Ann Glob Anal Geom (2013) 44:361–368
DOI 10.1007/s10455-013-9370-4

Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group. II

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Received: 3 December 2012 / Accepted: 27 February 2013 / Published online: 11 March 2013

Abstract The geodesic distance vanishes on the group $\operatorname{Diff}_c(M)$ of compactly supported diffeomorphisms of a Riemannian manifold M of bounded geometry, for the right invariant weak Riemannian metric which is induced by the Sobolev metric H^s of order $0 \leq s < \frac{1}{2}$ on the Lie algebra $\mathfrak{X}_c(M)$ of vector fields with compact support.

Keywords Diffeomorphism group · Geodesic distance · Sobolev metrics of non-integral order

Mathematics Subject Classification (1991) Primary 35Q31 · 58B20 · 58D05

1 Introduction

In the article [1] we studied right invariant metrics on the group $\operatorname{Diff}_c(M)$ of compactly supported diffeomorphisms of a manifold M, which are induced by the Sobolev metric H^s of order s on the Lie algebra $\mathfrak{X}_c(M)$ of vector fields with compact support. We showed that for $M = S^1$ the geodesic distance on $\operatorname{Diff}(S^1)$ vanishes if and only if $s \leq \frac{1}{2}$. For other manifolds, we showed that the geodesic distance on $\operatorname{Diff}_c(M)$ vanishes for $M = \mathbb{R} \times N$, $s < \frac{1}{2}$ and for $M = S^1 \times N$, $s \leq \frac{1}{2}$, with N being a compact Riemannian manifold.

Martin Bauer was supported by 'Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 24625'.

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Now we are able to complement this result by: *The geodesic distance vanishes on* $\mathrm{Diff}_c(M)$ *for any Riemannian manifold M of bounded geometry, if* $0 \le s < \frac{1}{2}$.

We believe that this result holds also for $s = \frac{1}{2}$, but we were able to overcome the technical difficulties only for the manifold $M = S^1$, in [1]. We also believe that it is true for the regular groups $\mathrm{Diff}_{\mathcal{H}^\infty}(\mathbb{R}^n)$ and $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ as treated in [8], and for all Virasoro groups, where we could prove it only for $s = 0$ in [2].

In Sect. 2, we review the definitions for Sobolev norms of fractional orders on diffeomorphism groups as presented in [1] and extend them to diffeomorphism groups of manifolds of bounded geometry. Section 3 is devoted to the main result.

2 Sobolev metrics H^s with $s \in \mathbb{R}$

2.1 Sobolev metrics H^s on $\mathbb{R}^n$

For $s \ge 0$ the Sobolev H^s-norm of an $\mathbb{R}^n$-valued function f on $\mathbb{R}^n$ is defined as

$$\|f\|^2_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\|^2_{L^2(\mathbb{R}^n)}, \tag{1}$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-i\langle x,\xi\rangle} f(x)\, \mathrm{d}x,$$

and ξ is the independent variable in the frequency domain. An equivalent norm is given by

$$\|f\|^2_{\overline{H}^s(\mathbb{R}^n)} = \|f\|^2_{L^2(\mathbb{R}^n)} + \||\xi|^s\mathcal{F}f\|^2_{L^2(\mathbb{R}^n)}. \tag{2}$$

The fact that both norms are equivalent is based on the inequality

$$\frac{1}{C}\left(1+\sum_j |\xi_j|^s\right) \le \left(1+\sum_j |\xi_j|^2\right)^{\frac{s}{2}} \le C\left(1+\sum_j |\xi_j|^s\right),$$

holding for some constant C. For $s > 1$ this says that all ℓ^s-norms on $\mathbb{R}^{n+1}$ are equivalent. But the inequality is true also for $0 < s < 1$, even though the expression does not define a norm on $\mathbb{R}^{n+1}$. Using any of these norms we obtain the Sobolev spaces with non-integral s

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^s(\mathbb{R}^n)} < \infty\}.$$

We will use the second version of the norm in the proof of the theorem, since it will make calculations easier.

2.2 Sobolev metrics for Riemannian manifolds of bounded geometry

Following [13, Section 7.2.1] we will now introduce the spaces $H^s(M)$ on a manifold M. If M is not compact we equip M with a Riemannian metric g of bounded geometry which exists by [5]. This means that

- (I) The injectivity radius of (M, g) is positive.
- (B_∞) Each iterated covariant derivative of the curvature is uniformly g-bounded: $\|\nabla^i R\|_g < C_i$ for $i = 0, 1, 2, \ldots$.

The following is a compilation of special cases of results collected in [3, Chapter 1], who treats Sobolev spaces only for integral order.

Proposition [4,6,10] *If (M, g) satisfies (I) and (B_∞) then the following holds:*

(1) *(M, g) is complete.*
(2) *There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there is a countable cover of M by geodesic balls $B_\varepsilon(x_\alpha)$ such that the cover of M by the balls $B_{2\varepsilon}(x_\alpha)$ is still uniformly locally finite.*
(3) *Moreover, there exists a partition of unity $1 = \sum_\alpha \rho_\alpha$ on M such that $\rho_\alpha \geq 0$, $\rho_\alpha \in C_c^\infty(M)$, $supp(\rho_\alpha) \subset B_{2\varepsilon}(x_\alpha)$, and $|D_u^\beta \rho_\alpha| < C_\beta$ where u are normal (Riemann exponential) coordinates in $B_{2\varepsilon}(x_\alpha)$.*
(4) *In each $B_{2\varepsilon}(x_\alpha)$, in normal coordinates, we have $|D_u^\beta g_{ij}| < C'_\beta$, $|D_u^\beta g^{ij}| < C''_\beta$, and $|D_u^\beta \Gamma_{ij}^m| < C'''_\beta$, where all constants are independent of α.*

We can now define the H^s-norm of a function f on M:

$$\begin{aligned}\|f\|^2_{H^s(M,g)} &= \sum_{\alpha=0}^{\infty} \|(\rho_\alpha f) \circ \exp_{x_\alpha}\|^2_{H^s(\mathbb{R}^n)} \\ &= \sum_{\alpha=0}^{\infty} \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}} \mathcal{F}((\rho_\alpha f) \circ \exp_{x_\alpha})\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

If M is compact the sum is finite. Changing the charts or the partition of unity leads to equivalent norms by the proposition above, see [13, Theorem 7.2.3]. For integer s we get norms which are equivalent to the Sobolev norms treated in [3, Chapter 2]. The norms depend on the choice of the Riemann metric g. This dependence is worked out in detail in [3].

For vector fields we use the trivialization of the tangent bundle that is induced by the coordinate charts and define the norm in each coordinate as above. This leads to a (up to equivalence) well-defined H^s-norm on the Lie algebra $\mathfrak{X}_c(M)$.

2.3 Sobolev metrics on $\mathrm{Diff}_c(M)$

A positive definite weak inner product on $\mathfrak{X}_c(M)$ can be extended to a right-invariant weak Riemannian metric on $\mathrm{Diff}_c(M)$. In detail, given $\varphi \in \mathrm{Diff}_c(M)$ and $X, Y \in T_\varphi \mathrm{Diff}_c(M)$ we define

$$G^s_\varphi(X, Y) = \langle X \circ \varphi^{-1}, Y \circ \varphi^{-1} \rangle_{H^s(M)}.$$

We are interested solely in questions of vanishing and non-vanishing of geodesic distance. These properties are invariant under changes to equivalent inner products, since equivalent inner products on the Lie algebra

$$\frac{1}{C}\langle X, Y\rangle_1 \leq \langle X, Y\rangle_2 \leq C\langle X, Y\rangle_1$$

imply that the geodesic distances will be equivalent metrics

$$\frac{1}{C}\mathrm{dist}_1(\varphi, \psi) \leq \mathrm{dist}_2(\varphi, \psi) \leq C\mathrm{dist}_1(\varphi, \psi).$$

Therefore the ambiguity—dependence on the charts and the partition of unity—in the definition of the H^s-norm is of no concern to us.

3 Vanishing geodesic distance

Theorem 3.1 (Vanishing geodesic distance) *The Sobolev metric of order s induces vanishing geodesic distance on* $\mathit{Diff}_c(M)$ *if:*

- $0 \leq s < \frac{1}{2}$ *and M is any Riemannian manifold of bounded geometry.*

This means that any two diffeomorphisms in the same connected component of $\mathit{Diff}_c(M)$ *can be connected by a path of arbitrarily short* G^s*-length.*

In the proof of the theorem we shall make use of the following lemma from [1].

Lemma 3.2 [1, Lemma 3.2] *Let* $\varphi \in \mathit{Diff}_c(\mathbb{R})$ *be a diffeomorphism satisfying* $\varphi(x) \geq x$ *and let* $T > 0$ *be fixed. Then for each* $0 \leq s < \frac{1}{2}$ *and* $\varepsilon > 0$ *there exists a time-dependent vector field* $u^\varepsilon_\mathbb{R}$ *of the form*

$$u^\varepsilon_\mathbb{R}(t, x) = 1_{[f^\varepsilon(t), g^\varepsilon(t)]} * G_\varepsilon(x),$$

with $f, g \in C^\infty([0, T])$*, such that its flow* $\varphi^\varepsilon(t, x)$ *satisfies—independently of* ε*—the properties* $\varphi^\varepsilon(0, x) = x$*,* $\varphi^\varepsilon(T, x) = \varphi(x)$ *and whose* H^s*-length is smaller than* ε*, i.e.,*

$$Len(\varphi^\varepsilon) = \int_0^T \|u^\varepsilon_\mathbb{R}(t, \cdot)\|_{H^s}\, \mathrm{d}t \leq C\|f^\varepsilon - g^\varepsilon\|_\infty \leq \varepsilon.$$

Furthermore $\{t : f^\varepsilon(t) < g^\varepsilon(t)\} \subseteq supp(\varphi)$ *and there exists a limit function* $h \in C^\infty([0, T])$*, such that* $f^\varepsilon \to h$ *and* $g^\varepsilon \to h$ *for* $\varepsilon \to 0$ *and the convergence is uniform in t.*

Here, $G_\varepsilon(x) = \frac{1}{\varepsilon}G_1(\frac{x}{\varepsilon})$ is a smoothing kernel, defined via a smooth bump function G_1 with compact support.

Proof of Theorem 3.1 Consider the connected component $\mathrm{Diff}_0(M)$ of Id, i.e. those diffeomorphisms of $\mathrm{Diff}_c(M)$, for which there exists at least one path, joining them to the identity. Denote by $\mathrm{Diff}_c(M)^{L=0}$ the set of all diffeomorphisms φ that can be reached from the identity by curves of arbitrarily short length, i.e., for each $\varepsilon > 0$ there exists a curve from Id to φ with length smaller than ε.

Claim A. $\mathrm{Diff}_c(M)^{L=0}$ *is a normal subgroup of* $\mathrm{Diff}_0(M)$.
Claim B. $\mathrm{Diff}_c(M)^{L=0}$ *is a non-trivial subgroup of* $\mathrm{Diff}_0(M)$.

By [12] or [7], the group $\mathrm{Diff}_0(M)$ is simple. Thus claims A and B imply $\mathrm{Diff}_c(M)^{L=0} = \mathrm{Diff}_0(M)$, which proves the theorem.

The proof of claim A can be found in [1, Theorem 3.1] and works without change in the case of M being an arbitrary manifold and hence we will not repeat it here. It remains to show that $\mathrm{Diff}_c(M)^{L=0}$ contains a diffeomorphism $\varphi \neq \mathrm{Id}$.

We shall first prove claim B for $M = \mathbb{R}^n$ and then show how to extend the arguments to arbitrary manifolds. Choose a diffeomorphism $\varphi_\mathbb{R} \in \mathrm{Diff}_c(\mathbb{R})$ with $\varphi_\mathbb{R}(x) \geq x$ and $\mathrm{supp}(\varphi_\mathbb{R}) \subseteq [1, \infty)$. Then let

$$u^\varepsilon_\mathbb{R}(t, x) := 1_{[f^\varepsilon(t), g^\varepsilon(t)]} * G_\varepsilon(x)$$

be the family of vector fields constructed in Lemma 3.2, whose flows at time T equal $\varphi_\mathbb{R}$. We extend the vector field $u^\varepsilon_\mathbb{R}$ to a vector field $u^\varepsilon_{\mathbb{R}^n}$ on $\mathbb{R}^n$ via

$$u^\varepsilon_{\mathbb{R}^n}(t, x_1, \ldots, x_n) := \left(u^\varepsilon_\mathbb{R}(t, |x|), 0, \ldots, 0\right).$$

The flow of this vector field is given by

$$\varphi^{\varepsilon}_{\mathbb{R}^n}(t, x_1, \dots, x_n) = \left(\varphi^{\varepsilon}_{\mathbb{R}}(t, |x|), x_2, \dots, x_n\right),$$

where $\varphi^{\varepsilon}_{\mathbb{R}}$ is the flow of $u^{\varepsilon}_{\mathbb{R}}$. In particular we see that at time $t = T$

$$\varphi^{\varepsilon}_{\mathbb{R}^n}(t, x_1, \dots, x_n) = (\varphi_{\mathbb{R}}(|x|), x_2, \dots, x_n),$$

the flow is independent of ε. So it remains to show that for the length of the path $\varphi^{\varepsilon}_{\mathbb{R}^n}(t, \cdot)$ we have

$$\mathrm{Len}(\varphi^{\varepsilon}_{\mathbb{R}^n}) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

We can estimate the length of this path via

$$\begin{aligned}\mathrm{Len}(\varphi^{\varepsilon}_{\mathbb{R}^n})^2 &= \left(\int_0^T \|u^{\varepsilon}_{\mathbb{R}^n}(t, .)\|_{H^s(\mathbb{R}^n)} \, \mathrm{d}t\right)^2 \le T \int_0^T \|u^{\varepsilon}_{\mathbb{R}^n}(t, .)\|^2_{H^s(\mathbb{R}^n)} \, \mathrm{d}t \\ &= T \int_0^T \|u^{\varepsilon}_{\mathbb{R}}(t, |\cdot|)\|^2_{H^s(\mathbb{R}^n)} \, \mathrm{d}t = T \int_0^T \|1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]} * G_{\varepsilon}(|x|)\|^2_{H^s(\mathbb{R}^n)} \, \mathrm{d}t \\ &\le C(G_1, T) \int_0^T \|1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)\|^2_{H^s(\mathbb{R}^n)} \, \mathrm{d}t,\end{aligned}$$

where the last estimate follows from

$$\begin{aligned}&\|1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]} * G_{\varepsilon}(|x|)\|^2_{H^s(\mathbb{R}^n)} \\ &\quad = \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) \left[\mathcal{F}\left(1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)\right)(\xi)\right]^2 \left[\mathcal{F}\left(G_{\varepsilon}(|\cdot|)\right)(\xi)\right]^2 \, \mathrm{d}\xi \\ &\quad = \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) \left[\mathcal{F}\left(1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)\right)(\xi)\right]^2 \left[\mathcal{F}\left(G_1(|\cdot|)\right)(\varepsilon\xi)\right]^2 \, \mathrm{d}\xi \\ &\quad \le \|\mathcal{F}G_1(|\cdot|)\|^2_{L^\infty} \cdot \|1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)\|^2_{H^s(\mathbb{R}^n)}.\end{aligned}$$

Hence it is sufficient to show that

$$\left\|1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)\right\|_{H^s(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in } t.$$

To compute the H^s-norm of $1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)$ we first Fourier transform it. The Fourier transform of a radially symmetric function $v(|\cdot|) \in L^1(\mathbb{R}^n)$ is again radially symmetric and given by the following formula, see [11, Theorem 3.3],

$$(\mathcal{F}v(|\cdot|))(\xi) = 2\pi|\xi|^{1-n/2} \int_0^\infty J_{n/2-1}(2\pi|\xi|s) v(s) s^{n/2} \, \mathrm{d}s,$$

with $J_{n/2-1}$ denoting the Bessel function of order $\frac{n}{2} - 1$. To simplify notation we will omit the dependence of the vector field $1_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)$ on t and ε. Changing coordinates, this becomes

$$(\mathcal{F}1_{[f,g]}(|\cdot|))(\xi) = (2\pi)^{-n/2}|\xi|^{-n} \int_{2\pi f|\xi|}^{2\pi g|\xi|} J_{n/2-1}(s) s^{n/2} \, \mathrm{d}s.$$

This integral can be evaluated explicitly using the following integral identity for Bessel functions from [9, (10.22.1)]

$$\int z^{\nu+1} J_\nu(z)\,\mathrm{d}z = z^{\nu+1} J_{\nu+1}(z), \quad \nu \neq -\tfrac{1}{2}.$$

This gives us

$$(\mathcal{F}1_{[f,g]}(|\cdot|))(\xi) = |\xi|^{-n/2}\left(J_{n/2}(2\pi g|\xi|)g^{n/2} - J_{n/2}(2\pi f|\xi|)f^{n/2}\right).$$

The H^s-norm of $1_{[f,g]}(|\cdot|)$ is given by

$$\left\|1_{[f,g]}(|\cdot|)\right\|^2_{H^s(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n}\left(1+|\xi|^{2s}\right)\mathcal{F}1_{[f,g]}(|\cdot|)(\xi)^2\,\mathrm{d}\xi.$$

We will only consider the term involving $|\xi|^{2s}$, since the L^2-term can be estimated in the same way by setting $s=0$. Transforming to polar coordinates we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^n}|\xi|^{2s}\left(\mathcal{F}1_{[f,g]}(|\cdot|)(\xi)\right)^2\,\mathrm{d}\xi &= \int\limits_{\mathbb{R}^n}|\xi|^{2s-n}\left(J_{n/2}(2\pi g|\xi|)g^{n/2} - J_{n/2}(2\pi f|\xi|)f^{n/2}\right)^2\,\mathrm{d}\xi\\ &= \mathrm{Vol}(S^{n-1})\int\limits_0^\infty r^{2s-1}\left(J_{n/2}(2\pi gr)g^{n/2} - J_{n/2}(2\pi fr)f^{n/2}\right)^2\,\mathrm{d}r.\end{aligned}$$

The above integral is non-zero only for those t, where $f^\varepsilon(t)\neq g^\varepsilon(t)$. From Lemma 3.2 and our assumptions on $\varphi_{\mathbb{R}}$ we know that

$$\{t\;:\;f^\varepsilon(t)<g^\varepsilon(t)\}\subseteq \mathrm{supp}(\varphi_{\mathbb{R}})\subseteq[1,\infty).$$

Thus both $f^\varepsilon(t)$ and $g^\varepsilon(t)$ are different and away from 0 and we can evaluate the above integral using the identity [9, (10.22.57)],

$$\int\limits_0^\infty \frac{J_\mu(at)J_\nu(at)}{t^\lambda}\,\mathrm{d}t = \frac{\left(\frac{1}{2}a\right)^{\lambda-1}\Gamma\left(\frac{\mu}{2}+\frac{\nu}{2}-\frac{\lambda}{2}+\frac{1}{2}\right)\Gamma(\lambda)}{2\Gamma\left(\frac{\lambda}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}\right)},$$

which holds for $\mathrm{Re}(\mu+\nu+1)>\mathrm{Re}\lambda>0$ and the identity [9, (10.22.56)],

$$\int\limits_0^\infty \frac{J_\mu(at)J_\nu(bt)}{t^\lambda}\,\mathrm{d}t = \frac{a^\mu\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1}\Gamma\left(\frac{\nu}{2}-\frac{\mu}{2}+\frac{\lambda}{2}+\frac{1}{2}\right)}\mathbf{F}\left(\tfrac{\nu}{2}+\tfrac{\mu}{2}-\tfrac{\lambda}{2}+\tfrac{1}{2},\,\tfrac{\mu}{2}-\tfrac{\nu}{2}-\tfrac{\lambda}{2}+\tfrac{1}{2};\,\mu+1;\,\tfrac{a^2}{b^2}\right),$$

which holds for $0<a<b$ and $\mathrm{Re}(\mu+\nu+1)>\mathrm{Re}\lambda>-1$. Here $\mathbf{F}(a,b;c;d)$ is the regularized hypergeometric function. Using these identities with $\lambda=1-2s$, $\mu=\nu=\frac{n}{2}$, $a=2\pi f$ and $b=2\pi g$ we obtain

$$\int\limits_0^\infty r^{2s-1}J_{n/2}(2\pi fr)^2\,\mathrm{d}r = \tfrac{1}{2}(\pi f)^{-2s}\frac{\Gamma\left(\frac{n}{2}+s\right)\Gamma(1-2s)}{\Gamma(1-s)^2\Gamma\left(\frac{n}{2}+1-s\right)}$$

and

$$\int\limits_0^\infty r^{2s-1}J_{n/2}(2\pi fr)J_{n/2}(2\pi gr)\,\mathrm{d}r = \frac{1}{2}(\pi g)^{-2s}\left(\frac{f}{g}\right)^{n/2}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)}\mathbf{F}\left(\tfrac{n}{2}+s,\,s;\,\tfrac{n}{2}+1;\,\tfrac{f^2}{g^2}\right).$$

Putting it together results in

$$\int_{\mathbb{R}^n} |\xi|^{2s} (\mathcal{F}1_{[f,g]}(|\cdot|))(\xi)^2 \, \mathrm{d}\xi = \mathrm{Vol}(S^{n-1}) \Bigg(\frac{f^{-2s} + g^{-2s}}{2\pi^{2s}} \frac{\Gamma\left(\frac{n}{2}+s\right)\Gamma(1-2s)}{\Gamma(1-s)^2\Gamma\left(\frac{n}{2}+1-s\right)} - \frac{g^{-2s}}{\pi^{2s}} \frac{f^{n/2}}{g^{n/2}} \frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)} \mathbf{F}\left(\frac{n}{2}+s, s; \frac{n}{2}+1; \frac{f^2}{g^2}\right) \Bigg).$$

In the limit $\varepsilon \to 0$ we know from Lemma 3.2 that $f^\varepsilon(t) \to h(t)$ and $g^\varepsilon(t) \to h(t)$ uniformly in t on $[0, T]$ and hence $\frac{f^\varepsilon(t)}{g^\varepsilon(t)} \to 1$. For the regularized hypergeometric function $\mathbf{F}(a, b; c; d)$ at $d = 1$ we have the identity [9, (15.4.20)]

$$\mathbf{F}(a, b; c; 1) = \frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

for $\mathrm{Re}(c - a - b) > 0$. Applying the identity with $a = \frac{n}{2} + s$, $b = s$ and $c = \frac{n}{2} + 1$ we get

$$\mathbf{F}\left(\frac{n}{2}+s, s; \frac{n}{2}+1; 1\right) = \frac{\Gamma(1-2s)}{\Gamma(1-s)\Gamma\left(\frac{n}{2}+1-s\right)}.$$

Using the continuity of the hypergeometric function it follows that

$$\int_{\mathbb{R}^n} |\xi|^{2s} \left(\mathcal{F}1_{[f,g]}(|\cdot|)(\xi)\right)^2 \, \mathrm{d}\xi \to 0,$$

as $\varepsilon \to 0$ and the convergence is uniform in t. This concludes the proof that

$$\left\| 1_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|) \right\|_{H^s(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in } t,$$

and hence we have established claim B for $\mathrm{Diff}_c(\mathbb{R}^n)$.

To prove this result for an arbitrary manifold M of bounded geometry we choose a partition of unity (τ_j) such that $\tau_0 \equiv 1$ on some open subset $U \subset M$, where normal coordinates centred at $x_0 \in M$ are defined. If $\varphi_{\mathbb{R}}$ is chosen with sufficiently small support, then the vector field $u^\varepsilon_{\mathbb{R}^n}$ has support in $\exp_{x_0}(U)$ and we can define the vector field $u^\varepsilon_M := (\exp_{x_0}^{-1})^* u^\varepsilon_{\mathbb{R}^n}$ on M. This vector field generates a path $\varphi^\varepsilon_M(t, \cdot) \in \mathrm{Diff}_0(M)$ with an endpoint $\varphi^\varepsilon_M(T, \cdot) = \varphi_M(\cdot)$ that does not depend on ε with arbitrarily small H^s-length since

$$\mathrm{Len}(\varphi^\varepsilon_M) \le C_1(\tau) \int_0^T \|u^\varepsilon_M\|_{H^s(M,\tau)} \, \mathrm{d}t = C_1(\tau) \int_0^T \| \exp^*_{x_0}(\tau_0.u^\varepsilon_M)\|_{H^s(\mathbb{R}^n)} \, \mathrm{d}t$$
$$= C_1(\tau) \int_0^T \|u^\varepsilon_{\mathbb{R}^n}\|_{H^s(\mathbb{R}^n)} \, \mathrm{d}t.$$

Thus we can reduce the case of arbitrary manifolds to $\mathbb{R}^n$ and this concludes the proof. □

References

1. Bauer, M., Bruveris, M., Harms, P., Michor, P.W.: Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group. Ann. Global Anal. Geom. (2012). doi:10.1007/s10455-012-9353-x

2. Bauer, M., Bruveris, M., Harms, P., Michor, P.W.: Vanishing geodesic distance for the Riemannian metric with geodesic equation the KdV-equation. Ann. Global Anal. Geom. **41**(4), 461–472 (2012)
3. Eichhorn, J.: Global Analysis on Open Manifolds. Nova Science Publishers Inc., New York (2007)
4. Eichhorn, J.: The boundedness of connection coefficients and their derivatives. Math. Nachr. **152**, 145–158 (1991)
5. Greene, R.E.: Complete metrics of bounded curvature on noncompact manifolds. Arch. Math. (Basel) **31**(1), 89–95 (1978/1979)
6. Kordyukov, Y.A.: L^p-theory of elliptic differential operators on manifolds of bounded geometry. Acta Appl. Math. **23**(3), 223–260 (1991)
7. Mather, J.N.: Commutators of diffeomorphisms. Comment. Math. Helv. **49**, 512–528 (1974)
8. Michor, P.W., Mumford, D.: A zoo of diffeomorphism groups on $\mathbb{R}^n$ (2012, in preparation)
9. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds.): NIST Handbook of Mathematical Functions. U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC (2010)
10. Shubin, M.A.: Spectral theory of elliptic operators on noncompact manifolds. Astérisque **207**(5), 35–108 (1992) [Méthodes semi-classiques, Vol. 1 (Nantes, 1991)]
11. Stein, E.M., Weiss, G.: Introduction to Fourier analysis on Euclidean spaces. N.J. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton (1971)
12. Thurston, W.: Foliations and groups of diffeomorphisms. Bull. Amer. Math. Soc. **80**, 304–307 (1974)
13. Triebel, H.: Theory of Function Spaces, II. Volume 84 of Monographs in Mathematics. Birkhäuser Verlag, Basel (1992)