langle \nabla \eta, \nabla h\rangle = 0.$$

By (2.6), we get

$$-\frac{1}{n-1}\int_M \eta^2|\nabla h|^2 - (n-1)\int_M \eta^2(|H|^2+1)h^2 + \frac{n-1}{n}\int_M \eta^2|\Phi|^2h^2$$
$$+\frac{(n-2)\sqrt{n(n-1)}}{n}\int_M |H||\Phi|h^2\eta^2 - \int_M \eta^2|\nabla h|^2 - 2\int_M \eta h\langle\nabla\eta,\nabla h\rangle \geq 0.$$

That is,

$$-2\int_M \eta h\langle\nabla\eta,\nabla h\rangle - \frac{n}{n-1}\int_M \eta^2|\nabla h|^2 - (n-1)\int_M \eta^2(|H|^2+1)h^2$$
$$+\frac{n-1}{n}\int_M \eta^2|\Phi|^2h^2 + \frac{(n-2)\sqrt{n(n-1)}}{n}\int_M |H||\Phi|h^2\eta^2 \geq 0. \tag{2.7}$$

Note that

$$\int_M |H||\Phi|h^2\eta^2 = \int_M (|H|\eta h)\cdot(|\Phi|\eta h)$$
$$\leq \frac{a}{2}\int_M |H|^2\eta^2h^2 + \frac{1}{2a}\int_M |\Phi|^2\eta^2h^2, \tag{2.8}$$

for any positive real number a. By (2.7) and (2.8), we obtain that

$$-2\int_M \eta h\langle\nabla\eta,\nabla h\rangle - \frac{n}{n-1}\int_M \eta^2|\nabla h|^2$$
$$-(n-1)\int_M \eta^2(|H|^2+1)h^2 + \frac{n-1}{n}\int_M \eta^2|\Phi|^2h^2$$
$$+\frac{(n-2)\sqrt{n(n-1)}}{n}\left[\frac{a}{2}\int_M |H|^2\eta^2h^2 + \frac{1}{2a}\int_M |\Phi|^2\eta^2h^2\right] \geq 0.$$

That is,

$$-2\int_M \eta h\langle\nabla\eta,\nabla h\rangle - \frac{n}{n-1}\int_M \eta^2|\nabla h|^2 + B(n,a)\int_M |\Phi|^2\eta^2h^2$$
$$+\int_M \left[-(n-1) + A(n,a)|H|^2\right]\eta^2h^2 \geq 0, \tag{2.9}$$

where

$$A(n,a) := -(n-1) + \frac{a(n-2)\sqrt{n(n-1)}}{2n}$$

and

$$B(n,a) := \frac{n-1}{n} + \frac{(n-2)\sqrt{n(n-1)}}{2an}.$$

Now we estimate the term $\int_M |\Phi|^2\eta^2h^2$: take $\phi(\eta) = \left(\int_{\mathrm{Supp}\eta} |\Phi|^n\right)^{\frac{1}{n}}$. Then

$$\begin{aligned}
\int\limits_M |\Phi|^2\eta^2h^2 &\le \left(\int\limits_{\mathrm{Supp}\eta} \left(|\Phi|^2\right)^{\frac{n}{2}}\right)^{\frac{2}{n}} \cdot \left(\int\limits_M \left(\eta^2h^2\right)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \\
&= \phi(\eta)^2 \cdot \left(\int\limits_M (\eta h)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\le \phi(\eta)^2 \cdot C_0 \left(\int\limits_M |\nabla(\eta h)|^2 + n^2 \int\limits_M (|H|^2+1)(\eta h)^2\right) \\
&\le \phi(\eta)^2 \cdot C_0 \left(\int\limits_M (1+\frac{1}{b})h^2|\nabla\eta|^2 + (1+b)\eta^2|\nabla h|^2 + n^2 \int_M (|H|^2+1)(\eta h)^2\right),
\end{aligned} \tag{2.10}$$

for any positive real number b, where the second inequality holds because of Proposition 2.2. Note that

$$-2\int\limits_M \eta h\langle\nabla\eta, \nabla h\rangle \le c\int\limits_M \eta^2|\nabla h|^2 + \frac{1}{c}\int\limits_M h^2|\nabla\eta|^2, \tag{2.11}$$

for any positive real number c. By (2.9)–(2.11), we have

$$\begin{aligned}
&c\int\limits_M \eta^2|\nabla h|^2 + \frac{1}{c}\int\limits_M h^2|\nabla\eta|^2 - \frac{n}{n-1}\int\limits_M \eta^2|\nabla h|^2 \\
&+ \int\limits_M \left[-(n-1) + A(n,a)|H|^2\right]\eta^2h^2 + C_0B(n,a)\phi(\eta)^2 \\
&\times \left(\int\limits_M (1+\frac{1}{b})h^2|\nabla\eta|^2 + (1+b)\eta^2|\nabla h|^2 + n^2\int\limits_M (|H|^2+1)(\eta h)^2\right) \ge 0.
\end{aligned}$$

That is,

$$C\int\limits_M \eta^2|\nabla h|^2 + D\int\limits_M |H|^2\eta^2h^2 + E\int\limits_M \eta^2h^2 \le F\int\limits_M h^2|\nabla\eta|^2, \tag{2.12}$$

where

$$\begin{aligned}
C &:= -c + \frac{n}{n-1} - C_0B(n,a)\phi(\eta)^2(1+b), \\
D &:= -A(n,a) - n^2C_0B(n,a)\phi(\eta)^2, \\
E &:= n-1-n^2C_0B(n,a)\phi(\eta)^2
\end{aligned}$$

and

$$F := \frac{1}{c} + \left(1 + \frac{1}{b}\right) C_0 B(n, a)\phi(\eta)^2.$$

Next, we prove there exists a positive constant δ such that if $\|\Phi\|_{L^n(M)} < \delta$ then C, D, E and F are positive. Obviously, $\phi(\eta) \leq \|\Phi\|_{L^n(M)} < \delta$. Choose $d \in (0, \frac{1}{2})$ and let $a = a(d)$, $\delta = \delta(d)$ such that

$$\begin{aligned} d + \frac{(n-1)d(1+d)}{n^2} &< \frac{n}{n-1}, \\ \frac{a(n-2)\sqrt{n(n-1)}}{2n} &< (n-1)d, \\ n^2 C_0 B(n, a)\delta^2 &< (n-1)d. \end{aligned}$$

Choosing $0 < c < d$ and $0 < b < d$, we obtain that

$$C > \frac{n}{n-1} - d - \frac{(n-1)d(1+d)}{n^2} > 0,$$

$$\begin{aligned} D &= (n-1) - \frac{a(n-2)\sqrt{n(n-1)}}{2n} - n^2 C_0 B(n, a)\phi(\eta)^2 \\ &> (n-1) - 2(n-1)d > 0, \end{aligned}$$

$$E = n - 1 - n^2 C_0 B(n, a)\phi(\eta)^2 > 0$$

and

$$F > 0.$$

Since the total curvature $\|\Phi\|_{L^n(M)}$ is finite, we can choose a fixed r_0 such that

$$\|\Phi\|_{L^n(M - B_{r_0})} < \delta.$$

Set

$$\begin{aligned} \tilde{C} &:= -c + \frac{n}{n-1} - C_0 B(n, a)\delta^2(1 + b), \\ \tilde{D} &:= -A(n, a) - n^2 C_0 B(n, a)\delta^2, \\ \tilde{E} &:= n - 1 + \tilde{D} \end{aligned}$$

and

$$\tilde{F} := \frac{1}{c} + \left(1 + \frac{1}{b}\right) C_0 B(n, a)\delta^2.$$

Thus,

$$\tilde{C} \int_M \eta^2 |\nabla h|^2 + \tilde{D} \int_M |H|^2 \eta^2 h^2 + \tilde{E} \int_M \eta^2 h^2 \leq \tilde{F} \int_M h^2 |\nabla \eta|^2, \tag{2.13}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$, where $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{F}$ are positive. Proposition 2.2 implies that

$$\begin{aligned}\frac{1}{C_0}\left(\int_M |\eta h|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\leq \int_M |\nabla(\eta h)|^2 + n^2\int_M (|H|^2+1)(\eta h)^2 \\ &\leq (1+s)\int_M \eta^2|\nabla h|^2 + (1+\frac{1}{s})\int_M h^2|\nabla\eta|^2 + n^2\int_M (|H|^2+1)\eta^2 h^2, \end{aligned} \tag{2.14}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$ and any positive real number s. Inequality (2.13) implies that

$$\begin{aligned}(1+s)\int_M \eta^2|\nabla h|^2 \leq \frac{(1+s)\tilde{F}}{\tilde{C}}\int_M h^2|\nabla\eta|^2 \\ -\frac{(1+s)\tilde{D}}{\tilde{C}}\int_M |H|^2\eta^2h^2 - \frac{(1+s)\tilde{E}}{\tilde{C}}\int_M \eta^2h^2.\end{aligned}$$

Combining with (2.14), we get

$$\begin{aligned}\frac{1}{C_0}\left(\int_M |\eta h|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\leq \left(1+\frac{1}{s}+\frac{\tilde{F}(1+s)}{\tilde{C}}\right)\int_M h^2|\nabla\eta|^2 \\ &+ (n^2 - \frac{(1+s)\tilde{D}}{\tilde{C}})\int_M |H|^2\eta^2h^2 + (n^2 - \frac{(1+s)\tilde{E}}{\tilde{C}})\int_M \eta^2h^2, \end{aligned} \tag{2.15}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$. Choose a sufficiently large s such that $n^2 - \frac{(1+s)\tilde{D}}{\tilde{C}} < 0$ and $n^2 - \frac{(1+s)\tilde{E}}{\tilde{C}} < 0$. Then (2.15) implies that

$$\left(\int_M (\eta h)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \tilde{A}\int_M h^2|\nabla\eta|^2, \tag{2.16}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$, where $\tilde{A}$ is a positive constant depending only on n. From now on, the proof follows standard techniques [for instance, see [1] after inequality (33)] and uses a Moser iteration argument and lemma 11 in [7]. We only include a concise proof here for the sake of completeness. Choose $r > r_0 + 1$ and $\eta \in C_0^\infty(M - B_{r_0})$ such that

$$\begin{cases} \eta = 0 \ \ on \ \ B_{r_0} \cup (M - B_{2r}), \\ \eta = 1 \ \ on \ \ B_r - B_{r_0+1}, \\ |\nabla\eta| < c_1 \ \ on \ \ B_{r_0+1} - B_{r_0}, \\ |\nabla\eta| \leq c_1 r^{-1} on \ \ B_{2r} - B_r, \end{cases}$$

for some positive constant c_1. Then (2.16) becomes that

$$\left(\int_{B_r - B_{r_0+1}} h^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \tilde{A}\int_{B_{r_0+1}-B_{r_0}} h^2 + \frac{\tilde{A}}{r^2}\int_{B_{2r}-B_r} h^2.$$

Letting $r \to \infty$ and noting that $|\omega| \in H^1(L^2(M))$, we obtain that

$$\left(\int_{M-B_{r_0+1}} h^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \tilde{A} \int_{B_{r_0+1}-B_{r_0}} h^2. \tag{2.17}$$

Combining with Hölder inequality , we have that

$$\int_{B_{r_0+2}} h^2 \leq (1 + \tilde{A} Vol(B_{r_0+2})^{\frac{2}{n}}) \int_{B_{r_0+1}} h^2. \tag{2.18}$$

Let

$$\Psi = |(n-1)(|H|^2+1) - \frac{n-1}{n}|\Phi|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi||.$$

Fix $x \in M$ and take $\tau \in C_0^1(B_1(x))$. (2.6) implies that

$$\begin{aligned} -2\int_{B_1(x)} \tau h^{p-1}\langle \nabla\tau, \nabla h\rangle &\geq (p-1+\frac{1}{n-1}) \int_{B_1(x)} \tau^2 h^{p-2}|\nabla h|^2 \\ &- \int_{B_1(x)} \tau^2 \Psi h^p. \end{aligned} \tag{2.19}$$

Note that

$$\begin{aligned} -2\tau h^{p-1}\langle \nabla\tau, \nabla h\rangle &= 2\tau\langle -h^{\frac{p}{2}}\nabla\tau, h^{\frac{p}{2}-1}\nabla h\rangle \\ &\leq (n-1)h^p|\nabla\tau|^2 + \frac{1}{n-1}h^{p-2}\tau^2|\nabla h|^2. \end{aligned}$$

Combining with (2.19), we obtain that

$$(p-1)\int_{B_1(x)} \tau^2 h^{p-2}|\nabla h|^2 \leq \int_{B_1(x)} \Psi\tau^2 h^p + (n-1)\int_{B_1(x)} |\nabla\tau|^2 h^p. \tag{2.20}$$

By Cauchy–Schwarz inequality and (2.20), we have

$$\int_{B_1(x)} |\nabla(\tau h^{\frac{p}{2}})|^2 \leq \int_{B_1(x)} \mathcal{A}\Psi\tau^2 h^p + \mathcal{B}|\nabla\tau|^2 h^p, \tag{2.21}$$

where $\mathcal{A} = \frac{p(p+1)}{4(p-1)} \leq p < 2n^2 p$ and $\mathcal{B} = p + 1 + (n-1)\mathcal{A} \leq 1 + np < 2n^2 p$. Choosing $f = \tau h^{\frac{p}{2}}$ in Proposition 2.2 and combining with (2.21), we have

$$\left(\int_{B_1(x)} (\tau h^{\frac{p}{2}})^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2}} \leq 2C_0 p n^2 \int_{B_1(x)} (\mathcal{C}\tau^2 + |\nabla\tau|^2) h^p, \tag{2.22}$$

where $\mathcal{C} = |H|^2 + 1 + \Psi$. Let $p_k = \frac{2n^k}{(n-2)^k}$ and $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$ for $k = 0, 1, 2, \ldots$. Take a function $\tau_k \in C_0^\infty(B_{\rho_k(x)})$ satisfying:

$$\begin{cases} 0 \leq \tau_k \leq 1, \\ \tau_k = 1 \ \ on \ \ B_{\rho_{k+1}}(x), \\ |\nabla \tau_k| \leq 2^{k+3}. \end{cases}$$

Choosing $p = p_k$ and $\tau = \tau_k$ in (2.22), we have

$$\left(\int\limits_{B_{\rho_{k+1}}(x)} h^{p_{k+1}} \right)^{\frac{1}{p_{k+1}}} \leq \left(p_k 4^{k+k_0} \right)^{\frac{1}{p_k}} \left(\int\limits_{B_{\rho_k}(x)} h^{p_k} \right)^{\frac{1}{p_k}}, \tag{2.23}$$

where k_0 is a positive integer such that $2C_0 n^2(4^3 + \sup_{B_1(x)} \mathcal{C}) \leq 4^{k_0}$. By recurrence, we have

$$\|h\|_{L^{p_{k+1}}(B_{\frac{1}{2}}(x))} \leq \prod_{i=0}^{k} p_i^{\frac{1}{p_i}} 4^{\frac{i}{p_i}} 4^{\frac{k_0}{p_i}} \|h\|_{L^2(B_1(x))} \leq \mathcal{D} \|h\|_{L^2(B_1(x))}, \tag{2.24}$$

where $\mathcal{D}$ is a positive constant depending only on n and $\sup_{B_1(x)} \Psi$. Letting $k \to \infty$, we get

$$\|h\|_{L^\infty(B_{\frac{1}{2}}(x))} \leq \mathcal{D} \|h\|_{L^2(B_1(x))}. \tag{2.25}$$

Now, choose $y \in \overline{B}_{r_0+1}$ so that $\sup_{B_{r_0+1}} h^2 = h(y)^2$. Note that $B_1(y) \subset B_{r_0+2}$. (2.25) implies that

$$\sup_{B_{r_0+1}} h^2 \leq \mathcal{D} \|h\|^2_{L^2(B_1(y))} \leq \mathcal{D} \|h\|^2_{L^2(B_{r_0+2})}. \tag{2.26}$$

By (2.18), we have

$$\sup_{B_{r_0+1}} h^2 \leq \mathcal{F} \|h\|^2_{L^2(B_{r_0+1})}, \tag{2.27}$$

where $\mathcal{F}$ depends only on n, $Vol(B_{r_0+2})$ and $\sup_{B_{r_0+2}} \Psi$. In order to show the finiteness of the dimension of $H^1(L^2(M))$, it suffices to prove that the dimension of any finite dimensional subspaces of $H^1(L^2(M))$ is bounded above by a fixed constant. By (2.27) and Lemma 11 in [7], we get $\dim H^1(L^2(M)) < +\infty$. By Proposition 2.1, we obtain that the number of non-parabolic ends of M is finite.

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