Ann Glob Anal Geom (2014) 45:287–302
DOI 10.1007/s10455-013-9401-1

Geometric flows and differential Harnack estimates for heat equations with potentials

Masashi Ishida

Received: 3 May 2013 / Accepted: 23 September 2013 / Published online: 26 October 2013

Abstract Let M be a closed Riemannian manifold with a Riemannian metric $g_{ij}(t)$ evolving by a geometric flow $\partial_t g_{ij} = -2S_{ij}$, where $S_{ij}(t)$ is a symmetric two-tensor on $(M, g(t))$. Suppose that S_{ij} satisfies the tensor inequality $2\mathcal{H}(S, X)+\mathcal{E}(S, X) \geq 0$ for all vector fields X on M, where $\mathcal{H}(S, X)$ and $\mathcal{E}(S, X)$ are introduced in Definition 1 below. Then, we shall prove differential Harnack estimates for positive solutions to time-dependent forward heat equations with potentials. In the case where $S_{ij} = R_{ij}$, the Ricci tensor of M, our results correspond to the results proved by Cao and Hamilton (Geom Funct Anal 19:983–989, 2009). Moreover, in the case where the Ricci flow coupled with harmonic map heat flow introduced by Müller (Ann Sci Ec Norm Super 45(4):101–142, 2012), our results derive new differential Harnack estimates. We shall also find new entropies which are monotone under the above geometric flow.

Keywords Ricci flow · Geometric flows · Differential Harnack Estimates · Heat equations with potentials

Mathematics Subject Classification (2000) 53C44 · 53C21

1 Introduction

The main purpose of the current article is to study time-dependent heat equations with potentials on closed Riemannian n-manifolds M evolving by the geometric flow

$$\frac{\partial}{\partial t} g_{ij} = -2S_{ij}, \tag{1}$$

where $S_{ij}(t)$ is a symmetric two-tensor on $(M, g(t))$. A typical example would be the case where $S_{ij} = R_{ij}$ is the Ricci tensor and $g(t)$ is a solution of the Ricci flow introduced by Hamilton [8]. We shall derive Li–Yau type differential Harnack inequalities [13] for positive

M. Ishida (✉)
Department of Mathematics, Graduate School of Science, Osaka University,
1-1, Machikaneyama, Toyonaka, Osaka 560-0043, Japan
e-mail: ishida@math.sci.osaka-u.ac.jp

solutions to the following heat equation with a potential term:

$$\frac{\partial f}{\partial t} = \Delta f + Sf, \tag{2}$$

where the symbol Δ stands for the Laplacian of the evolving metric $g(t)$ and $S = g^{ij} S_{ij}$ is the trace of S_{ij}. In the current article, we shall use the Einstein summation convention. For simplicity, we omit $g(t)$ in the above notation. All geometric operators are with respect to the evolving metric $g(t)$. Notice also that we have

$$\frac{\partial}{\partial t}\left(\int_M f \mathrm{d}\mu_g\right) = \int_M \left(\frac{\partial f}{\partial t} - S\right) \mathrm{d}\mu_g = \int_M \Delta f \mathrm{d}\mu_g = 0.$$

The main results of the current article are Theorems A, B, C, D and E stated below, which can be seen as natural generalizations of results proved by Cao and Hamilton [4]. See also Sect. 2 below.

The study of differential Harnack inequalities for parabolic equations originated with the work of Li and Yau [13]. They first proved a gradient estimate for the heat equation using the maximal principle. By integrating the gradient estimate along a space-time path, a classical Harnack inequality was derived. Therefore, Li–Yau type gradient estimate is often called differential Harnack inequality. Hamilton adapted similar techniques to prove Harnack inequalities for the Ricci flow [10] and the mean curvature flow [11]. Many authors used similar techniques to prove Harnack inequalities for geometric flows. For instance, see [1–3,5–7,12,17–19].

To state the main results of the current article, we shall introduce evolving tensor quantities associated with the tensor S_{ij}.

Definition 1 Suppose that $g(t)$ evolves by the geometric flow (1) and let $X = X^i \frac{\partial}{\partial x^i} \in \Gamma(TX)$ be a vector field on M. We define

$$\mathcal{I}(S, X) = (R^{ij} - S^{ij}) X_i X_j,$$

$$\mathcal{H}(S, X) = \frac{\partial S}{\partial t} + \frac{S}{t} + 2\nabla_i S X^i + 2S^{ij} X_i X_j,$$

$$\mathcal{E}(S, X) = \left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) - 2(2\nabla^i S_{i\ell} - \nabla_\ell S) X^\ell + 2\mathcal{I}(S, X),$$

where $R^{ij} = g^{ik} g^{j\ell} R_{k\ell}$, $S^{ij} = g^{ik} g^{j\ell} S_{k\ell}$, $S = g^{ij} S_{ij}$, $\nabla^i = g^{ij}\nabla_j$ and $X_k = g_{ik} X^i$.

We notice that these quantities are also introduced by Müller [15] to prove the monotonicity of Perelman type reduced volume under (1).

The first main result of the current article is as follows:

Theorem A *Suppose that $g(t)$ evolves by the geometric flow (1) on a closed oriented smooth n-manifold M and*

$$2\mathcal{H}(S, X) + \mathcal{E}(S, X) \geq 0 \tag{3}$$

holds for all vector fields X and all time $t \in [0, T)$ for which the flow exists. Let f be a positive solution to the heat equation (2), $u = -\log f$, and

$$H_S = 2\Delta u - |\nabla u|^2 - 3S - \frac{2n}{t}.$$

Then, $H_S \leq 0$ for all time $t \in (0, T)$.

In the case of the heat equation without the potential term, we shall also prove

Theorem B *Suppose that* $g(t)$ *evolves by the geometric flow* (1) *on a closed oriented smooth* n*-manifold* M *and*

$$\mathcal{I}(S, X) \geq 0 \tag{4}$$

holds for all vector fields X *and all time* $t \in [0, T)$ *for which the flow exists. Let* f (< 1) *be a positive solution to the heat equation* $\frac{\partial f}{\partial t} = \Delta f$, $u = -\log f$ *and*

$$H_S = |\nabla u|^2 - \frac{u}{t}.$$

Then, $H_S \leq 0$ *holds for all time* $t \in (0, T)$*. Hence we have the following on* $(0, T)$*,*

$$|\nabla f|^2 \leq -\frac{f^2}{t} \log f.$$

On the other hand, a similar technique with the proof of Theorem A also enables us to prove

Theorem C *Suppose that* $g(t)$ *evolves by the geometric flow* (1) *on a closed oriented smooth* n*-manifold* M *and* $2\mathcal{H}(S, X) + \mathcal{E}(S, X) \geq 0$ *holds for all vector fields* X *and all time* $t \in [0, T)$ *for which the flow exists. Let* f *be a positive solution to the heat equation* (2)*,* $v = -\log f - \frac{n}{2}\log(4\pi t)$ *and*

$$P_S = 2\Delta v - |\nabla v|^2 - 3S + \frac{v}{t} - d\frac{n}{t},$$

where d *is any constant. Then for all time* $t \in (0, T)$*,*

$$\begin{aligned}\frac{\partial}{\partial t}(tP_S) = \; & \Delta(tP_S) - 2\nabla^i(tP_S)\nabla_i v - 2t\left|\nabla_i\nabla_j v - S_{ij} - \frac{1}{2t}g_{ij}\right|^2 \\ & -t\left(2\mathcal{H}(S, \nabla v) + \mathcal{E}(S, \nabla v)\right)\end{aligned}$$

and $\max(tP_S)$ *is non-increasing.*

Moreover, inspired by the works of Perelman [17] and Cao and Hamilton [4], we shall introduce two entropies which are associated with the above Harnack quantities. The first one is associated with H_S.

Theorem D *Under the same assumption with Theorem A, we define*

$$\mathcal{F}_S = \int_M t^2 e^{-u} H_S \, d\mu_g.$$

Then for all time $t \in (0, T)$*, we have* $\mathcal{F}_S \leq 0$ *and*

$$\frac{d}{dt}\mathcal{F}_S \leq 0.$$

Moreover, suppose that $\mathcal{H}(S, X) \geq 0$ *and* $\mathcal{E}(S, X) \geq 0$ *holds for all vector fields* X *and all time* $t \in [0, T)$ *for which the flow exists. If* $\frac{d}{dt}\mathcal{F}_S = 0$ *holds for some time* t*, then the following holds:*

$$S_{ij} = -\frac{1}{t}g_{ij}, \quad \nabla u = 0, \; \mathcal{H}(S, \nabla u) = 0, \; \mathcal{E}(S, \nabla v) = 0 \tag{5}$$

The second one is associated with P_S as follows.

Theorem E *Under the same assumption with Theorem C, we define*

$$\mathcal{W}_S = \int_M t P_S (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-v} \mathrm{d}\mu_g.$$

Then for all time $t \in (0, T)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_S \leq 0.$$

Moreover, suppose that $\mathcal{H}(S, X) \geq 0$ and $\mathcal{E}(S, X) \geq 0$ holds for all vector fields X and all time $t \in [0, T)$ for which the flow exists. If $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_S = 0$ holds for some time t, then the following holds:

$$S_{ij} - \nabla_i \nabla_j v + \frac{1}{2t} g_{ij} = 0, \quad \mathcal{H}(S, \nabla v) = 0, \ \mathcal{E}(S, \nabla v) = 0. \tag{6}$$

2 Examples

2.1 The Ricci flow

Let $g(t)$ be a solution to the Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

Namely, we have $S_{ij} = R_{ij}$ and $S = R$ the scalar curvature. Notice that it is known that the scalar curvature R evolves by $\frac{\partial R}{\partial t} - \Delta R - 2|R_{ij}|^2 = 0$. Moreover, we have the twice contracted second Bianchi identity $2\nabla^i R_{i\ell} - \nabla_\ell R = 0$. Hence, we have $\mathcal{E}(S, X) = 0$ in this case. Therefore, (3) is equivalent to

$$\mathcal{H}(S, X) = \frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla_i R X^i + 2R^{ij} X_i X_j \geq 0.$$

This tells us that $g(t)$ has weakly positive curvature operator (see also [9]). Moreover, we have $\mathcal{I}(S, X) = 0$. Hence, (4) holds. Therefore, Theorems A, B, C, D and E in the case where $S_{ij} = R_{ij}$ just correspond to the results proved by Cao and Hamilton [4]. Notice also that (5) particularly tells us that $R_{ij} = -\frac{1}{t} g_{ij}$, i.e., $g(t)$ is Einstein. Similarly, (6) implies $R_{ij} + \nabla_i \nabla_j (-v) + \frac{1}{2t} g_{ij} = 0$. This tells us that $g(t)$ is an expanding gradient Ricci soliton. Since it is known [17] that any expanding Ricci soliton on a closed manifold must be Einstein, $g(t)$ is Einstein.

2.2 Bernhard List's flow

List [14] introduced a geometric flow closely related to the Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + 4\nabla_i \psi \nabla_j \psi,$$
$$\frac{\partial \psi}{\partial t} = \Delta \psi,$$

where $\psi : M \to \mathbb{R}$ is a smooth function. If we set $S_{ij} = R_{ij} - 2\nabla_i\psi\nabla_j\psi$, it is clear that the first of List's flow has the form (1). Notice also that we have $S = R - 2|\nabla\psi|^2$. List [14] pointed out that S satisfies the following evolution equation:

$$\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2 = 4|\Delta\psi|^2.$$

On the other hand, we get $2\nabla^i S_{i\ell} - \nabla_\ell S = 2\nabla^i(R_{i\ell} - 2\nabla_i\psi\nabla_\ell\psi) - \nabla_\ell(R - 2|\nabla\psi|^2) = -4\nabla^i(\nabla_i\psi\nabla_\ell\psi) - 2\nabla_\ell(\nabla^k\psi\nabla_k\psi) = -4\Delta\psi\nabla_\ell\psi$, where notice the twice contracted second Bianchi identity $2\nabla^i R_{i\ell} = \nabla_\ell R$. Therefore, we have

$$\mathcal{E}(S, X) = 4|\Delta\psi|^2 + 8\Delta\psi\nabla_\ell\psi X^\ell + 4\nabla^i\psi\nabla^j\psi X_i X_j = 4|\Delta\psi + \nabla_X\psi|^2 \geq 0. \quad (7)$$

In particular, (3) is particularly satisfied if

$$\mathcal{H}(S, X) \geq 0.$$

On the other hand, we have $\mathcal{I}(S, X) = \nabla^i\psi\nabla^j\psi X_i X_j = (\nabla_X\psi)^2 \geq 0$. Hence, (4) holds. Therefore, Theorems A, B, C, D and E just correspond to the result proved by Fang [6]. Notice also that, under the situation on Theorem D, we particularly have the following by (5) and (7):

$$R_{ij} - 2\nabla_i\psi\nabla_j\psi = -\frac{1}{t}g_{ij}, \quad \nabla u = 0, \ \Delta\psi + \nabla^i\psi\nabla_i u = 0.$$

Since it follows that $\Delta\psi = 0$, ψ must be a harmonic function on the closed manifold M. This implies that ψ is a constant for the time t. Therefore, we have $R_{ij} = -\frac{1}{t}g_{ij}$, i.e., M is Einstein.

2.3 Rent Müller's flow

Let (Y, h) be a fixed Riemannian manifold. Let $(g(t), \phi(t))$ be the couple consisting of a family of metric $g(t)$ on M and a family of maps $\phi(t)$ from M to Y. We call $(g(t), \phi(t))$ a solution of Rent Müller's flow [16] (also known as the Ricci flow coupled with harmonic map heat flow) with coupling function $\alpha(t) \geq 0$ if

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + 2\alpha(t)\nabla_i\phi\nabla_j\phi,$$
$$\frac{\partial\phi}{\partial t} = \tau_g\phi,$$

where $\tau_g\phi$ is the tension field of the map ϕ with respect to the metric $g(t)$. List's flow is a special case of this flow. If we set $S_{ij} = R_{ij} - \alpha(t)\nabla_i\phi\nabla_j\phi$, the first of Müller's flow has the form (1). Notice that $S = R - 2\alpha(t)|\nabla\phi|^2$ holds. Müller [16] proved that S satisfies

$$\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2 = 2\alpha(t)|\tau_g\phi|^2 - \left(\frac{\partial\alpha(t)}{\partial t}\right)|\nabla\phi|^2.$$

Since we are able to get $2\nabla^i S_{i\ell} - \nabla_\ell S = -2\alpha(t)\tau_g\phi\nabla_\ell\phi$, the following holds:

$$\begin{aligned}\mathcal{E}(S, X) &= 2\alpha(t)|\tau_g\phi|^2 - \left(\frac{\partial\alpha(t)}{\partial t}\right)|\nabla\phi|^2 + 4\alpha(t)\tau_g\phi\nabla_\ell\phi X^\ell + 2\alpha(t)\nabla^i\phi\nabla^j\phi X_i X_j \\ &= 2\alpha(t)\left|\tau_g\phi + \nabla_X\phi\right|^2 - \left(\frac{\partial\alpha(t)}{\partial t}\right)|\nabla\phi|^2.\end{aligned}$$

Therefore, $\mathcal{E}(S, X) \geq 0$ holds if $\alpha(t) \geq 0$ is non-increasing. In this case, (3) is particularly satisfied if

$$\mathcal{H}(S, X) \geq 0.$$

Notice also that $\mathcal{I}(S, X) = \alpha(t)\nabla^i\phi\nabla^j\phi X_i X_j = \alpha(t)(\nabla_X\phi)^2 \geq 0$. Hence, (4) holds. To the best of our knowledge, Theorems A, B, C, D and E in the case where $S_{ij} = R_{ij} - \alpha(t)\nabla_i\phi\nabla_j\phi$ are new.

On the other hand, under the situation on Theorem D, we have the following by (5) and the above computation if $\alpha(t)$ is constant:

$$R_{ij} - 2\nabla_i\phi\nabla_j\phi = -\frac{1}{t}g_{ij}, \quad \nabla u = 0, \quad \tau_g\phi + \nabla^i\phi\nabla_i u = 0.$$

Therefore, we have $R_{ij} - 2\nabla_i\phi\nabla_j\phi = -\frac{1}{t}g_{ij}$ and $\tau_g\phi = 0$. In particular, ϕ must be a harmonic map.

3 Proofs of Theorems A and B

Let f be a positive solution of the following heat equation with potential:

$$\frac{\partial f}{\partial t} = \Delta f - cSf, \tag{8}$$

where c is a constant. In what follows, let $u = -\log f$. By a direct computation, we are able to see that u satisfies

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 + cS. \tag{9}$$

Let us introduce the following:

Definition 2 Suppose that $g(t)$ evolves by (1) and let S be the trace of S_{ij}. Let $X = X^i\frac{\partial}{\partial x^i} \in \Gamma(TX)$ be a vector field on M. We define

$$\mathcal{D}_{(a,\alpha,\beta)}(S, X) = a\left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) + \alpha(2\nabla^i S_{i\ell} - \nabla_\ell S)X^\ell$$
$$+2\beta(R^{ij} - S^{ij})X_i X_j,$$

where a, α and β are constants.

Notice that we have $\mathcal{E}(S, X) = \mathcal{D}_{(1,-2,1)}(S, X)$.

Lemma 1 *Let $g(t)$ be a solution to the geometric flow (1) and u satisfies (9). Let*

$$H_S = \alpha\Delta u - \beta|\nabla u|^2 + aS - b\frac{u}{t} - d\frac{n}{t}, \tag{10}$$

where α, β, a, b and d are constants. Then, H_S satisfies

$$\frac{\partial H_S}{\partial t} = \Delta H_S - 2\nabla^i H_S\nabla_i u + 2(a - \beta c)\nabla^i S\nabla_i u - 2(\alpha - \beta)|\nabla\nabla u|^2$$
$$-2\alpha R^{ij}\nabla_i u\nabla_j u + 2\alpha S^{ij}\nabla_i\nabla_j u + \alpha c\Delta S - \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2}$$
$$+2a|S_{ij}|^2 + \mathcal{D}_{(a,\alpha,\beta)}(S, \nabla u).$$

Proof First of all, notice that we have the following three evolution equations, which follow from standard computation:

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta u) &= 2S^{ij}\nabla_i\nabla_j u + \Delta\left(\frac{\partial u}{\partial t}\right) - g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\nabla_k u,\\ \frac{\partial}{\partial t}(|\nabla u|^2) &= 2S^{ij}\nabla_i u\nabla_j u + 2\nabla^i\left(\frac{\partial u}{\partial t}\right)\nabla_i u,\\ g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right) &= -g^{k\ell}(2\nabla^i S_{i\ell} - \nabla_\ell S).\end{aligned}$$

By (9), (10) and these equations, we are able to obtain

$$\begin{aligned}\frac{\partial H_S}{\partial t} &= \alpha\frac{\partial}{\partial t}(\Delta u) - \beta\frac{\partial}{\partial t}(|\nabla u|^2) + a\frac{\partial S}{\partial t} - \frac{b}{t}\frac{\partial u}{\partial t} + \frac{b}{t^2}u + d\frac{n}{t^2}\\ &= \alpha\left(2S^{ij}\nabla_i\nabla_j u + \Delta\left(\frac{\partial u}{\partial t}\right) - g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\nabla_k u\right)\\ &\quad -\beta\left(2S^{ij}\nabla_i u\nabla_j u + 2\nabla^i\left(\frac{\partial u}{\partial t}\right)\nabla_i u\right) + a\frac{\partial S}{\partial t} - \frac{b}{t}\frac{\partial u}{\partial t} + \frac{b}{t^2}u + d\frac{n}{t^2}\\ &= \alpha(2S^{ij}\nabla_i\nabla_j u + \Delta(\Delta u - |\nabla u|^2 + cS) + g^{k\ell}(2\nabla^i S_{i\ell} - \nabla_\ell S)\nabla_k u)\\ &\quad -\beta(2S^{ij}\nabla_i u\nabla_j u + 2\nabla^i(\Delta u - |\nabla u|^2 + cS)\nabla_i u) + a\frac{\partial S}{\partial t}\\ &\quad -\frac{b}{t}(\Delta u - |\nabla u|^2 + cS) + \frac{b}{t^2}u + d\frac{n}{t^2}\\ &= 2\alpha S^{ij}\nabla_i\nabla_j u + \alpha\Delta(\Delta u) - \alpha\Delta(|\nabla u|^2) + \alpha c\Delta S + \alpha(2\nabla^i S_{i\ell} - \nabla_\ell S)\nabla^\ell u\\ &\quad -2\beta S^{ij}\nabla_i u\nabla_j u - 2\beta\nabla^i(\Delta u)\nabla_i u + 2\beta\nabla^i(|\nabla u|^2)\nabla_i u - 2\beta c\nabla^i S\nabla_i u\\ &\quad +\frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2} + a\frac{\partial S}{\partial t} - \frac{b}{t}\Delta u.\end{aligned}$$

On the other hand, we also have the following by (10):

$$\begin{aligned}\Delta H_S &= \alpha\Delta(\Delta u) - \beta\Delta(|\nabla u|^2) + a\Delta S - \frac{b}{t}\Delta u.\\ \nabla^i H_S &= \alpha\nabla^i(\Delta u) - \beta\nabla^i(|\nabla u|^2) + a\nabla^i S - \frac{b}{t}\nabla^i u\end{aligned}$$

Therefore, we get

$$\begin{aligned}\Delta H_S - 2\nabla^i H_S\nabla_i u &= \alpha\Delta(\Delta u) - \beta\Delta(|\nabla u|^2) + a\Delta S - \frac{b}{t}\Delta u\\ &\quad -2\alpha\nabla^i(\Delta u)\nabla_i u + 2\beta\nabla^i(|\nabla u|^2)\nabla_i u - 2a\nabla^i S\nabla_i u + \frac{2b}{t}|\nabla u|^2.\end{aligned}$$

Using this, we are able to obtain

$$\begin{aligned}\frac{\partial H_S}{\partial t} &= \Delta H_S - 2\nabla^i H\nabla_i u + 2\alpha S^{ij}\nabla_i\nabla_j u - 2\beta S^{ij}\nabla_i u\nabla_j u - (\alpha-\beta)\Delta(|\nabla u|^2)\\ &\quad +(\alpha c - a)\Delta S + \alpha(2\nabla^i S_{i\ell} - \nabla_\ell S)\nabla^\ell u + 2(\alpha-\beta)\nabla^i(\Delta u)\nabla_i u\\ &\quad +2(a-\beta c)\nabla^i S\nabla_i u - \frac{b}{t}|\nabla u|^2 + a\frac{\partial S}{\partial t} - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2}.\end{aligned}$$

On the other hand, we also have the following Bochner–Weitzenbock type formula:

$$\Delta(|\nabla u|^2) = 2|\nabla\nabla u|^2 + 2\nabla^i(\Delta u)\nabla_i u + 2R^{ij}\nabla_i u\nabla_j u.$$

Using this formula, we get

$$\begin{aligned}\frac{\partial H_S}{\partial t} &= \Delta H_S - 2\nabla^i H_S \nabla_i u + 2(a-\beta c)\nabla^i S \nabla_i u - 2(\alpha-\beta)|\nabla\nabla u|^2 \\ &\quad -2\alpha R^{ij}\nabla_i u \nabla_j u + 2\alpha S^{ij}\nabla_i\nabla_j u + \alpha c \Delta S - \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2} \\ &\quad +2a|S_{ij}|^2 + a\left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) + \alpha(2\nabla^i S_{i\ell} - \nabla_\ell S)\nabla^\ell u \\ &\quad +2\beta(R^{ij} - S^{ij})\nabla_i u \nabla_j u \\ &= \Delta H_S - 2\nabla^i H_S \nabla_i u + 2(a-\beta c)\nabla^i S \nabla_i u - 2(\alpha-\beta)|\nabla\nabla u|^2 \\ &\quad -2\alpha R^{ij}\nabla_i u \nabla_j u + 2\alpha S^{ij}\nabla_i\nabla_j u + \alpha c \Delta S - \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2} \\ &\quad +2a|S_{ij}|^2 + \mathcal{D}_{(a,\alpha,\beta)}(S, \nabla u),\end{aligned}$$

where notice that Definition 2. □

In particular, we shall use Lemma 1 to prove Theorem B. The following result is used to prove Theorem A.

Proposition 1 *The evolution equation in Lemma 1 can be rewritten as follows:*

$$\begin{aligned}\frac{\partial H_S}{\partial t} &= \Delta H_S - 2\nabla^i H_S \nabla_i u - 2(\alpha-\beta)\left|\nabla_i\nabla_j u - \frac{\alpha}{2(\alpha-\beta)}S_{ij} - \frac{\lambda}{2t}g_{ij}\right|^2 \\ &\quad +2(a-\beta c)\nabla^i u \nabla_i S - \frac{2(\alpha-\beta)}{\alpha}\frac{\lambda}{t}H_S + \frac{(\alpha-\beta)n\lambda^2}{2t^2} - \left(b + \frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla u|^2}{t} \\ &\quad +\left(2a + \frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2 + \left(\alpha\lambda - bc + \frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t} + \left(1 - \frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{b}{t^2}u \\ &\quad +\left(1 - \frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{d}{t^2}n + \alpha c\Delta S - 2\alpha R^{ij}\nabla_i u \nabla_j u + \mathcal{D}_{(a,\alpha,\beta)}(S, \nabla u),\end{aligned}$$

where λ is a constant, $\alpha \neq 0$ and $\alpha \neq \beta$.

Proof A direct computation implies

$$\begin{aligned}-2(\alpha-\beta)\left|\nabla_i\nabla_j u - \frac{\alpha}{2(\alpha-\beta)}S_{ij} - \frac{\lambda}{2t}g_{ij}\right|^2 &= -2(\alpha-\beta)|\nabla\nabla u|^2 + 2\alpha S^{ij}\nabla_i\nabla_j u \\ +2(\alpha-\beta)\frac{\lambda}{t}\Delta u - \frac{\lambda}{t}S - \frac{\alpha^2}{2(\alpha-\beta)}|S_{ij}|^2 - \frac{(\alpha-\beta)\lambda^2 n}{2t^2}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&-2(\alpha-\beta)|\nabla\nabla u|^2 + 2\alpha S^{ij}\nabla_i\nabla_j u + 2a|S_{ij}|^2 \\ &= -2(\alpha-\beta)\left|\nabla_i\nabla_j u - \frac{\alpha}{2(\alpha-\beta)}S_{ij} - \frac{\lambda}{2t}g_{ij}\right|^2 - 2(\alpha-\beta)\frac{\lambda}{t}\left(\Delta u - \frac{\alpha S}{2(\alpha-\beta)}\right) \\ &\quad + \frac{(\alpha-\beta)\lambda^2 n}{2t^2} + \left(2a + \frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2.\end{aligned}$$

By this and Lemma 1, we obtain

$$\begin{aligned}\frac{\partial H_S}{\partial t} &= \Delta H_S - 2\nabla^i H_S \nabla_i u - 2(\alpha-\beta)\left|\nabla_i\nabla_j u - \frac{\alpha}{2(\alpha-\beta)}S_{ij} - \frac{\lambda}{2t}g_{ij}\right|^2 \\ &\quad + 2(a-\beta c)\nabla^i S\nabla_i u - 2(\alpha-\beta)\frac{\lambda}{t}\left(\Delta u - \frac{\alpha S}{2(\alpha-\beta)}\right) + \frac{(\alpha-\beta)\lambda^2 n}{2t^2} \\ &\quad + \left(2a + \frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2 + \alpha c\Delta S - 2\alpha R^{ij}\nabla_i u\nabla_j u - \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS \\ &\quad + \frac{b}{t^2}u + d\frac{n}{t^2} + \mathcal{D}_{(a,\alpha,\beta)}(S,\nabla u).\end{aligned}$$

The desired result now follows from the above equation and the following:

$$\begin{aligned}&-2(\alpha-\beta)\frac{\lambda}{t}\left(\Delta u - \frac{\alpha S}{2(\alpha-\beta)}\right) - \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2} \\ &= -\frac{2(\alpha-\beta)}{\alpha}\frac{\lambda}{t}H_S - \left(b + \frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla u|^2}{t} + \left(1 - \frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{b}{t^2}u \\ &\quad + \left(\alpha\lambda - bc + \frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t} + \left(1 - \frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{d}{t^2}n.\end{aligned}$$

This equation also follows from a direct computation. □

As a corollary of the above proposition, we obtain the following result which is a key to prove Theorem A:

Corollary 1 *Suppose that $g(t)$ evolves by the geometric flow (1) on a closed oriented smooth n-manifold M. Let f be a positive solution to the heat equation (8) with $c=-1$, $u=-\log f$ and*

$$H_S = 2\Delta u - |\nabla u|^2 - 3S - \frac{2n}{t}.$$

Then,

$$\begin{aligned}\frac{\partial H_S}{\partial t} &= \Delta H_S - 2\nabla^i H_S\nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}H_S - \frac{2}{t}|\nabla u|^2 \\ &\quad - \left(2\mathcal{H}(S,\nabla u) + \mathcal{E}(S,\nabla u)\right).\end{aligned}$$

Proof By Proposition 1 in the case where $\alpha=2$, $\beta=1$, $a=-3$, $c=-1$, $\lambda=2$, $b=0$ and $d=2$, we get the desired result as follows:

$$\begin{aligned}\frac{\partial H_S}{\partial t} &= \Delta H_S - 2\nabla^i H_S\nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}H_S - \frac{2}{t}|\nabla u|^2 \\ &\quad -4\nabla_i S\nabla^i u - 4|S_{ij}|^2 - 2\frac{S}{t} - 2\Delta S - 4R^{ij}\nabla_i u\nabla_j u + \mathcal{D}_{(-3,2,1)}(S,\nabla u) \\ &= \Delta H_S - 2\nabla^i H_S\nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}H_S - \frac{2}{t}|\nabla u|^2 \\ &\quad -2\left(\frac{\partial S}{\partial t} + \frac{S}{t} + 2\nabla_i S\nabla^i u + 2S^{ij}\nabla_i u\nabla_j u\right) + 2\frac{\partial S}{\partial t} + 4(S^{ij} - R^{ij})\nabla_i u\nabla_j u \\ &\quad -4|S_{ij}|^2 - 2\Delta S + \mathcal{D}_{(-3,2,1)}(S,\nabla u)\end{aligned}$$

$$
\begin{aligned}
&= \Delta H_S - 2\nabla^i H_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}H_S - \frac{2}{t}|\nabla u|^2 \\
&\quad -2\left(\frac{\partial S}{\partial t} + \frac{S}{t} + 2\nabla_i S\nabla^i u + 2S^{ij}\nabla_i u\nabla_j u\right) + 2\left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) \\
&\quad -4(R^{ij} - S^{ij})\nabla_i u\nabla_j u - 3\left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) + 2(2\nabla^i S_{i\ell} - \nabla_\ell S)\nabla^\ell u \\
&\quad +2(R^{ij} - S^{ij})\nabla_i u\nabla_j u \\
&= \Delta H_S - 2\nabla^i H_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}H - \frac{2}{t}|\nabla u|^2 \\
&\quad -2\left(\frac{\partial S}{\partial t} + \frac{S}{t} + 2\nabla_i S\nabla^i u + 2S^{ij}\nabla_i u\nabla_j u\right) \\
&\quad -\left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) + 2(2\nabla^i S_{i\ell} - \nabla_\ell S)\nabla^\ell u - 2(R^{ij} - S^{ij})\nabla_i u\nabla_j u \\
&= \Delta H_S - 2\nabla^i H_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}H_S - \frac{2}{t}|\nabla u|^2 \\
&\quad -(2\mathcal{H}(S,\nabla u) + \mathcal{E}(S,\nabla u)),
\end{aligned}
$$

where we used Definition 2. □

We are now in a position to prove Theorem A. First of all, notice that, for t small enough, we get $H_S < 0$. Since we assumed that (3) holds, the maximal principle and Corollary 1 tell us that

$$H_S \le 0$$

for all time $t \in (0, T)$. Hence, we have proved Theorem A.
By Theorem A and integrating along a space-time path, we are able to get a classical Harnack inequality as follows:

Corollary 2 *Suppose that $g(t)$ evolves by the geometric flow (1) on a closed oriented smooth n-manifold M and $2\mathcal{H}(S, X) + \mathcal{E}(S, X) \ge 0$ holds for all vector fields X and all time $t \in [0, T)$ for which the flow exists. Let f be a positive solution to the heat equation (2). Assume that (x_1, t_1) and (x_2, t_2) are two points in $M \times (0, T)$, where $0 < t_1 < t_2$. Let*

$$L = \inf_{\ell} \int_{t_1}^{t_2} \left(|\dot{\ell}|^2 + S\right) \mathrm{d}t,$$

where ℓ is any space-time path joining (x_1, t_1) and (x_2, t_2). Then,

$$f(x_1, t_1) \le f(x_2, t_2)\left(\frac{t_2}{t_1}\right)^n \exp\left(\frac{L}{2}\right). \tag{11}$$

Proof The strategy of the proof is now standard. For the reader, let us include the proof. First of all, we have $H_S \le 0$ by Theorem A. And $u = -\log f$ satisfies (9) with $c = -1$, i.e.,

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 - S.$$

Therefore, we get

$$2\frac{\partial u}{\partial t} + |\nabla u|^2 - S - 2\frac{n}{t} = H_S \le 0. \tag{12}$$

Pick a space-time path $\ell(x,t)$ joining (x_1,t_1) and (x_2,t_2). Then, we obtain the following along the path $\ell(x,t)$ using (12):

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{\partial u}{\partial t} + \nabla u \cdot \dot{\ell}\\ &\le \frac{S}{2} + \frac{n}{t} - \frac{|\nabla u|^2}{2} + \nabla u \cdot \dot{\ell}\\ &\le \frac{n}{t} + \frac{1}{2}\left(|\dot{\ell}|^2 + S\right).\end{aligned}$$

This implies

$$u(x_2,t_1) - u(x_1,t_1) \le \frac{L}{2} + n\log\left(\frac{t_2}{t_1}\right).$$

This tells us that (11) holds. □

Let us close this section with the proof of Theorem B. Let f be a positive solution to linear heat equation $\frac{\partial f}{\partial t} = \Delta f$. Then, we may assume that $f < 1$ by the linearity. Then, $u = -\log f$ satisfies (9) with $c = 0$. Therefore, by taking $\alpha = 0$, $\beta = -1$, $a = c = 0$, $\lambda = 2$, $b = 1$ and $d = 0$ in Lemma 1, we have

$$H_S = |\nabla u|^2 - \frac{u}{t}$$

and

$$\begin{aligned}\frac{\partial H_S}{\partial t} &= \Delta H_S - 2\nabla^i H_S \nabla_i u - 2|\nabla\nabla u|^2 - \frac{1}{t}|\nabla u|^2 + \frac{1}{t^2}u + \mathcal{D}_{(0,0,-1)}\\ &= \Delta H_S - 2\nabla^i H_S \nabla_i u - \frac{1}{t}H_S - 2|\nabla\nabla u|^2 - 2\mathcal{I}(S,\nabla u).\end{aligned}$$

Notice that as t small enough, $H_S < 0$. By the maximal principle and this evolution equation, Theorem B follows as desired.

4 Proof of Theorem C

Let f be a positive solution of (8). In what follows, let $v = -\log f - \frac{n}{2}\log(4\pi t)$. By a direct computation, we see that v satisfies

$$\frac{\partial v}{\partial t} = \Delta v - |\nabla v|^2 + cS - \frac{n}{2t}. \tag{13}$$

Then, we have

Proposition 2 *Let $g(t)$ be a solution to the geometric flow (1) and v satisfies (13). Let*

$$P_S = \alpha\Delta v - \beta|\nabla v|^2 + aS - b\frac{v}{t} - d\frac{n}{t}, \tag{14}$$

where α, β, a, b and d are constants. Then, P_S satisfies

$$\begin{aligned}\frac{\partial P_S}{\partial t} &= \Delta P_S - 2\nabla^i P_S \nabla_i v - 2(\alpha-\beta)\left|\nabla_i\nabla_j v - \frac{\alpha}{2(\alpha-\beta)}S_{ij} - \frac{\lambda}{2t}g_{ij}\right|^2\\ &\quad + 2(a-\beta c)\nabla^i v\nabla_i S - \frac{2(\alpha-\beta)}{\alpha}\frac{\lambda}{t}P_S + \frac{(\alpha-\beta)n\lambda^2}{2t^2} - \left(b + \frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla v|^2}{t}\end{aligned}$$

$$+\left(2a+\frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2+\left(\alpha\lambda-bc+\frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t}+\left(1-\frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{b}{t^2}v$$
$$+\left(1-\frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{d}{t^2}n+\alpha c\Delta S-2\alpha R^{ij}\nabla_i v\nabla_j v+\frac{bn}{2t^2}+\mathcal{D}_{(a,\alpha,\beta)}(S,\nabla v),$$

where λ *is a constant,* $\alpha\neq 0$ *and* $\alpha\neq\beta$.

Proof A similar computation with Proposition 1 enables us to prove this result. In fact, notice that we have $v=u-\frac{n}{2}\log(4\pi t)$. Therefore, we get $\nabla u=\nabla v$ and $\Delta u=\Delta v$. We also have $P_S=H_S+\frac{bn}{2t}\log(4\pi t)$. Then, Proposition 1 and a direct computation imply

$$\begin{aligned}\frac{\partial P_S}{\partial t}&=\frac{\partial H_S}{\partial t}-\frac{bn}{2t^2}\log(4\pi t)+\frac{bn}{2t^2}\\&=\Delta P_S-2\nabla^i P_S\nabla_i v-2(\alpha-\beta)\left|\nabla_i\nabla_j v-\frac{\alpha}{2(\alpha-\beta)}S_{ij}-\frac{\lambda}{2t}g_{ij}\right|^2\\&\quad+2(a-\beta c)\nabla^i v\nabla_i S-\frac{2(\alpha-\beta)}{\alpha}\frac{\lambda}{t}P_S+\frac{(\alpha-\beta)n\lambda^2}{2t^2}-\left(b+\frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla v|^2}{t}\\&\quad+\left(2a+\frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2+\left(\alpha\lambda-bc+\frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t}+\left(1-\frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{b}{t^2}v\\&\quad+\left(1-\frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{d}{t^2}n+\alpha c\Delta S-2\alpha R^{ij}\nabla_i v\nabla_j v+\frac{bn}{2t^2}+\mathcal{D}_{(a,\alpha,\beta)}(S,\nabla v).\end{aligned}$$

Hence we obtained the desired result. □

As a special case of Proposition 2, we get

Corollary 3 *Suppose that* $g(t)$ *evolves by the geometric flow (1) on a closed oriented smooth n-manifold* M. *Let* f *be a positive solution to the heat equation (8) with* $c=-1$, $v=-\log f-\frac{n}{2}\log(4\pi t)$ *and*

$$P_S=2\Delta v-|\nabla v|^2-3S+\frac{v}{t}-d\frac{n}{t}.$$

Then,

$$\begin{aligned}\frac{\partial P_S}{\partial t}+\frac{1}{t}P_S&=\Delta P_S-2\nabla^i P_S\nabla_i v-2\left|\nabla_i\nabla_j v-S_{ij}-\frac{1}{2t}g_{ij}\right|^2\\&\quad-(2\mathcal{H}(S,\nabla v)+\mathcal{E}(S,\nabla v)).\end{aligned}$$

□

Proof By Proposition 2 in the case where $\alpha=2$, $\beta=1$, $a=-3$, $b=-1$, $c=-1$, $\lambda=1$, we obtain

$$\begin{aligned}\frac{\partial P_S}{\partial t}&=\Delta P_S-2\nabla^i P_S\nabla_i v-2\left|\nabla_i\nabla_j v-S_{ij}-\frac{1}{2t}g_{ij}\right|^2-\frac{1}{t}P_S\\&\quad-4\nabla_i S\nabla^i u-4|S_{ij}|^2-2\frac{S}{t}-2\Delta S-4R^{ij}\nabla_i u\nabla_j u\\&\quad+\mathcal{D}_{(-3,2,1)}(S,\nabla u)\\&=\Delta P_S-2\nabla^i P_S\nabla_i v-2\left|\nabla_i\nabla_j v-S_{ij}-\frac{1}{2t}g_{ij}\right|^2-\frac{1}{t}P_S\end{aligned}$$

$$
\begin{aligned}
&-2\left(\frac{\partial S}{\partial t}+\frac{S}{t}+2\nabla_i S\nabla^i u+2S^{ij}\nabla_i u\nabla_j u\right)+2\left(\frac{\partial S}{\partial t}-\Delta S-2|S_{ij}|^2\right)\\
&+4(S^{ij}-R^{ij})\nabla_i u\nabla_j u+\mathcal{D}_{(-3,2,1)}(S,\nabla u)\\
&=\Delta P_S-2\nabla^i P_S\nabla_i v-2\left|\nabla_i\nabla_j v-S_{ij}-\frac{1}{t}g_{ij}\right|^2-\frac{1}{t}P_S\\
&-2\left(\frac{\partial S}{\partial t}+\frac{S}{t}+2\nabla_i S\nabla^i v+2S^{ij}\nabla_i v\nabla_j v\right)+2\left(\frac{\partial S}{\partial t}-\Delta S-2|S_{ij}|^2\right)\\
&-4(R^{ij}-S^{ij})\nabla_i u\nabla_j v-3\left(\frac{\partial S}{\partial t}-\Delta S-2|S_{ij}|^2\right)+2(2\nabla^i S_{i\ell}-\nabla_\ell S)\nabla^\ell u\\
&+2(R^{ij}-S^{ij})\nabla_i u\nabla_j u\\
&=\Delta P_S-2\nabla^i P_S\nabla_i v-2\left|\nabla_i\nabla_j v-S_{ij}-\frac{1}{t}g_{ij}\right|^2-\frac{1}{t}P_S\\
&-2\left(\frac{\partial S}{\partial t}+\frac{S}{t}+2\nabla_i S\nabla^i v+2S^{ij}\nabla_i v\nabla_j v\right)\\
&-\left(\frac{\partial S}{\partial t}-\Delta S-2|S_{ij}|^2\right)+2(2\nabla^i S_{i\ell}-\nabla_\ell S)\nabla^\ell u-2(R^{ij}-S^{ij})\nabla_i u\nabla_j u\\
&=\Delta P_S-2\nabla^i P_S\nabla_i v-2\left|\nabla_i\nabla_j v-S_{ij}-\frac{1}{2t}g_{ij}\right|^2-\frac{1}{t}P_S\\
&-(2\mathcal{H}(S,\nabla v)+\mathcal{E}(S,\nabla v)).
\end{aligned}
$$

Therefore, the desired result follows. □

We shall prove Theorem C as follows. In fact, we are able to obtain the following by Corollary 3:

$$
\begin{aligned}
\frac{\partial}{\partial t}(tP_S) &= t\frac{\partial P_S}{\partial t}+P_S=\Delta(tP_S)-2\nabla^i(tP_S)\nabla_i v-2t\left|\nabla_i\nabla_j v-S_{ij}-\frac{1}{2t}g_{ij}\right|^2\\
&\quad -t\left(2\mathcal{H}(S,\nabla v)+\mathcal{E}(S,\nabla v)\right).
\end{aligned}
$$

Furthermore, the monotonicity of $\max(tP_S)$ follows from this equation and the maximal principle.

5 Proof of Theorem D

First of all, notice that $\mathcal{F}_S\le 0$ follows from the definition of $\mathcal{F}_S$ and $H_S\le 0$, where we used Theorem A. Let us consider the following quantity:

$$
A=2t\mathrm{e}^{-u}H_S-t^2\mathrm{e}^{-u}\frac{\partial u}{\partial t}H_S+t^2\mathrm{e}^{-u}\frac{\partial H_S}{\partial t}-St^2\mathrm{e}^{-u}H_S.
$$

On the other hand, a direct computation tells us that the following holds:

$$
\Delta(t^2\mathrm{e}^{-u}H_S)=t^2\mathrm{e}^{-u}(\Delta H_S-2\nabla^i H_S\nabla_i u-H_S\Delta u+H_S|\nabla u|^2). \tag{15}
$$

By (9) with $c = -1$, Corollary 1 and (15), we get

$$\begin{aligned}
A &= 2t\mathrm{e}^{-u} H_S - t^2\mathrm{e}^{-u}\left(\Delta u - |\nabla u|^2 - S\right) H_S \\
&\quad + t^2\mathrm{e}^{-u}\Bigg(\Delta H_S - 2\nabla^i H_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t} g_{ij}\right|^2 - \frac{2}{t} H_S - \frac{2}{t}|\nabla u|^2 \\
&\qquad - (2\mathcal{H}(S, \nabla u) + \mathcal{E}(S, \nabla u))\Bigg) \\
&\quad - S t^2 \mathrm{e}^{-u} H_S \\
&= \Delta(t^2\mathrm{e}^{-u} H_S) - 2t^2\mathrm{e}^{-u}\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t} g_{ij}\right|^2 - 2te^{-u}|\nabla u|^2 \\
&\quad - t^2\mathrm{e}^{-u}\left(2\mathcal{H}(S, \nabla u) + \mathcal{E}(S, \nabla u)\right).
\end{aligned}$$

On the other hand, notice that we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_S = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M t^2\mathrm{e}^{-u} H_S \mathrm{d}\mu_g\right) = \int_M A \mathrm{d}\mu_g.$$

Therefore, the following holds:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_S &= \int_M \Bigg(\Delta(t^2\mathrm{e}^{-u} H_S) - 2t^2\mathrm{e}^{-u}\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t} g_{ij}\right|^2 - 2te^{-u}|\nabla u|^2 \\
&\qquad - t^2\mathrm{e}^{-u}(2\mathcal{H}(S, \nabla u) + \mathcal{E}(S, \nabla u))\Bigg)\mathrm{d}\mu_g \\
&= -\int_M \Bigg(2t^2\mathrm{e}^{-u}\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t} g_{ij}\right|^2 + 2t\mathrm{e}^{-u}|\nabla u|^2 \\
&\qquad + t^2\mathrm{e}^{-u}(2\mathcal{H}(S, \nabla u) + \mathcal{E}(S, \nabla u))\Bigg)\mathrm{d}\mu_g \le 0.
\end{aligned}$$

Assume moreover that $\mathcal{H}(S, X) \ge 0$ and $\mathcal{E}(S, X) \ge 0$ holds. Suppose also that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_S = 0$ holds for some time t. Then, we obtain

$$\nabla_i\nabla_j u - S_{ij} - \frac{1}{t} g_{ij} = 0, \ \nabla u = 0, \ \mathcal{H}(S, \nabla u) = 0, \ \mathcal{E}(S, \nabla u) = 0.$$

These imply (5) as desired. We proved Theorem D.

6 Proof of Theorem E

Let us consider the following quantity:

$$B = \mathrm{e}^{-v} P_S + t\mathrm{e}^{-v}\frac{\partial P_S}{\partial t} - \frac{n}{2}\mathrm{e}^{-v} P_S - t\mathrm{e}^{-v}\frac{\partial v}{\partial t} P_S - S t\mathrm{e}^{-v} P_S.$$

A direct computation tells us that the following holds:

$$\Delta(t\mathrm{e}^{-v} P_S) = t\mathrm{e}^{-v}(\Delta P_S - 2\nabla^i P_S \nabla_i v - P_S \Delta v + P_S|\nabla v|^2) \tag{16}$$

By (13) with $c = -1$, Corollary 3 and (16), we obtain the following:

$$
\begin{aligned}
B &= \mathrm{e}^{-v} P_S + t\mathrm{e}^{-v}\left(\Delta P_S - 2\nabla^i P_S \nabla_i v - 2\left|\nabla_i\nabla_j v - S_{ij} - \frac{1}{2t} g_{ij}\right|^2 - \frac{1}{t} P_S\right. \\
&\quad - (2\mathcal{H}(S, \nabla v) + \mathcal{E}(S, \nabla v)) - \frac{n}{2}\mathrm{e}^{-v} P_S - t\mathrm{e}^{-v} P_S \left(\Delta v - |\nabla v|^2 - S - \frac{n}{2t}\right) \\
&\quad - St\mathrm{e}^{-v} P_S \\
&= \Delta(t\mathrm{e}^{-v} P_S) - 2t\mathrm{e}^{-v}\left|\nabla_i\nabla_j v - S_{ij} - \frac{1}{2t} g_{ij}\right|^2 - t\mathrm{e}^{-v}\left(2\mathcal{H}(S, \nabla v) + \mathcal{E}(S, \nabla v)\right).
\end{aligned}
$$

By a direct computation, we also have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_S = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M t P_S (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-v} \mathrm{d}\mu_g\right) = \int_M B (4\pi t)^{-\frac{n}{2}} \mathrm{d}\mu_g.
$$

Therefore, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_S &= \int_M \Big(\Delta(t\mathrm{e}^{-v} P_S) - 2t\mathrm{e}^{-v}\left|\nabla_i\nabla_j v - S_{ij} - \frac{1}{2t} g_{ij}\right|^2 \\
&\qquad - t\mathrm{e}^{-v}(2\mathcal{H}(S, \nabla v) + \mathcal{E}(S, \nabla v))\Big)(4\pi t)^{-\frac{n}{2}} \mathrm{d}\mu_g \\
&= -\int_M \Big(2t\mathrm{e}^{-v}\left|\nabla_i\nabla_j v - S_{ij} - \frac{1}{2t} g_{ij}\right|^2 \\
&\qquad + t\mathrm{e}^{-v}(2\mathcal{H}(S, \nabla v) + \mathcal{E}(S, \nabla v))\Big)(4\pi t)^{-\frac{n}{2}} \mathrm{d}\mu_g \le 0.
\end{aligned}
$$

Assume moreover that $\mathcal{H}(S, X) \ge 0$ and $\mathcal{E}(S, X) \ge 0$. Suppose also that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_S = 0$ for some time t. Then, we obtain

$$
S_{ij} - \nabla_i\nabla_j v + \frac{1}{2t} g_{ij} = 0, \quad \mathcal{H}(S, \nabla v) = 0, \quad \mathcal{E}(S, \nabla v) = 0.
$$

Hence, we have proved Theorem E.

Acknowledgments The author would like to thank the Max-Plank-Institut für Mathematik in Bonn for its hospitality during the inception of this work. This work was partially supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science, No. 20540090.

References

1. Andrews, B.: Harnack inequalities for evolving hypersurfaces. Math. Z. **217**, 179–197 (1994)
2. Cao, H.-D.: On Harnack's inequalities for the Kahler–Ricci flow. Invent. Math. **109**, 247–263 (1992)
3. Cao, H.-D., Ni, L.: Matrix Li–Yau–Hamilton estimates for the heat equation on Kahler manifolds. Math. Ann. **331**, 795–807 (2005)
4. Cao, H.-D., Hamilton, R.: Differential Harnack estimates for time-dependent heat equations with potentials. Geom. Funct. Anal. **19**, 983–989 (2009)
5. Chow, B.: On Harnack's inequality and entropy for the Gaussian curvature flow. Comm. Pure Appl. Math. **44**, 469–483 (1991)
6. Fang, S.: Differential Harnack inequalities for heat equations with potentials under the Bernhard List's flow. Geom. Dedicata. **161**, 11–22 (2012)

7. Guenther, C.M.: The fundamental solution on manifolds with time-dependent metrics. J. Geom. Anal. **12**, 425–436 (2002)
8. Hamilton, R.: Three manifolds with positive Ricci curvature. J. Differ. Geom. **17**, 255–306 (1982)
9. Hamilton, R.: Four-manifolds with positive curvature operator. J. Differ. Geom. **24**, 153–179 (1986)
10. Hamilton, R.: The Harnack estimate for the Ricci flow. J. Differ. Geom. **37**, 225–243 (1993)
11. Hamilton, R.: Harnack estimate for the mean curvature flow. J. Differ. Geom. **41**, 215–226 (1995)
12. Ni, L.: Monotonicity and Li–Yau–Hamilton inequalities. In: Surveys in differential geometry. Geometric flows, pp. 251–301, Surv. Differ. Geom., Vol. 12. Int. Press, Somerville (2008)
13. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrodinger operator. Acta Math. **156**, 153–201 (1986)
14. List, B.: Evolution of an extended Ricci flow system. PhD thesis, AEI Potsdam (2005)
15. Müller, R.: Monotone volume formulas for geometric flows. J. Reine Angew. Math. **643**, 39–57 (2010)
16. Müller, R.: Ricci flow coupled with harmonic map flow. Ann. Sci. Ec. Norm. Super. **45**(4), 101–142 (2012)
17. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications, math.DG/0211159 (2002)
18. Sun, J.: Gradient estimates for positive solutions of the heat equation under geometric flow. Pac. J. Math. **253**, 489–510 (2011)
19. Qi Zhang, S.: Some gradient estimates for the heat equation on domains and for an equation by Perelman. Int. Math. Res. Not. 2006, Art. ID 92314