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On the mean curvature of spacelike surfaces in certain three-dimensional Robertson–Walker spacetimes and Calabi–Bernstein's type problems

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Abstract Several uniqueness results for the spacelike slices in certain Robertson–Walker spacetimes are proved under boundedness assumptions either on the mean curvature function of the spacelike surface or on the restriction of the time coordinate on the surface when the mean curvature is a constant. In the nonparametric case, a uniqueness result and a nonexistence one are proved for bounded entire solutions of some constant mean curvature spacelike differential equations.

Keywords Mean curvature · Spacelike surface · Robertson–Walker spacetime · Calabi–Bernstein's problem

1 Introduction

A maximal surface in a three-dimensional Lorentzian manifold is a spacelike surface of zero mean curvature. The classical Calabi–Bernstein's theorem asserts that the only complete maximal surfaces in Lorentz–Minkowski spacetime $\mathbb{L}^3$ are the spacelike planes. This relevant uniqueness result was first proved by Calabi [8] and later extended for maximal hypersurfaces in $\mathbb{L}^{n+1}$ by Cheng and Yau [9]. It can also be stated in terms of the local complex representation of the surface [12,15]. There are even local estimates of the Gauss curvature which implies Calabi–Bernstein's theorem [4,13]. Moreover, a direct simple proof of the nonparametric version, inspired from [10], which uses only Liouville's theorem on harmonic functions on $\mathbb{R}^2$ was given in [21] (see also [3] for more details and related results).

As an application of the generalized maximum principle due to Omori and Yau [18,23] and of the Calabi–Bernstein's theorem, Aiyama [1] and Xin [22] (see also [19] for a first

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weaker version given by Palmer) obtained simultaneously and independently a characterization of spacelike hyperplanes as the only complete spacelike hypersurfaces with zero mean curvature in the Lorentz–Minkowski spacetime whose hyperbolic image is bounded. A second characterization for the spacelike hyperplanes as the only complete spacelike hypersufaces with constant mean curvature (CMC) in Lorentz–Minkowski spacetime, which are bounded between two parallel hyperplanes, has been given by Aledo and Alias [2], also as a consequence of the generalized maximum principle and the Calabi–Bernstein's theorem.

On the other hand, in a much more general setting, spacelike hypersurfaces with constant mean curvature have been extensively studied not only from their mathematical interest, but also because they are important in general relativity. Specially, several uniqueness results for CMC spacelike hypersurfaces in generalized Robertson–Walker spacetimes, and other spacetimes with certain symmetries, have been obtained [5,6], [7] and [17]. Recall that generalized Robertson–Walker spacetimes are warped product of a definite negative one-dimensional base, and a (general) Riemannian manifold as a fiber. This family includes classical Robertson–Walker (RW) spacetimes (i.e., the fiber has constant sectional curvature). Let us remark that in the first three references, the fiber of the ambient spacetime is assumed to be compact (i.e., the spatially closed cosmological case), which leads to complete spacelike hypersurfaces to be compact under suitable extra assumptions, and then several Minkowski's type integral formulas are used. In the last reference, neither compactness of the fiber nor completeness of the spacelike hypersurface are assumed, although the existence of a local maximum of some distinguished function on the spacelike hypersurface is used as assumption.

We are interested now in the case that the RW spacetime M has dimension three, the fiber is the Euclidean plane $\mathbb{R}^2$, and the sectional curvature of M is not zero on any proper open subset, i.e., the warping function is not locally constant, M is then said to be a proper RW spacetime, although the curvature of M satisfies certain natural geometric assumptions arising from relativity theory: the null convergence condition (NCC) or, a stronger assumption, the time convergence condition (TCC). Under the NCC, new Calabi–Bernstein's problems were solved in [16], which allowed to analyze the behavior of Calabi–Bernstein's properties with respect to some perturbations of Lorentz–Minkowski spacetime $\mathbb{L}^3$ (RW spacetimes with fiber $\mathbb{R}^2$ are natural deformations of $\mathbb{L}^3$ close to this spacetime if the warping function is close to the constant 1).

In any RW spacetime M, there is a natural foliation whose leaves, the level surfaces of the time coordinate of M, constitute a distinguished family of spacelike surfaces in M: its spacelike slices. This article is devoted to characterize this family from several points of view. The key starting point is the fact that on any spacelike surface S, the restriction, $f(t)$, of the warping function f of M satisfies a differential equation, (7), which, under suitable assumptions, leads to the function $f(t)(>0)$ on S to be superharmonic. Therefore, provided that S is parabolic, we can conclude that $f(t)$ is constant on S, which implies that t is constant on S when the RW spacetime is proper. Thus, we prove the following uniqueness results (Theorems 4.1, 4.5)

Let M be a proper RW spacetime with fiber $\mathbb{R}^2$ and which obeys the NCC (resp. TCC). The only complete spacelike surfaces S in M whose mean curvature H satisfies

$$H^2 \leq \frac{1}{\langle N, \partial_t\rangle^2}\frac{f'(t)^2}{f(t)^2}, \quad \left(resp. \quad H^2 \leq \frac{f'(t)^2}{f(t)^2}\right)$$

on all S, are the spacelike slices.

Note that in both results, H is not assumed to be constant. The function $-f'(t_0)/f(t_0)$ is, according to our sign choice for the unit normal vector field, the mean curvature of the spacelike slice $t = t_0$ and that $\langle N, \partial_t \rangle$ is the hyperbolic cosine of the hyperbolic angle between the normal vector field N and $-\partial_t$ at any point of S. The second inequality may be interpreted saying that at any $p \in S$, $\mid H(p) \mid$ does not exceed the analogous quantity for the spacelike slice $t = t(p)$. These results specialize to CMC spacelike surfaces; in particular, both extend a known result in the maximal case (Corollary 4.3 and Remark 4.6). Next, we study CMC spacelike surfaces which lie between two spacelike slices. First and in Theorem 5.2, a totally geodesic slice is characterized among all the complete CMC spacelike surfaces bounded between two spacelike slices. Later, after showing, Propositions 5.3, 5.4, that the inequality $H^2 \leq f'(t)^2/f(t)^2$ naturally holds under suitable assumptions on complete CMC spacelike surfaces which lie between two spacelike slices, we obtain (Theorem 5.5)

Let M be a proper RW spacetime with fiber $\mathbb{R}^2$ which obeys the TCC (resp. NCC) and with warping function satisfying either $f' > 0$ or $f' < 0$ (resp. $f'(t_0) = 0$ for some t_0). The only complete CMC spacelike surfaces S which lie between two spacelike slices are also the spacelike slices.

Finally, the last section deals with a Calabi–Bernstein's type problem for CMC spacelike graphs in RW spacetimes. Concretely, for the CMC spacelike differential equation,

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\right) = 2H - \frac{f'(u)}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\left(2 + \frac{\mid Du \mid^2}{f(u)^2}\right) \tag{E.1}$$

$$\mid Du \mid < \lambda f(u), \quad 0 < \lambda < 1, \tag{E.2}$$

where Du denotes the gradient of the function $u \in C^\infty(\Omega)$, $H \in \mathbb{R}$, and $f : I \to \mathbb{R}$ a smooth positive function such that $u(\Omega) \subset I$, the following uniqueness and nonexistence result (Theorem 6.1) is proved

Assume f is not locally constant, f' has no zero (resp. there exists t_0 such that $f'(t_0) = 0$) and $f'' \leq 0$ (resp. $(\log f)'' \leq 0$).

(a) *If $H \neq -\frac{f'(t)}{f(t)}$ for any $t \in I$, then there exists no entire bounded solution to the CMC spacelike differential equation* (E) *of mean curvature H in M.*
(b) *If $H = -\frac{f'(t_0)}{f(t_0)}$ for some $t_0 \in I$, then $u(x, y) = t_0$, is the unique entire bounded solution to the CMC spacelike differential equation (E) of mean curvature H in M.*

In view of the obtained results, it seems natural to wonder if our technique extends to spacelike hypersurfaces of $n(\geq 4)$-dimensional RW spacetimes with fiber $\mathbb{R}^{n-1}$. Recall that $\mathbb{R}^m$, $m \geq 3$ is not parabolic, and therefore, spacelike slices $\{t_0\} \times \mathbb{R}^m$ are not parabolic either. Since we always use paraboliticity to conclude that the spacelike surface must be a spacelike slice, the previously mentioned extension makes no sense.

2 Preliminaries

Let f be a positive smooth function defined on an open interval I of $\mathbb{R}$ and consider $M = I \times \mathbb{R}^2$ endowed with the Lorentzian metric

$$\langle\, ,\, \rangle = -\pi_I^*(\mathrm{d}t^2) + f(\pi_I)^2 \pi_{\mathbb{R}^2}^*(g_0), \tag{1}$$

where π_I and $\pi_{\mathbb{R}^2}$ denote the projections onto I and $\mathbb{R}^2$, respectively, and g_0 is the usual Riemannian metric of $\mathbb{R}^2$. The Lorentzian manifold $(M, \langle\, ,\, \rangle)$ is the warped product, in the sense of [20, p. 204], with base (I, $-\mathrm{d}t^2$), fiber $(\mathbb{R}^2, g_0)$ and warping function f. We will refer to M as a Robertson–Walker (RW) spacetime.

On M consider the vector field $\xi := f(\pi_I)\, \partial_t$, which is timelike and satisfies

$$\overline{\nabla}_X \xi = f'(\pi_I)\, X, \tag{2}$$

for any $X \in \mathfrak{X}(M)$, where $\overline{\nabla}$ denotes the Levi–Civita connection of the metric (1), [20, Cor. 7.35]. Thus, ξ is conformal with $\mathcal{L}_\xi \langle\, ,\, \rangle = 2\, f'(\pi_I)\, \langle\, ,\, \rangle$ and its metrically equivalent 1-form is closed.

Being three-dimensional M, its curvature is completely determined by its Ricci tensor, and this obviously depends on f; actually, M is flat if and only if f is constant [20, Cor. 7.43]. Here, we are interested in the case M is not flat but its curvature satisfies a natural geometric assumption arising from relativity theory. In fact, this condition on a spacetime is necessary in order that the spacetime obeys Einstein's equation (with zero cosmological constant). Therefore, we recall that a Lorentzian manifold obeys the NCC, when its Ricci tensor, $\overline{\mathrm{Ric}}$, satisfies

$$\overline{\mathrm{Ric}}(Z, Z) \geq 0,$$

for any null tangent vector Z, i.e., $Z \neq 0$ and such that $\langle Z, Z \rangle = 0$. Taking into account that the fiber of the RW spacetime M is flat, and making use again of [20, Cor. 7.43] we get,

$$\overline{\mathrm{Ric}}(Z, Z) = -(\log f)''\, \langle Z, \partial_t \rangle^2, \tag{3}$$

for any null tangent vector Z. Therefore, the RW spacetime M obeys the NCC if and only if its warping function satisfies

$$(\log f)'' \leq 0. \tag{4}$$

The more restrictive condition $f'' \leq 0$ holds when we have (3) for any timelike Z (and hence for any causal Z by continuity) is known as the TCC.

3 Set up

3.1 The restriction of the warping function on a spacelike surface

Let $x : S \longrightarrow M$ be a (connected) spacelike surface in M; that is, x is an immersion and induces a Riemannian metric on the two-dimensional manifold S from the Lorentzian metric (1). It should be noted that any spacelike surface in M is orientable and noncompact [5]. As usual, we agree to represent the induced metric with the same symbol as the metric (1) does. The unitary timelike vector field $\partial_t := \partial/\partial t \in \mathfrak{X}(M)$ determines a time orientation on M. Then, the time orientability of M allows us to consider $N \in \mathfrak{X}^\perp(S)$ as the only, globally defined, unitary timelike normal vector field on S in the same time orientation of $-\partial_t$. Thus, from the wrong way Cauchy–Schwarz inequality (see [20, Prop. 5.30]), for instance) we have $\langle N, \partial_t \rangle \geq 1$ and $\langle N, \partial_t \rangle = 1$ at a point p if and only if $N(p) = -\partial_t(p)$. We define spacelike slice to be a spacelike surface x such that $\pi_I \circ x$ is a constant. A spacelike surface is a spacelike slice if and only if it is orthogonal to ∂_t or, equivalently, orthogonal to ξ. Denote by $\partial_t^T := \partial_t + \langle N, \partial_t \rangle N$ the tangential component of ∂_t on S. It is not difficult to see

$$\nabla t = -\partial_t^T, \tag{5}$$

where ∇t is the gradient of $t := \pi_I \circ x$. Now, from the Gauss formula, taking into account $\xi^T = f(t)\, \partial_t^T$ and (5), the Laplacian of t satisfies

$$\Delta t = -\frac{f'(t)}{f(t)} \left\{ 2 + \mid \nabla t \mid^2 \right\} - 2H \, \langle N, \partial_t \rangle, \tag{6}$$

where $f(t) := f \circ t$, $f'(t) := f' \circ t$ and the function $H := -(1/2)\,\mathrm{trace}(A)$, where A is the shape operator associated to N, is called the mean curvature of S relative to N. A spacelike surface S with constant mean curvature is a critical point of the area functional under a certain volume constraint (see [11], for instance). A spacelike surface with $H = 0$ is called maximal. Note that with our choice of N, the shape operator of the spacelike slice $t = t_0$ is $A = (f'(t_0)/f(t_0))\, I$ and $H = -f'(t_0)/f(t_0)$.

A direct computation from (6) and (5) gives

$$\Delta f(t) = -2\, \frac{f'(t)^2}{f(t)} + f(t)(\log f)''(t) \mid \nabla t \mid^2 - 2 f'(t) H \langle N, \partial_t \rangle, \tag{7}$$

for any spacelike surface of the RW spacetime M.

3.2 The Gauss curvature of a spacelike surface

From the Gauss equation of a spacelike surface S in M and taking in mind the expression for the Ricci tensor of M [20, Prob. 7.13], the Gauss curvature K of S satisfies

$$K = \frac{f'(t)^2}{f(t)^2} - (\log f)''(t) \mid \nabla t \mid^2 - 2H^2 + \frac{1}{2}\,\mathrm{trace}(A^2), \tag{8}$$

where

$$\frac{f'(t)^2}{f(t)^2} - (\log f)''(t) \mid \nabla t \mid^2$$

is, at any $p \in S$, the sectional curvature in M of the tangent plane $\mathrm{d}x_{\mathrm{p}}(T_{\mathrm{p}}S)$.

4 Mean curvature and parabolicity

Consider a spacelike surface S with mean curvature H in a RW spacetime M, which obeys the NCC. Assume $H^2 \leq (f'(t)^2/f(t)^2)$. Then, using in (8) the classical Cauchy–Schwarz inequality, we have that the Gauss curvature of S satisfies $K \geq 0$. If in addition we suppose that S is complete, then we get that S is parabolic, making use of a classical result by Ahlfors and Blanc–Fiala–Huber, (see [14] for instance). Under a stronger assumption on H, we can derive the following uniqueness result

Theorem 4.1 *Let M be a proper RW spacetime with fiber $\mathbb{R}^2$ and which obeys the NCC. The only complete spacelike surfaces S in M, whose mean curvature H satisfies*

$$H^2 \leq \frac{1}{\langle N, \partial_t \rangle^2} \frac{f'(t)^2}{f(t)^2}$$

on all S, are the spacelike slices.

Proof From (6), using $-2ab \leq a^2 + b^2$ for $a, b \in \mathbb{R}$ with $a = \frac{f'(t)}{f(t)}$, $b = H\langle N, \partial_t \rangle$, we get

$$\begin{aligned} \frac{1}{f(t)} \Delta f(t) &\leq -2 \frac{f'(t)^2}{f(t)^2} + (\log f)''(t) \mid \nabla t \mid^2 + \frac{f'(t)^2}{f(t)^2} + H^2 \langle N, \partial_t \rangle^2 \\ &= H^2 \langle N, \partial_t \rangle^2 - \frac{f'(t)^2}{f(t)^2} + (\log f)''(t) \mid \nabla t \mid^2 \leq 0. \end{aligned}$$

Thus, $f(t)$ is a positive superharmonic function on the parabolic surface S. Therefore $f(t)$ must be constant and then S is necessarily a spacelike slice ([16], Lemma 2.1) □

Consequently, we have

Corollary 4.2 *The only CMC spacelike surfaces S in a proper RW spacetime with fiber $\mathbb{R}^2$, and which obeys the NCC, whose mean curvature H satisfies*

$$H^2 \leq \inf_S \left(\frac{1}{\langle N, \partial_t \rangle^2} \frac{f'(t)^2}{f(t)^2} \right)$$

are the spacelike slices.

In particular we have reproved [16, Cor. 5.1]

Corollary 4.3 *The only complete maximal surfaces in a proper RW spacetime, with fiber $\mathbb{R}^2$ and which obeys the NCC, are the spacelike slices $t = t_0$ with $f'(t_0) = 0$.*

Remark 4.4 Note that $\langle N, \partial_t \rangle^2 = \cosh^2 \theta$ where θ is, at any point of S, the hyperbolic angle between N and $-\partial_t$. Consider the family of proper RW spacetimes $\mathbb{R} \times \mathbb{R}^2$, with $f(t) = a\, e^{\rho t}$ $a \in \mathbb{R}^+$, $\rho \in \mathbb{R} \setminus \{0\}$. Clearly, each of its members M obeys the NCC. Let S be a complete CMC spacelike surface on M whose mean curvature never vanishes. If $\cosh^2 \theta \leq (\rho^2/H^2)$, then Theorem 4.1 gives that S is necessarily a spacelike slice.

Theorem 4.5 *Let M be a proper RW spacetime with fiber $\mathbb{R}^2$ and which obeys TCC. The only complete spacelike surfaces S in M whose mean curvature H satisfies*

$$H^2 \leq \frac{f'(t)^2}{f(t)^2}$$

on all S, are the spacelike slices.

Proof A similar reasoning as in Theorem 4.1 gives

$$\frac{1}{f(t)} \Delta f(t) \leq \left(H^2 - \frac{f'(t)^2}{f(t)^2} \right) \langle N, \partial_t \rangle^2 + \frac{f''(t)}{f(t)} \mid \nabla t \mid^2 .$$

Now we have that the Gauss curvature K of S satisfies $K \geq 0$, using (8). On the other hand, the first term of the right-hand side of this inequality is nonpositive and the second one is also nonpositive using TCC. This concludes the proof. □

Remark 4.6 Taking into account that, under the assumptions of Corollary 4.3, the NCC necessarily implies the TCC, we can also derive it as a consequence of Theorem 4.5. Moreover, an analogous result to Corollary 4.2 can be also stated from Theorem 4.5.

Remark 4.7 It should be noted that the assumptions on the mean curvature H, under the NCC (or the TCC), in Theorems 4.1, 4.5, and Corollary 4.2 not only lead to $K \geq 0$ (and so to the paraboliticity of the complete spacelike surface S), but they also imply that the restriction on S of the warping function, $f(t)$, is superharmonic. Even in the maximal case, in which the inequalities for H are trivially satisfied, we need the curvature assumption on the ambient spacetime to conclude that $f(t)$ is superharmonic. On the other hand, if we omit the assumptions on H, paraboliticity is proved as follows. A particular case of [5, Lemma 3.1] provides that a complete spacelike surface such that $f(t)$ is bounded must be diffeomorphic to $\mathbb{R}^2$. In this case, parabolicity can be achieved assuming $\int_S \max(0, -K)dS < \infty$, which is weaker than $K \geq 0$. However, we cannot prove that $f(t)$ is superharmonic; therefore, we need to include that as an assumption.

5 Bounded CMC spacelike surfaces

Recall the following generalized maximum principle for complete Riemannian manifolds due to Omori [18] and Yau [23].

Theorem 5.1 *Let S be a complete Riemannian manifold whose Ricci curvature is bounded from below and let $u : S \longrightarrow \mathbb{R}$ be a smooth function bounded from below on S (resp. bounded from above on S). Then, for each $\varepsilon > 0$, there exists a point $p_\varepsilon \in S$ such that*

(1) $|\nabla u(p_\varepsilon)| < \varepsilon$,

(2) $\Delta u(p_\varepsilon) > -\varepsilon$ (*resp.* $\Delta u(p_\varepsilon) < \varepsilon$),

(3) $\inf u \leq u(p_\varepsilon) \leq \inf u + \varepsilon$ (*resp.* $\sup u - \varepsilon < u(p_\varepsilon) \leq \sup u$).

Using this tool, we get

Theorem 5.2 *Let M be proper RW spacetime with fiber $\mathbb{R}^2$, which obeys the NCC, and assume there exists $t_0 \in I$ such that $f'(t_0) = 0$. Then, the only complete CMC spacelike surface S such that $\sup t(S) \geq t_0$, $\inf t(S) \leq t_0$ and which lies between the spacelike slices $t = t_1$ and $t = t_2$, with $t_1 \leq t_0 \leq t_2$, is the totally geodesic spacelike slice $t = t_0$.*

Proof From (6), we have

$$H = \frac{\frac{-f'(t)}{f(t)}\{2 + |\nabla t|^2\} - \Delta t}{2\langle N, \partial_t\rangle}. \tag{9}$$

The function $t (= \pi_I \circ x)$ is bounded from above by t_2, thus the Omori–Yau-generalized maximum principle says that for each $\varepsilon > 0$, there exists a $p_\varepsilon \in S$ such that $|\nabla t(p_\varepsilon)| < \varepsilon$, $\Delta t(p_\varepsilon) < \varepsilon$ and $\sup t - \varepsilon \leq t(p_\varepsilon) < \sup t$.

Therefore, (9) gives

$$H \geq \frac{\frac{-f'(t(p_\varepsilon))}{f(t(p_\varepsilon))}\{2 + |\nabla t|^2(p_\varepsilon)\} - \varepsilon}{2\langle N, \partial_t\rangle(p_\varepsilon)}. \tag{10}$$

Taking into account that M obeys the NCC,

$$\lim_{\varepsilon\to 0}\frac{-f'(t(p_\varepsilon))}{f(t(p_\varepsilon))}\{2+\mid\nabla t\mid^2(p_\varepsilon)\}-\varepsilon=2\frac{-f'(\sup t(S))}{f(\sup t(S))}\geq 0$$

therefore, we obtain from (10) that $H\geq 0$. On the other hand, the function t is also bounded from below by t_1; thus, the Omori–Yau-generalized maximum principle implies that for each $\varepsilon>0$, there exists a $p_\varepsilon\in S$ such that $\mid\nabla t(p_\varepsilon)\mid<\varepsilon$, $\Delta t(p_\varepsilon)>-\varepsilon$ and $\inf t\leq t(p_\varepsilon)<\inf t+\varepsilon$. Hence, making use again of (9), we get

$$H\leq\frac{\frac{-f'(t(p_\varepsilon))}{f(t(p_\varepsilon))}\{2+\mid\nabla t\mid^2(p_\varepsilon)\}+\varepsilon}{2\langle N,\partial_t\rangle(p_\varepsilon)},\tag{11}$$

and using again that M obeys the NCC, we get

$$\lim_{\varepsilon\to 0}\frac{-f'(t(p_\varepsilon))}{f(t(p_\varepsilon))}\{2+\mid\nabla t\mid^2(p_\varepsilon)\}+\varepsilon=2\frac{-f'(\inf t(S))}{f(\inf t(S))}\leq 0,$$

obtaining now from (11) the other inequality $H\leq 0$. The conclusion follows then from Corollary 4.3. □

According to the previous result, if there exists a complete CMC spacelike surface, with $H\neq 0$, which lies between two spacelike slices in M, then it must be contained either in the region $t\leq t_0$ or in $t\geq t_0$. Even more, we have the following

Proposition 5.3 *Let M be a proper RW spacetime with fiber $\mathbb{R}^2$, which obeys the NCC and assume that there exists $t_0\in I$ such that $f'(t_0)=0$. If a complete CMC spacelike surface S in M satisfies $t(S)\subset[t_0,t_2]$, $t_0<t_2$ (resp. $t(S)\subset[t_1,t_0]$, $t_1<t_0$), then $0\leq H\leq-\frac{f'(t)}{f(t)}$ (resp. $-\frac{f'(t)}{f(t)}\leq H\leq 0$) holds on S. In any case, we have*

$$H^2\leq\frac{f'(t)^2}{f(t)^2}$$

on S. Moreover, when $H\neq 0$ we have $\inf t(S)>t_0$ *(resp.* $\sup t(S)<t_0$*).*

Proof If M obeys the NCC and there exists $t_0\in I$ such that $f'(t_0)=0$, then t_0 is the unique critical point of f and $\sup f(I)=f(t_0)$, i.e., f attains its global maximum at t_0 [16]. Note that $f'(t)\leq 0$ in the first case. Being the function t bounded from above, we have $H\geq 0$.

On the other hand, as t is bounded from below by t_0, from the Omori–Yau-generalized maximum principle and using $\langle N,\partial_t\rangle\geq 1$, we have that for each $\varepsilon>0$, there exists $p_\varepsilon\in S$ such that

$$H\leq\frac{\frac{-f'(t(p_\varepsilon))}{f(t(p_\varepsilon))}\{2+\mid\nabla t\mid^2(p_\varepsilon)\}+\varepsilon}{2}.\tag{12}$$

If $\varepsilon\longrightarrow 0$ in (12), we get

$$H\leq\frac{-f'(\inf t(S))}{f(\inf t(S))}.$$

Even more, if $\inf t(S)=t_0$, then $H=0$ and S is the maximal slice. Now, the function $-f'(t)/f(t)$ is increasing because the NCC holds, and therefore,

$$H\leq\frac{-f'(t)}{f(t)}.$$

In the second case, we have $f'(t) \geq 0$, and as t is bounded from below for t_1, $H \leq 0$. Moreover, if S is not the maximal slice, the inequality is strict. Again from Theorem 5.1, for each $\varepsilon > 0$, there exists $p_\varepsilon \in S$ such that

$$H \geq \frac{\frac{-f'(t(p_\varepsilon))}{f(t(p_\varepsilon))}\{2+ \mid \nabla t \mid^2 (p_\varepsilon)\} - \varepsilon}{2}. \tag{13}$$

If $\varepsilon \longrightarrow 0$ in (13), we have

$$H \geq \frac{-f'(\sup t(S))}{f(\sup t(S))}.$$

If $\sup t(S) = t_0$, then $H = 0$ and S is the maximal slice. In a similar way as before, we conclude

$$\frac{-f'(t)}{f(t)} \leq H \leq 0,$$

which ends the proof. □

An analogous argument to the previous one gives

Proposition 5.4 *Let M be a proper RW spacetime with fiber $\mathbb{R}^2$ which obeys the NCC and with warping function satisfying either $f' > 0$ or $f' < 0$. If a complete CMC spacelike surface S lies between two spacelike slices, then $f'H < 0$ and*

$$H^2 \leq \frac{f'(t)^2}{f(t)^2}$$

holds on all S.

Theorem 5.5 *Let M be a proper RW spacetime with fiber $\mathbb{R}^2$ which obeys the TCC (resp. NCC) and with warping function satisfying either $f' > 0$ or $f' < 0$ (resp. $f'(t_0) = 0$ for some $t_0 \in I$). The only complete CMC spacelike surfaces S which lie between two spacelike slices are also the spacelike slices.*

Proof Observe that if $f'(t_0) = 0$ for some $t_0 \in I$, this t_0 is unique and the NCC implies in fact the TCC. On the other hand, the conclusion of Proposition 5.4 holds true, and then the proof follows from Theorem 4.5. □

6 A Calabi–Bernstein's type problem

In this section, we will deal with the CMC spacelike differential equation (E) announced in Sect. 1. The graph $\Sigma = \{(u(x, y), x, y) : (x, y) \in \Omega\}$ of any solution u to the equation (E) is a CMC spacelike surface, with constant mean curvature H in a RW spacetime defined as the warped product of base I, fiber Ω, and warping function f. We are interested in the entire solutions, i.e., defined on all $\mathbb{R}^2$, to equation (E). Put $\mathbf{B}_f = \{-f'(t)/f(t) : t \in I\} \subset \mathbb{R}$, and denote, as previously, by M the RW spacetime with base I, fiber $\mathbb{R}^2$ and warping function f.

Theorem 6.1 *Assume that f is not locally constant, f' has no zero (resp. there exists $t_0 \in I$ such that $f'(t_0) = 0$) and $f'' \leq 0$ (resp. $(\log f)'' \leq 0$).*

(a) *If $H \notin \mathbf{B}_f$, then there exists no entire bounded solution to the CMC spacelike differential equation* (E) *of mean curvature H in M.*

(b) *If $H \in \mathbf{B}_f$, then $u(x,y) = t_0$, where $H = -\frac{f'(t_0)}{f(t_0)}$, is the unique entire bounded solution to the CMC spacelike differential equation* (E) *of mean curvature H in M.*

Proof On the graph of a solution of (E), the constraint (E.2) may be expressed as follows:

$$\langle N, \partial_t \rangle < \frac{1}{\sqrt{1-\lambda^2}}. \tag{14}$$

The induced metric on the graph is represented on the plane $\mathbb{R}^2$ by

$$g_u = -\mathrm{d}u^2 + f(u)^2(\mathrm{d}x^2 + \mathrm{d}y^2),$$

and therefore, the assumption (14) gives

$$g_u\big((a,b),(a,b)\big) \geq (1-\lambda^2) f(u)^2 (a^2+b^2),$$

for all $(a,b) \in \mathbb{R}^2$. From our assumptions, we have $\inf f(u) > 0$, and therefore, the metric g_u is complete. Now, the result follows from Theorem 5.5. □

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