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New complete embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract We construct three kinds of complete embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The first is a simply connected, singly periodic, infinite total curvature surface. The second is an annular finite total curvature surface. These two are conjugate surfaces just as the helicoid and the catenoid are in $\mathbb{R}^3$. The third one is a finite total curvature surface which is conformal to $\mathbb{S}^2 \setminus \{p_1, \ldots, p_k\}$, $k \geq 3$.

Keywords Complete minimal surface · Finite total curvature · Product space

Mathematics Subject Classification (2000) Primary 53C42 · Secondary 53A35 · 53C40

1 Introduction

During recent years the theory of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ has been rapidly developed by many mathematicians. They found some interesting complete minimal surfaces as follows: the catenoid that is a surface of revolution about the $\mathbb{R}$ axis; the helicoid that is ruled by the horizontal geodesic; the Riemann type minimal surface that is foliated by horizontal circles and lines; the Scherk type minimal surface that is a minimal graph over an ideal polygon and is asymptotic to vertical planes (see [3,10,13,14]).

By Hauswirth and Rosenberg [4] some properties of complete minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ have been revealed. The vertical plane $\Gamma \times \mathbb{R}$, where Γ is a complete geodesic in $\mathbb{H}^2$, is clearly a complete minimal surface of finite total curvature. Apart from the vertical plane, the only such surface known to exist is the Scherk type minimal surface. Both

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surfaces are simply connected. So Hauswirth and Rosenberg [4] raised a natural question: is there a nonsimply connected complete minimal surface of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$? In particular, is there a minimal annulus of total curvature -4π? Note that the rotational catenoid has infinite total curvature.

In this article, using the conjugate surface method in $\mathbb{H}^2 \times \mathbb{R}$ we construct complete embedded minimal surfaces with k vertical planar ends and total curvature $-4(k-1)\pi$, giving an affirmative answer to Hauswirth and Rosenberg's [4] question.

The conjugate surface construction is initiated by Smyth [17], who constructed an embedded minimal disk in a tetrahedron $T \subset \mathbb{R}^3$ which is perpendicular to ∂T. Then Karcher, Rossman and others (see [8,9,12]) made some complete minimal surfaces by adopting this conjugate surface method.

Recently Daniel [2] and Hauswirth, Sa Earp and Toubiana [5] generalized the notion of conjugate surface to $\mathbb{H}^2 \times \mathbb{R}$. Our construction of new minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is based on their theory. We construct a minimal graph Δ^k over an infinite triangle in $\mathbb{H}^2$ such that Δ^k is asymptotic to a vertical plane and is bounded by two horizontal geodesics (one finite, the other infinite) making an angle of π/k and one vertical infinite geodesic (see Fig. 1). It turns out that the conjugate surface of Δ^k is also a minimal graph which is perpendicular along its boundary to a horizontal plane and the two vertical planes making an angle of π/k in $\mathbb{H}^2 \times \mathbb{R}$.

By reflecting the conjugate surface across these planes we can construct a nonsimply connected, genus zero, complete embedded minimal surface Σ_k with total curvature $-4(k-1)\pi$ which is asymptotic to k vertical planes, $k > 1$ (Theorem 4.1). This is similar to the k-noid of $\mathbb{R}^3$, but a remarkable difference is that Σ_k is embedded in $\mathbb{H}^2 \times \mathbb{R}$ whereas the k-noid has self intersection in $\mathbb{R}^3$ if $k \geq 3$.

If we extend the minimal graph Δ^2 by 180° rotations about the horizontal boundary geodesics, we obtain a minimal graph Δ_v which is bounded by two vertical geodesics (see Fig. 2). Rotating Δ_v by 180° about the vertical boundary geodesics repeatedly, we obtain a simply connected complete embedded minimal surface which is singly periodic. This surface is different from the ruled helicoid of $\mathbb{H}^2 \times \mathbb{R}$ because it is not ruled and because its fundamental piece has finite total curvature -4π whereas the fundamental piece of the ruled helicoid has infinite total curvature (see Theorem 3.2).

2 Preliminaries

In $\mathbb{H}^2 \times \mathbb{R}$ we consider the disk model for the hyperbolic plane $\mathbb{H}^2$ and solid cylinder model for whole space. Let x, y denote the coordinates in $\mathbb{H}^2$ and t denote the coordinate in $\mathbb{R}$. Let $\Omega \subset \mathbb{H}^2 \times \{0\}$ be a domain. In $\overline{\mathbb{H}}^2 \times \{0\}$, we denote $\partial\overline{\Omega} = \partial\Omega \cup \partial_\infty\Omega$, where the boundary part is $\partial\Omega \subset \mathbb{H}^2 \times \{0\}$ and the ideal boundary part is $\partial_\infty\Omega \subset \partial_\infty\mathbb{H}^2 \times \{0\}$. Consider a $\mathcal{C}^2$ function $t = u(x, y)$. The vertical minimal surface equation in $\mathbb{H}^2 \times \mathbb{R}$ is the following:

$$\mathrm{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_u}\right) = 0, \tag{2.1}$$

where $\mathrm{div}_{\mathbb{H}}$ and $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|^2_{\mathbb{H}}}$, $|\cdot|_{\mathbb{H}}$ being the norm in $\mathbb{H}^2$.

In the disk model for $\mathbb{H}^2$,

$$\mathbb{H}^2 = \left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\right\},$$

with the metric $ds^2 = \left(\frac{2}{1-x^2-y^2}\right)^2 \left(dx^2 + dy^2\right)$, the vertical minimal surface equation (2.1) becomes as follows:

$$\left(1 + D^2(x, y)u_y^2\right) u_{xx} + \left(1 + D^2(x, y)u_x^2\right) u_{yy} - 2D^2(x, y)u_x u_y u_{xy}$$
$$+D(x, y)\left(xu_x + yu_y\right)\left(u_x^2 + u_y^2\right) = 0,$$

where $D(x, y) = \frac{1-x^2-y^2}{2}$.

We refer to the existence theorem of minimal surfaces.

Theorem 2.1 (Corollary 4.1 of [15]) *Let $\Omega \subset \mathbb{H}^2 \times \{0\}$ be a domain and let $g : \partial\Omega \cup \partial_\infty \Omega \to \mathbb{R}$ be a bounded function everywhere continuous except perhaps at a finite set $S \subset \partial\Omega \cup \partial_\infty\Omega$. Assume that the finite boundary $\partial\Omega$ is convex. Then g admits an extension $u : \overline{\Omega} \setminus S \to \mathbb{R}$ satisfying the vertical minimal surface equation (2.1). Furthermore, the total boundary of the graph of u (that is the finite and ideal boundary) is the union of the graph of g on $(\partial\Omega \cup \partial_\infty\Omega \setminus S)$ with the vertical segments*

$$\left\{(q, t) | t \in \left[A := \lim_{x\to q} \inf_{x\neq q} g(x),\ B := \lim_{x\to q} \sup_{x\neq q} g(x)\right],\ x \in \partial\Omega \cup \partial_\infty\Omega\right\}$$

at any $q \in S$.

Theorem 2.2 (Monotone convergence theorem of [1]) *Let $\{u_n\}$ be a monotone sequence of solutions of* (2.1) *in Ω. If the sequence $\{|u_n|\}$ is bounded at one point of Ω, then there is a non-empty open set $U \subset \Omega$* (*the convergence set*) *such that $\{u_n\}$ converges to a solution of* (2.1) *in U. The convergence is uniform on compact subsets of U and the divergence is uniform on compact subsets of $\Omega - U = V$. V is called the divergence set.*

The following well-known theorems are the maximum principle for minimal surfaces. It is a special case of a lemma by Schoen [16], and is proven there.

Theorem 2.3 (Maximum principle)

(1) (*Interior maximum principle) Let Σ_1 and Σ_2 be minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Suppose p is an interior point of both Σ_1 and Σ_2, and suppose $T_p(\Sigma_1) = T_p(\Sigma_2)$. If Σ_1 lies on one side of Σ_2 near p, then $\Sigma_1 = \Sigma_2$;*

(2) (*Boundary point maximum principle*) *Suppose Σ_1, Σ_2 have C^2-boundaries C_1, C_2. Furthermore, suppose the tangent planes of both Σ_1, Σ_2 and C_1, C_2 agree at p, i.e. suppose $T_p(\Sigma_1) = T_p(\Sigma_2)$, $T_p(C_1) = T_p(C_2)$. If, near p, Σ_1 lies to one side of Σ_2, then $\Sigma_1 = \Sigma_2$.*

By Daniel [2] and by Hauswirth, Sa Earp and Toubiana [5], we have the following two equivalent concept of associate and conjugate surfaces. Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a surface equipped with a connection ∇. Let N denote its unit normal vector field, J denote the rotation by angle $\frac{\pi}{2}$ on $T\Sigma$, and S denote a field of symmetric operator $S_y : T_y\Sigma \to T_y\Sigma$ for each $y \in \Sigma$. Let T be the projection of the vertical vector $\frac{\partial}{\partial t}$ onto the tangent space $T\Sigma$ of Σ and $\nu = \left\langle N, \frac{\partial}{\partial t}\right\rangle$. We have $|T|^2 + \nu^2 = 1$. Let $TC(\Sigma)$ denote the total curvature of Σ, $TC(\Sigma) = \int_\Sigma K dA$, where $K(p) = \det S_p - \left(1 - |T_p|^2\right)$ (see, for instance, Daniel [2] or Hauswirth and Rosenberg [4] for details). We set

$$S_\theta = e^{\theta J} S = (\cos\theta)S + (\sin\theta)JS,$$
$$T_\theta = e^{\theta J} T = (\cos\theta)T + (\sin\theta)JT.$$

Theorem 2.4 (Conjugate minimal surface I, [2]) *Let Σ be a simply connected surface and $X : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ a conformal minimal immersion. Let N be the normal, S be the symmetric operator on Σ induced by the shape operator of $X(\Sigma)$. Let T and ν be defined as above. Let $z_0 \in \Sigma$. Then there exists a unique family $(X_\theta)_{\theta\in\mathbb{R}}$ of conformal minimal immersions $X_\theta : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ such that*

(1) $X_\theta(z_0) = X(z_0)$ *and* $dX_\theta(z_0) = dX(z_0)$,
(2) *the metrics induced on Σ by X and X_θ are the same,*
(3) *the symmetric operator on Σ induced by the shape operator of X_θ is S_θ,*
(4) $\frac{\partial}{\partial t} = dX_\theta(T_\theta) + \nu N_\theta$, *where N_θ is the unit normal to X_θ.*

Moreover the family X_θ is continuous with respect to θ, and $X_0 = X$. The family of immersions $(X_\theta)_{\theta\in\mathbb{R}}$ is called the associate family of the immersion X. In particular the immersion $X_{\frac{\pi}{2}}$ is called the conjugate immersion of the immersion X.

Let $X = (\varphi, h) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion. Then φ is a harmonic map to $\mathbb{H}^2$ and h is a harmonic function. The Hopf differential of φ is the following holomorphic 2-form:

$$Q\varphi = 4\left\langle \frac{\partial\varphi}{\partial z}, \frac{\partial\varphi}{\partial z}\right\rangle dz^2.$$

Because of conformality of $X, Q\varphi = -4\left(\frac{\partial h}{\partial z}\right)^2 dz^2$, where $z = x + iy$ is a local coordinate on Σ and $h = \pm\mathrm{Re}\int 2i\sqrt{Q\varphi}dz$.

Theorem 2.5 (Conjugate minimal surface II, [2,5]) *Let $X = (\varphi, h) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion, and $X_\theta = (\varphi_\theta, h_\theta)$ its associate family of conformal minimal immersions. In particular the immersion $X_{\frac{\pi}{2}}$ is called the conjugate immersion of the immersion X. Let $h_{\frac{\pi}{2}}$ be the harmonic conjugate of h. Then we have*

$$Q\varphi_\theta = e^{-2\sqrt{-1}\theta} Q\varphi, \quad h_\theta = (\cos\theta)h + (\sin\theta)h_{\frac{\pi}{2}}.$$

Now, we refer to Krust's type theorem for minimal vertical graphs and associate family of surfaces in $\mathbb{H}^2 \times \mathbb{R}$. We call that G is a vertical graph in $\mathbb{H}^2 \times \mathbb{R}$ if G is graph of g, where $g : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$.

Theorem 2.6 (Krust's type theorem, Theorem 14 of [5]) *Let $X(\Omega)$ be a minimal vertical graph on a convex domain $\Omega \subset \mathbb{H}^2$. Then the associate surface $X_\theta(\Omega), \theta \in \mathbb{R}$ is also a vertical graph.*

We can extend a minimal surface across its special boundary.

Theorem 2.7 (Schwarz reflection principle, [11]) *Suppose a minimal surface $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ containing a curve Υ as its boundary.*

(1) *Υ is a horizontal or vertical geodesic line then Σ can be extended smoothly across Υ by* 180° *rotation about Υ.*
(2) *Υ lies in a plane. Υ is a geodesic of Σ and it is not a horizontal or vertical geodesic line, and Σ meets orthogonal to the plane along Υ then Σ can be extended smoothly across Υ by reflection through the plane containing Υ.*

3 Simply connected complete embedded minimal surface

Lemma 3.1 *Let* $0 < \alpha < 1$ *and integer* $k \geq 2$ *be given. Let* D *be a domain in* $\mathbb{H}^2$ *with* $2k$ *vertices* $p_{2m-1} = \alpha e^{\sqrt{-1}\frac{(2m-2)\pi}{k}}$, $p_{2m} = e^{\sqrt{-1}\frac{(2m-1)\pi}{k}}$, $m = 1, \ldots, k$ *and* $2k$ *sides* A_m *be a geodesic from* p_{2m-1} *to* p_{2m}, B_m *be a geodesic from* p_{2m} *to* p_{2m+1}, $m = 1, \ldots, k$ *and* $p_1 = p_{2k+1}$.

Then there exists a unique (up to a vertical translation) embedded minimal surface $\Sigma(\alpha, k)$ *which has vertical geodesic lines* V_m *through the* p_{2m-1}, $m = 1, \ldots, k$, *as its boundary and the surface is of finite absolute total curvature at most* $-\int K dA \leq (2k-2)\pi$. *More precisely,* $\Sigma(\alpha, k)$ *is the graph of a function* $u : D \to \mathbb{R}$ *with* $u|_{A_m} = +\infty$ *and* $u|_{B_m} = -\infty$, $m = 1, \ldots, k$.

Proof Let L_1 be a geodesic segment from the origin 0 of $\mathbb{H}^2$ to p_1, L_2 a geodesic ray from 0 to p_2 and Γ a geodesic ray from p_1 to p_2. Let Ω be a convex domain bounded by L_1, L_2 and Γ. Let $\widetilde{\Gamma}$ be the complete geodesic containing Γ. For each $n \in \mathbb{N}$, let g be a function on $\partial\Omega$ such that $g = 0$ on $L_1 \cup L_2$ and $g = n$ on Γ. By Theorem 2.1, there is a unique function $u_n : \Omega \to \mathbb{R}$ satisfying $u_n|_{L_i} = 0$, $i = 1, 2$ and $u_n|_\Gamma = n$ and the minimal surface equation (2.1). By the maximum principle, $\{u_n\}$ is a monotone increasing sequence with respect to n.

To show that the limit of the sequence $\{u_n\}$ exists, we need to find a suitable barrier. Let E be the component of $\mathbb{H}^2 \setminus \widetilde{\Gamma}$ which contains the domain Ω. There exists a function $v \geq 0$ defined on E, asymptotic to $+\infty$ on $\widetilde{\Gamma}$ and to zero on $\partial_\infty(E)$ and v satisfies the minimal surface equation (2.1) (see [1,13]). So v is a suitable barrier for the sequence $\{u_n\}$.

By the monotone convergence theorem we can find the limit function u over Ω of the sequence $\{u_n\}$ such that $u|_\Gamma = +\infty$, $u|_{L_i} = 0$, $i = 1, 2$ and at p_1: Δ^k, the graph of u, has a vertical geodesic ray as its boundary (see Fig. 1).

Since Δ^k lies between the graph of v and the vertical plane $\widetilde{\Gamma} \times \mathbb{R}$, outside of a compact part of Δ^k uniformly converges to a vertical plane. We prove in the following that Δ^k has finite total curvature.

More precisely, let $\{q_j | q_j \in \Gamma\}$ be a sequence such that $|q_j - p_2|_{\mathbb{R}^2}$, the Euclidean distance between q_j and p_2 on the disk, is monotone decreasing to zero. Let $\Omega(j)$ be a compact domain of the convex domain Ω with $\partial\Omega(j) = \overline{0p_1} \cup \overline{p_1q_j} \cup \overline{0q_j}$, where $\overline{ab}$ indicates the geodesic line segment in $\mathbb{H}^2$ from a to b. For $n, j \in \mathbb{N}$ fixed, we denote by $u_n(j)$ the function on $\Omega(j)$ which equals n on $\overline{p_1q_j}$ and zero on $\overline{0p_1} \cup \overline{0q_j}$ and satisfies (2.1). Let P be a

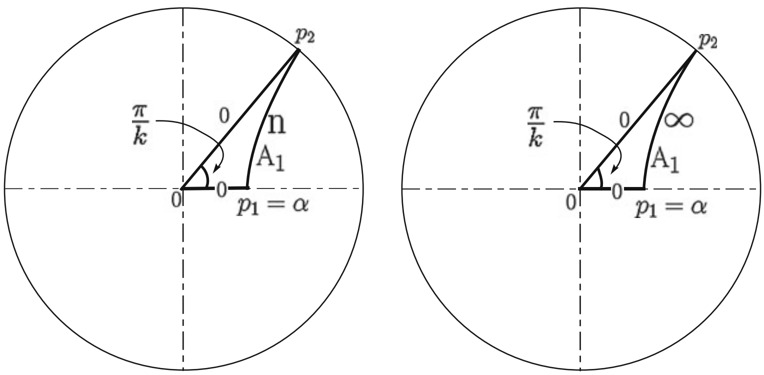


Fig. 1 *Left* The graph of u_n; *Right* The graph of u

point in $\mathbb{H}^2 \times \mathbb{R}$, we denote $P = (p, t)$, where p is the complex coordinate of $\mathbb{H}^2$ and t is the coordinate of $\mathbb{R}$. Denote $v_1(j) = (0, 0)$, $v_2(j) = (p_1, 0)$, $v_3(j) = (p_1, n)$, $v_4(j) = (q_j, n)$ and $v_5(j) = (q_j, 0)$, and $\gamma_i(j)$ be the geodesic segment in $\mathbb{H}^2 \times \mathbb{R}$ from $v_i(j)$ to $v_{i+1}(j)$, $i = 1, \ldots, 5$ and $v_6(j) = v_1(j)$. The graph of $u_n(j)$ denote by $\Delta_n^k(j)$, is bounded by $\gamma_i(j)$, $i = 1, \ldots, 5$. Applying the Gauss–Bonnet formula to $\Delta_n^k(j)$, we have:

$$\int_{\Delta_n^k(j)} K_n(j) + \sum_{i=1}^{5} \int_{\gamma_i(j)} k_{g,i}(j) + \sum_{i=1}^{5} \theta_i(j) = 2\pi,$$

where $K_n(j)$ is the Gauss curvature function of $\Delta_n^k(j)$ on the domain $\Omega(j)$ and 0 on $\Omega - \Omega(j)$, and $k_{g,i}(j)$ is a geodesic curvature function of $\gamma_i(j)$, $i = 1, \ldots, 5$, and $\theta_i(j)$ is the exterior angle at $v_i(j)$, $i = 1, \ldots, 5$. By the Gauss equation, the Gauss curvature function K is nonpositive for any minimal surfaces in $\mathbb{H} \times \mathbb{R}$ (see [4]). Since $k_{g,i}(j)$ is identically zero and $\theta_1(j) = \frac{(k-1)}{k}\pi + \angle p_2 0 q_j$, $\theta_i(j) = \frac{\pi}{2}$, $i = 2, \ldots, 5$, the total curvature of $\Delta_n^k(j)$ is $\frac{(1-k)}{k}\pi - \angle p_2 0 q_j$. As j goes to infinity, the sequence of $\{u_n(j)\}$ converges monotonically to the previous function u_n on Ω and the sequence of $\left\{\angle p_2 0 q_j\right\}$ converges to zero. By theorem 2.2, the sequence of $\{u_n(j)\}$ converges uniformly on compact sets of Ω to u_n. By Fatou's lemma, the absolute value of total curvature of $\Delta_n^k(j)$ is at most $\left|\frac{(1-k)}{k}\pi\right|$ for any n.

Similarly, as n goes to infinity the absolute value of total curvature of Δ^k is at most $\left|\frac{(1-k)}{k}\pi\right|$.

Using the Schwarz reflection principle, we extend Δ^k about the geodesic L_2, extend again about the image of L_1, again about the image of L_2, and so forth. After $2k$ extensions we get $\Sigma(\alpha, k)$ which is an embedded minimal surface with $2k$ congruent pieces and k vertical geodesics passing p_{2m-1}, $m = 1, \ldots, k$. So the absolute total curvature of $\Sigma(\alpha, k)$ is at most $(2k - 2)\pi$. By denoting u a extended function defined on D of the previous function u, the proof is completed. □

In case of $k = 2$, the $\Delta_v = \Sigma(\alpha, 2)$ has two vertical geodesic lines, V_1 and V_2. By the Schwarz reflection principle, we can extend the Δ_v to the complete minimal surface $\Sigma(\alpha)$ which is singly periodic. Hence the following theorem holds (see Fig. 2).

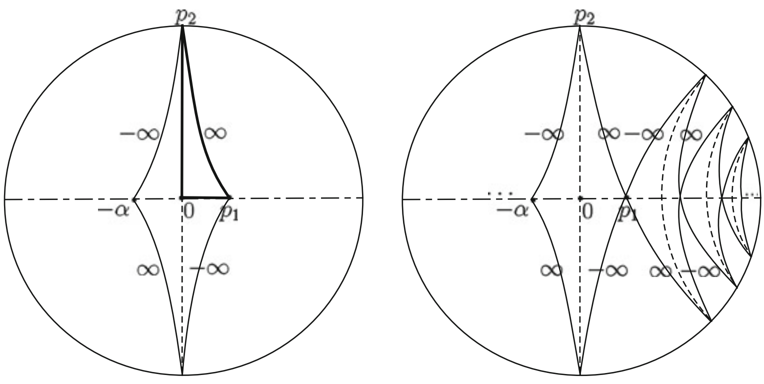


Fig. 2 *Left* The graph of $\Delta_v = \Sigma(\alpha, 2)$; *Right* Simply-connected complete embedded minimal surface $\Sigma(\alpha)$

Theorem 3.2 *In* $\mathbb{H}^2 \times \mathbb{R}$, *there is a simply connected complete embedded minimal surface* $\Sigma(\alpha)$ *which is singly periodic under the horizontal hyperbolic translation* T_α, $|T_\alpha|_{\mathbb{H}} = 4|\overline{0\alpha}|_{\mathbb{H}}$. *And the fundamental piece* $\Sigma(\alpha)/T_\alpha$ *has finite total curvature* -4π.

Remark 3.3 (1) We will compute the total curvature of $\Sigma(\alpha)/T_\alpha$ in the next section.
(2) Let $\Delta^k = \Delta(\alpha, k)$ be a fundamental piece of $\Sigma(\alpha, k)$. If $\alpha_1 \neq \alpha_2$, $\Delta(\alpha_1, k)$ and $\Delta(\alpha_2, k)$ can not be conformally equivalent. So we have one-parameter family of $\Sigma(\alpha)$ for α.

4 Nonsimply connected complete embedded minimal surface

Theorem 4.1 *For each integer* $k \geq 2$, *there exists a nonsimply connected complete embedded minimal surface* $\Sigma(k) \subset \mathbb{H}^2 \times \mathbb{R}$ *satisfying the following:*

(1) $\Sigma(k)$ *has finite total curvature* $-4(k-1)\pi$;
(2) $\Sigma(k)$ *is conformal to a k-punctured two-dimensional sphere*;
(3) $\Sigma(k)$ *is symmetric about k vertical planes and one horizontal plane.*

Proof We take the fundamental piece $\Delta = \Delta^k$ of $\Sigma(\alpha, k)$ in Lemma 3.1, if $k \geq 3$ we assume that $\alpha \geq \alpha(k)$, where $\alpha(k)$ is the value that $\overline{0p_1}$ is perpendicular to $\overline{p_1p_2}$ at p_1. The Δ is the graph of u over Ω and is bounded by two geodesic rays and one geodesic segment. Let R_1 be a vertical geodesic ray from $(p_1, 0)$, R_2 a horizontal geodesic segment from $(p_1, 0)$ to the origin $(0, 0)$ of $\mathbb{H}^2 \times \mathbb{R}$, and R_3 a horizontal geodesic ray from $(0, 0)$ to $(p_2, 0)$. Here we use the same notations as in Lemma 3.1 except L_i, $i = 1, 2$. Since we consider geodesics in $\mathbb{H}^2 \times \mathbb{R}$, we write R_{i+1}, $i = 1, 2$ instead of L_i, $i = 1, 2$.

Let N be a normal vector field of Δ. Since Δ is a simply connected, we write its immersion by $X = (\varphi, h) : D^S = D \setminus \{S, segment\} \to \Delta$, where D is a closed unit disk in the complex plane $\mathbb{C}$. Let $\theta_1, \theta_2 \in (0, 2\pi)$ be such that $X\left(e^{\sqrt{-1}\theta_1}\right) = (p_1, 0)$ and $X\left(e^{\sqrt{-1}\theta_2}\right) = (0, 0)$. And let $c_1 = \left\{e^{\sqrt{-1}\theta} | 0 < \theta \leq \theta_1\right\}$, $c_2 = \left\{e^{\sqrt{-1}\theta} | \theta_1 \leq \theta \leq \theta_2\right\}$ and $c_3 = \left\{e^{\sqrt{-1}\theta} | \theta_2 \leq \theta < \theta_3\right\}$ such that $X(c_i) = R_i$, $i = 1, 2, 3$ and the segment S is $\left\{e^{\sqrt{-1}\theta} | \theta_3 \leq \theta \leq 2\pi\right\}$.

Let $\Delta_{\frac{\pi}{2}}$ be a conjugate surface of Δ with its immersion denoted by $X_{\frac{\pi}{2}} = \left(\varphi_{\frac{\pi}{2}}, h_{\frac{\pi}{2}}\right) : D^S \to \Delta_{\frac{\pi}{2}}$. Let γ_1 be an arclength parametrization of R_1. Note that the tangent vector field γ_1' is identically e_3 along R_1 and $h_{\frac{\pi}{2}}$ is a harmonic conjugate of h. By Theorem 2.5, on c_1 we have

$$1 = dh\left(\frac{\partial}{\partial s}\right) = dh_{\frac{\pi}{2}}\left(J\frac{\partial}{\partial s}\right).$$

This means that the conormal vector field of $\Delta_{\frac{\pi}{2}}$ along $\widetilde{R}_1$, the conjugate image of R_1, is identically e_3. So $\widetilde{R}_1$ lies on a horizontal plane $\mathbb{H}^2 \times \{t\}$ and $\Delta_{\frac{\pi}{2}}$ is orthogonal to the horizontal plane along $\widetilde{R}_1$. Without loss of generality we assume $t = 0$.

Let γ_2 be a parametrization of the horizontal geodesic segment R_2 and $\widetilde{\gamma}_2$ a parametrization of curve $\widetilde{R}_2$, the conjugate image of R_2. Since the conjugate transformation preserves the metric, $\widetilde{R}_2$ is also a geodesic curve on $\Delta_{\frac{\pi}{2}}$. By (3) of Theorem 2.4, we have

$$0 = \left\langle \overline{\nabla}_{\gamma_2'}\gamma_2', N \right\rangle = \langle S\gamma_2', \gamma_2' \rangle = \left\langle S_{\frac{\pi}{2}}\widetilde{\gamma}_2', J\widetilde{\gamma}_2' \right\rangle,$$

where $\overline{\nabla}$ is the connection in $\mathbb{H}^2 \times \mathbb{R}$. This means that $\widetilde{\gamma}_2$ is a line of curvature. On c_2 we have

$$0 = dh\left(\frac{\partial}{\partial s}\right) = dh_{\frac{\pi}{2}}\left(J\frac{\partial}{\partial s}\right).$$

So the conormal vector field of $\Delta_{\frac{\pi}{2}}$ along $\widetilde{R}_2$ is orthogonal to $\frac{\partial}{\partial t}$ and T, the tangential part of $\frac{\partial}{\partial t}$. As a result, the curve $\widetilde{\gamma}_2$ is a line of curvature associated to the field T.

Lemma 4.2 (See the proof of Proposition 15 of [18]) *Let $\Sigma \subset \mathbb{H}^2\times\mathbb{R}$ be a surface transversal to each slice $\mathbb{H}^2\times\{t\}$. Let N be a normal field of Σ, T a vector field on Σ such that $dX(T)$ is the projection of $\frac{\partial}{\partial t}$ onto tangent plane $TX(\Sigma)$ and $\nu = \langle N, \frac{\partial}{\partial t}\rangle$. Let $c : \tau \in I \subset \mathbb{R} \to c(\tau) \in \Sigma$ be a line of curvature associated to the vector field T. Then $c(I)$ is contained in a vertical totally geodesic plane.*

By Lemma 4.2, $\widetilde{R}_2$ is contained in a vertical plane Π_1. Using an isometry in $\mathbb{H}^2 \times \mathbb{R}$ we can assume that $\Pi_1 = \Gamma_1 \times \mathbb{R}$, where Γ_1 is a geodesic in $\mathbb{H}^2$ through zero. Let $N_{\frac{\pi}{2}}$ be a normal vector field of $\Delta_{\frac{\pi}{2}}$ and N_1 the unit normal vector of Π_1. Since $\widetilde{\gamma}_2$ is a line of curvature and $\widetilde{\gamma}_2' \subset T_{\widetilde{\gamma}_2}\Pi_1$, we have

$$\frac{d}{ds}\left\langle N_{\frac{\pi}{2}}, N_1\right\rangle = \left\langle \overline{\nabla}_{\widetilde{\gamma}_2'} N_{\frac{\pi}{2}}, N_1\right\rangle = \langle \mu\widetilde{\gamma}_2', N_1\rangle = 0,$$

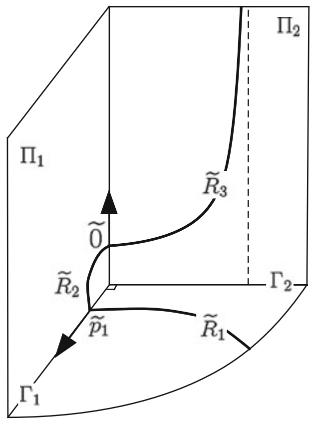
where μ is a real valued function. So $\left\langle N_{\frac{\pi}{2}}, N_1\right\rangle = C_0$, where C_0 is a constant. Since $\nu = \left\langle N, \frac{\partial}{\partial t}\right\rangle = 1$ at $(0, 0)$ and is preserved by the conjugate transformation, $\nu = 1$ at $\widetilde{0}$, the conjugate image of $(0, 0)$, *i.e.* $N_{\frac{\pi}{2}} = \frac{\partial}{\partial t}$. So $C_0 = \left\langle \frac{\partial}{\partial t}, N_1\right\rangle = 0$. Hence $\Delta_{\frac{\pi}{2}}$ meets Π_1 orthogonally. By the same argument $\widetilde{R}_3$, the conjugate image of R_3, is also contained in a vertical plane Π_2 and $\Delta_{\frac{\pi}{2}}$ meets Π_2 orthogonally. Since $\widetilde{0} \in \widetilde{R}_2$ and $\widetilde{0} \in \widetilde{R}_3$, $\Pi_1 \cap \Pi_2 \neq \emptyset$. So we can assume that $\Pi_2 = \Gamma_2 \times \mathbb{R}$, where Γ_2 is a geodesic in $\mathbb{H}^2$ through zero.

Because $\nu = 1$ at $\widetilde{0}$ and the angle between R_2 and R_3 is $\frac{\pi}{k}$, the angle between $\widetilde{R}_2$ and $\widetilde{R}_3$ is also $\frac{\pi}{k}$. This implies that the angle between Π_1 and Π_2 is $\frac{\pi}{k}$. Because Δ is a graph over the convex domain Ω, $\Delta_{\frac{\pi}{2}}$ is also a graph over $\Lambda \subset \mathbb{H}^2 \times \{0\}$ by Krust's type theorem. This theorem implies that $\Delta_{\frac{\pi}{2}}$ is embedded.

We claim that $\Delta_{\frac{\pi}{2}}$ is bounded by Π_1, Π_2 and $\mathbb{H}^2 \times \{0\}$.

We first focus on $\widetilde{R}_2$. Define $d_i(q) = \mathrm{dist}_{\mathbb{R}^2}(q, \Gamma_i)$, the Euclidean distance from q to Γ_i, $i = 1, 2$ on Π_1, Π_2. Since $\Delta_{\frac{\pi}{2}}$ is a graph over Λ, $\widetilde{R}_2$ is also graph over $\Lambda \cap \Gamma_1$. We claim that $d_1(q)$ on Π_1 cannot have any interior critical point. Suppose $d_1(q)$ has a local maximum or minimum at $q_1 \in \widetilde{R}_2$. We have $\nu(q_1) = 1$. Let $Q_1 \in R_2$ be a preimage of q_1. Since ν is preserved by the conjugate transformation, $\nu(Q_1)$ is also one. That is, at Q_1 the normal vector of Δ is e_3. Extend the Δ along R_2, the Q_1 is an interior point of the extended minimal surface. The normal vector of the extended minimal surface at Q_1 coincides with the one of the horizontal plane $\mathbb{H}^2 \times \{0\}$ and the intersection curve between the extended minimal surface and $\mathbb{H}^2 \times \{0\}$ is just a line. This contradicts to the interior maximum principle.

Fig. 3 In case of $m = 2$, the boundary behavior of $\Delta_{\frac{\pi}{2}}$



Since ν of Δ varies from 0 to 1 as Q varies from $(p_1, 0)$ to $(0, 0)$, ν of $\Delta_{\frac{\pi}{2}}$ varies from 0 to 1 as q varies from $\widetilde{p}_1$ to $\widetilde{0}$ along $\widetilde{R}_2$. Here $\widetilde{p}_1$ is the conjugate image of $(p_1, 0)$.

Similarly $\widetilde{R}_3$ is also a graph over $\Lambda \cap \Gamma_2$ and also $d_2(q)$ on Π_2 cannot have any interior local maximum or minimum. Since the ν converges to 0 as $Q \in R_3$ moves toward $(p_2, 0)$, the ν also converges to 0 as $q \in \widetilde{R}_3$ varies far away from $\widetilde{0}$. So the $\widetilde{R}_3$ asymptotically approaches to a vertical geodesic which is orthogonal to Γ_2.

Now we consider the behavior of $\widetilde{R}_1$. The curve $\widetilde{R}_1$ cannot intersect with Γ_1 and Γ_2. Suppose not $\widetilde{R}_1$ intersects Γ_1 at s_1. We extend $\Delta_{\frac{\pi}{2}}$ with respect to Π_1. Then the extended surface $\widetilde{\Delta}_{\frac{\pi}{2}} = \Delta_{\frac{\pi}{2}} \cup \Delta^*_{\frac{\pi}{2}}$ has a self-intersection, where $\Delta^*_{\frac{\pi}{2}}$ is the mirror image of $\Delta_{\frac{\pi}{2}}$ with respect to Π_1. But $\widetilde{\Delta}_{\frac{\pi}{2}}$ is the conjugate surface of the graph on the convex domain $\Omega \cup \Omega^*_{L_1}$, where $\Omega^*_{L_1}$ is the $180°$ rotated domain about the L_1. This contradicts to the Krust's type theorem. Similarly, $\widetilde{R}_1$ cannot intersect with Γ_2. So $\Delta_{\frac{\pi}{2}}$ is bounded by Π_1, Π_2 and $\mathbb{H}^2 \times \{0\}$.

We claim that $\widetilde{R}_1$ is not convex with respect to Λ at any point. Suppose $\widetilde{R}_1$ is convex at q_0. Because $\Delta_{\frac{\pi}{2}}$ is a graph over the domain $\Lambda \subset \mathbb{H}^2 \times \{0\}$, near q_0, $\Delta_{\frac{\pi}{2}}$ lies on one side of a vertical plane or a vertical strip of a vertical plane. We extend $\Delta_{\frac{\pi}{2}}$ with respect to $\mathbb{H}^2 \times \{0\}$, then the extended surface intersects with the vertical plane or the vertical strip of a vertical plane along a point or a line. This contradicts to the interior maximum principle. Since the $\widetilde{R}_1$ is not convex with respect to Λ and the length of $\widetilde{R}_1$ is infinite, the only option is that $\widetilde{R}_1$ goes to the ideal boundary $\partial_\infty \mathbb{H}^2 \times \{0\}$ (see Fig. 3).

By the Schwarz reflection principle, we extend $\Delta_{\frac{\pi}{2}}$ which is of finite absolute total curvature at most $\frac{k-1}{k}\pi$ inductively about $\mathbb{H}^2 \times \{0\}$, Π_1 and its rotation around $\{0\} \times \mathbb{R}$ axis by $\frac{m}{k}\pi$ degrees, $m = 1, \ldots, k-1$. In particular, Π_2 is the rotation of Π_1 around $\{0\} \times \mathbb{R}$ axis by $\frac{\pi}{k}$ degrees. Finally, we get $\Sigma(k)$, a complete embedded surface of finite total curvature with $4k$ congruent fundamental pieces.

First, by Huber's theorem $\Sigma(k)$ is conformal to a k-punctured two-dimensional sphere (see [4,6]). This implies that the segment S is nothing but a point. Second, we apply Hauswirth and Rosenberg's curvature estimation [4] to say that the Hopf map extends meromorphically to each puncture. Moreover, the degree of each pole depends in the number of curves are intersecting horizontal section at infinity. Since this number is one, the degree is zero. So the total curvature of $\Sigma(k)$ is $-4(k-1)\pi$. Because the fundamental piece of $\Sigma(k)$ is $-\frac{k-1}{k}\pi$, the total curvature of $\Sigma(\alpha)/T_\alpha$ is -4π. Third, $\mathbb{H}^2 \times \{0\}$, Π_1 and its rotation around $\{0\} \times \mathbb{R}$

axis by $\frac{m}{k}\pi$ degrees, $m = 1, \ldots, k-1$ are symmetric planes. By [4] each *end, a conformal parametrization of the punctured disk is asymptotic to a vertical plane.* □

Remark 4.3 (1) The $\Sigma(k)$ is similar to the Jorge–Meeks k-noid in $\mathbb{R}^3$ [7]. And as in Remark 3.3, we have one-parameter family of minimal surfaces $\Sigma(k)$ with respect to α.

(2) The $\Sigma(\alpha)$ [resp. $\Sigma(2)$] is quite similar to the Euclidean helicoid (resp. Euclidean catenoid). The $\Sigma(2)$ and the period $\Sigma(\alpha)/T_\alpha$ are conjugate minimal surfaces, in the sense of Theorems 2.4 or 2.5.

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