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Smooth distributions are finitely generated

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Abstract A subbundle of variable dimension inside the tangent bundle of a smooth manifold is called a smooth distribution if it is the pointwise span of a family of smooth vector fields. We prove that all such distributions are finitely generated, meaning that the family may be taken to be a finite collection. Further, we show that the space of smooth sections of such distributions need not be finitely generated as a module over the smooth functions. Our results are valid in greater generality, where the tangent bundle may be replaced by an arbitrary vector bundle.

Keywords Distributions · Foliations

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1 Introduction

Let M be a smooth manifold, and let L be a distribution on M, i.e., a subbundle of the tangent bundle TM. The well-known Frobenius theorem states that L determines a foliation of M if and only if L is involutive. Recall that the hypotheses of the Frobenius theorem require that the dimension of the subspace L_p is a constant function of $p \in M$.

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In many fields, for example control theory and Poisson geometry, one encounters *generalized distributions*, where the subspace L_p can have different dimensions at different points. We call a distribution of constant rank a *regular* distribution. Sussmann [7] and Stefan [6] extended the Frobenius theorem to generalized distributions (see Michor [4, Chap. 1, Sect. 3] for a nice exposition). Sussmann and Stefan considered a distribution L to be smooth if for each $p \in M$, there are locally defined vector fields that are sections of L whose values at p span L_p.

On the other hand, in the theory of exterior differential systems one encounters generalized distributions that are defined as the kernels of families of one-forms. We call such a distribution *cosmooth*. The subspaces of the cotangent spaces spanned by these one-forms determine a generalized subbundle of the cotangent bundle. This motivates studying generalized subbundles of arbitrary vector bundles.

If E is a vector bundle over M, a generalized subbundle G of E is an assignment $p \mapsto G_p$ of a subspace G_p of the fiber E_p of E over p, for each point $p \in M$. The interesting cases are where G is smooth or cosmooth. A section s of E is said to be a *section of* G if $s(p) \in G_p$ for all $p \in M$.

The book [1] by Bullo and Lewis has an interesting discussion of generalized distributions and generalized subbundles; their book was an inspiration for this article.

If F is a smooth k-dimensional subbundle of E in the usual sense, then every point has a neighborhood U on which there are k smooth sections $s_1, s_2, \dots, s_k$ whose values form a basis for F_p at every point $p \in U$. Every smooth section of F over U can be written as a combination of the sections $s_1, \dots, s_k$ with smooth coefficients. To put it in other words, the set of smooth sections of F over U is a module over the ring of smooth functions on U; this module is finitely generated with generators $s_1, \dots, s_k$.

In the case of a generalized subbundle G, we can find sections of G on an open set U that form a basis at each point only if the dimension of G is constant on U. We can generalize what happens in the regular case in two directions.

First, following the terminology in Bullo and Lewis [1], we say that a generalized subbundle G of E is *finitely generated* over an open set U if there are smooth sections $s_1, s_2, \dots s_k$ of G over U so that the values $s_1(p), s_2(p), \dots, s_k(p)$ *span* G_p for each $p \in U$. We say that $s_1, \dots, s_k$ are generators for G over U. One can ask if such generators always exist, either locally or globally.

Second, we can consider the set of sections of G over U as a module over the ring of smooth functions on U and ask if there are sections $s_1, \dots, s_k$ as above that form a finite set of generators for this module. If the $s_1, \dots, s_k$ generate the module, Bullo and Lewis [1] call them *nondegenerate* generators for G over U. One can ask if the module of sections is finitely generated, either globally, i.e., when $U = M$, or locally, i.e., for some neighborhood U of each point.

Bullo and Lewis [1] have a discussion that shows that every point of a real analytic generalized subbundle has a neighborhood U on which there are nondegenerate generators. This follows from the fact that the ring of convergent power series is Noetherian.

In this article, we study these questions in the smooth case. For the first question, we show that every generalized subbundle of a vector bundle is *globally finitely generated*, that is, there are finitely many globally defined sections whose values span the generalized subbundle at each point. Other researchers have conjectured that this result is not true, even locally (see, for example, Bullo and Lewis [1, p. 125]).

We obtain a negative answer to the second question. We give an example which shows that the module of sections of a generalized subbundle (even a tangent distribution) need not be finitely generated, even locally.

2 Precise formulations

Let M be a manifold. We will assume that all of our manifolds and maps are smooth.

If V is a vector space, we denote the space of smooth functions $M \to V$ by $C^\infty(M; V)$. In case $V = \mathbb{R}$, we write this space as $C^\infty(M)$.

The space of smooth sections of a vector bundle E over an open set U is denoted $\Gamma(U; E)$, and the space of smooth globally defined sections of E is denoted $\Gamma(E)$. A *local section* of E is a smooth section of E defined on some open set. We denote by $\Gamma_{\text{loc}}(E)$ the set of local sections of E. Thus, $\Gamma_{\text{loc}}(E)$ is the union of the spaces $\Gamma(U; E)$ as U ranges over all open sets. If $s \in \Gamma_{\text{loc}}(E)$, we denote the domain of s by $\text{dom}(s)$.

A *generalized subbundle* G of E is an assignment of a subspace $G_p \subseteq E_p$ for each point $p \in M$. We do not assume the subspaces G_p vary continuously with p or have constant dimension.

If $U \subseteq M$, we say a local section s of E over U *belongs to* G or *is a section of* G if $s(p) \in G_p$ for all $p \in U$. The set of sections of E over U that belong to G is denoted by $\Gamma(U; G)$. The set of local sections of E that belong to G on their domains is denoted $\Gamma_{\text{loc}}(G)$.

Given a generalized subbundle G, there need not be any nonzero smooth sections of G. The following condition on G insures a supply of sections of G.

Definition 2.1 A generalized subbundle G of a vector bundle E is *smooth* if for every point p we can find a family of sections $s_1, \ldots, s_k \in \Gamma_{\text{loc}}(G)$ which contain p in the intersection of their domains such that

$$G_p = \text{span}\{s_1(p), \ldots, s_k(p)\}.$$

The next proposition gives an equivalent definition of smoothness; the elementary proof is omitted.

Proposition 2.2 *A generalized subbundle of G of a vector bundle E is smooth if and only if for every point p and every $v \in G_p$ there is some section $s \in \Gamma_{\text{loc}}(G)$ such that $s(p) = v$.*

Definition 2.3 If $\mathcal{F} \subseteq \Gamma_{\text{loc}}(E)$, we define a generalized subbundle $\text{Span}(\mathcal{F})$ of E by

$$\text{Span}(\mathcal{F})_p = \text{span}\{s(p) : s \in \mathcal{F},\ p \in \text{dom}(s)\}.$$

We follow the convention that if V is a vector space, the span of $\emptyset \subseteq V$ is $\{0\} \subseteq V$. Thus, if p is not in the domain of any element of $\mathcal{F}$, then $\text{Span}(\mathcal{F})_p = \{0\}$.

For any family $\mathcal{F} \subseteq \Gamma_{\text{loc}}(E)$, the subbundle $\text{Span}(\mathcal{F})$ is smooth by definition. Note that a generalized subbundle G is smooth if and only if $G = \text{Span}(\Gamma_{\text{loc}}(G))$.

The following observation will be important. Let G be smooth. For any point p, we can find sections $s_1 \ldots s_k$ of G defined on some open neighborhood U of p such that $s_1(p), \ldots, s_k(p)$ form a basis of G_p. These sections will remain linearly independent on some open neighborhood $V \subseteq U$ of p. Because these sections belong to G, at a point $q \in V$ other than p, the set $\{s_1(q), \ldots, s_k(q)\}$ is a linearly independent set in G_q. Thus, $\dim(G_q) \geq k = \dim(G_p)$. This implies that $\dim(p) = \dim(G_p)$ is a lower semicontinuous function of $p \in M$.

We say that $p \in M$ is a *regular point* of G if dim is constant on a neighborhood of p. The set of regular points is open and dense, as is well known.

Definition 2.4 If $\mathcal{F}$ is a family of local sections of the dual bundle E^*, we define a generalized subbundle $\text{Ker}(\mathcal{F})$ of E by

$$\text{Ker}(\mathcal{F})_p = \{v \in E_p : s(p)(v) = 0 \text{ for all } s \in \mathcal{F} \text{ with } p \in \text{dom}(s)\}.$$

Note that we can always add the globally defined zero section of E^* to $\mathcal{F}$ without changing $\mathrm{Ker}(\mathcal{F})$.

Definition 2.5 A generalized subbundle G of E is *cosmooth* if $G = \mathrm{Ker}(\mathcal{F})$ for some family $\mathcal{F} \subseteq \Gamma_{\mathrm{loc}}(E^*)$.

Definition 2.6 If F is a generalized subbundle of E^*, we define a generalized subbundle $F^\perp$ of E by

$$F_p^\perp = \left\{ v \in E_p : \lambda(v) = 0 \text{ for all } \lambda \in F_p \right\}.$$

Observe that $\mathrm{Ker}(\mathcal{F}) = (\mathrm{Span}(\mathcal{F}))^\perp$.

Theorem 2.7 *A generalized subbundle G of E is cosmooth if and only if $G = F^\perp$ for some smooth generalized subbundle F of E^*.*

Proof Suppose G is cosmooth, so that $G = \mathrm{Ker}(\mathcal{F}) = (\mathrm{Span}(\mathcal{F}))^\perp$ for some family $\mathcal{F}$ of local sections of E^*. Since $\mathrm{Span}(\mathcal{F})$ is by definition a smooth generalized subbundle of E^*, the conclusion follows.

Conversely, suppose that $G = F^\perp$ for some smooth generalized subbundle F of E^*. Let $\mathcal{F} = \Gamma_{\mathrm{loc}}(F)$. We claim that $F^\perp = \mathrm{Ker}(\mathcal{F})$.

Let $v \in F_p^\perp$. If $s \in \mathcal{F}$ and $p \in \mathrm{dom}(\mathcal{F})$, then $s(p) \in F_p$ and $s(p)(v) = 0$. Thus, $v \in \mathrm{Ker}(\mathcal{F})$, and we conclude $F^\perp \subseteq \mathrm{Ker}(\mathcal{F})$.

Next, take $v \in \mathrm{Ker}(\mathcal{F})_p$. Let $\lambda \in F_p$. Since F is smooth, Proposition 2.2 shows there is an $s \in \Gamma_{\mathrm{loc}}(F)$ with $s(p) = \lambda$. Then $\lambda(v) = s(p)(v) = 0$, and thus, $v \in F_p^\perp$. Therefore, $\mathrm{Ker}(\mathcal{F}) \subseteq F^\perp$.

If F is a smooth generalized subbundle of E^*, then the function $p \mapsto \dim(F_p)$ is lower semicontinuous. The dimension of $F_p^\perp$ is $\dim(E_p) - \dim(F_p)$. Hence the function $p \mapsto \dim(F_p^\perp)$ is *upper* semicontinuous. Thus, a cosmooth generalized subbundle need not be smooth.

Example 2.8 Let E and F be vector bundles over M and let $\varphi\colon E \to F$ be a vector bundle map. It is well known that if φ has constant rank, the image and kernel of φ are regular subbundles. If φ does not have constant rank, the image of φ is a smooth generalized subbundle of F and the kernel of φ is a cosmooth generalized subbundle of E.

Definition 2.9 Let E be a vector bundle over a manifold M and let G be a generalized subbundle of E. If $U \subseteq M$ is an open set, we say *G is finitely generated over U* (or, *is finitely spanned over U*) if there are a finite number of sections $s_1, \ldots, s_k \in \Gamma(U; G)$ so that, for all $p \in U$,

$$G_p = \mathrm{span}\left\{s_1(p), \ldots, s_k(p)\right\}.$$

We say that $s_1, \ldots, s_k$ *generate G over U*. If we can take $U = M$, we say that G is *globally finitely generated*.

3 Fréchet spaces of smooth functions

We now introduce the function spaces we need. We denote by $|\cdot|$ the Euclidean norm on any of the spaces $\mathbb{R}^n$.

If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a bounded function, let

$$\|f\| = \sup\left\{|f(x)| : x \in \mathbb{R}^n\right\}$$

be the supremum norm.

We say that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is in $\mathrm{BC}^\infty(\mathbb{R}^n;\mathbb{R}^m)$ if f is C^∞ and f and all of its partial derivatives are bounded on $\mathbb{R}^n$. If f is such a function, we can define seminorms p_k for $k = 0, 1, 2, \ldots$ by

$$p_k(f) = \max\left\{ \left\| \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right\| : |\alpha| = k \right\},$$

where the maximum is taken over all multi-indices α of order k. We combine the p_k's to form seminorms $\|\cdot\|_k$ by

$$\|f\|_k = \sum_{j=0}^{k} p_j(f).$$

Thus, $\|f\|_k \le \|f\|_{k+1}$. Note that these seminorms are actually norms.

These norms $\|\cdot\|_k$ (or, equivalently, the seminorms p_k) induce a topology on $\mathrm{BC}^\infty(\mathbb{R}^n;\mathbb{R}^m)$ which makes $\mathrm{BC}^\infty(\mathbb{R}^n;\mathbb{R}^m)$ into a Fréchet space.

Recall that a sequence $\{f_i\}_{i=1}^\infty$ in $\mathrm{BC}^\infty(\mathbb{R}^n;\mathbb{R}^m)$ is Cauchy if it is Cauchy with respect to the norm $\|\cdot\|_k$ for each k. Therefore, to show that a series $\sum_{i=0}^\infty f_i$ converges in $\mathrm{BC}^\infty(\mathbb{R}^n;\mathbb{R}^m)$, it suffices to show that each of the series

$$\sum_{i=1}^\infty \|f_i\|_k, \qquad k = 0, 1, 2, \ldots,$$

converges. If $\sum_{i=1}^\infty f_i$ converges in $\mathrm{BC}^\infty(\mathbb{R}^n;\mathbb{R}^m)$ then $f := \sum_{i=1}^\infty f_i$ can be evaluated pointwise, because convergence in $\mathrm{BC}^\infty(\mathbb{R}^n;\mathbb{R}^m)$ implies pointwise convergence.

4 The main theorem

Theorem 4.1 *Let M be a connected manifold and let E be a vector bundle over M. Let G be a smooth generalized subbundle of E. Then G is globally finitely generated.*

As we will see, it is possible to give an explicit upper bound on the number of global sections needed to generate G under appropriate assumptions. The remainder of this section contains the proof of this theorem. Before we begin the proof, we note this important corollary.

Corollary 4.2 *Let M and E be as in Theorem 4.1. If G is a cosmooth generalized subbundle of E, then there are finitely many globally defined sections $s_1, \ldots, s_k$ of E^* such that for each $p \in M$,*

$$G_p = \left\{v \in E_p : s_1(p)(v) = 0, \ldots, s_k(p)(v) = 0\right\}.$$

In other words, G is defined as the kernel of a finite collection of global sections of E^*.

We now prove Theorem 4.1. Let M be a connected manifold and let E be a vector bundle over M. The fiber dimension of E will be denoted by $\mathrm{rk}(E)$. Let G be a smooth generalized subbundle of E.

We begin by making two reductions in the problem. First, we invoke the theorem that every vector bundle over a connected manifold is isomorphic to a subbundle of a trivial bundle. Thus, for our problem, we may assume that E is a subbundle of a trivial bundle $\Theta^m(M) = M \times \mathbb{R}^m$ for some integer m. This theorem, without an estimate on m, is well known in the case where M is compact. The proof in the noncompact case, which gives an estimate of m, uses topological dimension theory. A reference for this material is Greub et al. [3, p. 77], but we will need to use a more refined treatment of the dimension theory, as in Munkres [5] or Engelking [2]. The main point for our purposes is that an upper bound for m is $\mathrm{rk}(E)(\dim(M) + 1)$.

Our generalized subbundle $G \subseteq E$ is now contained in $\Theta^m(M)$ and is smooth when considered as a generalized subbundle of $\Theta^m(M)$. Thus, to prove the theorem, it will suffice to consider a smooth generalized subbundle G of a trivial bundle $\Theta^m(M)$. We identify each subspace G_p with a subspace of $\mathbb{R}^m$ and identify sections of $\Theta^m(M)$, and hence sections of G, with $\mathbb{R}^m$-valued functions. We will switch between these points of view as convenient.

Next, we can properly embed M in $\mathbb{R}^n$ for some n. This is not really necessary for our proof to work, but it makes dealing with the functions spaces involved simpler. If $f: U \to \mathbb{R}^m$ is a smooth function defined on an open subset U of M, we can find a smooth extension of f to a function $\tilde{f}: \widetilde{U} \to \mathbb{R}^m$, where $\widetilde{U}$ is an open subset of $\mathbb{R}^n$ such that $\widetilde{U} \cap M = U$. This can be done by a partition of unity argument, but perhaps the fastest proof is to note that the tubular neighborhood theorem says there is an open set $\mathcal{O}$ in $\mathbb{R}^n$ containing M and a smooth retraction $r: \mathcal{O} \to M$ (i.e., $r(p) = p$ for $p \in M$). We define $\widetilde{U} = r^{-1}(U)$ and $\tilde{f} = f \circ r$.

Because G is smooth, it is the span of the family $\mathcal{F} = \Gamma_{\mathrm{loc}}(G)$ of local sections of $\Theta^m(M)$. Considering the elements of $\mathcal{F}$ to be locally defined vector-valued functions on M, we can extend them to locally defined vector-valued functions on $\mathbb{R}^n$. This gives us a family $\widetilde{\mathcal{F}} = \{\tilde{f} : f \in \mathcal{F}\}$ of locally defined vector-valued functions whose restriction to M is $\mathcal{F}$.

Considering $\widetilde{\mathcal{F}}$ as a subset of $\Gamma_{\mathrm{loc}}(\Theta^m(\mathbb{R}^n))$, we define a generalized subbundle $\widetilde{G}$ of $\Theta^m(\mathbb{R}^n)$ by $\widetilde{G} = \mathrm{Span}(\widetilde{\mathcal{F}})$. For each $p \in M$, $\widetilde{G}_p = G_p$, and thus, the restriction of $\widetilde{G}$ to M is G.

Given a set of global generators for $\widetilde{G}$, the restriction of these generators to M determines a set of global generators for G. Thus, to prove Theorem 4.1, it suffices to prove the following proposition.

Proposition 4.3 *If G is a smooth generalized subbundle of the trivial bundle $\Theta^m(\mathbb{R}^n)$, then G is globally finitely generated.*

The proof of this proposition will occupy most of the rest of this section. To begin, we adopt some notation and terminology.

If B is a Euclidean ball in $\mathbb{R}^n$, we denote by $2B$ the ball with the same center and twice the radius. Let $\mathcal{B}$ denote the set of all balls of rational radius centered at points that have rational coordinates; $\mathcal{B}$ is a countable basis for the topology of $\mathbb{R}^n$.

For $0 \leq d \leq m$, let

$$\Sigma_d = \left\{ p \in \mathbb{R}^n : \dim(G_p) = d \right\}.$$

Fix $d \geq 1$ such that $\Sigma_d \neq \emptyset$. Our goal now is to construct finitely many globally defined sections which span G at each point of Σ_d. Note that spanning is automatic for points in Σ_0.

The usual Euclidean metric on $\mathbb{R}^m$ induces a metric on the bundle $\Theta^m(\mathbb{R}^n)$; we use this metric throughout the rest of the proof. For each $p \in \mathbb{R}^n$, let Q_p denote the orthogonal projection operator on $\Theta^m(\mathbb{R}^n)_p$ whose image is G_p.

Let p be a point of Σ_d. We can find sections $s_1, \ldots, s_d$ of G defined on some open neighborhood of p such that $s_1(p), \ldots, s_d(p)$ is a basis of G_p. These sections are linearly independent on some open neighborhood U of p. At each point $q \in U$, define $D_q \subseteq G_q$ to be the span of $s_1(q), \ldots, s_d(q)$. For each $q \in U$, let $P(q)$ be the orthogonal projection operator on $\Theta^m(\mathbb{R}^n)_q$ whose image is D_q. At any point q in $U \cap \Sigma_d$, we have $D_q = G_q$, and so $P(q) = Q_q$ at such points.

We can think of P as a vector bundle mapping of $\Theta^m(U)$ to itself, or as a section of the bundle $\mathrm{Hom}(\Theta^m(U), \Theta^m(U))$, whose fiber at q is the vector space $\mathrm{Hom}(\Theta^m(U)_q, \Theta^m(U)_q)$ of linear maps $\Theta^m(U)_q \to \Theta^m(U)_q$. Because $\Theta^m(U)$ is a trivial bundle, P can be thought of as a map $U \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m)$. The mapping $q \mapsto P(q)$ is smooth. Indeed, if we think of $P(q)$ as a linear map on $\mathbb{R}^m$, its matrix with respect to the standard basis can be explicitly constructed by applying the Gram-Schmidt process to the vectors $s_1(q), \ldots, s_d(q)$, which shows that the matrix entries are smooth functions of q.

The following lemma summarizes the discussion above.

Lemma 4.4 *For each $p \in \Sigma_d$, choose a ball $B \in \mathcal{B}$ such that $p \in B$ and $2B \subseteq U$. There exists a smooth vector bundle map P of $\Theta^m(2B)$ to itself with the following properties: For each $q \in 2B$,*

(1) *$P(q)$ is an orthogonal projection operator.*
(2) $\mathrm{im}(P(q))$*, the image of $P(q)$, is contained in G_q.*
(3) $\mathrm{im}(P(q))$ *has dimension d.*
(4) *If $q \in 2B \cap \Sigma_d$ then $im(P(q)) = G_q$ and so $P(q) = Q_q$.*

Since $\mathcal{B}$ is countable, the lemma shows that we can find a countable collection of balls $\{B\}_{i \in I}$ that covers Σ_d, where for each ball B_i there is a vector bundle map P_i over $2B_i$ with the properties in the lemma. There may be many such vector bundle maps over a given ball $2B_i$; we only need one, so we just pick one arbitrarily.

Let $e_1, \ldots, e_m$ denote the standard basis of $\mathbb{R}^m$. Let $E_1, \ldots, E_m$ be the corresponding constant sections of $\Theta^m(\mathbb{R}^n)$.

For each $i \in I$, choose a smooth bump function ψ_i on $\mathbb{R}^n$ so that $0 \leq \psi_i \leq 1$, $\psi_i = 1$ on B_i and $\mathrm{supp}(\psi_i) \subset 2B_i$.

On $2B_i$ we define smooth sections $P_i E_\alpha$ of G by $p \mapsto P_i(p)E_\alpha(p)$ for $\alpha = 1, \ldots, m$. Multiplying by ψ_i we get sections $\psi_i P_i E_\alpha$ such that $\mathrm{supp}(\psi_i P_i E_\alpha) \subset 2B_i$. We extend the section $\psi_i P_i E_\alpha$ smoothly to $\mathbb{R}^n$ by defining it to be zero outside $2B_i$. We use the same notation for the extended sections. We also adopt the notational convention that

$$0 \cdot (\text{undefined}) = 0 \tag{4.1}$$

in this context, as many authors do implicitly.

Let us deal first with the case where our collection of balls $\{B_i\}_{i \in I}$ is countably infinite, in which case we can assume the index set I is the natural numbers.

Since ψ_i has compact support, ψ_i and its derivatives are bounded, so $\psi_i \in \mathrm{BC}^\infty(\mathbb{R}^n; \mathbb{R})$. Similarly, the sections $\psi_i P_i E_\alpha$ have compact support, so we can view them as vector-valued functions in $\mathrm{BC}^\infty(\mathbb{R}^n; \mathbb{R}^m)$.

For each i, we can find a constant $c_i > 0$ so that

$$c_i \|\psi_i\|_i \leq \frac{1}{2^i},$$
$$c_i \|\psi_i P_i E_\alpha\|_i \leq \frac{1}{2^i}, \qquad \alpha = 1, \ldots, m.$$

Note that the order of the seminorm is the same as the index here.

Now define functions $\varphi_i = c_i \psi_i$, so we can rewrite the inequalities above as

$$\|\varphi_i\|_i \le \frac{1}{2^i},$$
$$\|\varphi_i P_i E_\alpha\|_i \le \frac{1}{2^i}, \qquad \alpha = 1, \ldots, m.$$

We now attempt to define a smooth function φ and smooth sections S_α of G by

$$\varphi = \sum_{i=1}^{\infty} \varphi_i,$$
$$S_\alpha = \sum_{i=1}^{\infty} \varphi_i P_i E_\alpha.$$

To do this, we must show these series are convergent in the appropriate function spaces.

To show that the series $\sum_i \varphi_i$ is convergent in $\mathrm{BC}^\infty(\mathbb{R}^n; \mathbb{R})$, it suffices to show that the series

$$\sum_{i=1}^{\infty} \|\varphi_i\|_k \tag{4.2}$$

converges for each k. To show that the series (4.2) converges, it suffices to show for fixed k that the tail

$$\sum_{i=k}^{\infty} \|\varphi_i\|_k$$

of the series converges. Since the norms $\|\cdot\|_j$ are increasing in j, we have

$$\sum_{i=k}^{\infty} \|\varphi_i\|_k \le \sum_{i=k}^{\infty} \|\varphi_i\|_i \le \sum_{i=k}^{\infty} \frac{1}{2^i} = 2^{-k+1}.$$

We conclude that the series $\sum_i \varphi_i$ converges in $\mathrm{BC}^\infty(\mathbb{R}^n; \mathbb{R})$, so $\varphi = \sum_i \varphi_i$ is a smooth function. As previously mentioned, we can evaluate this series pointwise, so at any point $p \in M$ we have

$$\varphi(p) = \sum_{i=1}^{\infty} \varphi_i(p). \tag{4.3}$$

It follows that $\varphi > 0$ on Σ_d, since a point $p \in \Sigma_p$ is in one of the balls B_j and $\varphi_j = c_j \psi_j = c_j > 0$ on B_j. Since all of the terms in the sum (4.3) are nonnegative, we conclude that $\varphi(p) \ge c_j > 0$.

Similarly, we can show that the series $\sum_i \varphi_i P_i E_\alpha$ is convergent in $\mathrm{BC}^\infty(\mathbb{R}^n; \mathbb{R}^m)$. As above, the sum

$$\sum_{i=1}^{\infty} \|\varphi_i P_i E_\alpha\|_k$$

converges since we have

$$\sum_{i=k}^{\infty} \|\varphi_i P_i E_\alpha\|_k \le \sum_{i=k}^{\infty} \|\varphi_i P_i E_\alpha\|_i \le \sum_{i=k}^{\infty} \frac{1}{2^i} = 2^{-k+1}.$$

Thus, we have smooth vector-valued functions, or to look at it another way, sections of the trivial bundle, defined by

$$S_\alpha = \sum_{i=1}^{\infty} \varphi_i P_i E_\alpha.$$

We can evaluate this sum pointwise and write

$$S_\alpha(p) = \sum_{i=1}^{\infty} \varphi_i(p) P_i(p) E_\alpha(p),$$

using the convention (4.1). Since the image of each P_i is in G, the sum on the right-hand side of the equation above is a convergent series in the closed subspace $G_p \subseteq \Theta^m(\mathbb{R}^n)_p$, so $S_\alpha(p) \in G_p$. Thus, S_α is a section of G.

We now claim that if $p \in \Sigma_d$, then the sections $S_\alpha(p)$ span G_p. Recall from Lemma 4.4 that if $p \in 2B_i$ then $\operatorname{im}(P_i(p)) = G_p$ and $P_i(p) = Q_p$. We then have

$$\begin{aligned} S_\alpha(p) &= \sum_{i=1}^{\infty} \varphi_i(p) P_i(p) E_\alpha(p) \\ &= \sum_{i=1}^{\infty} \varphi_i(p) Q_p E_\alpha(p) \\ &= \sum_{i=1}^{\infty} Q_p[\varphi_i(p) E_\alpha(p)] = Q_p[\varphi(p) E_\alpha(p)]. \end{aligned}$$

For each p, $\{E_1(p), \ldots, E_m(p)\}$ is a basis of $\Theta^m(\mathbb{R}^n)_p$. Since $\varphi(p) \neq 0$,

$$\varphi(p)E_1(p), \ldots, \varphi(p)E_m(p)$$

also form a basis. Thus,

$$\{Q_p\varphi(p)E_1(p), \ldots, Q_p\varphi(p)E_m(p)\}$$

spans G_p, and thus, the vectors $S_\alpha(p)$ span G_p.

This completes the construction of a finite number of generators for $G|_{\Sigma_d}$ in the case where we have a countably infinite collection of balls. In the case where our collection of balls $\{B_i\}_{i \in I}$ is finite, we can dispense with the convergence questions and just define

$$\begin{aligned} \varphi &= \sum_{i \in I} \psi_i \\ S_\alpha &= \sum_{i \in I} \psi_i P_i E_\alpha, \end{aligned}$$

where these are finite sums. A similar analysis shows that the sections S_α span G_p at every $p \in \Sigma_d$.

Finally, to complete the proof of Proposition 4.3, we apply this construction for each d such that $\Sigma_d \neq \emptyset$. For each such integer d, we get m sections. Putting all these sections together we get a finite set of globally defined sections that span G at each point, i.e., a finite set of global generators for G □.

To finish this section, we discuss the number of sections this construction yields and the case of disconnected manifolds.

If we denote by $\operatorname{maxdim}(G)$ the maximum dimension of the fibers of G, the above construction will yield m sections for every integer d, $1 \le d \le \operatorname{maxdim}(G)$ such that $\Sigma_d \neq \emptyset$, and so a maximum of $m \operatorname{maxdim}(G)$ sections. If desired, we can get exactly $m \operatorname{maxdim}(G)$ sections by adding in m copies of the zero section for each d such that $\Sigma_d = \emptyset$.

Recall that our original vector bundle E is isomorphic to a subbundle of the trivial bundle $\Theta^m(M)$. As mentioned above, an upper estimate on m is $(\dim(M) + 1)\operatorname{rk}(E)$. In the case where $E = TM$, we can get a better estimate. By the hard Whitney embedding theorem, M can be embedded in $\mathbb{R}^n$ where $n = 2\dim(M)$. Then TM is isomorphic to a subbundle of the restriction of $T\mathbb{R}^n$ to M. Since $T\mathbb{R}^n$ is canonically trivial, we see that TM is isomorphic to a subbundle of a trivial bundle of fiber dimension n, so we can take $m = n = 2\dim(M)$ in this case.

Different authors use slightly different definitions of manifolds and vector bundles. One can define a manifold so that the dimension is allowed to be different at different points. In this case, the dimension is locally constant, and so must be constant on components. With this definition, a manifold M that is not connected can have components of different dimensions. If the number of components is infinite, it is conceivable the dimensions of the components could be unbounded, although one might be hesitant to use the word "manifold" in that case.

Similarly, one can give a definition of the concept of a vector bundle E over a manifold M that allows the fiber dimension to vary with the point. The local triviality condition makes the fiber dimension locally constant, and so constant on the components of M. This point of view is taken in some of the foundational literature behind this article, such as Swan [8]. If M has infinitely many components, E could have fibers of arbitrarily large dimension.

For each component C of M, we can find an m_C so that $E_C = E\mid_C$ is isomorphic to a subbundle of $\Theta^{m_C}(C)$. If we have an upper bound on the dimension of the components of M and on the dimension of the fibers of E, we can get an upper bound m on m_C. Then, for each C, E_C is isomorphic to a subbundle of $\Theta^m(C)$. Since we have an upper bound of the dimension of the fibers of E, there is an upper bound on the dimension of the fibers of G. As above, we can construct on each C a finite set of global generators S_j^C for $j = 1, \dots, m \operatorname{maxdim}(G)$. We can then define global sections S_j, $j = 1, \dots, m \operatorname{maxdim}(G)$ by defining $S_j(p) = S_j^C(p)$, where C is the component containing p. Thus, we will still have a finite number of global generators in this case.

5 Modules of sections

Let E be a vector bundle over M of fiber dimension k. For any open set $U \subseteq M$, the space $\Gamma(U; E)$ of sections of E over U is a module over the ring $C^\infty(U)$ of smooth functions on U. Every point $p \in M$ has a neighborhood U on which there are sections $s_1, \dots, s_k$ such that $s_1(q), \dots, s_k(q)$ form a basis of E_q for all $q \in U$. Thus, for an arbitrary section $s \in \Gamma(U; E)$, we have $s(q) = f_1(q)s_1(q) + \ldots + f_k(q)s_k(q)$ for some uniquely determined functions $f_1, \dots, f_k$. The definition of vector bundle shows that these functions are smooth. As elements of the module $\Gamma(U; E)$ we have $s = f_1 s_1 + \ldots + f_k s_k$, so $s_1, \dots, s_k$ is set of generators for $\Gamma(U; E)$.

The space $\Gamma(E)$ of global sections of E is a module over the ring $C^\infty(M)$ of smooth functions on M. This module is also finitely generated. This fact is part of the proof that E is isomorphic to a subbundle of a trivial bundle; see Greub et al. [p. 77][3] and Swan [8].

One can ask the same questions about the modules of sections of a generalized subbundle $G \subseteq E$. The fact that we can find a finite set of global generators for G might lead one to hope that the module $\Gamma(G)$ is finitely generated. Alas, we will give an example to show that $\Gamma(G)$ is not in general finitely generated and that there may be arbitrarily small neighborhoods U of a point p such that the module $\Gamma(U; G)$ is not finitely generated.

To begin the construction of our example, we introduce some notation. Let $J = (-a, a) \subseteq \mathbb{R}$ be an open interval, where we allow the case $a = \infty$. Let $\mathcal{I}(J) \subseteq \mathrm{C}^{\infty}(J)$ be the space of smooth functions on J that are zero on $(-a, 0]$. Recall that one can construct a function $\psi \in \mathcal{I}(\mathbb{R})$ by

$$\psi(x) = \begin{cases} \mathrm{e}^{-1/x}, & 0 < x < \infty \\ 0, & -\infty < x \leq 0, \end{cases}$$

for example. The restriction of ψ to $J = (-a, a)$ is an element of $\mathcal{I}(J)$.

We require the following lemmas. The first lemma follows from standard one-variable calculus.

Lemma 5.1 *Let $J = (-a, a)$ be an open interval around* 0.

(1) *If f is a smooth function on J such that $f = 0$ on $(-a, 0]$, then*

$$\lim_{x \to 0} \frac{f(x)}{x^n} = 0, \qquad n = 0, 1, 2, \ldots. \tag{5.1}$$

(2) *Let g be a smooth function on $(0, a)$ and suppose that*

$$\lim_{x \to 0^+} \frac{g(x)}{x^n} = 0, \qquad n = 0, 1, 2, \ldots. \tag{5.2}$$

Then, the function f defined by

$$f(x) = \begin{cases} g(x), & 0 < x < a \\ 0, & -a < x \leq 0 \end{cases} \tag{5.3}$$

is smooth.

Lemma 5.2 *Let $h \in \mathcal{I}(J)$ be strictly positive on $(0, a)$. Then, the function $\sqrt{h}$ is in $\mathcal{I}(J)$.*

Proof Clearly $\sqrt{h} = 0$ on $(-a, 0]$ and $\sqrt{h}$ is smooth on $(0, a)$. We have

$$\lim_{x \to 0^+} \frac{\sqrt{h(x)}}{x^n} = \lim_{x \to 0^+} \sqrt{\frac{h(x)}{x^{2n}}} = \sqrt{0} = 0, \qquad n = 0, 1, 2, 3, \ldots,$$

so $h \in \mathcal{I}(J)$ by Lemma 5.1. □

Proposition 5.3 *Let G be the generalized subbundle of $\Theta^1(\mathbb{R})$ given by*

$$G_x = \begin{cases} \Theta^1(\mathbb{R})_x, & x > 0, \\ \{0\} \subset \Theta^1(\mathbb{R})_x, & x \leq 0. \end{cases}$$

This is a smooth generalized subbundle of $\Theta^1(\mathbb{R})$; indeed it is spanned by the single smooth section ψ.

The module of sections $\Gamma(G)$ is not finitely generated, and $\Gamma(J; G)$ is not finitely generated for any interval $J = (-a, a)$, $a > 0$.

Proof Regarding sections of the trivial bundle as functions, we have $\Gamma(J; G) = \mathcal{I}(J)$. Clearly $\mathcal{I}(J)$ is an ideal in the ring $\mathrm{C}^{\infty}(J)$, and our assertion is that $\mathcal{I}(J)$ is not finitely generated.

Suppose, for a contradiction, that $g_1, \ldots, g_k$ is a finite set of generators for $\mathcal{I}(J)$. Thus, if f is any function in $\mathcal{I}(J)$,

$$f = \sum_{i=1}^{k} a_i g_i \tag{5.4}$$

for some functions $a_1, \ldots, a_k$ belonging to $C^\infty(J)$.

We claim that $g_1, \ldots, g_k$ have no common zero in $(0, a)$. Indeed, if all the g_i's vanish at $p \in (0, a)$, then (5.4) shows that $f(p) = 0$ for all $f \in \mathcal{I}(J)$. But there is no such point. For example, $\psi\,|_J$ is an element of $\mathcal{I}(J)$ that does not vanish at any point in $(0, a)$.

If we define $h = g_1^2 + g_2^2 + \ldots, g_k^2$ then $h \in \mathcal{I}(J)$ and $h \geq 0$. Since the g_i's have no common zero in $(0, a)$, h is strictly positive on $(0, a)$. It follows that $\sqrt{h} \in \mathcal{I}(J)$. Since $\sqrt{h}$ is strictly positive on $(0, a)$ we have $h^{1/4} = \sqrt{\sqrt{h}} \in \mathcal{I}(J)$.

Since the g_i's generate $\mathcal{I}(J)$, we have

$$h^{1/4} = \sum_{i=1}^{k} b_i g_i \tag{5.5}$$

for some b_i's in $C^\infty(J)$. Applying the Cauchy-Schwartz inequality to (5.5) we have

$$h^{1/4} = \left|\sum_{i=1}^{k} b_i g_i\right| \leq \left[\sum_{i=1}^{k} b_i^2\right]^{1/2} \left[\sum_{i=1}^{k} g_i^2\right]^{1/2} = \left[\sum_{i=1}^{k} b_i^2\right]^{1/2} \sqrt{h} \tag{5.6}$$

If we restrict x to $(0, a)$ so that $h(x) > 0$, we can divide both sides of this inequality by $\sqrt{h(x)}$ to get

$$\left[\sum_{i=1}^{k} b_i(x)^2\right]^{1/2} \geq \frac{1}{[h(x)]^{1/4}}. \tag{5.7}$$

But, if we let x go to zero from the right, the right-hand side of (5.7) goes to $+\infty$ while the the continuity of the b_i's implies that the left-hand side approaches some finite value.

This contradiction shows that $\mathcal{I}(J)$ has no finite set of generators. □

Proposition 5.3 implies that $C^\infty(J)$ is not noetherian, which is no surprise.

Because $T\mathbb{R} \cong \Theta^1(\mathbb{R})$, the proposition above shows that tangent distributions are no better behaved than generalized subbundles of arbitrary vector bundles in this respect.

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