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Energy functionals and soliton equations for G_2-forms

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Abstract We extend short-time existence and stability of the Dirichlet energy flow as proven in a previous article by the authors to a broader class of energy functionals. Furthermore, we derive some monotonely decreasing quantities for the Dirichlet energy flow and investigate an equation of soliton type. In particular, we show that nearly parallel G_2-structures satisfy this soliton equation and study their infinitesimal soliton deformations.

Keywords G_2-manifolds · Geometric evolution equations

1 Introduction

In the quest for 'special' metrics, variational principles play an important rôle. A prominent example is the total scalar curvature functional on the space of Riemannian metrics, whose critical points are Ricci-flat metrics. In this article, we consider various functionals defined on $\Omega^3_+(M)$, the space of *positive* 3-*forms* on a compact, seven-dimensional spin manifold M. These forms are sections of the fibre bundle $\Lambda^3_+T^*M \to M$ whose fibre is the *open* orbit $\mathrm{GL}(7)_+/\mathrm{G}_2$ of $\mathrm{GL}(7)_+$ acting on $\Lambda^3\mathbb{R}^{7*}$. Furthermore, such a section Ω induces a Riemannian metric g_Ω on M. We also refer to Ω as a G_2-*structure* on M. The importance of this notion stems from the fact the only (irreducible) odd-dimensional instance of special holonomy comes from metrics of the form g_Ω. A central problem is to find conditions which ensure the existence of a holonomy G_2-metric provided necessary topological conditions are met. Such a theorem would yield an analogue of Yau's celebrated theorem [18] which asserts

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the existence of a metric with holonomy $\mathrm{SU}(m)$ on a Kähler manifold M^{2m} whose first Chern class vanishes.

The quantity we seek to extremalise is the *intrinsic torsion* of a positive 3-form Ω which can be thought of as an endomorphism of TM (cf. Sect. 2 for a definition). To see what this means concretely we recall that by a result of Fernández and Gray [9], Ω is *torsion-free*, i.e. its intrinsic torsion vanishes, if and only if $d\Omega = 0$ and $\delta_\Omega \Omega = 0$ (here, δ_Ω denotes the codifferential induced by g_Ω). This, in turn, is equivalent for the holonomy of g_Ω to be contained in G_2. In [17] we show that the critical points of the *Dirichlet energy functional*

$$\mathcal{D} : \Omega^3_+(M) \to \mathbb{R}, \quad \Omega \mapsto \frac{1}{2}\int_M \left(|d\Omega|^2_\Omega + |\delta_\Omega \Omega|^2_\Omega\right) \mathrm{vol}_\Omega$$

(with $\mathrm{vol}_\Omega = \Omega \wedge \star_\Omega \Omega/7$) are precisely the torsion-free forms. Since these are absolute minimisers of $\mathcal{D}$, it is natural to consider the negative gradient flow

$$\frac{\partial}{\partial t}\Omega_t = -\mathrm{grad}\,\mathcal{D}(\Omega_t) =: Q(\Omega_t) \tag{DF}$$

for $t \in [0, T)$, subject to some initial condition $\Omega_0 \in \Omega^3_+(M)$. Here, $-\mathrm{grad}$ denotes the negative L^2-gradient determined by $D_\Omega \mathcal{D}(\dot\Omega) = -\langle Q(\Omega), \dot\Omega\rangle_\Omega = -\int_M Q(\Omega) \wedge \star_\Omega \dot\Omega$ for all $\dot\Omega \in \Omega^3(M)$. The principal results of [17] are these:

Theorem 1.1 (Short-time existence) *The Dirichlet energy flow $\partial_t \Omega_t = Q(\Omega_t)$ has a unique short-time solution for any initial condition $\Omega_0 \in \Omega^3_+(M)$.*

In particular, for any initial condition, there exists a unique solution to (DF) on a maximal time interval $[0, T_{\max})$ where $T_{\max} \in (0, \infty]$.

Theorem 1.2 (Stability) *Let $\bar\Omega \in \Omega^3_+(M)$ be torsion-free. Then for any initial condition sufficiently close to $\bar\Omega$ in the C^∞-topology, the Dirichlet energy flow exists for all times and converges modulo diffeomorphisms to a torsion-free* G_2*-structure.*

In this article, we analyse the flow (DF) further. Firstly, we derive various monotonely decreasing quantities. In particular, we show that the $W^{1,2}$-Sobolev norm $\|\Omega_t\|^2_{W^{1,2}_{\Omega_t}}$ is bounded by a monotonely decreasing bound C_t. Moreover, $\frac{d}{dt}C_t = 0$ if and only if Ω_t is torsion-free. The proof involves the functional

$$\mathcal{C}(\Omega) = \frac{1}{2}\int_M |\nabla^\Omega \Omega|^2_\Omega \,\mathrm{vol}_\Omega,$$

where ∇^Ω is the Levi–Civita connection induced by g_Ω. Its critical points are again the torsion-free positive forms, and the associated negative gradient flow has properties very similar to (DF). In fact, both $\mathcal{D}$ and $\mathcal{C}$ are special instances of a whole family of energy functionals. To discuss these in general, we first recall that any $\Omega \in \Omega^3_+(M)$ induces a G_2-decomposition of p-forms $\Lambda^p = \oplus_q \Lambda^p_q$ into irreducible modules, where q is the rank of the module. The corresponding module of sections will be denoted by $\Omega^p_q(M)$ (this is analogous to the decomposition into (p, q)-forms over an almost-complex manifold). For example,

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \quad \text{and} \quad \Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}. \tag{1}$$

Of course, Λ^3_1 is spanned by the invariant form Ω. Furthermore, Λ^1 is irreducible. Since the induced Hodge-star operator $\star_\Omega$ is a G_2-equivariant isomorphism $\Lambda^p \to \Lambda^{7-p}$, we immediately get the decomposition of Λ^p for $p = 4$, 5 and 6. In particular, we can decompose

$d\Omega$ and $d\star_\Omega\Omega$ into irreducible components. Using various G_2-equivariant isomorphisms, we can write

$$d\Omega = \tau_0 \star_\Omega \Omega + 3\tau_1 \wedge \Omega + \star_\Omega \tau_3 \tag{2}$$

and

$$d \star_\Omega \Omega = 4\tau_1 \wedge \star_\Omega \Omega + \tau_2 \wedge \Omega \tag{3}$$

(see e.g. Proposition 1 in [5]) for uniquely determined *torsion forms* $\tau_0 \in \Omega^0_1(M)$, $\tau_1 \in \Omega^1_7(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$. These forms depend on Ω and can be thought of as maps from $\Omega^3_+(M)$ to Ω^p_q. The $\tau_k(\Omega)$ vanish identically for all k if and only if Ω is closed and coclosed, that is, if Ω is torsion-free. Note in passing that it is not obvious that τ_1 appears twice in both $d\Omega$ and $d\star_\Omega\Omega$, cf. [4]. Here, this will be a consequence of a Bianchi-type identity for Ω, see the remark after Lemma 3.3. We now define the energy functionals

$$\mathcal{D}_\nu := \sum_{i=0}^{3} \nu_i \mathcal{D}_i$$

with

$$\mathcal{D}_i(\Omega) := \frac{1}{2}\int_M |\tau_i|^2_\Omega \operatorname{vol}_\Omega .$$

and $\nu = (\nu_0, \nu_1, \nu_2, \nu_3) \in \mathbb{R}^4$. If $\nu \in \mathbb{R}^4_+$, that is, all entries in ν are positive, then we can prove Theorems 1.1 and 1.2 for the *generalised Dirichlet energy flow*

$$\frac{\partial}{\partial t}\Omega_t = Q_\nu(\Omega_t), \tag{DF$_\nu$}$$

see Theorems 2.9 and 2.10. The flow (DF) is just the special case for $\nu = (7, 84, 1, 1)$. However, we shall write $\mathcal{D}$ and Q for $\mathcal{D}_\nu$ and Q_ν in this case to be consistent with [17].

To obtain concrete solutions to (DF$_\nu$), we consider the equation

$$Q_\nu(\Omega_0) = \mu_0\Omega_0 + \mathcal{L}_{X_0}\Omega_0$$

for some real constant μ_0 and vector field X_0 (with $\mathcal{L}_{X_0}$ the Lie derivative along X_0). In analogy with Ricci-flow, we call this the *$\mathcal{D}_\nu$-soliton equation*. A $\mathcal{D}$-soliton, where $\mathcal{D}$ is the original Dirichlet energy functional, will be simply called a *G_2-soliton*. For a $\mathcal{D}_\nu$-soliton Ω_0 as initial condition, the solution to (DF$_\nu$) has the form $\Omega_t = \mu(t)\Omega_0$ with $\mu(t) \searrow 0$ as $t \nearrow T_{\max}$, and so becomes singular. As in the Ricci-flow case, one expects G_2-solitons to play a major rôle in the study of finite time singularities. We first show that any G_2-soliton is necessarily of the form $Q(\Omega) = \mu\Omega$. This is precisely the condition to be a critical point for $\mathcal{D}$ subject to the constraint that the total volume $\int_M \operatorname{vol}_\Omega$ equals 1. Furthermore, any such G_2-soliton is either steady, i.e. $\mu_0 = 0$, in which case the flow is constant and thus exists trivially for all times, or shrinking, i.e. $\mu_0 < 0$. In this case, the flow collapses in finite time. Our main result is that *nearly parallel G_2-structures* (i.e. G_2-structures for which all torsion forms but τ_0 vanish) are G_2-solitons in the sense above (cf. Theorem 4.1). For example, the 7-sphere with the round metric is nearly parallel. In general, nearly parallel G_2-structures induce Einstein metrics with positive Einstein constant. However, we do not know whether a soliton is necessarily of this type. Finally, we investigate the premoduli space of G_2-soliton deformations at a nearly parallel G_2-structure. As in the Einstein case, we can prove that the premoduli space is a real-analytic subset of some finite-dimensional real analytic submanifold (cf. Theorem 5.7). Any infinitesimal Einstein deformation of a nearly parallel G_2-structure gives an infinitesimal soliton deformation, but again we do not know whether the converse holds.

1.1 Conventions

(i) In this article, we shall only encounter irreducible G_2-representation spaces of dimension equal or less than 27. In this range, an irreducible G_2-representation is uniquely determined by its dimension q. For instance, the space of symmetric 2-tensors $\odot^2\mathbb{R}^{7*}$ can be decomposed into the line spanned by the identity and the 27-dimensional irreducible space of tracefree 2-tensors $\odot_0^2\mathbb{R}^{7*}$, which is thus isomorphic to $\Lambda^3_{27}\mathbb{R}^{7*}$. Consequently, the module of endomorphisms can be decomposed into

$$\mathbb{R}^{7*}\otimes\mathbb{R}^{7*} = \odot^2\mathbb{R}^{7*}\oplus\Lambda^2\mathbb{R}^{7*} = \Lambda^3_0\oplus\Lambda^3_{27}\oplus\Lambda^3_7\oplus\Lambda^2_{14}. \tag{4}$$

We denote projection onto irreducible components by $[\,\cdot\,]_q$. For example, a 3-form $\dot\Omega\in\Omega^3(M)$ can be decomposed into $\dot\Omega = [\dot\Omega]_1\oplus[\dot\Omega]_7\oplus[\dot\Omega]_{27}$ and an endomorphism $\dot A$ into $[\dot A]_1\oplus[\dot A]_7\oplus[\dot A]_{14}\oplus[\dot A]_{27}$.

(ii) If $F:\Omega^3_+(M)\to E$ is a smooth map between Fréchet spaces, then we often write $\dot F_\Omega$ for $D_\Omega F(\dot\Omega)$, the linearisation of F at Ω evaluated in $\dot\Omega\in\Omega^3(M)$. For example, for the map $\Theta:\Omega^3_+(M)\to\Omega^4(M)$ which sends Ω to $\Theta(\Omega)=\star_\Omega\Omega$, we get

$$\dot\Theta_\Omega = \star_\Omega p_\Omega(\dot\Omega) \tag{5}$$

with

$$p_\Omega(\dot\Omega) = \frac{4}{3}[\dot\Omega]_1 + [\dot\Omega]_7 - [\dot\Omega]_{27}.$$

Another example is $Q:\Omega^3_+(M)\to\Omega^3(M)$, the negative gradient of $\mathcal{D}$, given by

$$Q(\Omega) = -\delta_\Omega d\Omega - p_\Omega(d\delta_\Omega\Omega) - q_\Omega(\nabla^\Omega\Omega), \tag{6}$$

where q_Ω is determined by the identities

$$\langle\dot\Omega, q_\Omega(\nabla^\Omega\Omega)\rangle_\Omega = \frac{1}{2}\big(\langle\dot\star_\Omega d\Omega, \star_\Omega d\Omega\rangle_\Omega + \langle\dot\star_\Omega d\star_\Omega\Omega, \star_\Omega d\star_\Omega\Omega\rangle_\Omega\big) \tag{7}$$

to hold for all $\dot\Omega\in\Omega^3(M)$.

2 The Dirichlet energy and the Hitchin functional

2.1 The torsion forms of a positive 3-form

Recall that $\nabla^\Omega\Omega$ is a section of $\Lambda^1\otimes\Lambda^3_7$ and hence may be written as $\nabla^\Omega\Omega = T(\Omega)$ for a uniquely determined tensor field $T\in\Gamma(\Lambda^1\otimes\Lambda^2_7)$, the *intrinsic torsion* of the G_2-structure (cf. for example [5]). Here the Λ^2_7 factor of T acts, seen as an element in $\Lambda^2\cong\mathfrak{so}(7)$, the Lie algebra of SO(7), equivariantly in the standard way on Ω and gives an element in Λ^3_7. The module $\Lambda^1_7\otimes\Lambda^3_7$ decomposes as $\Lambda^0_1\oplus\Lambda^1_7\oplus\Lambda^2_{14}\oplus\Lambda^3_{27}$ into G_2-irreducible ones. Hence

$$\nabla^\Omega\Omega = \xi_1+\xi_7+\xi_{14}+\xi_{27},$$

where ξ_i denotes the projection of $\xi := \nabla^\Omega \Omega$ onto the corresponding irreducible summand. The ξ_k are thus the irreducible components of the intrinsic torsion T under the embedding $T \mapsto T(\Omega)$.

Proposition 2.1 *Let $\Omega \in \Omega^3_+(M)$ be a positive 3-form. Then the following holds:*

(i) *One has*

$$|d\Omega|^2_\Omega = 7\tau_0^2 + 36|\tau_1|^2_\Omega + |\tau_3|^2_\Omega$$

and

$$|\delta_\Omega \Omega|^2_\Omega = 48|\tau_1|^2_\Omega + |\tau_2|^2_\Omega.$$

In particular,

$$|d\Omega|^2_\Omega + |\delta_\Omega \Omega|^2_\Omega = 7\tau_0^2 + 84|\tau_1|^2_\Omega + |\tau_2|^2_\Omega + |\tau_3|^2_\Omega. \tag{8}$$

(ii) *One has*

$$|\nabla^\Omega \Omega|^2_\Omega = \frac{7}{4}\tau_0^2 + 24|\tau_1|^2_\Omega + 2|\tau_2|^2_\Omega + 2|\tau_3|^2_\Omega. \tag{9}$$

Proof (i) Clearly

$$|d\Omega|^2_\Omega = \tau_0^2|\star_\Omega \Omega|^2_\Omega + 9|\tau_1 \wedge \Omega|^2_\Omega + |\tau_3|^2_\Omega,$$

which using $|\star_\Omega \Omega|^2_\Omega = |\Omega|^2_\Omega = 7$ and $|\tau_1 \wedge \Omega|^2_\Omega = 4|\tau_1|^2_\Omega$ (cf. for instance Eq. (15) in [17]) yields the first equation. Similarly,

$$|\delta_\Omega \Omega|^2_\Omega = |d \star_\Omega \Omega|^2_\Omega = 16|\tau_1 \wedge \star_\Omega \Omega|^2_\Omega + |\tau_2 \wedge \Omega|^2_\Omega$$

as $|\tau_1 \wedge \star_\Omega \Omega|^2_\Omega = 3|\tau_1|^2_\Omega$ (cf. Eq. (15) in [17]) and $|\tau_2 \wedge \Omega|^2_\Omega = |\tau_2|^2_\Omega$, for $\Lambda^2_{14} = \{\alpha \in \Lambda^2 \mid \alpha \wedge \Omega = -\star_\Omega \alpha\}$.

(ii) Let $\varepsilon : \Lambda^1 \otimes \Lambda^k \to \Lambda^{k+1}$ and $\iota : \Lambda^1 \otimes \Lambda^k \to \Lambda^{k-1}$ denote exterior resp. interior multiplication. Then $d\Omega = \epsilon(\xi)$ and $\delta_\Omega \Omega = -\iota(\xi)$. Since ε and ι are GL-equivariant, one has more precisely

$$d\Omega = \varepsilon(\xi_1) + \varepsilon(\xi_7) + \varepsilon(\xi_{27})$$

and

$$\delta_\Omega \Omega = -\iota(\xi_7) - \iota(\xi_{14}).$$

We need to calculate the length distortion of the maps ξ and ι on the irreducible summands. We claim that

$$|\varepsilon(\xi_1)|^2_\Omega = 4|\xi_1|^2_\Omega, \quad |\varepsilon(\xi_7)|^2_\Omega = \frac{3}{2}|\xi_7|^2_\Omega, \quad |\varepsilon(\xi_{27})|^2_\Omega = \frac{1}{2}|\xi_{27}|^2_\Omega$$

and

$$|\iota(\xi_7)|^2_\Omega = 2|\xi_7|^2_\Omega, \quad |\iota(\xi_{14})|^2_\Omega = \frac{1}{2}|\xi_{14}|^2_\Omega.$$

To establish these we consider the map $f : \Lambda^1 \otimes \Lambda^1 \to \Lambda^1 \otimes \Lambda^3_7$ which to $v \otimes w$ assigns $v \otimes (w \llcorner \star_\Omega \Omega)$. The module of symmetric endomorphisms $\odot^2$ which is spanned by $v \otimes w + w \otimes v$ can be decomposed into the tracefree endomorphisms $\odot^2_0$ and multiples of the identity. A (GL(7)-)equivariant projection $\pi_0 : \odot^2 \to \odot^2_0$ is given by $\pi_0(a) = a - \mathrm{Tr}(a)\mathrm{Id}/7$. We want to compute $|\varepsilon(f(\pi_0(a)))|^2$ and $|f(\pi_0(a))|^2$ for $a \in \odot^2$. It suffices to do this for elements of the form $e_i \otimes e_j + e_j \otimes e_i$ for some orthonormal basis $e_1, \dots, e_7$ of Λ^1. Furthermore, since G_2 acts transitively on pairs of orthonormal vectors, we need to consider the element $e_1 \otimes e_2 + e_2 \otimes e_1$ only, which is already in $\odot_0^2$. Thus,

$$|f(e_1 \otimes e_2 + e_2 \otimes e_1)|^2 = |e_1 \otimes (e_2 \llcorner \star_\Omega \Omega) + e_2 \otimes (e_1 \llcorner \star_\Omega \Omega)|^2 = 8$$

while

$$|\varepsilon(f(e_1 \otimes e_2 + e_2 \otimes e_1))|^2 = |e_1 \wedge (e_2 \llcorner \star_\Omega \Omega) + e_2 \wedge (e_1 \llcorner \star_\Omega \Omega)|^2 = 4,$$

whence the distortion factor $1/2$ as claimed above. In the same vein, consider the projection $\pi_{14}^2 : \Lambda^2 \to \Lambda_{14}^2$ given by $\pi_{14}^2(\alpha) = (2\alpha - \star_\Omega(\alpha \wedge \Omega))/3$. Then

$$|f(e_1 \otimes e_2 - e_2 \otimes e_1)|^2 = |e_1 \otimes (e_2 \llcorner \star_\Omega \Omega) - e_2 \otimes (e_1 \llcorner \star_\Omega \Omega)|^2 = 8$$

and

$$|\iota(f(e_1 \otimes e_2 - e_2 \otimes e_1))|^2 = |e_1 \llcorner (e_2 \llcorner \star_\Omega \Omega) - e_2 \llcorner (e_1 \llcorner \star_\Omega \Omega)|^2 = 4,$$

giving again the distortion factor $1/2$. Either by proceeding as before or by using the transitivity of G_2 on the sphere of its vector representation we deduce the remaining coefficients. Therefore

$$|d\Omega|_\Omega^2 = 4|\xi_1|_\Omega^2 + \frac{3}{2}|\xi_7|_\Omega^2 + \frac{1}{2}|\xi_{27}|_\Omega^2$$

and

$$|\delta_\Omega \Omega|_\Omega^2 = 2|\xi_7|_\Omega^2 + \frac{1}{2}|\xi_{14}|_\Omega^2.$$

Comparing this with the formulæ (2) and (3) we get:

$$|\xi_1|_\Omega^2 = \frac{7}{4}\tau_0^2, \quad |\xi_7|_\Omega^2 = 24|\tau_1|_\Omega^2, \quad |\xi_{14}|_\Omega^2 = 2|\tau_2|_\Omega^2, \quad |\xi_{27}|_\Omega^2 = 2|\tau_3|_\Omega^2.$$

Since clearly

$$|\nabla^\Omega \Omega|_\Omega^2 = |\xi_1|_\Omega^2 + |\xi_7|_\Omega^2 + |\xi_{14}|_\Omega^2 + |\xi_{27}|_\Omega^2$$

the result follows. □

Remark The previous proposition provides an alternative proof of the result of Fernández and Gray mentioned in the introduction: For $\Omega \in \Omega_+^3(M)$ one has $\nabla^\Omega \Omega = 0$ if and only if $d\Omega = \delta_\Omega \Omega = 0$, since both equations are equivalent to $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$. By standard holonomy theory, $\nabla^\Omega \Omega = 0$ is equivalent to g_Ω having holonomy contained in G_2.

2.2 Monotone quantities

For any smooth family Ω_t, we can write

$$\partial_t \Omega_t = 3 f_t \Omega_t + \star_{\Omega_t}(\alpha_t \wedge \Omega_t) + \gamma_t$$

for uniquely determined quantities $f_t \in C^\infty(M)$, $\alpha_t \in \Omega^1(M)$ and $\gamma_t \in \Omega_{27,\Omega_t}^3(M)$ depending smoothly on t. These are called the *deformation forms* of Ω_t. In particular, the evolution of the associated volume form is given by

$$\partial_t \operatorname{vol}_{\Omega_t} = 7 f_t \operatorname{vol}_{\Omega_t},$$

see e.g. [5]. For a solution Ω_t to (DF), we have

$$g_{\Omega_t}(Q(\Omega_t),\Omega_t) = 3f_t g_{\Omega_t}(\Omega_t,\Omega_t) = 21 f_t$$

and hence

$$\partial_t \operatorname{vol}_{\Omega_t} = \tfrac{1}{3} g_{\Omega_t}(Q(\Omega_t),\Omega_t)\operatorname{vol}_{\Omega_t}. \tag{10}$$

Alternatively, use that the differential of the map $\phi : \Lambda^3_+ \to \Lambda^7$ sending Ω to $\operatorname{vol}_\Omega$ is given by

$$D_\Omega\phi(\dot{\Omega}) = \tfrac{1}{3}\dot{\Omega}\wedge\star_\Omega\Omega, \tag{11}$$

cf. [12]. The *Hitchin functional* is defined by

$$\mathcal{H} : \Omega^3_+(M) \to \mathbb{R}, \quad \Omega \mapsto \int_M \operatorname{vol}_\Omega,$$

i.e. it associates with $\Omega \in \Omega^3_+(M)$ its total volume. We find that the value of the Hitchin functional is monotone and convex along a solution to the Dirichlet energy flow:

Proposition 2.2 *If* $(\Omega_t)_{t\in[0,T)}$ *is a solution to* (DF)*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\Omega_t) \leq 0 \quad \textit{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}(\Omega_t) \geq 0$$

for all $t \in [0,T)$*. Further,* $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0}\mathcal{H}(\Omega_t) = 0$ *if and only if* Ω_{t_0} *is torsion-free.*

Proof Using Eq. (10) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\Omega_t) &= \int_M \frac{\partial}{\partial t}\operatorname{vol}_{\Omega_t}\\ &= \frac{1}{3}\int_M g_{\Omega_t}(Q(\Omega_t),\Omega_t)\operatorname{vol}_{\Omega_t}\\ &= -\tfrac{1}{3}D_{\Omega_t}\mathcal{D}(\Omega_t)\end{aligned}$$

Since $\mathcal{D}$ is positively homogeneous, i.e. $\mathcal{D}(\lambda\Omega) = \lambda^{5/3}\mathcal{D}(\Omega)$ for $\lambda > 0$, one has $D_\Omega\mathcal{D}(\Omega) = \frac{5}{3}\mathcal{D}(\Omega)$ by Euler's formula, cf. the proof of Corollary 4.3 in [17]. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\Omega_t) = -\tfrac{5}{9}\mathcal{D}(\Omega_t) \leq 0 \tag{12}$$

with equality if and only if Ω_t is torsion-free. Furthermore,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}(\Omega_t) = -\tfrac{5}{9}D_{\Omega_t}\mathcal{D}(Q(\Omega_t)) = \tfrac{5}{9}\|Q(\Omega_t)\|^2_{\Omega_t}$$

which is always non-negative. □

Equation (12) has the following noteworthy consequence for a long-time solution to the Dirichlet energy flow: Suppose that Ω_t is a solution to (DF) on $[0,\infty)$. Then, since $\mathcal{D}(\Omega_t)$ is monotonely decreasing, the limit

$$\mathcal{D}_\infty := \lim_{t\to\infty}\mathcal{D}(\Omega_t) \geq 0$$

exists. In fact, we have

Corollary 2.3 *If* $(\Omega_t)_{t\in[0,\infty)}$ *is a solution to* (DF)*, then* $\mathcal{D}_\infty = 0$.

Proof Assume to the contrary that $\mathcal{D}_\infty > 0$. Then $\mathcal{D}(\Omega_t) \geq \mathcal{D}_\infty > 0$ for all $t \in [0,\infty)$. Hence, by Eq. (12), $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\Omega_t) \leq -\frac{5}{9}\mathcal{D}_\infty < 0$ for all t, and therefore

$$\mathcal{H}(\Omega_t) \leq \mathcal{H}(\Omega_0) - \frac{5}{9}\mathcal{D}_\infty t.$$

In particular, $\mathcal{H}(\Omega_t)$ becomes negative in finite time. Contradiction! □

Remark As an example communicated to us by Joel Fine shows, long-time existence is not sufficient to imply convergence to a critical point (cf. Fine, Pers. commun.). It is obtained by restricting the Dirichlet energy functional $\mathcal{D}$ to the space of SO(4)-invariant forms on $\mathbb{R}^4 \times SO(3)$. Using Lemma 3.1, the flow equations can be reduced to a system of nonlinear ODEs which can be explicitly solved and whose solutions project down to $T^4 \times SO(3)$. This is related to the failure of the Dirichlet energy functional to satisfy the Palais–Smale condition. If, however, $\lim_{t\to\infty}\Omega_t = \Omega_\infty \in \Omega^3_+(M)$, say w.r.t. the C^1-topology, then Corollary 2.3 suffices to conclude that Ω_∞ is torsion-free.

As for the Dirichlet energy functional, we may set

$$\mathcal{H}_\infty := \lim_{t\to\infty}\mathcal{H}(\Omega_t) \geq 0$$

for a solution Ω_t to (DF) on $[0,\infty)$. Here two cases may occur:

(1) $\mathcal{H}_\infty > 0$
(2) $\mathcal{H}_\infty = 0$

A prototypical example for the first case is a solution converging to a torsion-free G_2-structure as $t \to \infty$. Such solutions exist as a consequence of Theorem 1.2, our stability result for the Dirichlet energy flow. A solution fitting into the second case is provided by Fine's example (cf. Fine, Pers. commun.).

A further consequence of Eq. (12) is that the value of the Hitchin functional decays at most linearly along a solution to the Dirichlet energy flow:

Corollary 2.4 *If* $(\Omega_t)_{t\in[0,T)}$ *is a solution to* (DF)*, then*

$$\mathcal{H}(\Omega_0) \geq \mathcal{H}(\Omega_t) \geq \mathcal{H}(\Omega_0) - \tfrac{5}{9}\mathcal{D}(\Omega_0)t$$

for all $t \in [0,T)$. *In particular, for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\mathcal{H}(\Omega_t) \geq \delta$ *for all* $t \in [0, t_0 - \varepsilon]$ *with* $t_0 = \min\{T, \frac{9}{5}\frac{\mathcal{H}(\Omega_0)}{\mathcal{D}(\Omega_0)}\}$.

Proof Since the Dirichlet energy flow is the negative gradient flow of $\mathcal{D}$, one clearly has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(\Omega_t) \leq 0$$

for all $t \in [0,T)$, in particular $\mathcal{D}(\Omega_t) \leq \mathcal{D}(\Omega_0)$. Hence by Eq. (12)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\Omega_t) \geq -\frac{5}{9}\mathcal{D}(\Omega_0),$$

and the claim follows by integration. □

Remark If one knew exponential decay of $\mathcal{D}(\Omega_t)$ for a solution to (DF) on $[0,\infty)$ beforehand, then $\mathcal{H}(\Omega_t)$ would be bounded from below: Assuming $\mathcal{D}(\Omega_t) \le Ce^{-\lambda t}$ for constants $C, \lambda > 0$ and using Eq. (12) once again, one gets

$$\begin{aligned}\mathcal{H}(\Omega_t) &\ge \mathcal{H}(\Omega_0) - \frac{5}{9}\int_0^t Ce^{-\lambda\tau}\,\mathrm{d}\tau \\ &= \mathcal{H}(\Omega_0) - \frac{5}{9}\frac{C}{\lambda}(1-e^{-\lambda t}) \\ &\ge \mathcal{H}(\Omega_0) - \frac{5}{9}\frac{C}{\lambda}\end{aligned}$$

for all $t \in [0,\infty)$. This would be particularly useful if one could choose C and λ in such a way that $\delta := \mathcal{H}(\Omega_0) - \frac{5}{9}\frac{C}{\lambda} > 0$.

In [5] it is shown that the scalar curvature of the metric g_Ω is given by

$$s_{g_\Omega} = 12\delta_\Omega\tau_1 + \frac{21}{8}\tau_0^2 + 30|\tau_1|_\Omega^2 - \frac{1}{2}|\tau_2|_\Omega^2 - \frac{1}{2}|\tau_3|_\Omega^2$$

(cf. (4.28) loc. cit.). Thus, by Stokes' theorem, the total scalar curvature

$$\mathcal{S}(\Omega) := \int_M s_{g_\Omega}\,\mathrm{vol}_\Omega$$

of g_Ω is given by

$$\mathcal{S}(\Omega) = \int_M \left(\frac{21}{8}\tau_0^2 + 30|\tau_1|_\Omega^2 - \frac{1}{2}|\tau_2|_\Omega^2 - \frac{1}{2}|\tau_3|_\Omega^2\right)\mathrm{vol}_\Omega\,. \tag{13}$$

On the other hand, by Proposition 2.1, we have

$$\mathcal{D}(\Omega) = \int_M \left(\frac{7}{2}\tau_0^2 + 42|\tau_1|_\Omega^2 + \frac{1}{2}|\tau_2|_\Omega^2 + \frac{1}{2}|\tau_3|_\Omega^2\right)\mathrm{vol}_\Omega\,.$$

Comparing coefficients immediately yields

Lemma 2.5 *Let $\Omega \in \Omega_+^3(M)$ be a positive 3-form. Then $|\mathcal{S}(\Omega)| \le \mathcal{D}(\Omega)$.*

Using the monotonicity of $\mathcal{D}$ and Corollary 2.3 we obtain

Corollary 2.6 *The absolute value of the total scalar curvature $\mathcal{S}(\Omega_t)$ is bounded by a monotonely decreasing quantity along a solution $(\Omega_t)_{t\in[0,T)}$ to (DF). If Ω_t is defined on $[0,\infty)$, then $\lim_{t\to\infty}\mathcal{S}(\Omega_t) = 0$.*

If we define

$$\mathcal{C}(\Omega) := \frac{1}{2}\int_M |\nabla^\Omega\Omega|_\Omega^2\,\mathrm{vol}_\Omega,$$

then we get from Eqs. (8) and (9)

$$\begin{aligned}\mathcal{C}(\Omega) &= \int_M \Big(\frac{7}{8}\tau_0^2 + 12|\tau_1|_\Omega^2 + |\tau_2|_\Omega^2 + |\tau_3|_\Omega^2\Big)\,\mathrm{vol}_\Omega \\ &= \mathcal{D}(\Omega) + \int_M \Big(-\frac{21}{8}\tau_0^2 - 30|\tau_1|_\Omega^2 + \frac{1}{2}|\tau_2|_\Omega^2 + \frac{1}{2}|\tau_3|_\Omega^2\Big)\,\mathrm{vol}_\Omega \\ &= \mathcal{D}(\Omega) - \mathcal{S}(\Omega).\end{aligned}$$

Furthermore, we remark that

$$2\mathcal{C}(\Omega) + 7\mathcal{H}(\Omega) = \|\Omega\|^2_{W^{1,2}_\Omega},$$

whence

$$0 \le \|\Omega\|^2_{W^{1,2}_\Omega} \le 4\mathcal{D}(\Omega) + 7\mathcal{H}(\Omega) \le 8\|\Omega\|^2_{W^{1,2}_\Omega}.$$

In particular, we find along a solution to the Dirichlet energy flow

Proposition 2.7 *Let $(\Omega_t)_{t\in[0,T)}$ be a solution to (DF). Then*

$$\|\Omega_t\|^2_{W^{1,2}_{\Omega_t}} \le C_t \le C_0$$

for the monotonely decreasing bound $C_t := 4\mathcal{D}(\Omega_t) + 7\mathcal{H}(\Omega_t)$. Furthermore, one has $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} C_t = 0$ if and only if Ω_{t_0} is torsion-free.

Proof The first assertion follows directly from the discussion above. Secondly, $\frac{\mathrm{d}}{\mathrm{d}t}C_t = 4\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(\Omega_t) + 7\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\Omega_t) \le 0$ with equality if and only if $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(\Omega_t) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\Omega_t) = 0$, whence the result by Proposition 2.2. □

2.3 The generalised Dirichlet energy flow

The energy functionals $\mathcal{D}$ and $\mathcal{C}$ considered above are special instances of the functional

$$\mathcal{D}_\lambda := \sum_{i=0}^{3} \nu_i \mathcal{D}_i$$

with

$$\mathcal{D}_i(\Omega) := \frac{1}{2}\int_M |\tau_i|^2_\Omega\,\mathrm{vol}_\Omega$$

and $\nu = (\nu_0, \nu_1, \nu_2, \nu_3) \in \mathbb{R}^4$. More specifically, one has

$$\mathcal{D} = 7\mathcal{D}_0 + 84\mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3$$

and

$$\mathcal{C} = \frac{7}{4}\mathcal{D}_0 + 24\mathcal{D}_1 + 2\mathcal{D}_2 + 2\mathcal{D}_3.$$

We call the functional $\mathcal{D}_\nu$ the *generalised Dirichlet energy functional* associated with the parameter $\nu \in \mathbb{R}^4$. The aim of this section is to further analyse this family of functionals. In particular, we prove generalised versions of Theorems 1.1 and 1.2 for $\mathcal{D}_\nu$ for $\nu \in \mathbb{R}^4_+$.

Set $Q_i(\Omega) := -\mathrm{grad}\,\mathcal{D}_i(\Omega)$, $i = 0, 1, 2, 3$ and $Q_\nu(\Omega) := -\mathrm{grad}\,\mathcal{D}_\nu(\Omega)$ for $\nu \in \mathbb{R}^4$. The functional $\mathcal{D}_\nu$ shares the same basic properties with $\mathcal{D}$: It is $\mathrm{Diff}(M)_+$-invariant and positively homogeneous, i.e. $\mathcal{D}_\nu(\mu\Omega) = \mu^{\frac{5}{3}}\mathcal{D}_\nu(\Omega)$ for $\mu \in \mathbb{R}_+$.

Next we consider the negative gradient flow of the generalised Dirichlet energy functional

$$\frac{\partial}{\partial t}\Omega_t = Q_\nu(\Omega_t) \qquad (\mathrm{DF}_\nu)$$

for $\nu \in \mathbb{R}^4$, subject to some initial condition $\Omega_0 \in \Omega^3_+(M)$. We call the flow equation (DF_ν) the *generalised Dirichlet energy flow*.

For $\nu \in \mathbb{R}^4_+$, the generalised Dirichlet energy flow behaves much like the ordinary Dirichlet energy flow. In this case, Euler's formula implies as for Q (corresponding to $\mathcal{D}$) that $Q_\nu(\Omega) = 0$ holds if and only if Ω is torsion-free. As a first result, we have

Lemma 2.8 *The flow equation* (DF_ν) *is weakly parabolic for* $\nu \in \mathbb{R}^4_{\geq 0}$, *i.e.*

$$-g_\Omega(\sigma(D_\Omega Q_\nu)(x,\xi)\dot{\Omega}, \dot{\Omega}) \geq 0$$

for all $x \in M$, $\xi \in T^*_x M$ *and* $\dot{\Omega} \in \Lambda^3 T^*_x M$.

Proof According to Proposition 2.1 one has

$$|[\mathrm{d}\Omega]_1|^2_\Omega = 7\tau_0^2, \quad |[\mathrm{d}\Omega]_7|^2_\Omega = 36|\tau_1|^2_\Omega, \quad |[\mathrm{d}\Omega]_{27}|^2_\Omega = |\tau_3|^2_\Omega$$

and

$$|[\delta_\Omega\Omega]_7|^2 = 48|\tau_1|^2_\Omega, \quad |[\delta_\Omega\Omega]_{14}|^2 = |\tau_2|^2_\Omega.$$

Therefore

$$7\cdot\mathcal{D}_0(\Omega) = \frac{1}{2}\int_M |[\mathrm{d}\Omega]_1|^2_\Omega \,\mathrm{vol}_\Omega,$$

$$36\cdot\mathcal{D}_1(\Omega) = \frac{1}{2}\int_M |[\mathrm{d}\Omega]_7|^2_\Omega \,\mathrm{vol}_\Omega,$$

$$\mathcal{D}_3(\Omega) = \frac{1}{2}\int_M |[\mathrm{d}\Omega]_{27}|^2_\Omega \,\mathrm{vol}_\Omega$$

and

$$48\cdot\mathcal{D}_1(\Omega) = \frac{1}{2}\int_M |[\delta_\Omega\Omega]_7|^2_\Omega \,\mathrm{vol}_\Omega, \quad \mathcal{D}_2(\Omega) = \frac{1}{2}\int_M |[\delta_\Omega\Omega]_{14}|^2_\Omega \,\mathrm{vol}_\Omega\,.$$

Linearising as in [17] we get

$$-\sigma(D_\Omega Q_0)(x,\xi)\dot{\Omega} = \frac{1}{7}\xi\llcorner[\xi\wedge\dot{\Omega}]_1, \quad -\sigma(D_\Omega Q_1)(x,\xi)\dot{\Omega} = \frac{1}{36}\xi\llcorner[\xi\wedge\dot{\Omega}]_7,$$

$$-\sigma(D_\Omega Q_2)(x,\xi)\dot{\Omega} = p_\Omega(\xi\wedge[\xi\llcorner p_\Omega\dot{\Omega}]_{14}), \quad -\sigma(D_\Omega Q_3)(x,\xi)\dot{\Omega} = \xi\llcorner[\xi\wedge\dot{\Omega}]_{27}.$$

Now for $k = 1, 7, 27$ we have for $\xi \in T^*_x M$

$$g_\Omega(\xi\llcorner[\xi\wedge\dot{\Omega}]_k, \dot{\Omega}) = |[\xi\wedge\dot{\Omega}]_k|^2_\Omega \geq 0$$

and for $k = 14$

$$g_\Omega(p_\Omega(\xi \wedge [\xi \llcorner p_\Omega \dot\Omega]_{14}), \dot\Omega) = |[\xi \llcorner p_\Omega \dot\Omega]_{14}|^2_\Omega \geq 0.$$

Since $D_\Omega Q_\nu = \sum_{i=0}^{3} \nu_i D_\Omega Q_i$, the result follows. □

Breaking the diffeomorphism invariance one gets:

Theorem 2.9 *The generalised Dirichlet energy flow $\partial_t \Omega_t = Q_\nu(\Omega_t)$ has a unique short-time solution for $\nu \in \mathbb{R}^4_+$ and any initial condition $\Omega_0 \in \Omega^3_+(M)$.*

Proof We employ DeTurck's trick as in [17]. Given some background G_2-structure $\bar\Omega \in \Omega^3_+(M)$ (e.g. the initial condition Ω_0) we consider the vector field

$$X(\Omega) = -(\delta_{\bar\Omega}\Omega) \llcorner \bar\Omega.$$

For $\varepsilon(\nu) = \min_{i=0,1,2,3} \nu_i/36$, we set $\Lambda(\Omega) := \mathcal{L}_{X(\Omega)}\Omega$ and

$$\widetilde{Q}_\nu(\Omega) := Q_\nu(\Omega) + \varepsilon(\nu)\Lambda(\Omega).$$

Then, $D_\Omega \widetilde{Q}_\nu = D_\Omega Q_\nu + \varepsilon(\nu) D_\Omega \Lambda$. For $\xi \in T^*_x M$ with $|\xi|_\Omega = 1$, we find that

$$\begin{aligned} -g_\Omega(\sigma(D_\Omega Q_\nu)(x,\xi)\dot\Omega, \dot\Omega) &= -\sum_{i=0}^{3} \nu_i g_\Omega(\sigma(D_\Omega Q_i)(x,\xi)\dot\Omega, \dot\Omega) \\ &\geq -\varepsilon(\nu) g_\Omega(\sigma(D_\Omega Q)(x,\xi)\dot\Omega, \dot\Omega) \end{aligned}$$

and hence

$$\begin{aligned} & - g_\Omega(\sigma(D_\Omega \widetilde{Q}_\nu)(x,\xi)\dot\Omega, \dot\Omega) \\ \geq & - \varepsilon(\nu) g_\Omega(\sigma(D_\Omega Q)(x,\xi)\dot\Omega, \dot\Omega) - \varepsilon(\nu) g_\Omega(\sigma(D_\Omega \Lambda)(x,\xi)\dot\Omega, \dot\Omega) \\ = & - \varepsilon(\nu) g_\Omega(\sigma(D_\Omega \widetilde{Q})(x,\xi)\dot\Omega, \dot\Omega) \geq \varepsilon(\nu)|\dot\Omega|^2_\Omega, \end{aligned}$$

where the last line follows from Lemma 5.7 in [17].

This shows that the flow equation $\partial_t \widetilde\Omega_t = \widetilde{Q}_\nu(\widetilde\Omega_t)$ is strongly parabolic. Standard methods, see for instance [16], now yield a unique short-time solution $\widetilde\Omega_t$. A short-time solution Ω_t for the original flow equation $\partial_t \Omega_t = Q_\nu(\Omega_t)$ is then obtained by integrating the time-dependent vector field $X(\widetilde\Omega_t)$ and pulling back $\widetilde\Omega_t$ by the corresponding family of diffeomorphisms, cf. [17] for details.

The proof of uniqueness given in [17] for the Dirichlet energy flow applies without change to yield uniqueness of the solution Ω_t on short time intervals. □

Finally, as in [17] we also get a stability result:

Theorem 2.10 *Let $\bar\Omega \in \Omega^3_+(M)$ be torsion-free. Then for any initial condition sufficiently close to $\bar\Omega$ in the C^∞-topology the solution to (DF$_\nu$) for $\nu \in \mathbb{R}^4_+$ exists for all times and converges modulo diffeomorphisms to a torsion-free* G_2*-structure.*

Proof Let $\Omega \in \Omega^3_+(M)$ be torsion-free, i.e. $d\Omega = \delta_\Omega \Omega = 0$. Then

$$\begin{aligned} (D_\Omega Q_0)\dot\Omega &= -\frac{1}{7}\delta_\Omega [d\dot\Omega]_1, \quad (D_\Omega Q_1)\dot\Omega = -\frac{1}{36}\delta_\Omega [d\dot\Omega]_7 \\ (D_\Omega Q_2)\dot\Omega &= -p_\Omega(d[\delta_\Omega p_\Omega \dot\Omega]_{14}), \quad (D_\Omega Q_3)\dot\Omega = -\delta_\Omega [d\dot\Omega]_{27} \end{aligned}$$

and

$$(D_\Omega \Lambda)(\dot{\Omega}) = -3\mathrm{d}[\delta_\Omega \dot{\Omega}]_7.$$

We set $L_\nu := D_\Omega \widetilde{Q}_\nu$ and $L := D_\Omega \widetilde{Q}$ as in [17]. Then we get

$$L_\nu = -\nu_0 \frac{1}{7}\delta_\Omega[\mathrm{d}\dot{\Omega}]_1 - \nu_1 \frac{1}{36}\delta_\Omega[\mathrm{d}\dot{\Omega}]_7 - \nu_2 p_\Omega(\mathrm{d}[\delta_\Omega p_\Omega \dot{\Omega}]_{14}) - \nu_3 \delta_\Omega[\mathrm{d}\dot{\Omega}]_{27} - 3\varepsilon(\nu) d[\delta_\Omega \dot{\Omega}]_7$$

and hence

$$\langle -L_\nu \dot{\Omega}, \dot{\Omega}\rangle_{L^2_\Omega} \geq \varepsilon(\nu)\langle -L\dot{\Omega}, \dot{\Omega}\rangle_{L^2_\Omega} \quad \forall \dot{\Omega} \in \Omega^3(M)$$

with $\varepsilon(\nu) = \min_{i=0,1,2,3} \nu_i/36$ as above. In particular, L_ν is non-positive and the Gårding inequality holds. The proof then proceeds along the same lines as the one given in [17] for the Dirichlet energy flow. □

3 G_2-solitons

3.1 Symmetries

We recall that one has a natural $\mathrm{Diff}(M)_+$-action on $\Omega^3_+(M)$ given by pullback and that $\mathcal{D}$ is $\mathrm{Diff}(M)_+$-invariant, i.e. $\mathcal{D}(\varphi^*\Omega) = \mathcal{D}(\Omega)$ for all $\varphi \in \mathrm{Diff}(M)_+$. This implies that

$$\varphi^* Q(\Omega) = Q(\varphi^*\Omega). \tag{14}$$

Further, any symmetry of the initial condition Ω_0 is preserved by the Dirichlet energy flow:

Lemma 3.1 *Let $(\Omega_t)_{t\in[0,T)}$ be a solution to (DF) with initial condition Ω_0. If $\varphi^*\Omega_0 = \Omega_0$ for some $\varphi \in \mathrm{Diff}(M)_+$, then $\varphi^*\Omega_t = \Omega_t$ for all $t \in [0, T)$.*

Proof Using Eq. (14) one gets that $(\varphi^*\Omega_t)_{t\in[0,T)}$ is a solution to (DF) with initial condition $\varphi^*\Omega_0$. Since $\varphi^*\Omega_0 = \Omega_0$, uniqueness of the Dirichlet energy flow implies that $\varphi^*\Omega_t = \Omega_t$ for all $t \in [0, T)$. □

Secondly, one has a natural $\mathbb{R}_+$-action on $\Omega^3_+(M)$ given by scaling with respect to which $\mathcal{D}$ is positively homogeneous:

$$\mathcal{D}(\lambda\Omega) = \lambda^{\frac{5}{3}}\mathcal{D}(\Omega) \tag{15}$$

for all $\lambda \in \mathbb{R}_+$.

Lemma 3.2 *One has $Q(\lambda\Omega) = \lambda^{\frac{1}{3}} Q(\Omega)$ for all $\lambda \in \mathbb{R}_+$.*

Proof Using Eq. (15) we calculate

$$\begin{aligned} D_{\lambda\Omega}\mathcal{D}(\dot{\Omega}) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{D}(\lambda\Omega + t\dot{\Omega}) \\ &= \lambda^{\frac{5}{3}} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{D}(\Omega + t\lambda^{-1}\dot{\Omega}) \\ &= \lambda^{\frac{5}{3}} D_\Omega\mathcal{D}(\lambda^{-1}\dot{\Omega}) = \lambda^{\frac{2}{3}} D_\Omega\mathcal{D}(\dot{\Omega}). \end{aligned}$$

Hence

$$D_{\lambda\Omega}\mathcal{D}(\dot{\Omega}) = \lambda^{\frac{2}{3}} D_\Omega \mathcal{D}(\dot{\Omega}) = \lambda^{\frac{2}{3}} \int_M g_\Omega(\operatorname{grad}\mathcal{D}(\Omega), \dot{\Omega}) \operatorname{vol}_\Omega$$

and on the other hand

$$D_{\lambda\Omega}\mathcal{D}(\dot{\Omega}) = \int_M g_{\lambda\Omega}(\operatorname{grad}\mathcal{D}(\lambda\Omega), \dot{\Omega}) \operatorname{vol}_{\lambda\Omega} = \lambda^{\frac{1}{3}} \int_M g_\Omega(\operatorname{grad}\mathcal{D}(\lambda\Omega), \dot{\Omega}) \operatorname{vol}_\Omega .$$

Here, we have used the fact that $\operatorname{vol}_{\lambda\Omega} = \lambda^{\frac{7}{3}} \operatorname{vol}_\Omega$ and $g_{\lambda\Omega} = \lambda^{-2} g_\Omega$ on 3-forms. Comparing these two expressions, we get the result. □

Remark As a consequence of the preceding lemma, if Ω_t is a solution to (DF) on $[0, T)$ and $\lambda > 0$, then the space–time rescaling $\Omega_t^\lambda := \lambda\Omega_{\lambda^{-2/3}t}$ is again a solution to (DF), defined on $[0, \lambda^{2/3}T)$.

3.2 A Bianchi-type identity

For some fixed background G_2-structure Ω, consider the operator

$$\lambda_\Omega^* : \mathcal{X}(M) \to \Omega^3(M), \quad X \mapsto \mathcal{L}_X\Omega$$

and its formal adjoint with respect to $L^2_{g_\Omega}$, namely

$$\lambda_\Omega : \Omega^3(M) \to \mathcal{X}(M), \quad \dot{\Omega} \mapsto -X_\Omega(\dot{\Omega}) - \dot{\Omega} \llcorner d\Omega,$$

where $X_\Omega(\dot{\Omega}) = -\delta_\Omega \dot{\Omega} \llcorner \Omega$. As usual, we identify 1-forms and vector fields using g_Ω. Recall that we have an L^2-orthogonal decomposition

$$\Omega^3(M) = \ker \lambda_\Omega \oplus \operatorname{im} \lambda_\Omega^*, \tag{16}$$

where the second summand is tangent to the $\operatorname{Diff}(M)_+$-orbit through Ω, see Proposition 5.6 and Lemma 7.3 in [17].

Lemma 3.3 *For all $\Omega \in \Omega^3_+(M)$, we have $\lambda_\Omega(Q(\Omega)) = 0$ and $\lambda_\Omega\Omega = 0$.*

Proof The proof proceeds along the same lines as Kazdan's derivation of the usual Bianchi identity in [13]: If $\mathcal{F} : \Omega^3_+(M) \to \mathbb{R}$ is a $\operatorname{Diff}(M)_+$-invariant functional, then $\lambda_\Omega(\operatorname{grad}\mathcal{F}(\Omega)) = 0$, since the level-set $\mathcal{F}^{-1}(\mathcal{F}(\Omega))$ contains the $\operatorname{Diff}(M)_+$-orbit through Ω. Now by definition, $Q(\Omega) = -\operatorname{grad}\mathcal{D}(\Omega)$, which yields $\lambda_\Omega(Q(\Omega)) = 0$. Secondly, from Eq. (11) it follows that

$$\operatorname{grad}\mathcal{H}(\Omega) = \tfrac{1}{3}\Omega$$

which gives $\lambda_\Omega\Omega = 0$. □

Remark The equation $\lambda_\Omega\Omega = 0$ is equivalent to $\tau_1 = \tilde{\tau}_1$, where in light of the definition of the torsion forms, one has

$$d\Omega = \tau_0 \star_\Omega \Omega + 3\tau_1 \wedge \Omega + \star_\Omega \tau_3$$

and

$$d \star_\Omega \Omega = 4\tilde{\tau}_1 \wedge \star_\Omega\Omega + \tau_2 \wedge \Omega$$

for $\tilde{\tau}_1$ a priori different from τ_1. Indeed, $\lambda_\Omega \Omega = (\delta_\Omega \Omega) \llcorner \Omega - \Omega \llcorner \mathrm{d}\Omega = 0$ is equivalent to

$$([\delta_\Omega \Omega]_7) \llcorner \Omega = \Omega \llcorner ([\mathrm{d}\Omega]_7). \tag{17}$$

Substituting $[\delta_\Omega \Omega]_7 = -4 \star_\Omega \tilde{\tau}_1 \wedge \star_\Omega \Omega$ and $[\mathrm{d}\Omega]_7 = 3\tau_1 \wedge \Omega$ we obtain that Eq. (17) is equivalent to

$$-4 \star_\Omega (\tilde{\tau}_1 \wedge \star_\Omega \Omega) \llcorner \Omega = 3\Omega \llcorner (\tau_1 \wedge \Omega). \tag{18}$$

A routine calculation establishes for $\xi \in \Omega^1(M)$ the identities $\Omega \llcorner (\xi \wedge \Omega) = -4\xi$ and $\star_\Omega (\xi \wedge \star_\Omega \Omega) \llcorner \Omega = 3\xi$. Hence, the left-hand side of Eq. (18) equals $-12\tilde{\tau}_1$, whereas the right-hand side equals $-12\tau_1$.

Corollary 3.4 *If $\Omega \in \Omega^3_+(M)$ satisfies $Q(\Omega) = f \cdot \Omega$ for $f \in C^\infty(M)$, then f is constant, i.e. $Q(\Omega) = \lambda \Omega$ for $\lambda \in \mathbb{R}$.*

Proof Applying λ_Ω to the equation $Q(\Omega) = f \cdot \Omega$ yields the equation $\lambda_\Omega(f\Omega) = 0$ using Lemma 3.3. On the other hand

$$\begin{aligned}\lambda_\Omega(f\Omega) &= -\delta_\Omega(f\Omega) \llcorner \Omega - f\Omega \llcorner \mathrm{d}\Omega \\ &= (\mathrm{d}f \llcorner \Omega - f\delta_\Omega \Omega) \llcorner \Omega - f\Omega \llcorner \mathrm{d}\Omega \\ &= (\mathrm{d}f \llcorner \Omega) \llcorner \Omega - f\lambda_\Omega \Omega = (\mathrm{d}f \llcorner \Omega) \llcorner \Omega,\end{aligned}$$

where we have again used Lemma 3.3 in the last line. Now since $(\xi \llcorner \Omega) \llcorner \Omega = 3\xi$ for all $\xi \in \Omega^1(M)$ we conclude that $df = 0$, i.e. f is constant. □

Next we consider the operator $\widetilde{Q}_{\bar{\Omega}}(\Omega) = Q(\Omega) + \lambda^*_\Omega(X_{\bar{\Omega}}(\Omega))$, $\Omega, \bar{\Omega} \in \Omega^3_+(M)$, defined in [17].

Corollary 3.5 *If $\Omega \in \Omega^3_+(M)$ satisfies $\widetilde{Q}_{\bar{\Omega}}(\Omega) = 0$, then $Q(\Omega) = 0$, i.e. Ω is torsion-free.*

Proof Applying λ_Ω to the equation

$$\widetilde{Q}_{\bar{\Omega}}(\Omega) = Q(\Omega) + \lambda^*_\Omega(X_{\bar{\Omega}}(\Omega)) = 0 \tag{19}$$

yields the equation $\lambda_\Omega \lambda^*_\Omega(X_{\bar{\Omega}}(\Omega)) = 0$ using Lemma 3.3. Hence, $\lambda^*_\Omega(X_{\bar{\Omega}}(\Omega)) = 0$ and therefore $Q(\Omega) = 0$. □

Remark Note that if M has finite fundamental group or more generally satisfies $H^1(M; \mathbb{R}) = \{0\}$, then $\widetilde{Q}_{\bar{\Omega}}(\Omega) = 0$ also implies $X_{\bar{\Omega}}(\Omega) = 0$. Indeed, since $Q(\Omega) = 0$, Ω is torsion-free and $\mathcal{L}_{X_{\bar{\Omega}}(\Omega)}\Omega = 0$. Hence, g_Ω is Ricci-flat and $X_{\bar{\Omega}}(\Omega)$ is Killing. But this implies that $X_{\bar{\Omega}}(\Omega)$ is parallel and therefore its dual 1-form is harmonic. In general, a parallel Killing vector field has no zeros unless it is identically vanishing. Hence, the dual of $X_{\bar{\Omega}}(\Omega)$ is a closed, nowhere vanishing 1-form. By Tischler's theorem [15], M must globally fibre over the circle. Note, however, that non-trivial parallel Killing vector fields can exist: If X is a Calabi–Yau threefold, then the product $X \times S^1$ admits a natural torsion-free G_2-structure for which the coordinate vector field ∂_t on S^1 is a parallel Killing vector field. Conversely, by standard holonomy theory, (cf. for instance [2]), a torsion-free G_2-manifold (M, Ω) with non-trivial parallel Killing vector field is reducible, i.e. *locally* of the form $X \times S^1$ for X a Calabi–Yau manifold.

3.3 The soliton equation

Definition 3.6 A triple (Ω_0, X_0, μ_0) with $\Omega_0 \in \Omega^3_+(M)$, $X_0 \in \mathcal{X}(M)$ a vector field and $\mu_0 \in \mathbb{R}$, which satisfy the equation

$$Q(\Omega_0) = \mu_0\Omega_0 + \mathcal{L}_{X_0}\Omega_0$$

is called a G_2-*soliton structure*. A solution to (DF) of the form

$$\Omega_t = \mu(t)\varphi_t^*\Omega_0$$

for some function $\mu(t)$ and a family of orientation-preserving diffeomorphisms φ_t is called a G_2-*soliton solution.*

A particular case of a soliton structure is a G_2-structure Ω_0 satisfying the equation $Q(\Omega_0) = \mu_0 \cdot \Omega_0$ for some constant $\mu_0 \in \mathbb{R}$. The ansatz

$$\Omega_t = \mu(t)\Omega_0, \quad \mu(0) = 1$$

yields using Lemma 3.2

$$\partial_t\Omega_t = \mu'(t)\Omega_0$$
$$Q(\Omega_t) = \mu(t)^{\frac{1}{3}}\mu_0\Omega_0$$

and hence, the ODE

$$\mu'(t) = \mu_0\mu(t)^{\frac{1}{3}}, \quad \mu(0) = 1. \tag{20}$$

The solution of (20) is given by

$$\mu(t) = \left(\frac{2\mu_0}{3}t + 1\right)^{\frac{3}{2}}$$

on some maximal time interval $[0, T_{\max})$. As in the Ricci-flow case, one has more generally:

Lemma 3.7 *Let* (Ω_0, X_0, μ_0) *be a* G_2-*soliton structure. Then*

$$\Omega_t := \mu(t)\varphi_t^*\Omega_0 \tag{21}$$

is a G_2-*soliton solution on* $[0, T_{\max})$ *for* $\mu(t) = (\frac{2\mu_0}{3}t + 1)^{\frac{3}{2}}$ *and* φ_t *the flow of the time-dependent vector field* $\mu(t)^{-\frac{2}{3}}X_0$. *The associated metric flow is given by*

$$g_t = \mu(t)^{\frac{2}{3}}\varphi_t^*g_0.$$

Conversely, if $\Omega_t = \mu(t)\varphi_t^*\Omega_0$ *is a* G_2-*soliton solution on* $[0, T_{\max})$, *then* (Ω_0, X_0, μ_0) *with* $X_0 = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varphi_t$ *and* $\mu_0 = \mu(0)$ *is a* G_2-*soliton structure.*

Proof Differentiating Eq. (21) we get

$$\partial_t\Omega_t = \varphi_t^*\left(\mu(t)^{\frac{1}{3}}\mathcal{L}_{X_0}(\Omega_0) + \mu'(t)\Omega_0\right)$$
$$Q(\Omega_t) = \varphi_t^*\mu(t)^{\frac{1}{3}}Q(\Omega_0)$$

which yields the claim upon substituting (20). The evolution of the associated metric g_t immediately follows from its scaling behaviour. □

Remark By the preceding lemma, a G_2-soliton structure and a G_2-soliton solution are essentially the same thing. We will therefore simply refer to both the G_2-soliton structure or the corresponding soliton solution as a G_2-*soliton*.

Definition 3.8 A G_2-soliton (Ω_0, X_0, μ_0) is called *expanding*, if $\mu_0 > 0$; *steady*, if $\mu_0 = 0$; and *shrinking*, if $\mu_0 < 0$. It is called *trivial* if $Q(\Omega_0) = \mu_0\Omega_0$.

Using this terminology we can state the following:

Proposition 3.9 *Let* (Ω_0, X_0, μ_0) *be a* G_2*-soliton. Then the following holds:*
(i) *Any* G_2*-soliton* (Ω_0, X_0, μ_0) *is trivial, i.e. already satisfies* $Q(\Omega_0) = \mu_0\Omega_0$.
(ii) *One has* $\mu_0 \leq 0$*, i.e. there are no expanding* G_2*-solitons.*
(iii) *If* Ω_t *denotes the corresponding soliton solution, then* $T_{\max} = \infty$ *in the steady case and* $T_{\max} = -\frac{3}{2\mu_0}$ *in the shrinking case.*

Proof To prove the first assertion we apply λ_{Ω_0} to the equation

$$Q(\Omega_0) = \mu_0\Omega_0 + \mathcal{L}_{X_0}\Omega_0 = \mu_0\Omega_0 + \lambda^*_{\Omega_0}X_0.$$

This gives, using Lemma 3.3, the equation $\lambda_{\Omega_0}\lambda^*_{\Omega_0}X_0 = 0$, hence, $\mathcal{L}_{X_0}\Omega_0 = 0$.
Secondly, for $\mu_0 > 0$ we would have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(\Omega_t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(\mu(t)\Omega_0) = \frac{5}{3}\mu_0\mu(t)\mathcal{D}(\Omega_0) > 0$$

which is incompatible with the monotonicity of $\mathcal{D}$. The remaining statements follow from the behaviour of the solution of the ODE (20). □

Remark For a shrinking soliton one clearly has $\lim_{t\to T_{max}}\mu(t) = 0$ and therefore $\lim_{t\to T_{\max}}\mathcal{H}(\Omega_t) = \lim_{t\to T_{\max}}\mathcal{D}(\Omega_t) = 0$. This follows easily from the scaling behaviour of these functionals.

3.4 A constrained variational principle

Next we ask for critical points of $\mathcal{D}$ under the constraint $\mathcal{H}(\Omega) = 1$. Let $\Omega^3_{+,1}(M)$ be the submanifold of $\Omega^3_+(M)$ consisting of positive 3-forms of total volume 1. Its tangent space at Ω is $\ker D_\Omega\mathcal{H}$. Now by (11), $\dot{\mathcal{H}}_\Omega = \langle\dot\Omega, \Omega\rangle/3$ so that $T_\Omega\Omega^3_{+,1}(M) = \Omega^\perp$, the 3-forms which are perpendicular to Ω with respect to the natural L^2-product. On the other hand, we need $\operatorname{grad}\mathcal{D} = -Q$ to be orthogonal to $T_\Omega\Omega^3_{+,1}(M)$, hence, a constrained critical point Ω satisfies $Q(\Omega) = \mu_0\Omega$ for some constant $\mu_0 \in \mathbb{R}$. In view of Proposition 3.9, we obtain an alternative characterisation of G_2-solitons.

Corollary 3.10 *A positive* 3*-form* Ω *is a* G_2*-soliton if and only if* Ω *is a critical point of* $\mathcal{D}$ *subject to* $\mathcal{H} \equiv 1$.

Remark The results of this section apply mutatis mutandis to the generalised Dirichlet energy functionals $\mathcal{D}_\nu$, $\nu \in \mathbb{R}^4_+$. More precisely, we say that (Ω_0, X_0, μ_0) is a $\mathcal{D}_\nu$-*soliton* if the equation $Q_\nu(\Omega_0) = \mu_0\Omega_0 + \mathcal{L}_{X_0}\Omega_0$ holds. Since $\mathcal{D}_\nu$ shares the same symmetries with $\mathcal{D}$, we obtain the Bianchi identity $\lambda_\Omega(Q_\nu(\Omega)) = 0$. Hence, we may deduce that any $\mathcal{D}_\nu$-soliton is trivial with $\mu_0 \leq 0$. The explicit solution to the soliton equation remains unchanged.

4 Examples

4.1 Homogeneous spaces

Consider a compact homogeneous space $M = G/H$. Then G acts on M via diffeomorphisms coming from left translations. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the decomposition at Lie algebra level from the inclusion $H \hookrightarrow G$, where $\mathfrak{m}$ is some complement invariant under the isotropy action of H (the adjoint action of G restricted to H). The space of G-invariant G_2-forms is precisely the space of H-invariant G_2-forms in $\Lambda^3\mathfrak{m}^*$. Since invariant critical points can be obtained by restricting the functional to invariant G_2-forms, we are left with a finite-dimensional variational problem. We will illustrate this procedure for the Dirichlet energy functional $\mathcal{D}$.

4.1.1 The round sphere

We think of S^7 as the homogeneous space $\mathrm{Spin}(7)/\mathrm{G}_2$. Then $\mathfrak{spin}(7) = \Lambda^2\mathbb{R}^{7*} = \mathfrak{g}_2 \oplus \mathfrak{m}$ by (1), where $\mathfrak{m}$ is isomorphic to the 7-dimensional irreducible vector representation of G_2. Hence, $\Lambda^3\mathfrak{m}^* \cong \mathbf{1} \oplus \mathfrak{m} \oplus \odot^2_0\mathfrak{m}$ (also cf. our first convention at the end of Sect. 1) is a decomposition into irreducible G_2-modules, and we find a one-dimensional space of $\mathrm{Spin}(7)$-invariant G_2-forms spanned by Ω_0. In fact, if we think of S^7 as the unit octonians with induced metric g_0 (the round metric), then at $p \in S^7$, $\Omega_{0,p}(u, v, w) = g_{0,p}\big(p, u \cdot (\bar{v} \cdot w) - w \cdot (\bar{v} \cdot u)\big)$ (here $^-$ and $\cdot$ denote conjugation and multiplication on $\mathbb{O}$). Since $Q(\Omega_0)$ must be also $\mathrm{Spin}(7)$-invariant by Lemma 3.1, we deduce $Q(\Omega_0) = c\Omega_0$ for some nonpositive constant c. Furthermore, $H^3(S^7; \mathbb{R}) = 0$ so that Ω_0 cannot be torsionfree, whence, $Q(\Omega_0) \neq 0$.

4.1.2 The squashed sphere

Now consider S^7 as the homogeneous space $G/H = \mathrm{Sp}(2) \times \mathrm{Sp}(1)/\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ defined by the embedding

$$(a, b) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1) \mapsto \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, b\right).$$

The complex irreducible representations of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ are obtained from the symmetric powers $\sigma_p = \odot^p\mathbb{C}^2$ of the standard vector representation on $\mathbb{C}^2$. Endowed with some negative multiple of the Killing form G/H becomes a normal Riemannian homogeneous space (cf. Definition 7.86 in [2]) with orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. As an $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$-space, $\mathfrak{m} = \mathbf{1} \otimes \sigma_2 \oplus \sigma_1 \otimes \sigma_1 =: \mathfrak{m}' \oplus \mathfrak{m}''$. Here, by abuse of notation, $\sigma_1 \otimes \sigma_1$ (which is of real type) also denotes the underlying real representation. In the resulting decomposition of $\Lambda^3\mathfrak{m}^*$, we find two trivial representations, namely $\Lambda^3\mathfrak{m}'^* \cong \mathbb{R}$ and one in $\mathfrak{m}'^* \otimes \Lambda^2\mathfrak{m}''^*$ (cf. [1]). If f_1, f_2 and f_3 denotes an orthonormal basis of $\mathfrak{m}'$, then the first one is spanned by[1] $\Omega_1 = f^{123}$. For the second invariant form Ω_2, we note that $\Lambda^2\mathfrak{m}''^* = \mathbf{1} \otimes \sigma_2 \oplus \sigma_2 \otimes \mathbf{1}$ which is just the decomposition into self- and antiselfdual forms. Consequently, if $e_1, \ldots, e_4$ is an orthonormal basis for $\mathfrak{m}''$, then $\Omega_1 = \sum_k f^k \wedge \omega_k$ where

$$\omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}.$$

The G-invariant forms

$$\mathcal{I} = \left\{\Omega_{a,b} := -a^3\Omega_1 + ab^2\Omega_2 \,|\, a,\ b > 0\right\}$$

[1] Here and in the sequel, f^{123} will be shorthand for $f^1 \wedge f^2 \wedge f^3$.

are of G_2-type and compatible with the natural orientation. To compute the G-invariant critical points we must compute $\mathcal{D}$ on $\mathcal{I}$. We first note that $\Omega_{a,b}$ induces the metric $g_{a,b} = -a^2B|_{\mathfrak{m}'} - b^2B|_{\mathfrak{m}''}$ so that $\mathrm{vol}_{a,b} = a^3b^4e^{1234} \wedge f^{123}$ and

$$\star_{a,b}\Omega_{a,b} = -b^4e^{1234} + a^2b^2\left(f^{23} \wedge \omega_1 - f^{13} \wedge \omega_2 + f^{12} \wedge \omega_3\right).$$

We compute the commutators $[\cdot,\cdot]_{\mathfrak{m}}$ and thus the exterior differentials of $e_1, \ldots, f_3$. Upon suitably rescaling B we find

$$\mathrm{d}\Omega_{a,b} = 12ab^2e^{1234} + (10ab^2 + 2a^3)\left(-f^{23} \wedge \omega_1 + f^{13} \wedge \omega_2 - f^{12} \wedge \omega_3\right)$$

and $\mathrm{d} \star_{a,b} \Omega_{a,b} = 0$. Consequently, $|\mathrm{d}\Omega_{a,b}|^2 = 24(7a^2b^{-4} + 25a^{-2} + 10b^{-2})$, whence,

$$\mathcal{D}(\Omega_{a,b}) = 12(7a^5 + 10a^3b^2 + 25ab^4)\mathrm{Vol},$$

with Vol the total volume of G/H with respect to $\mathrm{vol}_{1,1} = e^{1234} \wedge f^{123}$. Subject to the constraint $a^3b^4 = 1$ the critical point equations read

$$7a^4 + 6a^2b^2 + 5b^4 = 3\mu a^2b^4, \quad a^2 + 5b^2 = \mu a^2b^2, \quad a^3b^4 = 1$$

for some constant τ. Substituting $u = a^2$ and $v = b^2$ shows that $u = v$ and $\mu = 6/v$. Hence, $a = 1, b = 1$ and $\mu = 6$ is the unique solution which gives the soliton $\Omega_{1,1}$. The resulting metric is the so-called *squashed* metric.

4.2 Nearly parallel G_2-structures

The previous two examples define in fact *nearly parallel* G_2*-structures* (see for instance [10]). These were first investigated by Gray [11] (who called them weak holonomy G_2-structures). This is a G_2-structure given by a G_2-form Ω satisfying

$$\mathrm{d}\Omega = \tau_0 \star_\Omega \Omega$$

for some constant $\tau_0 \neq 0$. In particular, $\mathrm{d}\star_\Omega \Omega = 0$ so that alternatively, we may characterise nearly parallel G_2-structures as those for which all torsion forms but τ_0 do vanish. By abuse of language, we refer to such an Ω itself as a nearly parallel G_2-structure. The associated metric is necessarily Einstein with positive constant scalar curvature $s_\Omega = \frac{21}{7}\tau_0^2$.

Theorem 4.1 *If* Ω *is a nearly parallel* G_2*-structure, then*

$$Q_\nu(\Omega) = -\tfrac{5}{42}\nu_0\tau_0^2(\Omega)\Omega \tag{22}$$

for all $\nu = (\nu_0, \nu_1, \nu_2, \nu_3) \in \mathbb{R}^4_+$*. In particular,* Ω *is a* G_2*-soliton.*

Proof First we note that $D_\Omega\mathcal{D}_k(\dot\Omega) = \int_M \dot\tau_{k,\Omega} \wedge \star_\Omega\tau_k(\Omega) + \frac{1}{2}\int_M \tau_k(\Omega) \wedge \dot\star_\Omega\tau_k(\Omega)$. But for a nearly parallel G_2-form Ω we have $\tau_k = 0, k \neq 0$, so that $\mathrm{grad}\,\mathcal{D}_k(\Omega) = 0$ and in particular $Q_\nu(Q) = -\nu_0\mathrm{grad}\,\mathcal{D}_0(\Omega)$. We contend that for general $\Omega \in \Omega^3_+(M)$,

$$\mathrm{grad}\,\mathcal{D}_0(\Omega) = -\tfrac{1}{6}\tau_0^2\Omega + \tfrac{2}{7}\tau_0 \star_\Omega \mathrm{d}\Omega + \tfrac{1}{7} \star_\Omega (\mathrm{d}\tau_0 \wedge \Omega). \tag{23}$$

If this is true, then $\mathrm{grad}\,\mathcal{D}_0(\Omega) = \frac{5}{42}c^2\Omega$ for nearly parallel Ω, whence the result. It remains to show (23). We first determine $\dot\star_\Omega$, the derivative of the map $\Lambda^3_+ \to \mathrm{Hom}(\Lambda^0, \Lambda^7)$ which sends Ω to $\star_\Omega$. As this is a pointwise computation, we can write $\dot\Omega = \dot A^*\Omega$, where $\dot A = \dot A_0$ for a smooth curve $A_t \subset \mathrm{GL}(7)$ with $A_0 = \mathrm{Id}$. Then,

$$\dot\star_\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\star_{A_t^*\Omega} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}A_t^* \star_\Omega A_t^{-1*} = \dot A^*\star_\Omega,$$

for GL(7) acts trivially on 0-forms. In general, if $v, w \in \Lambda^1$, the action is given by $(v \otimes w)^* \alpha^p = v \wedge (w \llcorner \alpha^p)$ for $\alpha^p \in \Lambda^p$. Using the standard formulæ $\star_\Omega(v \llcorner \alpha^p) = (-1)^{p+1} v \wedge \star_\Omega \alpha^p$ and $\star_\Omega(v \wedge \alpha^p) = (-1)^p v \llcorner \star_\Omega \alpha^p$ we get

$$\dot{A}^* \star_\Omega = \mathrm{Tr}(\dot{A}) \star_\Omega - \star_\Omega (\dot{A}^t)^* = \mathrm{Tr}(\dot{A}) \star_\Omega .$$

On the other hand, we have $\dot{A}^*\Omega = \dot{A}_1^*\Omega + \dot{A}_7^*\Omega + \dot{A}_{27}^*\Omega$ where we used the decomposition of $\dot{A} \in \Lambda^1 \otimes \Lambda^1$ given by (4). Since $\dot{A}_1 = \frac{3}{7}\mathrm{Tr}(\dot{A})\,\mathrm{id}$, we have

$$\dot{A}_1^*\Omega = \tfrac{3}{7}\mathrm{Tr}(\dot{A})\Omega. \tag{24}$$

Hence,

$$\dot{\star}_\Omega \tau_0 = \tau_0 \mathrm{Tr}(\dot{A}) \star_\Omega 1 = \tfrac{1}{7}\tau_0 \mathrm{Tr}(\dot{A})\Omega \wedge \star_\Omega \Omega = \tfrac{1}{3}\tau_0 \dot{\Omega} \wedge \star_\Omega \Omega.$$

To compute the linearisation of $\tau_0(\Omega) = \star_\Omega(\mathrm{d}\Omega \wedge \Omega)/7$ we note that $\star_\Omega^2 = \mathrm{id}$ implies $\star_\Omega \dot{\star}_\Omega = -\dot{\star}_\Omega \star_\Omega$, whence,

$$\begin{aligned}\dot{\tau}_{0,\Omega} &= \dot{\star}_\Omega\big(\star_\Omega \tau_0(\Omega)\big) + \tfrac{1}{7} \star_\Omega (\mathrm{d}\dot{\Omega} \wedge \Omega + \dot{\Omega} \wedge \mathrm{d}\Omega)\\ &= -\tfrac{1}{3}\tau_0(\Omega) \star_\Omega (\dot{\Omega} \wedge \star_\Omega \Omega) + \tfrac{1}{7} \star_\Omega (\mathrm{d}\dot{\Omega} \wedge \Omega + \dot{\Omega} \wedge \mathrm{d}\Omega).\end{aligned}$$

From

$$\langle \mathrm{grad}\, \mathcal{D}_0(\Omega), \dot{\Omega}\rangle_\Omega = \int_M \tau_0 \dot{\tau}_{0\Omega}\, \mathrm{vol}_\Omega + \tfrac{1}{6} \int_M \tau_0^2 \dot{\Omega} \wedge \star_\Omega \Omega$$

Equation (23) easily follows. □

Remark The factor appearing in the soliton equation (22) can also be computed using the homogeneity of $\mathcal{D}_\nu$: If $\mathrm{d}\Omega = \tau_0 \star_\Omega \Omega$, then by Euler's rule

$$\langle Q_\nu(\Omega), \Omega\rangle_\Omega = -D_\Omega \mathcal{D}_\nu(\Omega) = -\tfrac{5}{3}\mathcal{D}_\nu(\Omega) = -\tfrac{5}{42}\nu_0 \tau_0^2 \langle \Omega, \Omega\rangle_\Omega.$$

In particular, it follows that

$$\tau_0^2(\Omega) = \frac{2}{\nu_0} \cdot \frac{\mathcal{D}(\Omega)}{\mathcal{H}(\Omega)}. \tag{25}$$

Corollary 4.2 *Let $\Omega \in \Omega_+^3(M)$ be torsion-free. Then there exists a neighbourhood of Ω in $\Omega_+^3(M)$ with respect to the C^∞-topology which does not contain any shrinking $\mathcal{D}_\nu$-solitons, and in particular no nearly parallel* G_2*-structures.*

Proof Choose a neighbourhood $\mathcal{U} \subset \Omega_+^3(M)$ such that for any initial condition $\Omega_0 \in \mathcal{U}$ the conclusion of Theorem 1.2 holds. Now if Ω_0 were a shrinking $\mathcal{D}_\nu$-soliton, then $T_{\max} < \infty$ according to Proposition 3.9, which is impossible. □

Remark The previous corollary should be compared with Theorem 1.2 in [6] which asserts that a Ricci-flat metric which admits nonzero parallel spinors (as it is the case for g_Ω with Ω torsion-free) cannot be smoothly deformed into a metric of positive scalar curvature.

5 Soliton deformations

Let $\bar{\Omega} \in \Omega^3_+(M)$ be a fixed nearly parallel G_2-structure, i.e. $d\bar{\Omega} = \bar{\tau}_0 \star_{\bar{\Omega}} \bar{\Omega}$ for some constant $\bar{\tau}_0 \neq 0$. In this final section, we linearise the G_2-soliton equation

$$S_{\bar{\Omega}}(\Omega) := Q(\Omega) + \tfrac{5}{6}\bar{\tau}_0^2 \Omega = 0 \tag{26}$$

at $\bar{\Omega}$ and study the premoduli space of G_2-soliton deformations.

5.1 The linearised soliton equation

In order to linearise the G_2-soliton equation, we need a lemma first. Recall the map

$$\Theta : \Omega^3_+(M) \to \Omega^4(M), \quad \Omega \mapsto \star_\Omega \Omega$$

from Convention (ii) in Sect. 1. Its linearisation at Ω is given by $\dot{\Theta}_\Omega = \star_\Omega p_\Omega(\dot{\Omega})$ where $p_\Omega(\dot{\Omega}) := \frac{4}{3}[\dot{\Omega}]_1 + [\dot{\Omega}]_7 - [\dot{\Omega}]_{27}$.

Lemma 5.1 *Let $\Omega \in \Omega^3_+(M)$. For $x \in M$, let $\Omega_t = A_t^*\Omega_x$ for a curve $A_t \subset \mathrm{GL}(7)$ such that $A_0 = \mathrm{Id}_{T_xM}$. If we define $s_\Omega(\dot{\Omega}) := [\dot{\Omega}]_1 - [\dot{\Omega}]_7 + [\dot{\Omega}]_{27}$, then for the second derivative $\ddot{\Theta}_\Omega := \frac{d^2}{dt^2}\big|_{t=0}\Theta(\Omega_t)$ at x we find*

$$\begin{aligned}\ddot{\Theta}_\Omega =& \frac{1}{3} g(\Omega, \dot{\Omega}) \star_\Omega (p_\Omega - s_\Omega)\dot{\Omega} + 2 \star_\Omega (\dot{A}^t)^{*2}\Omega - \star_\Omega s_\Omega \ddot{\Omega} \\ &+ \frac{1}{3}\big(g(\ddot{\Omega}, \Omega) - g(s_\Omega \dot{\Omega}, \dot{\Omega})\big) \star_\Omega \Omega.\end{aligned}$$

In particular, we have

$$\ddot{\Theta}_\Omega = \frac{1}{3} g(\Omega, \dot{\Omega}) \star_\Omega (p_\Omega - s_\Omega)\dot{\Omega} + 2 \star_\Omega (\dot{A}^t)^{*2}\Omega - \frac{1}{3} g(s_\Omega \dot{\Omega}, \dot{\Omega}) \star_\Omega \Omega.$$

for $\ddot{\Omega} = 0$.

Proof Writing $A_t = A_t A_{t_0}^{-1} A_{t_0}$ we get

$$\frac{d}{dt}\Big|_{t=t_0} A_t^* \Theta(\Omega) = A_{t_0}^* \frac{d}{dt}\Big|_{t=t_0} \left(A_t A_{t_0}^{-1}\right)^* \Theta(\Omega) = A_{t_0}^* \left(\dot{A}_{t_0} A_{t_0}^{-1}\right)^* \Theta(\Omega)$$

and hence,

$$\frac{d^2}{dt^2}\Big|_{t=t_0} \Theta(\Omega_t) = \big((\dot{A}^*)^2 + \ddot{A}^* - (\dot{A}^2)^*\big)\Theta(\Omega).$$

In the same way, we obtain

$$\ddot{\Omega} = ((\dot{A}^*)^2 + \ddot{A}^* - (\dot{A}^2)^*)\Omega. \tag{27}$$

Now

$$\begin{aligned}(\dot{A}^*)^2\Theta(\Omega) &= \dot{A}^*(\dot{A}^* \star_\Omega \Omega) \\ &= \dot{A}^*(\mathrm{Tr}\,\dot{A} \star_\Omega \Omega - \star_\Omega (\dot{A}^t)^*\Omega) \\ &= \mathrm{Tr}\,\dot{A}\big(\mathrm{Tr}\,\dot{A} \star_\Omega \Omega - \star_\Omega(\dot{A}^t)^*\Omega)\big) - \mathrm{Tr}\,\dot{A} \star_\Omega (\dot{A}^t)^*\Omega + \star_\Omega(\dot{A}^t)^{*2}\Omega \\ &= \mathrm{Tr}\,\dot{A} \star_\Omega (p_\Omega - s_\Omega)\dot{\Omega} + \star_\Omega(\dot{A}^t)^{*2}\Omega,\end{aligned}$$

where we have used $\operatorname{Tr} \dot{A} \star_\Omega \Omega - \star_\Omega(\dot{A}^t)^*\Omega = \dot{\Theta}_\Omega$ and $(\dot{A}^t)^*\Omega = s_\Omega\dot{\Omega}$. Similarly,

$$\ddot{A}^*\Theta(\Omega) = \ddot{A}^* \star_\Omega \Omega = \operatorname{Tr} \ddot{A} \star_\Omega \Omega - \star_\Omega(\ddot{A}^t)^*\Omega$$

and

$$-(\dot{A}^2)^*\Theta(\Omega) = -\operatorname{Tr} \dot{A}^2 \star_\Omega \Omega + \star_\Omega(\dot{A}^2)^{t*}\Omega.$$

Finally, using (27)

$$\begin{aligned}&\big((\dot{A}^*)^2 + \ddot{A}^* - (\dot{A}^2)^*\big)\Theta(\Omega)\\ &= \operatorname{Tr} \dot{A} \star_\Omega (p_\Omega - s_\Omega)\dot{\Omega} + (\operatorname{Tr} \ddot{A} - \operatorname{Tr} \dot{A}^2) \star_\Omega \Omega + 2 \star_\Omega ((\dot{A}^t)^*)^2\Omega - \star_\Omega s_\Omega \ddot{\Omega}.\end{aligned}$$

Next we need to compute the expression $\operatorname{Tr}(\ddot{A} - \dot{A}^2)$. By (24) $\operatorname{Tr} \dot{A} = \frac{1}{3}g(\Omega, \dot{\Omega})$ and similarly $\operatorname{Tr} \ddot{A} = \frac{1}{3}g(\Omega, \ddot{A}^*\Omega)$. Write $(\ddot{A} - \dot{A}^2)^*\Omega = \ddot{\Omega} - (\dot{A}^*)^2\Omega$. Then

$$\operatorname{Tr}(\ddot{A} - \dot{A}^2) = \tfrac{1}{3}g(\Omega, (\ddot{A} - \dot{A}^2)^*\Omega) = \tfrac{1}{3}g(\Omega, \ddot{\Omega} - (\dot{A}^*)^2\Omega).$$

Furthermore,

$$\begin{aligned}[\dot{A}_1^*\dot{\Omega}]_1 &= \tfrac{1}{7}g(\dot{A}_1^*\dot{\Omega}, \Omega)\Omega = \tfrac{1}{7}g(\dot{\Omega}, \dot{A}_1^*\Omega)\Omega = \tfrac{1}{7}|[\dot{\Omega}]_1|^2\Omega\\ [\dot{A}_7^*\dot{\Omega}]_1 &= \tfrac{1}{7}g(\dot{A}_7^*\dot{\Omega}, \Omega)\Omega = -\tfrac{1}{7}g(\dot{\Omega}, \dot{A}_7^*\Omega)\Omega = -\tfrac{1}{7}|[\dot{\Omega}]_7|^2\Omega\\ [\dot{A}_{27}^*\dot{\Omega}]_1 &= \tfrac{1}{7}g(\dot{A}_{27}^*\dot{\Omega}, \Omega)\Omega = \tfrac{1}{7}g(\dot{\Omega}, \dot{A}_{27}^*\Omega)\Omega = \tfrac{1}{7}|[\dot{\Omega}]_{27}|^2\Omega.\end{aligned}$$

Hence,

$$\begin{aligned}[\dot{A}^*\dot{A}^*\Omega]_1 &= [\dot{A}^*\dot{\Omega}]_1\\ &= [\dot{A}_1^*\dot{\Omega}]_1 + [\dot{A}_7^*\dot{\Omega}]_1 + [\dot{A}_{27}^*\dot{\Omega}]_1\\ &= \tfrac{1}{7}\left(|[\dot{\Omega}]_1|^2 - |[\dot{\Omega}]_7|^2 + |[\dot{\Omega}]_{27}|^2\right)\Omega\\ &= \tfrac{1}{7}g(s_\Omega\dot{\Omega}, \dot{\Omega})\Omega\end{aligned}$$

and in turn

$$\operatorname{Tr}(\ddot{A} - \dot{A}^2) = \tfrac{1}{3}g(\Omega, \ddot{\Omega}) - \tfrac{1}{3}g(s_\Omega\dot{\Omega}, \dot{\Omega}),$$

which yields the assertion. □

Proposition 5.2 *Let* $\Omega \in \Omega^3_+(M)$ *be a nearly parallel* G_2*-structure and define* $r_\Omega(\dot{\Omega}) := (\mathrm{id} - p_\Omega)(\dot{\Omega})$. *Then*

$$\begin{aligned}D_\Omega Q(\dot{\Omega}) &= -\delta_\Omega d\dot{\Omega} - p_\Omega d\delta_\Omega p_\Omega\dot{\Omega} - \tau_0(\star_\Omega d r_\Omega + r_\Omega \star_\Omega d)\dot{\Omega}\\ &\quad + \tau_0^2\left(\tfrac{1}{18}[\dot{\Omega}]_1 + \tfrac{1}{6}[\dot{\Omega}]_7 - \tfrac{23}{6}[\dot{\Omega}]_{27}\right)\\ &= -p_\Omega d(p_\Omega d)^*\dot{\Omega} - (\star_\Omega d + \tau_0 r_\Omega)^2\dot{\Omega} + \tfrac{1}{6}\tau_0^2\dot{\Omega}\end{aligned}$$

for $\tau_0 = \tau_0(\Omega)$ *and* $\dot{\Omega} \in \Omega^3(M)$.

Proof We compute the linearisation by starting from Eqs. (6) and (7). First,

$$\begin{aligned}D_\Omega(\Omega \mapsto \delta_\Omega \mathrm{d}\Omega)(\dot{\Omega}) &= \dot{\star}_\Omega \mathrm{d} \star_\Omega \mathrm{d}\Omega + \star_\Omega \mathrm{d}\dot{\star}_\Omega \mathrm{d}\Omega + \star_\Omega \mathrm{d} \star_\Omega \mathrm{d}\dot{\Omega}\\ &= \tau_0^2\dot{\star}_\Omega \star_\Omega \Omega + \tau_0 \star_\Omega \mathrm{d}\dot{\star}_\Omega \star_\Omega \Omega + \star_\Omega \mathrm{d} \star_\Omega \mathrm{d}\dot{\Omega}\\ &= \tau_0^2 r_\Omega\dot{\Omega} + \tau_0 \star_\Omega \mathrm{d} r_\Omega\dot{\Omega} + \delta_\Omega \mathrm{d}\dot{\Omega}.\end{aligned}$$

Second,

$$D_\Omega(\Omega \mapsto p_\Omega \mathrm{d}\delta_\Omega\Omega)(\dot\Omega) = -\dot p_\Omega(\mathrm{d}\star_\Omega \mathrm{d}\star_\Omega\Omega) - p_\Omega(\mathrm{d}\dot\star_\Omega \mathrm{d}\star_\Omega\Omega) - p_\Omega(\mathrm{d}\star_\Omega \mathrm{d}\dot\Theta_\Omega)$$
$$= p_\Omega \mathrm{d}\delta_\Omega p_\Omega\dot\Omega.$$

Third we note that $q_\Omega(\nabla^\Omega) = q_\Omega(\mathrm{d}\Omega) + q_\Omega(\delta_\Omega\Omega)$, where $q_\Omega(\mathrm{d}\Omega)$ and $q_\Omega(\delta_\Omega\Omega)$ are determined by the identities

$$q_\Omega(\mathrm{d}\Omega) \wedge \star_\Omega\Omega' = \tfrac{1}{2}(\star'_\Omega \mathrm{d}\Omega) \wedge \mathrm{d}\Omega \tag{28}$$

and

$$q_\Omega(\delta_\Omega\Omega) \wedge \star_\Omega\Omega' = \tfrac{1}{2}(\star'_\Omega \mathrm{d}\star_\Omega\Omega) \wedge \mathrm{d}\star_\Omega\Omega \tag{29}$$

(with $\star'_\Omega = D_\Omega(\Omega \mapsto \star_\Omega)(\Omega')$) valid for all $\Omega' \in \Omega^3(M)$. It follows that $q_\Omega(\mathrm{d}\Omega) = -\frac{1}{6}\tau_0^2\Omega$. Indeed, the left hand side of (28) is twice $\tau_0^2\star'_\Omega\Theta(\Omega) \wedge \Theta(\Omega)$. Now $\Omega = \star_\Omega\Theta(\Omega)$ so that $\Omega' = \star'_\Omega\Theta(\Omega) + \star_\Omega\Theta'_\Omega$. Hence, $[\star'_\Omega\Theta(\Omega)]_1 = -[\Omega']_1/3$ which is the only component which survives wedging by $\Theta(\Omega)$. Differentiating Eq. (28) therefore implies

$$\begin{aligned}&D_\Omega(\Omega \mapsto q_\Omega(\mathrm{d}\Omega))(\dot\Omega) \wedge \star_\Omega\Omega'\\ &\quad= \tfrac{1}{2}(D^2_\Omega\star)(\dot\Omega, \Omega')\mathrm{d}\Omega \wedge \mathrm{d}\Omega + \star'_\Omega \mathrm{d}\Omega \wedge \mathrm{d}\dot\Omega - q_\Omega(\mathrm{d}\Omega) \wedge \dot\star_\Omega\Omega'\\ &\quad= \tfrac{1}{2}\tau_0^2(D^2_\Omega\star)(\dot\Omega, \Omega')\star_\Omega\Omega \wedge \star_\Omega\Omega + \tau_0\star'_\Omega\star_\Omega\Omega \wedge \mathrm{d}\dot\Omega - \tfrac{1}{6}\tau_0^2 r_\Omega\dot\Omega \wedge \star_\Omega\Omega'.\end{aligned}$$

On the other hand, differentiating the equation $\Omega = \star_\Omega\Theta(\Omega)$ gives $\ddot\Omega = \ddot\star_\Omega\Theta(\Omega) + 2\dot\star_\Omega\dot\Theta_\Omega + \star_\Omega\ddot\Theta_\Omega$. Without loss of generality we may assume that $\Omega_t = (1+t)\Omega$, so in particular $\ddot\Omega = 0$ and hence, $\ddot\star_\Omega\Theta_\Omega = -2\dot\star_\Omega - \star_\Omega\ddot\Theta_\Omega$. From Lemma 5.1, we deduce

$$\begin{aligned}&\tfrac{1}{2}\tau_0^2(D^2_\Omega\star)(\dot\Omega, \Omega')\star_\Omega\Omega \wedge \star_\Omega\Omega\\ &\quad= \tau_0^2\big(-\dot\star_\Omega\dot\Theta_\Omega - \tfrac{1}{6}g_\Omega(\Omega, \dot\Omega)(p_\Omega - s_\Omega)\dot\Omega - \big((\dot A^t)^{*2}\Omega + \tfrac{1}{6}g_\Omega(s_\Omega\dot\Omega, \dot\Omega)\Omega\big) \wedge \star_\Omega\Omega'\big).\end{aligned}$$

Furthermore, the identities

$$\begin{aligned}&-((\dot A^t)^*)^2\Omega \wedge \star_\Omega\Omega = -(\dot A^t)^*\Omega \wedge \star_\Omega\dot A^*\Omega = -s_\Omega\dot\Omega \wedge \star_\Omega\dot\Omega\\ &-\dot\star_\Omega\dot\Theta_\Omega \wedge \star_\Omega\Omega = -r_\Omega p_\Omega\dot\Omega \wedge \star_\Omega\dot\Omega\\ &-\tfrac{1}{6}g_\Omega(\Omega, \dot\Omega)(p_\Omega - s_\Omega)\dot\Omega \wedge \star_\Omega\Omega = -\tfrac{7}{18}[\dot\Omega]_1 \wedge \star_\Omega\dot\Omega\\ &\tfrac{1}{6}g_\Omega(s_\Omega\dot\Omega, \dot\Omega)\Omega \wedge \star_\Omega\Omega = \tfrac{7}{6}s_\Omega\dot\Omega \wedge \star_\Omega\dot\Omega\end{aligned}$$

imply

$$\tfrac{1}{2}\tau_0^2(D^2_\Omega\star)(\dot\Omega, \Omega')\star_\Omega\Omega \wedge \star_\Omega\Omega = \tau_0^2\left(\tfrac{2}{9}[\dot\Omega]_1 - \tfrac{1}{6}[\dot\Omega]_7 + \tfrac{13}{6}[\dot\Omega]_{27}\right) \wedge \star_\Omega\Omega'.$$

Hence, using

$$\tau_0\star'_\Omega\star_\Omega \wedge \mathrm{d}\dot\Omega = -\tau_0\star_\Omega\star'_\Omega\Omega \wedge \mathrm{d}\dot\Omega = \tau_0 r_\Omega\Omega' \wedge \mathrm{d}\dot\Omega = \tau_0 r_\Omega\star_\Omega \mathrm{d}\dot\Omega \wedge \star_\Omega\Omega'$$

we arrive at

$$\begin{aligned}&D_\Omega(\Omega \mapsto q_\Omega(\mathrm{d}\Omega))(\dot\Omega) \wedge \star_\Omega\Omega'\\ &\quad= \tau_0^2\left(\tfrac{2}{9}[\dot\Omega]_1 - \tfrac{1}{6}[\dot\Omega]_7 + \tfrac{13}{6}[\dot\Omega]_{27}\right) \wedge \star_\Omega\Omega' + \tau_0 r_\Omega\star_\Omega \mathrm{d}\dot\Omega \wedge \star_\Omega\Omega' - \tfrac{1}{6}\tau_0^2 r_\Omega\dot\Omega \wedge \star_\Omega\Omega'\\ &\quad= \tau_0^2\left(\tfrac{5}{18}[\dot\Omega]_1 - \tfrac{1}{6}[\dot\Omega]_7 + \tfrac{11}{6}[\dot\Omega]_{27}\right) \wedge \star_\Omega\Omega' + \tau_0 r_\Omega\star_\Omega \mathrm{d}\dot\Omega \wedge \star_\Omega\Omega'.\end{aligned}$$

Similarly, differentiating Eq. (29) we get

$$\begin{aligned} &D_\Omega(\Omega \mapsto q_\Omega(\delta_\Omega\Omega))(\dot\Omega) \wedge \star_\Omega\Omega' \\ &\quad = \tfrac{1}{2}(D^2_\Omega\star)(\dot\Omega, \Omega')\mathrm{d}\Theta(\Omega) \wedge \mathrm{d}\Theta(\Omega) + \star'_\Omega\mathrm{d}\Theta(\Omega) \wedge \mathrm{d}\dot\Theta_\Omega - q_\Omega(\delta_\Omega\Omega) \wedge \dot\star_\Omega\Omega' \\ &\quad = 0, \end{aligned}$$

for $\mathrm{d}\Theta(\Omega) = q_\Omega(\delta_\Omega\Omega) = 0$. Hence,

$$\begin{aligned} D_\Omega(\Omega \mapsto q_\Omega(\nabla^\Omega))(\dot\Omega) &= D_\Omega(\Omega \mapsto q_\Omega(\mathrm{d}\Omega))(\dot\Omega) \\ &= \tau_0 r_\Omega \star_\Omega \mathrm{d}\dot\Omega + \tau_0^2\left(\tfrac{5}{18}[\dot\Omega]_1 - \tfrac{1}{6}[\dot\Omega]_7 + \tfrac{11}{6}[\dot\Omega]_{27}\right). \end{aligned}$$

Summing up we obtain

$$\begin{aligned} (D_\Omega Q)(\dot\Omega) =& -\delta_\Omega\mathrm{d}\dot\Omega - p_\Omega\mathrm{d}\delta_\Omega p_\Omega\dot\Omega - \tau_0 \star_\Omega \mathrm{d}r_\Omega\dot\Omega - \tau_0^2 r_\Omega\dot\Omega \\ &- \tau_0 r_\Omega \star_\Omega \mathrm{d}\dot\Omega - \tau_0^2\left(\tfrac{5}{18}[\dot\Omega]_1 - \tfrac{1}{6}[\dot\Omega]_7 + \tfrac{11}{6}[\dot\Omega]_{27}\right) \\ =& -\delta_\Omega\mathrm{d}\dot\Omega - p_\Omega\mathrm{d}\delta_\Omega p_\Omega\dot\Omega - \tau_0(\star_\Omega\mathrm{d}r_\Omega + r_\Omega\star_\Omega \mathrm{d})\dot\Omega \\ &+ \tau_0^2\left(\tfrac{1}{18}[\dot\Omega]_1 + \tfrac{1}{6}[\dot\Omega]_7 - \tfrac{23}{6}[\dot\Omega]_{27}\right), \end{aligned}$$

which is the desired result. □

Remark In particular, we see that $D_\Omega Q(\Omega) = -\frac{5}{18}\tau_0^2\Omega$ which, of course, follows directly from differentiating $Q((1+t)\Omega) = (1+t)^{1/3}Q(\Omega)$ at $t = 0$ (cf. Lemma 3.2).

As a corollary to Proposition 5.2, we immediately obtain the linearisation of the operator $S_{\bar\Omega}$ at $\bar\Omega$:

Corollary 5.3 *Let $\bar\Omega \in \Omega^3_+(M)$ be a nearly parallel* G_2*-structure. Then*

$$D_{\bar\Omega}S_{\bar\Omega}(\dot\Omega) = -p_{\bar\Omega}\mathrm{d}(p_{\bar\Omega}\mathrm{d})^*\dot\Omega - (\star_{\bar\Omega}\mathrm{d} + \bar\tau_0 r_{\bar\Omega})^2\dot\Omega + \bar\tau_0^2\dot\Omega$$

for $\bar\tau_0 = \tau_0(\bar\Omega)$ and $\dot\Omega \in \Omega^3(M)$.

5.2 The premoduli space

As above, let $\bar\Omega \in \Omega^3_+(M)$ be a fixed nearly parallel G_2-structure on M. We wish to study the space of G_2-soliton deformations of $\bar\Omega$, i.e. solutions $\Omega \in \Omega^3_+(M)$ to the soliton equation (26) close to $\bar\Omega$ modulo the action of diffeomorphisms. Towards that end, we first investigate the linear equation $D_{\bar\Omega}S_{\bar\Omega}(\dot\Omega) = 0$. As this parallels the corresponding theory for the Einstein premoduli space as developed by Koiso, we follow [2,3] and only sketch the main points. Recall the L^2-orthogonal decomposition

$$\Omega^3(M) = \mathrm{im}\,\lambda^*_{\bar\Omega} \oplus \ker\lambda_{\bar\Omega}.$$

given in (16). By Ebin's slice theorem [8], $\ker\lambda_{\bar\Omega} = T_{\bar\Omega}\mathcal{S}_{\bar\Omega}$ integrates to a slice $\mathcal{S}_{\bar\Omega}$ for the *Diff*$_0(M)$-action. Hence, the space $\sigma(\bar\Omega)$ of *infinitesimal soliton deformations* of $\bar\Omega$ consists of $\dot\Omega \in \Omega^3(M)$ satisfying the equations

$$D_{\bar\Omega}S_{\bar\Omega}(\dot\Omega) = 0 \quad \text{and} \quad \lambda_{\bar\Omega}(\dot\Omega) = 0.$$

The *premoduli space* $\mathcal{M}(\bar\Omega)$ of G_2-soliton deformations at $\bar\Omega$ is the set of G_2-solitons in the slice $\mathcal{S}_{\bar\Omega}$ near $\bar\Omega$. To investigate the structure of $\sigma(\bar\Omega)$ and $\mathcal{M}(\bar\Omega)$ further we introduce the linear operator

$$P_{\bar\Omega} : \Omega^3(M) \to \Omega^3(M),\;\; P_{\bar\Omega}(\dot\Omega) := D_{\bar\Omega}S_{\bar\Omega}(\dot\Omega) - \lambda^*_{\bar\Omega}\lambda_{\bar\Omega}(\dot\Omega),$$

which is clearly symmetric.

Lemma 5.4 *The operator $P_{\bar\Omega}$ is elliptic.*

Proof The operator $P_{\bar\Omega}$ differs from the linearisation of the Dirichlet–DeTurck operator only in the lower order terms, cf. in particular Eq. (32) in [17]. Hence, it has the same symbol and the claim follows from Lemma 5.7 in [17]. □

Since any infinitesimal soliton deformation of $\bar\Omega$ lies in the kernel of $P_{\bar\Omega}$, we immediately conclude from ellipticity:

Corollary 5.5 *The space $\sigma(\bar\Omega)$ is finite dimensional.*

To discuss the structure of the premoduli space we first prove the following

Lemma 5.6 *The restricted linear operator $D_{\bar\Omega}S_{\bar\Omega} : T_{\bar\Omega}\mathcal{S}_{\bar\Omega} \to \Omega^3(M)$ has closed[2] image.*

Proof Clearly, $P_{\bar\Omega}(T_{\bar\Omega}\mathcal{S}_{\bar\Omega}) = D_{\bar\Omega}S_{\bar\Omega}(T_{\bar\Omega}\mathcal{S}_{\bar\Omega})$. As an elliptic operator, $P_{\bar\Omega}$ has closed image. Furthermore, $\lambda_{\bar\Omega} \circ P_{\bar\Omega} = \lambda_{\bar\Omega}\lambda^*_{\bar\Omega} \circ \lambda_{\bar\Omega}$ and thus

$$P_{\bar\Omega}(T_{\bar\Omega}\mathcal{S}_{\bar\Omega}) \subset P_{\bar\Omega}(\Omega^3(M)) \cap \ker\lambda_{\bar\Omega} \subset P_{\bar\Omega}\left(\lambda^{-1}_{\bar\Omega}\left(\ker\lambda_{\bar\Omega}\lambda^*_{\bar\Omega}\right)\right).$$

Now $L_{\bar\Omega} := \lambda_{\bar\Omega}\lambda^*_{\bar\Omega}$ is elliptic. Indeed, for the principal symbol applied to a covector $\xi \in T^*_xM$ we find that $\sigma(L_{\bar\Omega})(x,\xi)v = i(v\otimes\xi)^*\bar\Omega$. This is injective, for $(v\otimes\xi)^*\bar\Omega = 0$ implies $v\otimes\xi \in \Lambda^2 \subset \Lambda^1\otimes\Lambda^1$ on representation theoretic grounds, that is, $v\otimes\xi$ is skew. But this is impossible for a decomposable endomorphism unless $v = 0$. Hence, $g_{\bar\Omega}(\sigma(L_{\bar\Omega})(x,\xi)v, v) = -|\sigma(\lambda^*_{\bar\Omega})(x,\xi)v|^2_{\bar\Omega}$ is negative-definite. Consequently, $\ker L_{\bar\Omega}$ is finite-dimensional and so $T_{\bar\Omega}\mathcal{S}_{\bar\Omega}$ is of finite codimension in $\lambda^{-1}_{\bar\Omega}(\ker\lambda_{\bar\Omega}\lambda^*_{\bar\Omega})$. Since $T_{\bar\Omega}\mathcal{S}_{\bar\Omega}$ is also closed, $P_{\bar\Omega}(T_{\bar\Omega}\mathcal{S}_{\bar\Omega})$ is closed in $P_{\bar\Omega}\big(\lambda^{-1}_{\bar\Omega}(\ker\lambda_{\bar\Omega}\lambda^*_{\bar\Omega})\big)$. As a result, $P_{\bar\Omega}(T_{\bar\Omega}\mathcal{S}_{\bar\Omega})$ is closed in $P_{\bar\Omega}(\Omega^3(M)) \cap \ker\lambda_{\bar\Omega}$ and thus in $\Omega^3(M)$. □

Let $p : \Omega^3(M) \to D_{\bar\Omega}S_{\bar\Omega}(T_{\bar\Omega}\mathcal{S}_{\bar\Omega})$ be the orthogonal projection. By the previous lemma, $p \circ S_{\bar\Omega} : \mathcal{S}_{\bar\Omega} \to D_{\bar\Omega}S_{\bar\Omega}(T_{\bar\Omega}\mathcal{S})$ is a submersion at $\bar\Omega$. It is also a real analytic map, since $g_{\bar\Omega}$ is Einstein (hence, real analytic in harmonic coordinates, cf. [7]) and $\Delta_{g_{\bar\Omega}}\bar\Omega = \bar\tau_0^2\bar\Omega$ (so that $\bar\Omega$ is real analytic as a solution of an elliptic PDE with real analytic coefficients). As a consequence, $Z := p\circ S^{-1}_{\bar\Omega}(0)$ is a real analytic submanifold with tangent space $\ker D_{\bar\Omega}S_{\bar\Omega} \cap T_{\bar\Omega}\mathcal{S}_{\bar\Omega} = \sigma(\bar\Omega)$. Restricted to Z, $S_{\bar\Omega}$ is also real analytic so that $(S_{\bar\Omega}|_Z)^{-1}(0)$, the premoduli space of solitons, is a real analytic subset. We thus arrive at the following conclusion (compare with Koiso's work [14] in the Einstein case).

Theorem 5.7 *The slice $\mathcal{S}_{\bar\Omega}$ contains a finite-dimensional real analytic submanifold Z such that Z contains $\mathcal{M}(\bar\Omega)$ as a real analytic subset and $T_{\bar\Omega}Z = \sigma(\bar\Omega)$.*

Example Consider the spaces

$$\sigma_1 = \{\gamma \in \Omega^3_{27}(M) \mid \star_{\bar\Omega}\, d\gamma = -\bar\tau_0\gamma\},$$
$$\sigma_2 = \{\gamma \in \Omega^3_{27}(M) \mid \star_{\bar\Omega}\, d\gamma = -3\bar\tau_0\gamma\},$$
$$\sigma_3 = \{\gamma \in \Omega^3_{27}(M) \mid \star_{\bar\Omega}\, d\gamma = -3\bar\tau_0^2\gamma\}.$$

[2] Here and thereafter, this refers to the natural extension of $D_{\bar\Omega}S_{\bar\Omega}$ to Sobolev- or Hölder-spaces.

Any $\gamma \in \sigma_{1,2}$ is coclosed. Since $d\bar{\Omega} = \tau_0 \star_{\bar{\Omega}} \Omega$, we also have $\gamma \llcorner d\Omega = 0$. Furthermore, any $\gamma \in \sigma_3$ is closed, hence, $[\delta_{\bar{\Omega}}\gamma]_7 = 0$ (see the proofs of Lemma 3.3 and Proposition 5.3 in [1]). Therefore, $\lambda_{\bar{\Omega}}(\gamma) = 0$ in all three cases. It is straightforward to check that $P_{\bar{\Omega}}\gamma = 0$ for $\gamma \in \sigma_{1,2,3}$. By Theorem 6.2 in [1] these spaces correspond to the infinitesimal Einstein deformations of $\bar{\Omega}$. We do not know whether they exhaust all of $\sigma(\bar{\Omega})$.

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