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A quasi-linear elliptic equation with critical growth on compact Riemannian manifold without boundary

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Abstract Let (M, g) be an N-dimensional compact Riemannian manifold without boundary. When m is a positive integer strictly smaller than N, we prove that

$$\sup_{\|u\|_{m,N/m}\le 1} \int_M e^{\alpha_{N,m}|u|^{N/(N-m)}} \mathrm{d}v_g < \infty,$$

where $\|u\|_{m,N/m}$ is the usual Sobolev norm of $u \in W^{m,N/m}(M)$, and $\alpha_{N,m}$ is the best constant in Adams' original inequality (Ann Math 128:385–398, 1988). This is a modified version of Adams' inequality on compact Riemannian manifold which has been proved by Fontana (Comment Math Helv 68:415–454, 1993). Using the above inequality in the case when $m = 1$, we establish sufficient conditions under which the quasi-linear equation

$$-\Delta_N u + \tau|u|^{N-2}u = f(x, u)$$

has a nontrivial positive weak solution in $W^{1,N}(M)$, where $-\Delta_N u = -\operatorname{div}(|\nabla u|^{N-2}\nabla u)$, $\tau > 0$, and $f(x, u)$ behaves like $e^{\gamma|u|^{N/(N-1)}}$ as $|u| \to \infty$ for some $\gamma > 0$.

Keywords Trudinger–Moser inequality · Adams inequality · Critical growth · N-Laplacian

Mathematics Subject Classification (2000) 58J05 · 58E30 · 35J60 · 35B33

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1 Introduction and main results

Let (M, g) be a compact Riemannian manifold of dimension $N(N \geq 2)$ without boundary. Assume that m is a positive integer strictly smaller than N. Take $W^{m,N/m}(M)$ the usual Sobolev space, the completion of $C^\infty(M)$ under the norm

$$\|u\|_{m,N/m} = \left(\int_M \left(|\nabla^m u|^{N/m} + |u|^{N/m} \right) \mathrm{d}v_g \right)^{m/N}, \tag{1.1}$$

where $\nabla^m u = \Delta_g^{m/2} u$ if m is even, $\nabla \Delta_g^{(m-1)/2} u$ if m is odd, ∇, Δ_g are the gradient operator and the Laplace–Beltrami operator, respectively, $\mathrm{d}v_g$ is the volume element of (M, g). Precisely in local coordinates $\{x^i\}_{i=1}^N$, $g = g_{ij}(x)\mathrm{d}x^i \mathrm{d}x^j$, $\mathrm{d}v_g = \sqrt{g}\,\mathrm{d}x^1 \cdots \mathrm{d}x^N$,

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}, \quad \Delta_g f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{g} \frac{\partial f}{\partial x^i} \right)$$

for all $f \in C^\infty(M)$, where $(g^{ij}) = (g_{ij})^{-1}$, the inverse of the matrix (g_{ij}), and $\sqrt{g} = \sqrt{\det(g_{ij})}$. Here, we have used the repeated summation convention.

In a celebrated paper [10], Fontana obtained the following estimates:

$$\sup_{\int_M u \mathrm{d}v_g = 0, \|\nabla^m u\|_{N/m} \leq 1} \int_M e^{\alpha_{N,m} |u|^{N/(N-m)}} \mathrm{d}v_g < \infty, \tag{1.2}$$

where $\|\cdot\|_{N/m}$ denotes the $L^{N/m}(M)$ norm and

$$\alpha_{N,m} = \begin{cases} \frac{N}{\omega_{N-1}} \left(\frac{\pi^{N/2} 2^m \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{N-m+1}{2}\right)} \right)^{\frac{N}{N-m}} & \text{if } m \text{ is odd} \\ \frac{N}{\omega_{N-1}} \left(\frac{\pi^{N/2} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{N-m}{2}\right)} \right)^{\frac{N}{N-m}} & \text{if } m \text{ is even.} \end{cases} \tag{1.3}$$

If $\alpha_{N,m}$ is replaced by any larger number, the integral in (1.2) is still finite, but cannot be bounded uniformly by any constant. Inequality (1.2) is a manifold case of the well-known Adams inequality [1], which is the generalization of the Trudinger–Moser inequality [13,15,16]. Adams' approach to the problem is to express u as the Riesz Potential of its gradient of order m and then use the symmetrization to reduce the problem to one-dimensional case. By estimating the asymptotic express of the Green function of Δ_g^m, Fontana was able to find the counterpart of Adams' approach on (M, g).

Replacing the hypothesis $\int_M u \mathrm{d}v_g = 0$, $\|\nabla^m u\|_{N/m} \leq 1$ by $\|u\|_{m,N/m} \leq 1$, we will show (1.2) is still valid. More generally, if (1.1) is replaced by an equivalent Sobolev norm

$$\|u\|_{S_{m,\tau}} := \left(\int_M \left(|\nabla^m u|^{N/m} + \tau |u|^{N/m} \right) \mathrm{d}v_g \right)^{m/N} \tag{1.4}$$

for any $\tau > 0$, we have the following:

Theorem 1.1 *Let (M, g) be a compact Riemannian manifold of dimension N without boundary and m is a positive strictly smaller than N. Then, for any $\tau > 0$*

$$\sup_{u \in W^{m,\frac{N}{m}}(M),\ \|u\|_{S_{m,\tau}} \le 1} \int_M e^{\alpha_{N,m}|u|^{N/(N-m)}} \mathrm{d}v_g < \infty, \tag{1.5}$$

where $\alpha_{N,m}$ is defined by (1.3). Furthermore, this inequality is sharp: when $\alpha_{N,m}$ is replaced by any larger number, the integral in (1.5) is still finite, but the supremum is infinity.

Theorem 1.1 is a modification of Fontana's result. But nevertheless, the inequality (1.5) will be more natural when we consider related partial differential equations. We remark that Theorem 1.1 is a generalization of our recent result [18]. The proof of Theorem 1.1 is based on (1.2) and the Young inequality in a nontrivial way. Similar idea has been used by Adimurthi and the second named author [3]. A special case of Theorem 1.1 is $m = 1$, which is also known by Li [11], namely

Theorem 1.2 *Let (M, g) be a compact Riemannian manifold of dimension N without boundary. Then for any $\tau > 0$*

$$\sup_{u \in W^{1,N}(M),\ \|u\|_{S_{1,\tau}} \le 1} \int_M e^{\alpha_N |u|^{N/(N-1)}} \mathrm{d}v_g < \infty, \tag{1.6}$$

where $\|u\|_{S_{1,\tau}}$ is defined by (1.4), $\alpha_N = \alpha_{N,1} = N\omega_{N-1}^{1/(N-1)}$, ω_{N-1} is the volume of the unit sphere $\mathbb{S}^{N-1}$. Furthermore this inequality is sharp: when α_N is replaced by any larger number, the integral in (1.6) is still finite, but the supremum is infinity.

Next, we study the existence of solutions to the following quasi-linear equation:

$$\begin{cases} -\Delta_N u + \tau |u|^{N-2} u = f(x, u) & \text{in } M \\ u \ge 0 & \text{in } M, \end{cases} \tag{1.7}$$

where $-\Delta_N u = -\mathrm{div}_g(|\nabla u|^N \nabla u)$, the nonlinearity $f(x, u)$ has the maximal growth on u which allows us to treat problem (1.7) variationally in the Sobolev space $W^{1,N}(M)$. Motivated by pioneer works of Adimurthi [2], de Figueiredo et al. [8,7], and do Ó [9], we say that a function $f : M \times \mathbb{R} \to \mathbb{R}$ has subcritical growth on M if for any $\alpha > 0$

$$\lim_{|s| \to \infty} \frac{f(x, s)}{e^{\alpha |s|^{N/(N-1)}}} = 0 \quad \text{uniformly for } x \in M; \tag{1.8}$$

and f has critical growth on M if there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \to \infty} \frac{|f(x, s)|}{e^{\alpha |s|^{N/(N-1)}}} = \begin{cases} 0 \text{uniformly for } x \in M, & \forall \alpha > \alpha_0 \\ \infty, & \forall \alpha < \alpha_0. \end{cases} \tag{1.9}$$

In order to study the existence of solutions to Eq. 1.7, we assume f satisfies the following:

(H_1) $f : M \times \mathbb{R} \to \mathbb{R}$ is continuous.
(H_2) There exist $R > 0$ and $A > 0$ such that for all $s \ge R$ and all $x \in M$,

$$0 < F(x, s) = \int_0^s f(x, t) \mathrm{d}t \le A f(x, s).$$

(H_3) $f(x, s) \ge 0$ for all $(x, s) \in M \times [0, \infty)$ and $f(x, 0) = 0$ for all $x \in M$.

The existence results of Eq. 1.7 in the subcritical case and critical case can be stated, respectively, as below.

Theorem 1.3 (The subcritical case) *Assume* (H_1), (H_2), (H_3), *and that* f *has subcritical growth. Furthermore suppose that*

(H_4)

$$\limsup_{s\to 0^+} \frac{NF(x,s)}{s^N} < \tau \quad \text{uniformly for } x \in M.$$

Then, Eq. 1.7 *has a nontrivial solution.*

Theorem 1.4 (The critical case) *Assume* (H_1), (H_2), (H_3) *and that* f *has critical growth. Furthermore, suppose* (H_4) *and*

(H_5)

$$sf(x,s)e^{-\alpha_0 s^{N/(N-1)}} \to +\infty \quad \text{as} \quad s \to +\infty \quad \text{uniformly for } x \in M.$$

Then, Eq. 1.7 *has a nontrivial solution.*

Let us explain the relation between Theorem 1.2 and Theorems 1.3 and 1.4. Solutions to Eq. 1.7 are critical points of the functional

$$J(u) := \frac{1}{N}\int_M \left(|\nabla u|^N + \tau|u|^N\right) \mathrm{d}v_g - \int_M F(x,u)\mathrm{d}v_g, \tag{1.10}$$

where $F(x,s) = \int_0^s f(x,t)dt$ for all $x \in M$ and $s \in \mathbb{R}$. In view of the structure of J, particularly its first term $\int_M (|\nabla u|^N + \tau|u|^N)\mathrm{d}v_g$, it is reasonable to use Theorem 1.2 instead of Fontana's original inequality (1.2) to study the compactness of the Palais–Smale sequence of J. This is exactly our motivation of establishing Theorem 1.2, more generally Theorem 1.1.

The proofs of Theorems 1.3 and 1.4 are based on the Mountain Pass theory. Similar idea has been used by de Figueiredo et al. [7] to establish the same results in the case when (M,g) is replaced by any smooth bounded domain in $\mathbb{R}^2$.

The remaining part of this article is organized as following: in Sect. 2, we prove Theorem 1.1, particularly Theorem 1.2. As an application of Theorem 1.2, Theorems 1.3 and 1.4 will be proved in Sect. 3.

2 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. The method we used here is combining Fontana's inequality (1.2) and the Young inequality. The proof is straightforward and divided into two steps:

*Step*1 : *For any* $0 < m < N$, $\alpha_{N,m}$ *is the largest possible constant such that the integrals in* (1.5) *are uniformly bounded.*

Based on Fontana's result, the integral in (1.5) in our case is still finite if $\alpha_{N,m}$ is replaced by any larger number. However we are left to prove $\alpha_{N,m}$ is the largest possible constant such that the integrals in (1.5) are uniformly bounded under the hypothesis $\|u\|_{S_{m,\tau}} \leq 1$. Following Adams [1] and Fontana [10], we distinguish two cases:

Case 1: $m=1$. In this case, $\alpha_{N,1}=N\omega_{N-1}^{\frac{1}{N-1}}$. For some point $p\in M$, let $r=r(x)=d_g(p,x)$ be the geodesic distance between x and p. Without loss of generality, we assume the injective radius of (M,g) is strictly larger than 1. Set

$$\phi_\delta(x)=\begin{cases}1, & \text{when} \quad r<\delta\\ \left(\log\frac{1}{\delta}\right)^{-1}\log\frac{1}{r}, & \text{when} \quad \delta\le r\le 1\\ 0, & \text{when} \quad r>1.\end{cases}$$

Then $\phi_\delta\in W^{1,N}(M)$ and for any $\tau>0$

$$\int_M \left(|\nabla\phi_\delta|^N+\tau|\phi_\delta|^N\right)\mathrm{d}v_g=\left(\log\frac{1}{\delta}\right)^{1-N}\omega_{N-1}\left(1+O\left(\frac{1}{\log\delta}\right)\right).$$

Denote $\widetilde{\phi}_\delta=\phi_\delta/\|\phi_\delta\|_{S_{1,\tau}}$. Then we have on the geodesic ball $B_p(\delta)\subset M$,

$$|\widetilde{\phi}_\delta|^{\frac{N}{N-1}}=\left(\log\frac{1}{\delta}\right)\omega_{N-1}^{-\frac{1}{N-1}}\left(1+O\left(\frac{1}{\log\delta}\right)\right).$$

It follows immediately that for any $\gamma>N\omega_{N-1}^{1/(N-1)}$, as $\delta\to 0$,

$$\int_M e^{\gamma|\widetilde{\phi}_\delta|^{\frac{N}{N-1}}}\mathrm{d}v_g\ge\int_{B_p(\delta)} e^{\gamma|\widetilde{\phi}_\delta|^{\frac{N}{N-1}}}\mathrm{d}v_g\to+\infty.$$

Case 2 : $m>1$. Let $\Phi\in C^\infty[0,1]$ be such that

$$\Phi(0)=\Phi'(0)=\cdots=\Phi^{(m-1)}(0)=0,\quad \Phi(1)=\Phi'(1)=1$$

and if $m>2$,

$$\Phi''(1)=\cdots=\Phi^{(m-1)}(1)=0.$$

For any fixed small $\epsilon>0$, we set

$$H(t)=\begin{cases}\epsilon\Phi\left(\frac{t}{\epsilon}\right) & \text{when} \quad 0\le t\le\epsilon\\ t & \text{when} \quad \epsilon<t\le 1-\epsilon\\ 1-\epsilon\Phi\left(\frac{1-t}{\epsilon}\right) & \text{when} \quad 1-\epsilon<t\le 1\\ 1 & \text{when} \quad t>1.\end{cases}$$

For $0<\delta<1$, $0<t<1$, we define

$$\Psi(t)=H\left(\left(\log\frac{1}{\delta}\right)^{-1}\log\frac{1}{t}\right).$$

For any fixed point $p\in M$, denote the distance between p and x by $r=r(x)=d_g(p,x)$, then the function

$$\phi_\delta(x)=\Psi(r)\in C^m(B_p(1)).$$

By a delicate calculation of Fontana ([10], pp. 441–443),

$$\int_M|\nabla^m\phi_\delta|^{\frac{n}{m}}\mathrm{d}v_g\le c(m,N)^{\frac{N}{m}}\omega_{N-1}\left(1+C\epsilon+O\left(\frac{1}{\log\delta}\right)\right)\left(\log\frac{1}{\delta}\right)^{-(N-m)/m},$$

where

$$c(m,N)=\begin{cases}2^{\frac{m-2}{2}}\Gamma\left(\frac{m}{2}\right)(N-m)(N-m+2)\cdots(N-2) & \text{for } m \text{ even}\\ 2^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2}\right)(N-m+1)(N-m+3)\cdots(N-2) & \text{for } m \text{ odd.}\end{cases}$$

We are left to estimate

$$\int_M |\phi_\delta|^{\frac{N}{m}}\mathrm{d}v_g=\int_0^1\left(H\left(\frac{\log s}{\log\delta}\right)\right)^{\frac{N}{m}}\omega_{N-1}s^{N-1}(1+O(s))\mathrm{d}s.$$

Since $H(t)\le Ct$, we obtain

$$\int_M |\phi_\delta|^{\frac{N}{m}}\mathrm{d}v_g=O\left(\left(\log\frac{1}{\delta}\right)^{-\frac{N}{m}}\right).$$

Define $\widetilde{\phi}_\delta=\phi_\delta/\|\phi_\delta\|_{S_{m,\tau}}$. Then we have on the geodesic ball $B_p(\delta)$,

$$|\widetilde{\phi}_\delta|^{\frac{N}{N-m}}\ge\left(\log\frac{1}{\delta}\right)\frac{1-C\epsilon+O\left(\frac{1}{\log\delta}\right)}{\omega_{N-1}^{\frac{m}{N-m}}c(m,N)^{\frac{N}{N-m}}}\left(1+O\left(\frac{1}{\log\delta}\right)\right).$$

It is easy to see that for any $\gamma>N\omega_{N-1}^{\frac{m}{N-m}}c(m,N)^{\frac{N}{N-m}}=\alpha_{N,m}$,

$$\int_M e^{\gamma|\widetilde{\phi}_\delta|^{\frac{N}{N-m}}}\mathrm{d}v_g\ge\int_{B_p(\delta)}e^{\gamma|\widetilde{\phi}_\delta|^{\frac{N}{N-m}}}\mathrm{d}v_g\to+\infty$$

as $\delta\to 0$, provided that ϵ is chosen sufficiently small. This completes the proof of Step 1.

Step 2 : *The modified Adams inequality (1.5) holds.*

In view of Fontana's inequality, to conclude (1.5), one only needs to prove

$$\sup_{u\in W^{m,\frac{N}{m}}(M),\|u\|_{S_{m,\tau}}\le 1}\int_{|u-\overline{u}|\ge\overline{u}>0}e^{\alpha_{N,m}|u|^{N/(N-m)}}\mathrm{d}v_g<\infty.$$

Assume $\|u\|_{S_{m,\tau}}\le 1$. Denote $\overline{u}=\frac{1}{Vol(M)}\int_M u\mathrm{d}v_g$ and write $u=(u-\overline{u})+\overline{u}$. Clearly $\overline{u}$ is bounded. Using an elementary inequality $(a+b)^p\le b^p+(2^p-1)b^{p-1}a$ for $0\le a\le b$ and $p>1$, one has by employing the Young inequality

$$(a+b)^p\le(1+\gamma)b^p+c(p)\frac{a^p}{\gamma^{p-1}},\quad\forall\gamma>0,$$

where $c(p)$ is a constant depending only on p. Taking $a=\overline{u}$, $b=|u-\overline{u}|$, $p=N/(N-m)$, γ satisfies

$$1+\gamma=\left(\int_M|\nabla^m u|^{\frac{N}{m}}\mathrm{d}v_g\right)^{1-p}$$

and $w=(1+\gamma)^{1/p}(u-\overline{u})$. Then one can see

$$\int_M|\nabla^m w|^{\frac{N}{m}}\mathrm{d}v_g=1,\quad\int_M w\mathrm{d}v_g=0.$$

Since

$$\gamma = \left(1 - \tau \int\limits_M |u|^{\frac{N}{m}} \mathrm{d}v_g\right)^{1-p} - 1 \geq \tau(p-1) \int\limits_M |u|^{\frac{N}{m}} \mathrm{d}v_g$$

and on the set $\{x \in M : |u(x) - \overline{u}| \geq \overline{u} > 0\}$,

$$|u|^p \leq |w|^p + c(p)\frac{\overline{u}^p}{\gamma^{p-1}},$$

one ends step 2 by using Fontana's inequality and completes the proof of Theorem 1.1. □

3 Applications of Theorem 1.1

In this section, we will use the Mountain Pass theory to establish Theorems 1.3 and 1.4. To this end, we begin with constructing a functional closely related to Eq. 1.7.

For $m \in \mathbb{N}$, $0 < m < N$, we assume $f : M \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist constants $\beta > 0$, $C > 0$ such that

$$|f(x,s)| \leq Ce^{\beta|s|^{\frac{N}{N-m}}}, \quad \forall (x,s) \in M \times \mathbb{R}. \tag{3.1}$$

Let $F(x,s) = \int_0^s f(x,t)dt$. For $0 < m < N$ and $\tau > 0$, we define functionals

$$J_{m,\tau}(u) = \frac{m}{N}\int\limits_M \left(|\nabla^m u|^{N/m} + \tau|u|^{N/m}\right)\mathrm{d}v_g - \int\limits_M F(x,u)\mathrm{d}v_g, \quad \forall u \in W^{m,\frac{N}{m}}(M).$$

In view of Theorem 1.1, $J_{m,\tau}$ is well defined on $W^{m,\frac{N}{m}}(M)$. When $m = 1$, $J_{1,\tau}$ is exactly J defined by (1.10). Clearly, $J \in C^1(W^{1,N}(M), \mathbb{R})$ and (3.1) becomes

$$|f(x,s)| \leq Ce^{\beta|s|^{\frac{N}{N-1}}}, \quad \forall (x,s) \in M \times \mathbb{R}. \tag{3.2}$$

3.1 The geometry of the functional J

Define two functions

$$\widetilde{f}(x,s) = \begin{cases} f(x,s) & \text{when} \quad (x,s) \in M \times (0,\infty) \\ 0 & \text{when} \quad (x,s) \in M \times (-\infty, 0] \end{cases}$$

and $\widetilde{F}(x,s) = \int_0^s \widetilde{f}(x,t)dt$. If f satisfies $(H_1) - (H_5)$, then so does $\widetilde{f}$. Moreover if $u \in W^{1,N}(M)$ is a solution of

$$\begin{cases} -\Delta_N u + \tau|u|^{N-2}u = \widetilde{f}(x,u) & \text{in} \quad M \\ u \geq 0 & \text{in} \quad M, \end{cases}$$

then it is also a solution of (1.7). Without loss of generality, we can assume henceforth that $f(x,s) \equiv 0$ for all $(x,s) \in M \times (-\infty, 0]$.

Lemma 3.1 *Assume* (H_1), (H_2), (H_3), *and* (3.2). *Then* $J(tu) \to -\infty$ *as* $t \to +\infty$, *for all* $u \in W^{1,N}(M) \setminus \{0\}$ *with* $u \geq 0$.

Proof Assume $u \in W^{1,N}(M) \setminus \{0\}$ with $u \geq 0$. By (H_2), for $p > N$, there exist two positive constants c_1 and c_2 such that

$$F(x,u) \geq c_1 u^p - c_2.$$

Hence

$$J(tu) \leq \frac{t^N}{N} \int_M (|\nabla u|^N + |u|^N) \mathrm{d}v_g - c_1 t^p \int_M |u|^p \mathrm{d}v_g + c_2.$$

Since $p > N$, $J(tu) \to -\infty$ as $t \to +\infty$. □

Lemma 3.2 *Assume* (H_1), (H_4)*, and* (3.2)*. Then there exist* $\delta, \sigma > 0$ *such that*

$$J(u) \geq \delta \quad \text{if} \quad \|u\|_{S_{1,\tau}} = \sigma.$$

Proof By (H_1), (H_4), and (3.2), there exists some $\lambda < \tau$ such that for $q > N$

$$F(x,u) \leq \frac{1}{N}\lambda |u|^N + C|u|^q e^{\beta |u|^{\frac{N}{N-1}}} \quad \text{for all} \quad (x,u) \in M \times \mathbb{R}. \tag{3.3}$$

By Theorem 1.1 and the Hölder inequality,

$$\int_M |u|^q e^{\beta|u|^{\frac{N}{N-1}}} \mathrm{d}v_g \leq \left(\int_M e^{p'\beta|u|^{\frac{N}{N-1}}} \mathrm{d}v_g \right)^{\frac{1}{p'}} \left(\int_M |u|^{qp} \mathrm{d}v_g \right)^{\frac{1}{p}}$$
$$\leq C \left(\int_M |u|^{qp} \mathrm{d}v_g \right)^{\frac{1}{p}}, \tag{3.4}$$

provided that $\|u\|_{S_{1,\tau}} \leq \varrho$, where $p'\beta\varrho^{\frac{N}{N-1}} \leq \alpha_N$ and $\frac{1}{p'} + \frac{1}{p} = 1$. Obviously,

$$\int_M |u|^N \mathrm{d}v_g \leq \frac{1}{\tau} \|u\|_{S_{1,\tau}}^N, \quad \forall u \in W^{1,N}(M) \setminus \{0\}.$$

This together with (3.3) and (3.4) implies that

$$J(u) \geq \frac{1}{N}\left(1 - \frac{\lambda}{\tau}\right) \|u\|_{S_{1,\tau}}^N - C\|u\|_{S_{1,\tau}}^q.$$

Thus, we can further choose $\sigma < \varrho$ and $\delta > 0$ such that $J(u) \geq \delta$ if $\|u\|_{S_{1,\tau}} = \sigma$. □

3.2 Minimax level

To get a more precise information of the minimax level obtained by the mountain pass theorem, we employ the Moser function sequence

$$\mathbf{M}_n(x,r) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log n)^{(N-1)/N} & \text{when} \quad r \leq R/n, \\ (\log n)^{-1/N} \log(R/r) & \text{when} \quad R/n \leq r \leq R, \\ 0 & \text{when} \quad r \geq R, \end{cases}$$

where $0 < R < \mathrm{inj}(M)$, $\mathrm{inj}(M)$ is the injective radius of (M, g), and $r = r(x)$ denotes the geodesic distance between x and a fixed point $O \in M$.

Lemma 3.3 *Assume (H_1), (H_2), (H_3), and (H_5) hold. Then there exists $n \in \mathbb{N}$ such that*

$$\max_{t \geq 0} J(t\mathbf{M}_n) < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof Suppose not. Then, we have for all n

$$\max_{t \geq 0} J(t\mathbf{M}_n) \geq \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.5}$$

By Lemma 3.1, there exists $t_n > 0$ for any fixed n such that

$$J(t_n\mathbf{M}_n) = \frac{1}{N} t_n^N \|\mathbf{M}_n\|_{S_{1,\tau}}^N - \int_M F(x, t_n\mathbf{M}_n)\mathrm{d}v_g = \max_{t \geq 0} J(t\mathbf{M}_n). \tag{3.6}$$

Since $F(x, s) \geq 0$ for all $(x, s) \in M \times \mathbb{R}$, we get by combining (3.5) and (3.6) that

$$t_n^N \|\mathbf{M}_n\|_{S_{1,\tau}}^N \geq \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.7}$$

By (3.6), we arrive at $\frac{\mathrm{d}}{\mathrm{d}t} J(t\mathbf{M}_n) = 0$ at $t = t_n$, or equivalently

$$t_n^N \|\mathbf{M}_n\|_{S_{1,\tau}}^N = \int_M t_n\mathbf{M}_n f(x, t_n\mathbf{M}_n)\mathrm{d}v_g. \tag{3.8}$$

By (H_5), $\forall \rho > 0$, $\exists R_\rho > 0$ such that for all $s \geq R_\rho$, there holds

$$sf(x, s) \geq \rho e^{\alpha_0 s^{\frac{N}{N-1}}}. \tag{3.9}$$

Choosing a normal coordinate system near the point O, we calculate

$$\begin{aligned}\int_M |\nabla \mathbf{M}_n|^N \mathrm{d}v_g &= \frac{1}{\omega_{N-1}\log n} \int_{\frac{R}{n}}^{R} \frac{\omega_{N-1}}{r}(1 + O(r^2))\mathrm{d}r \\ &= 1 + \frac{O(R^2)}{\log n},\end{aligned}$$

and similarly

$$\int_M \tau |\mathbf{M}_n|^N \mathrm{d}v_g = \frac{1}{\log n}\left(o_n(1) + O(R^2)\right),$$

where $o_n(1) \to 0$ as $n \to \infty$ and $|O(R^2)| \leq CR^2$. Hence we get

$$\|\mathbf{M}_n\|_{S_{1,\tau}}^N = 1 + \frac{1}{\log n}\left(o_n(1) + O(R^2)\right). \tag{3.10}$$

Thus (3.7) becomes

$$t_n^N \geq \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}\left(1 + \frac{o_n(1) + O(R^2)}{\log n}\right). \tag{3.11}$$

This together with (3.8) and (3.9) implies

$$
\begin{aligned}
t_n^N \|\mathbf{M}_n\|_{S_{1,\tau}}^N &\geq \rho \int\limits_{B_{R/n}(O)} e^{\alpha_0 |t_n \mathbf{M}_n|^{\frac{N}{N-1}}} \mathrm{d}v_g \\
&= \rho \frac{\omega_{N-1}}{N} \left(\frac{R}{n}\right)^N e^{\alpha_0 t_n^{\frac{N}{N-1}} \omega_{N-1}^{-\frac{1}{N-1}} \log n} \left(1 + O\left(\frac{R^2}{n^2}\right)\right) \qquad (3.12)
\end{aligned}
$$

for sufficiently large n. The power of this inequality is evident. Since $\|\mathbf{M}_n\|_{S_{1,\tau}}^N$ is bounded and $\rho > 0$, it is easy to see from (3.12) that t_n is a bounded sequence. Notice that $t_n^{N/(N-1)} > \alpha_N/\alpha_0$ implies $\alpha_0 t_n^{N/(N-1)} \omega_{N-1}^{-1/(N-1)} > N$, then it follows from (3.11) and (3.12) that

$$
\lim_{n \to \infty} t_n^N = \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \qquad (3.13)
$$

It follows from (3.11) and (3.12) that

$$
\begin{aligned}
t_n^N \|\mathbf{M}_n\|_{S_{1,\tau}}^N &\geq \rho \frac{\omega_{N-1}}{N} \left(\frac{R}{n}\right)^N e^{N \log n} \left(1 + o_n(1) + O(R^2)\right) \\
&= \rho \frac{\omega_{N-1}}{N} R^N \left(1 + o_n(1) + O(R^2)\right).
\end{aligned}
$$

By (3.10) and (3.13), letting $n \to \infty$ in the above inequality, we obtain

$$
\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \geq \rho \frac{\omega_{N-1}}{N} R^N \left(1 + O(R^2)\right).
$$

This is impossible when ρ is chosen sufficiently large and completes the proof of the Lemma. □

3.3 Palais–Smale sequences

We state a manifold version of Lemma 2.1 in [7] as below. Since the proof is almost the same, we omit the details.

Lemma 3.4 *Let $u_n \to u$ in $L^1(M)$. Assume that $f(x, u_n(x))$ and $f(x, u(x))$ are also $L^1(M)$ functions. If $\int_M |f(x, u_n(x)) u_n(x)| \mathrm{d}v_g \leq C$, then $f(x, u_n) \to f(x, u)$ in $L^1(M)$.*

The following result can be found in [5,6].

Lemma 3.5 (Cherrier) *Let W be any compact N-dimensional Riemannian manifold with smooth boundary ∂W. Then for any $\alpha < \alpha_N/2^{1/(N-1)}$,*

$$
\sup_{\|\nabla v\|_{L^N(W)} \leq 1,\ \int_W v \mathrm{d}v_g = 0} \int_W e^{\alpha |v|^{\frac{N}{N-1}}} \mathrm{d}v_g < \infty.
$$

Moreover when $\alpha > \alpha_N/2^{1/(N-1)}$, the above integral is still finite, but the supremum is infinite.

We remark that the significance of Lemma 3.5 is that the best constant $\alpha_N/2^{1/(N-1)}$ depends only on the dimension of W. When $N = 2$, this result has been strengthened by the second author in [17].

Lemma 3.6 *Assume f satisfies (H_1), (3.2), and there exist $R_0 > 0$, $\mu > N$ such that*

$$0 \leq \mu F(x,s) \leq sf(x,s), \quad \forall |s| \geq R_0, \quad \forall x \in M. \tag{3.14}$$

Let $(u_n) \subset W^{1,N}(M)$ be a Palais–Smale sequence of any level, i.e., $J(u_n) \to c$, $J'(u_n) \to 0$ in $W^{-1,\frac{N}{N-1}}(M)$ as $n \to \infty$. Then there exists a subsequence of (u_n), still denoted by (u_n), and $u \in W^{1,N}(M)$ such that

$$\begin{cases} f(x,u_n) \to f(x,u) & \text{in } \ L^1(M) \\ \nabla u_n(x) \to \nabla u(x) & \text{for almost all } \ x \in M \\ |\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u & \text{weakly in } \ (L^{N/(N-1)}(M))^N. \end{cases}$$

Proof Assume $(u_n) \subset W^{1,N}(M)$ be a Palais–Smale sequence of any level, i.e.,

$$\frac{1}{N}\int_M \left(|\nabla u_n|^N + \tau |u_n|^N\right) dv_g - \int_M F(x,u_n) dv_g \to c, \tag{3.15}$$

$$\left|\langle J'(u_n), \varphi\rangle\right| \leq \tau_n \|\varphi\|_{S_{1,\tau}}, \quad \forall \varphi \in W^{1,N}(M), \tag{3.16}$$

where $\tau_n \to 0$ as $n \to \infty$. Multiplying (3.15) by μ and subtracting (3.16) with $\varphi = u_n$, we obtain

$$\left(\frac{\mu}{N} - 1\right) \|u_n\|_{S_{1,\tau}}^N - \int_M (\mu F(x,u_n) - u_n f(x,u_n)) dv_g \leq C + \tau_n \|u_n\|_{S_{1,\tau}}$$

for some constant C. By (3.14) and (H_1), the second term in the above inequality has lower bound, and thus u_n is bounded in $W^{1,N}(M)$. It then follows that

$$\int_M \left||\nabla u_n|^{N-2}\nabla u_n\right|^{\frac{N}{N-1}} dv_g \leq C, \quad \int_M F(x,u_n) dv_g \leq C, \quad \text{and} \quad \int_M f(x,u_n)u_n dv_g \leq C.$$

Moreover, up to a subsequence, we may assume

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,N}(M), \quad u_n \to u \quad \text{a. e. in} \quad M$$
$$u_n \to u \quad \text{strongly in} \quad L^q(M), \quad \forall q \geq 1.$$

The assumption (3.14) implies that $sf(x,s) = |sf(x,s)|$ for all $s \geq R_0$, and thus

$$\int_M |f(x,u_n)u_n| dv_g \leq C.$$

It then follows from Lemma 3.4 that $f(x,u_n) \to f(x,u)$ in $L^1(M)$.

Next, we will prove $\nabla u_n(x) \to \nabla u(x)$ almost everywhere. Up to a subsequence, we can define an energy concentration set for some $\delta > 0$ to be determined later,

$$\Sigma_\delta = \left\{ x \in M : \lim_{r\to 0}\lim_{n\to\infty} \int_{B_r(x)} (|\nabla u_n|^N + \tau |u_n|^N) dv_g \geq \delta \right\}.$$

Since (u_n) is bounded in $W^{1,N}(M)$, Σ_δ must be a finite set. For any $x^* \in M \setminus \Sigma_\delta$, there exists $r : 0 < r < dist(x^*, \Sigma_\delta)$ such that

$$\lim_{n\to\infty} \int_{B_r(x^*)} (|\nabla u_n|^N + \tau |u_n|^N) dv_g < \delta.$$

It follows that for large n,

$$\int\limits_{B_r(x^*)} (|\nabla u_n|^N + \tau |u_n|^N) \mathrm{d}v_g < \delta. \tag{3.17}$$

Let $\overline{u_n} = \int\limits_{B_r(x^*)} u_n \mathrm{d}v_g$. It is easy to see from (3.17) that $|\overline{u_n}| \leq \delta^{1/N}(Vol(M))^{1-1/N}$, and thus

$$\begin{aligned}\int\limits_{B_r(x^*)} e^{\beta |u_n|^{\frac{N}{N-1}}} \mathrm{d}v_g &\leq \int\limits_{B_r(x^*)} e^{2^{\frac{N}{N-1}}\beta |u_n - \overline{u_n}|^{\frac{N}{N-1}} + 2^{\frac{N}{N-1}}\beta |\overline{u_n}|^{\frac{N}{N-1}}} \mathrm{d}v_g \\ &\leq C \int\limits_{B_r(x^*)} e^{2^{\frac{N}{N-1}}\beta |u_n - \overline{u_n}|^{\frac{N}{N-1}}} \mathrm{d}v_g.\end{aligned}$$

Now, we choose δ such that $2^{\frac{N}{N-1}}\beta\delta^{\frac{1}{N-1}} < \alpha_N/2^{\frac{1}{N-1}}$. Then $e^{\beta|u_n|^{\frac{N}{N-1}}}$ is bounded in $L^q(B_r(x^*))$ for some $q > 1$, thanks to Lemma 3.5. By (3.2), $f(x, u_n)$ is also bounded in $L^q(B_r(x^*))$. For any $\eta > 0$, denote

$$A_\eta = \{x \in B_r(x^*) : |u(x)| \geq \eta\}.$$

We estimate

$$\begin{aligned}\int\limits_{A_\eta} |f(x, u_n) - f(x, u)||u| \mathrm{d}v_g &\leq \left(\int\limits_{A_\eta} |f(x, u_n) - f(x, u)|^q \mathrm{d}v_g\right)^{1/q} \left(\int\limits_{A_\eta} |u|^{q'}\right)^{1/q'} \\ &\leq C \left(\int\limits_{A_\eta} |u|^{q'}\right)^{1/q'},\end{aligned}$$

where $1/q + 1/q' = 1$, since $f(x, u_n)$ is bounded in $L^q(B_r(x^*))$. Hence for any $\nu > 0$,

$$\int\limits_{A_\eta} |f(x, u_n) - f(x, u)||u| \mathrm{d}v_g < \nu, \tag{3.18}$$

provided that η is chosen sufficiently large. Since $f(x, u_n) \to f(x, u)$ in $L^1(M)$,

$$\lim_{n\to\infty} \int\limits_{B_r(x^*)\setminus A_\eta} |f(x, u_n) - f(x, u)||u| \mathrm{d}v_g = 0. \tag{3.19}$$

Combining (3.18) and (3.19), we have

$$\lim_{n\to\infty} \int\limits_{B_r(x^*)} |f(x, u_n) - f(x, u)||u| \mathrm{d}v_g \leq \nu.$$

Since $\nu > 0$ is arbitrary, we obtain

$$\lim_{n\to\infty} \int\limits_{B_r(x^*)} |f(x, u_n) - f(x, u)||u| \mathrm{d}v_g = 0. \tag{3.20}$$

On the other hand, we have by using the Hölder inequality,

$$\int_{B_r(x^*)} |f(x,u_n)||u_n-u|\mathrm{d}v_g \le \|f(x,u_n)\|_{L^q(B_r(x^*))}\,\|u_n-u\|_{L^{q'}(M)} \to 0, \quad (3.21)$$

where $1/q + 1/q' = 1$. Combining (3.20) and (3.21), we immediately get

$$\lim_{n\to\infty} \int_{B_r(x^*)} |f(x,u_n)u_n - f(x,u)u|\mathrm{d}v_g = 0.$$

A covering argument implies that for any compact set $K \subset\subset M \setminus \Sigma_\delta$,

$$\lim_{n\to\infty} \int_K |f(x,u_n)u_n - f(x,u)u|\mathrm{d}v_g = 0.$$

Now we are proving for any compact set $K \subset\subset M \setminus \Sigma_\delta$,

$$\lim_{n\to\infty} \int_K |\nabla u_n - \nabla u|^N \mathrm{d}v_g = 0. \quad (3.22)$$

It suffices to prove for any $x^* \in M \setminus \Sigma_\delta$, and $r : 0 < r < dist(x^*, \Sigma_\delta)$ given in (3.17), there holds

$$\lim_{n\to\infty} \int_{B_{r/2}(x^*)} |\nabla u_n - \nabla u|^N \mathrm{d}x = 0. \quad (3.23)$$

For this purpose, we take $\phi \in C_0^\infty(B_r(x^*))$ with $0 \le \phi \le 1$ and $\phi \equiv 1$ on $B_{r/2}(x^*)$. Obviously, ϕu_n is a bounded sequence in E. Inserting $\varphi = \phi u_n$ and $\varphi = \phi u$ into (3.16) respectively, we have

$$\begin{aligned}&\int_{B_r(x^*)} \phi(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u)(\nabla u_n - \nabla u)\mathrm{d}v_g \\ &\le \int_{B_r(x^*)} |\nabla u_n|^{N-2}\nabla u_n \nabla\phi(u-u_n)\mathrm{d}v_g + \int_{B_r(x^*)} \phi|\nabla u|^{N-2}\nabla u(\nabla u - \nabla u_n)\mathrm{d}v_g \\ &\quad + \int_{B_r(x^*)} \phi(u_n-u)f(x,u_n)\mathrm{d}v_g + \tau_n\|\phi u_n\|_{S_{1,\tau}} + \tau_n\|\phi u\|_{S_{1,\tau}}. \end{aligned} \quad (3.24)$$

The integrals on the right side of this inequality can be estimated as below. Since $u_n \to u$ in $L^p(M)(\forall p \ge 1)$, we have by the Hölder inequality

$$\lim_{n\to\infty} \int_{B_r(x^*)} |\nabla u_n|^{N-2}\nabla u_n \nabla\phi(u-u_n)\mathrm{d}x = 0. \quad (3.25)$$

Since $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L^N(M))^N$, there holds

$$\lim_{n\to\infty} \int_{B_r(x^*)} \phi|\nabla u|^{N-2}\nabla u(\nabla u - \nabla u_n)\mathrm{d}x = 0. \quad (3.26)$$

From (3.21), we see $\int_{B_r(x^*)} \phi(u_n - u) f(x, u_n) \mathrm{d}v_g \to 0$ as $n \to \infty$, which together with (3.25), (3.26), and $\tau_n \to 0$ implies that the first integral sequence of (3.24) tends to zero as $n \to \infty$. Therefore we derive (3.23) from (3.24) and an elementary inequality

$$2^{2-N}|b-a|^N \le \langle |b|^{N-2}b - |a|^{N-2}a, b-a\rangle, \quad \forall a,\, b \in \mathbb{R}^N.$$

Since $x^* \in M \setminus \Sigma_\delta$ is arbitrary, a covering argument and (3.23) implies (3.22), which yields that ∇u_n, up to a subsequence, converges to ∇u almost everywhere in M.

Let (u_n) be a sequence such that $\nabla u_n(x) \to \nabla u(x)$ for almost every $x \in M$. Recall that $|\nabla u_n|^{N-2}\nabla u_n$ is bounded in $(L^{\frac{N}{N-1}}(M))^N$, we can assume $|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup V$ weakly in $(L^{\frac{N}{N-1}}(M))^N$. Then V must be $|\nabla u|^{N-2}\nabla u$, thanks to the almost everywhere convergence of ∇u_n. This completes the proof of the Lemma. □

3.4 Proof of Theorems 1.3 and 1.4

From Lemma 3.1 and Lemma 3.2, we can see that J satisfies the following properties:

(i) $J \in \mathcal{C}^1(W^{1,N}(M), \mathbb{R})$, $J(0) = 0$;

(ii) There exist $\delta, \sigma > 0$ such that $J(u) \ge \delta$ if $\|u\|_{S_{1,\tau}} = \sigma$.

(iii) There exists $\varphi \in W^{1,N}(M)$ such that $J(\varphi) < \delta$.

Now we can apply the Mountain Pass Lemma [4] to obtain a positive level c and a Palais–Smale sequence (u_n) satisfying (3.15) and (3.16), where

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma} J(u) \ge \delta, \quad \Gamma = \left\{\gamma \in \mathcal{C}\left([0,1], W^{1,N}(M)\right) : \gamma(0) = 0, \gamma(1) = \varphi\right\}.$$

Thanks to Lemma 3.6, (u_n) is bounded a sequence in $W^{1,N}(M)$, and

$$\int_M F(x, u_n)\mathrm{d}v_g \le C, \quad \int_M f(x, u_n)u_n \mathrm{d}v_g \le C.$$

Up to a subsequence, we can assume that

$$\begin{aligned} &u_n \rightharpoonup u_0 \quad \text{weakly} \quad \text{in} \quad W^{1,N}(M), \quad u_n \to u_0 \quad \text{a.e.} \quad \text{in} \quad M \\ &u_n \to u_0 \quad \text{strongly} \quad \text{in} \quad L^q(M), \quad \forall q \ge 1. \end{aligned}$$

From (H_2) and Lemma 3.6, we have

$$F(x, u_n) \to F(x, u_0) \quad \text{in} \quad L^1(M), \tag{3.27}$$

thanks to the generalized Lebesgue dominated convergence theorem, namely *assume* (g_n), (h_n) *are two measurable function sequences on* (M, g). *Moreover* $|g_n| \le h_n$, *a.e.* $(n = 1, 2, \cdots)$; $g_n \to g$, *a.e.;* $h_n \to h$, *a.e.;* $\int_M h_n(x)\mathrm{d}v_g \to \int_M h(x)\mathrm{d}v_g < \infty$. *Then there holds*

$$\lim_{n\to\infty} \int_M g_n(x)\mathrm{d}v_g = \int_M g(x)\mathrm{d}v_g.$$

Thus we obtain by (3.15) and (3.27)

$$\lim_{n\to\infty} \int_M |\nabla u_n|^N \mathrm{d}v_g = N\left(c + \int_M F(x, u_0)\mathrm{d}v_g\right). \tag{3.28}$$

Notice that (3.16) and Lemma 3.6 lead to

$$\int_M |\nabla u_0|^{N-2}\nabla u_0 \nabla v \mathrm{d}v_g - \int_M f(x,u_0)v\mathrm{d}v_g = 0, \quad \forall v \in C^\infty(M).$$

Since $C^\infty(M)$ is dense in $W^{1,N}(M)$, the above identity holds for all $v \in W^{1,N}(M)$. Hence u_0 is a weak solution of problem (1.7). Finally, we will prove that u_0 is nontrivial. Suppose $u_0 \equiv 0$. Then (3.28) gives

$$\lim_{n\to\infty} \int_M |\nabla u_n|^N \mathrm{d}v_g = Nc. \tag{3.29}$$

To proceed, we distinguish two cases:

Case 1: *f is subcritical.*

By definition of subcritical function (1.8), $\forall \alpha : 0 < \alpha < \frac{\alpha_N}{Nc}$, there exists a constant C such that

$$|f(x,u_n)| \le C + e^{\alpha |u_n|^{\frac{N}{N-1}}} \quad \text{for all } n.$$

Take $q > 1$ such that $q\alpha Nc < \alpha_N$. Then

$$\begin{aligned}\int_M |f(x,u_n(x))|^q \mathrm{d}v_g &\le C + C\int_M e^{q\alpha |u_n|^{\frac{N}{N-1}}}\mathrm{d}v_g \\ &\le C + C\int_M e^{q\alpha \|u_n\|_{S_{1,\tau}}^{\frac{N}{N-1}} \left|\frac{u_n}{\|u_n\|_{S_{1,\tau}}}\right|^{\frac{N}{N-1}}}\mathrm{d}v_g \\ &\le C,\end{aligned}$$

thanks to Theorem 1.1. Let $v = u_n$ in (3.16), we have by using the above estimate and $u_n \to 0$ in $L^p(M)$ for all $p \ge 1$,

$$\begin{aligned}\|u_n\|_{S_{1,\tau}}^N &\le \tau_n \|u_n\|_{S_{1,\tau}} + \int_M |f(x,u_n)u_n|\mathrm{d}v_g \\ &\le \tau_n \|u_n\|_{S_{1,\tau}} + \left(\int_M |f(x,u_n(x))|^q \mathrm{d}v_g\right)^{\frac{1}{q}} \left(\int_M |u_n|^{q'}\mathrm{d}v_g\right)^{q'} \\ &\le \tau_n \|u_n\|_{S_{1,\tau}} + C\|u_n\|_{q'} \to 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. This contradicts (3.29). Hence $u_0 \not\equiv 0$.

Case 2: *f is critical.*

By definition of critical function (1.9), $\forall \epsilon > 0, \exists C_\epsilon$ such that

$$|f(x,s)| \le C_\epsilon + e^{(\alpha_0+\epsilon)|s|^{\frac{N}{N-1}}} \quad \text{for all} \quad (x,s) \in M\times\mathbb{R}.$$

By Lemma 3.3, $c < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)$. Clearly $\|u_n\|_{S_{1,\tau}} \to Nc$ thanks to (3.29) and $u_n \to 0$ in $L^p(M)$ for all $p \ge 1$. We choose $\epsilon > 0$ sufficiently small and $q > 1$ sufficiently close to 1 such that $q(\alpha_0+\epsilon)\|u_n\|_{S_{1,\tau}}^{\frac{N}{N-1}} < \alpha_N$ for sufficiently large n. Then

$$
\begin{aligned}
\int\limits_M |f(x, u_n(x))|^q \mathrm{d}v_g &\le 2^q C_\epsilon^q + 2^q \int\limits_M e^{q(\alpha_0+\epsilon)|u_n|^{\frac{N}{N-1}}} \mathrm{d}v_g \\
&\le 2^q C_\epsilon^q + 2^q \int\limits_M e^{q\alpha \|u_n\|_{S_{1,\tau}}^{\frac{N}{N-1}} \left|\frac{u_n}{\|u_n\|_{S_{1,\tau}}}\right|^{\frac{N}{N-1}}} \mathrm{d}v_g \\
&\le C.
\end{aligned}
$$

As in Case 1, we obtain $\|u_n\|_{S_{1,\tau}} \to 0$ which contradicts (3.29). Hence $u_0 \not\equiv 0$. This completes the proof of Theorems 1.3 and 1.4.

3.5 An example of maximizer

In this subsection, we will give an example of maximizer. In view of Theorem 1.1, one has for all $\alpha \le \alpha_{N,m}$

$$
\Lambda_\alpha = \sup_{\|u\|_{S_{m,\tau}} \le 1} \int\limits_M e^{\alpha |u|^{\frac{N}{N-m}}} \mathrm{d}v_g < \infty.
$$

Furthermore we have the following:

Proposition 3.7 *Assume $0 < m < N$. For any $\alpha : 0 < \alpha < \alpha_{N,m}$, there exists a function $u_\alpha \in W^{m,N/m}(M)$ with $\|u\|_{S_{m,\tau}} \le 1$ such that*

$$
\int\limits_M e^{\alpha |u_\alpha|^{\frac{N}{N-m}}} \mathrm{d}v_g = \sup_{\|u\|_{S_{m,\tau}} \le 1} \int\limits_M e^{\alpha |u|^{\frac{N}{N-m}}} \mathrm{d}v_g .
$$

Moreover u_α is a weak solution of the equation

$$
\begin{cases}
-\Delta^{k-1}\left(\mathrm{div}\left(|\nabla\Delta^{k-1}u_\alpha|^{\frac{N}{m}-2}\nabla\Delta^{k-1}u_\alpha\right)\right) + \tau |u_\alpha|^{\frac{N}{m}-2}u_\alpha \\
\qquad = \frac{1}{\lambda_\alpha}|u_\alpha|^{\frac{N}{N-m}-2}u_\alpha e^{\alpha|u_\alpha|^{\frac{N}{N-m}}} \quad \text{when} \quad m = 2k-1,\ k = 1, 2, \ldots, \\
\Delta^k\left(|\Delta^k u_\alpha|^{\frac{N}{m}-2}\Delta^k u_\alpha\right) + \tau |u_\alpha|^{\frac{N}{m}-2}u_\alpha \\
\qquad = \frac{1}{\lambda_\alpha}|u_\alpha|^{\frac{N}{N-m}-2}u_\alpha e^{\alpha|u_\alpha|^{\frac{N}{N-m}}} \quad \text{when} \quad m = 2k,\ k = 1, 2, \ldots, \\
\lambda_\alpha = \int_M |u_\alpha|^{\frac{N}{N-m}} e^{\alpha|u_\alpha|^{\frac{N}{N-m}}} \mathrm{d}v_g, \quad \|u_\alpha\|_{S_{m,\tau}} = 1.
\end{cases}
\tag{3.30}
$$

In particular, when $m = 1$, u_α can be further chosen nonnegative and thus satisfies

$$
-\Delta_N u_\alpha + \tau u_\alpha^{N-1} = \frac{1}{\lambda_\alpha} u_\alpha^{\frac{1}{N-1}} e^{\alpha u_\alpha^{\frac{N}{N-1}}} \quad \text{in} \quad M.
$$

Remark 3.8 In the case when $m = 1$, it is easy to see that $0 < \lambda_\alpha < \alpha_N$ for any $0 < \alpha < \alpha_N$. Proposition 3.8 particularly gives a positive solution of the N-Laplacian equation

$$
-\Delta_N u + \tau |u|^{N-2} u = f(x, u(x)) \quad \text{in} \quad M,
$$

where $f(x, u) = \frac{1}{\lambda_\alpha}|u|^{\frac{1}{N-1}-1} u e^{\alpha|u|^{\frac{N}{N-1}}}$ is critical, λ_α is defined by (3.30). We calculate for all $s > 0$,

$$
F(x, s) = \int\limits_0^s f(x, t)\mathrm{d}t = \frac{N-1}{\alpha\lambda_\alpha N}\left(e^{\alpha s^{\frac{N}{N-1}}} - 1\right).
$$

It can be easily checked that f satisfies (H_1), (H_2), and (H_3). But when $N \geq 3$,

$$\frac{NF(x,s)}{s^N} \to +\infty \quad \text{as} \quad s \to 0+,$$

thus (H_4) does not hold. This possibly yields a new method of studying positive solutions of the above N-Laplacian equation with $f(x,u)$ behaves like $e^{\alpha|u|^{\frac{N}{N-1}}}$ as $|u| \to \infty$.

Remark 3.9 For compactness analysis of the above equations, particularly for extremal functions of the Trudinger–Moser inequality on manifolds, we refer the reader to [11,12].

Proof of Proposition 3.7: It is easy to see that

$$\sup_{\|u\|_{S_{m,\tau}}=1} \int_M e^{\alpha|u|^{\frac{N}{N-m}}} \mathrm{d}v_g = \sup_{\|u\|_{S_{m,\tau}}\leq 1} \int_M e^{\alpha|u|^{\frac{N}{N-m}}} \mathrm{d}v_g = \Lambda_\alpha. \tag{3.31}$$

Take a function sequence u_k with $\|u_k\|_{S_{m,\tau}} = 1$ such that

$$\int_M e^{\alpha|u_k|^{\frac{N}{N-m}}} \mathrm{d}v_g \to \Lambda_\alpha \quad \text{as} \quad k \to \infty.$$

Up to a subsequence, we can assume

$$\begin{aligned} &u_k \rightharpoonup u_\alpha \quad \text{weakly in} \quad W^{m,\frac{N}{m}}(M) \\ &u_k \to u_\alpha \quad \text{strongly in} \quad L^p(M),\ \forall p \geq 1 \\ &u_k \to u_\alpha \quad \text{a.e. in} \quad M. \end{aligned}$$

It follows that

$$\begin{aligned} \int_M |\nabla^m u_\alpha|^{\frac{N}{m}} \mathrm{d}v_g &= \lim_{k\to\infty} \int_M |\nabla^m u_\alpha|^{\frac{N}{m}-2} \nabla^m u_\alpha \nabla^m u_k \mathrm{d}v_g \\ &\leq \limsup_{k\to\infty} \left(\int_M |\nabla^m u_\alpha|^{\frac{N}{m}} \mathrm{d}v_g \right)^{\frac{N-m}{N}} \left(\int_M |\nabla^m u_k|^{\frac{N}{m}} \mathrm{d}v_g \right)^{\frac{m}{N}} \\ &\leq \left(\int_M |\nabla^m u_\alpha|^{\frac{N}{m}} \mathrm{d}v_g \right)^{\frac{N-m}{N}}. \end{aligned}$$

Hence, we obtain $\|u_\alpha\|_{S_{m,\tau}} \leq 1$, thanks to $u_k \to u_\alpha$ strongly in $L^p(M)$ for all $p \geq 1$. On the other hand, the mean value theorem implies that

$$e^{\alpha|u_k|^{\frac{N}{N-m}}} - e^{\alpha|u_\alpha|^{\frac{N}{N-m}}} = e^{\xi}\alpha \left(|u_k|^{\frac{N}{N-m}} - |u_\alpha|^{\frac{N}{N-m}} \right)$$

for some $\xi(x)$ lies between $|u_k(x)|$ and $|u_\alpha(x)|$, and that

$$|u_k|^{\frac{N}{N-m}} - |u_\alpha|^{\frac{N}{N-m}} = \frac{N}{N-m} \zeta^{\frac{m}{N-m}} (|u_k| - |u_\alpha|)$$

for some $\zeta(x)$ lies between $|u_k(x)|$ and $|u_\alpha(x)|$. Notice that u_k is bounded in $L^q(M)$, $u_k \to u_\alpha$ in $L^q(M)$ for all $q \geq 1$, and $e^{\alpha|u_k|^{\frac{N}{N-1}}}$ is bounded in $L^r(M)$ for some $r > 1$, applying the Hölder inequality to the above two equalities, one can derive that

$$\int_M e^{\alpha|u_\alpha|^{\frac{N}{N-m}}} \mathrm{d}v_g = \lim_{k\to\infty} \int_M e^{\alpha|u_k|^{\frac{N}{N-m}}} \mathrm{d}v_g = \Lambda_\alpha.$$

Hence, we obtain by (3.31)

$$\int_M e^{\alpha|u_\alpha|^{\frac{N}{N-m}}} \mathrm{d}v_g = \sup_{\|u\|_{S_{m,\tau}}\le 1} \int_M e^{\alpha|u|^{\frac{N}{N-m}}} \mathrm{d}v_g \tag{3.32}$$

and $\|u_\alpha\|_{S_{m,\tau}} = 1$. Clearly u_α is a critical point of the functional $J_\alpha(u) = \int_M e^{\alpha|u|^{\frac{N}{N-m}}} \mathrm{d}v_g$ under the constraint $\|u\|_{S_{m,\tau}} = 1$. A straightforward calculation shows u_α satisfies the Euler–Lagrange Eq. 3.30.

When $m = 1$, notice that $u \in W^{1,N}(M)$ implies $|u| \in W^{1,N}(M)$ and $\||u|\|_{S_{1,\tau}} \le \|u\|_{S_{1,\tau}}$. If u_α satisfies (3.32) and $\|u_\alpha\|_{S_{1,\tau}} = 1$, then so does $|u_\alpha|$. Hence, u_α can be chosen such that $u_\alpha \ge 0$. □

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