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Curvature homogeneous Lorentzian three-manifolds

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Abstract We study three-dimensional curvature homogeneous Lorentzian manifolds. We prove that for all Segre types of the Ricci operator, there exist examples of nonhomogeneous curvature homogeneous Lorentzian metrics in $\mathbb{R}^3$.

Keywords Lorentzian manifolds · Curvature homogeneity · Segre types

Mathematics Subject Classification (2000) 53C50 · 53C20 · 53C25

1 Introduction and main results

A pseudo-Riemannian manifold (M, g) is called *curvature homogeneous up to order k* if, for any points $p, q \in M$, there exists a linear isometry $\phi : T_pM \to T_qM$ such that $\phi^*(\nabla^i R(q)) = \nabla^i R(p)$ for all $i \leq k$. When $k = 0$, (M, g) is simply called a *curvature homogeneous space*. A locally homogeneous space is curvature homogeneous of any order k. Conversely, curvature homogeneity up to order k implies local homogeneity when k is sufficiently high. This result was proved by Singer [12] for Riemannian manifolds and extended to the pseudo-Riemannian case through the equivalence theorem for G-structures due to Cartan and Sternberg [13].

If $\dim M = 2$, then curvature homogeneity (up to order 0) already implies local homogeneity. However, when $\dim M \geq 3$, a curvature homogeneous space needs not to be locally homogeneous. Three-dimensional spaces are natural candidates for a deep investigation about curvature homogeneity, because in dimension three the curvature tensor is completely determined by the Ricci tensor, and curvature homogeneity is equivalent to requiring that there exists, at least locally, a pseudo-orthonormal frame field with respect to which the Ricci components are constant.

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In dimension three, the Riemannian case and the Lorentzian one split apart. In fact, K. Sekigawa [11] proved that a three-dimensional Riemannian manifold, which is curvature homogeneous up to order one, is locally homogeneous. On the other hand, Bueken and Vanhecke [4] found the first examples of nonhomogeneous Lorentzian three-manifolds which are curvature homogeneous up to order one. The full classification of three-dimensional Lorentzian manifolds curvature homogeneous up to order one was obtained by Bueken and Djorić in [3], where they also proved that curvature homogeneity up to order two implies local homogeneity for a three-dimensional Lorentzian manifold.

Differences arising between the Riemannian and the Lorentzian cases are essentially due to the different behavior of self-adjoint operators in these frameworks. Because of the symmetries of the curvature tensor, the Ricci tensor ϱ is symmetric. Hence, the *Ricci operator* Q, defined by $g(QX, Y) = \varrho(X, Y)$, is self-adjoint. Consequently, at each point of a Riemannian manifold there exists an orthonormal basis diagonalizing Q, while for a Lorentzian manifold four different cases can occur [3,10], known as *Segre types*, which depend on the multiplicity of the Ricci eigenvalues and on the dimension of the corresponding eigenspaces. The possible cases are the following:

1. *Segre type* $\{11, 1\}$: the Ricci operator itself is symmetric and so, diagonalizable. The comma is used to separate the spacelike and timelike eigenvectors. In the degenerate case, at least two of the Ricci eigenvalues coincide.
2. *Segre type* $\{1z\bar{z}\}$: the Ricci operator has one real and two complex conjugate eigenvalues.
3. *Segre type* $\{21\}$: the Ricci operator has two real eigenvalues (coinciding in the degenerate case), one of which has multiplicity two and each associated to a one-dimensional eigenspace.
4. *Segre type* $\{3\}$: the Ricci operator has three equal eigenvalues, associated to a one-dimensional eigenspace.

In particular, at each point $p \in M$ there exists a pseudo-orthogonal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, such that Q takes one of the following forms:

$$\text{Segre type } \{11, 1\} : \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \text{Segre type } \{1z\bar{z}\} : \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \ c \neq 0, \qquad (1.1)$$

$$\text{Segre type } \{21\} : \begin{pmatrix} a & 0 & 0 \\ 0 & b & \varepsilon \\ 0 & -\varepsilon & b - 2\varepsilon \end{pmatrix}, \ \varepsilon = \pm 1, \quad \text{Segre type } \{3\} : \begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix}, \ a \neq 0.$$

When (M, g) is a curvature homogeneous Lorentzian three-space, starting from a pseudo-orthonormal basis $\{(e_i)_p\}$ at a fixed point p, we can use the linear isometries from T_pM into the tangent spaces at any other point, to construct a pseudo-orthonormal frame field $\{e_i\}$, such that the components of ϱ with respect to $\{e_i\}$ remain constant along M. Hence, Q has constant eigenvalues and the same Segre type at any point $p \in M$. A natural question to ask is the following:

Do there exist nonhomogeneous curvature homogeneous Lorentzian three-manifolds for all Segre types of the Ricci operator?

A negative answer holds for Lorentzian three-manifolds curvature homogeneous up to order one: non-homogeneous examples only occur for degenerate Segre types $\{11, 1\}$ and $\{21\}$ (see [3]). For the case when the Ricci operator is of Segre type either $\{1z\bar{z}\}$ or $\{3\}$, Bueken and Djorić [3] wrote: "We do not know, however, if there exist non-homogeneous curvature homogeneous three-dimensional Lorentzian manifolds whose Ricci operator is of this type or curvature homogeneity is sufficient to guarantee local homogeneity of the manifolds of this type." Also three-dimensional Einstein-like curvature homogeneous Lorentzian metrics do not provide nonhomogeneous examples for these Segre types of the Ricci operator [7].

The aim of this article is to answer the question above, by proving the following:

Maim Theorem *Three-dimensional nonhomogeneous curvature homogeneous Lorentzian metrics exist for all different Segre types of the Ricci operator (except in the degenerate diagonal case with three equal Ricci eigenvalues, when the manifold has necessarily constant sectional curvature).*

To prove our Main Theorem, we shall take into account the previous results on curvature homogeneous Lorentzian three-spaces. Bueken provided curvature homogeneous examples in the diagonal case with two distinct Ricci eigenvalues [1] and for case of degenerate Segre type {21} [2]. Some examples with diagonalizable Ricci operator and constant Ricci eigenvalues were described in [8]. In [6], the author generalized to pseudo-Riemannian manifolds the powerful technique introduced by Kowalski and Prüfer in [9], to build examples with diagonal Ricci operator and constant distinct Ricci eigenvalues.

To our knowledge, references above cover the known examples of curvature homogeneous Lorentzian three-spaces. These examples focus on the diagonal case, except for [2], where the Ricci operator is assumed to be of degenerate Segre type {21} (all the Ricci eigenvalues coincide and the corresponding eigenspace is two-dimensional).

In this article, we shall provide some families of explicit examples of nonhomogeneous curvature homogeneous Lorentzian metrics on $\mathbb{R}^3$, whose Ricci operator is either of Segre type $\{1z\bar{z}\}$, of Segre type {3}, or of nondegenerate Segre type {21}. Together with the above-cited results, these new examples complete the proof of our Main Theorem.

For the different Segre types, a different approach will be used to determine these examples. After giving a general description of curvature homogeneous Lorentzian three-spaces in Sect. 2, examples with Ricci operator of Segre type {3} and nondegenerate {21} will be given in Sect. 3 inside the class of Lorentzian three-spaces admitting a parallel degenerate line field [8]. On the other hand, the case of Ricci operator of Segre type $\{1z\bar{z}\}$ will be dealt with in Sect. 4, by describing these spaces via a system of differential equations and finding explicit solutions.

2 Curvature homogeneous Lorentzian 3-spaces: a general description

Let (M, g) be a connected three-dimensional Lorentzian manifold. We denote by ∇ the Levi Civita connection of (M, g) and by R its curvature tensor, taken with the sign convention

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

Since M is three-dimensional, R is completely determined by the Ricci tensor ϱ, defined by

$$\varrho(X, Y) = \sum_{i=1}^{3} \varepsilon_i g(R(X, e_i)Y, e_i), \tag{2.1}$$

for any vector fields X, Y, where $\{e_1, e_2, e_3\}$ is a (local) pseudo-orthonormal basis of T_pM with e_3 timelike, that is, $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$. As we already pointed out, if (M, g) is curvature homogeneous, then the Ricci operator Q has the same Segre type at any point $p \in M$ and has constant eigenvalues. Let $\{e_i\}$ denote a local pseudo-orthonormal frame field on (M, g), with respect to which the Ricci components are constant. Following [5], we then put

$$\nabla_{e_i} e_j = \sum_k \varepsilon_j b^i_{jk} e_k, \tag{2.2}$$

for all indeces i, j. Clearly, the functions b^i_{jk} determine completely the Levi Civita connection, and conversely. Note that from $\nabla g = 0$ it follows at once

$$b^i_{kj} = -b^i_{jk}, \tag{2.3}$$

for all i, j, k. In particular,

$$b^i_{jj} = 0 \tag{2.4}$$

for all indices i and j. We now put

$$\begin{aligned} &b^1_{12} = \alpha, \quad b^1_{13} = \beta \quad b^1_{23} = \gamma, \quad b^2_{12} = \kappa, \quad b^2_{13} = \mu, \quad b^2_{23} = \nu, \quad b^3_{12} = \sigma, \\ &b^3_{13} = \tau, \quad b^3_{23} = \psi. \end{aligned} \tag{2.5}$$

By (2.2–2.5) we get that the Levi Civita connection ∇ of (M, g) is completely determined by

$$\begin{aligned} &\nabla_{e_1} e_1 = \alpha\, e_2 + \beta\, e_3, \quad \nabla_{e_2} e_1 = \kappa\, e_2 + \mu\, e_3, \quad \nabla_{e_3} e_1 = \sigma\, e_2 + \tau\, e_3, \\ &\nabla_{e_1} e_2 = -\alpha\, e_1 + \gamma\, e_3, \nabla_{e_2} e_2 = -\kappa\, e_1 + \nu\, e_3, \nabla_{e_3} e_2 = -\sigma\, e_1 + \psi\, e_3, \\ &\nabla_{e_1} e_3 = \beta\, e_1 + \gamma\, e_2, \quad \nabla_{e_2} e_3 = \mu\, e_1 + \nu\, e_2, \quad \nabla_{e_3} e_3 = \tau\, e_1 + \psi\, e_2. \end{aligned} \tag{2.6}$$

In particular, from (2.6) we get at once

$$\begin{aligned} &[e_1, e_2] = -\alpha\, e_1 - \kappa\, e_2 + (\gamma - \mu)\, e_3, \\ &[e_1, e_3] = \beta\, e_1 + (\gamma - \sigma)\, e_2 - \tau e_3, \\ &[e_2, e_3] = (\mu + \sigma)\, e_1 + \nu\, e_2 - \psi\, e_3. \end{aligned} \tag{2.7}$$

Conversely, functions (b^i_{jk}) are completely determined by the Lie brackets of vector fields e_1, e_2, e_3. In fact, the well-known *Koszul formula* [10] yields

$$2\varepsilon_j \varepsilon_k b^i_{jk} = 2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) - g([e_j, e_k], e_i) + g([e_k, e_i], e_j). \tag{2.8}$$

We can now compute the components of curvature tensor with respect to $\{e_i\}$. Starting from (2.6), standard calculations give

$$\begin{cases} R_{1212} = e_2(\alpha) - e_1(\kappa) - \alpha^2 - \kappa^2 + \beta\nu - \gamma\mu + \sigma(\gamma - \mu), \\ R_{1313} = e_1(\tau) - e_3(\beta) - \beta^2 + \tau^2 - \alpha\psi + \gamma\sigma - \mu(\gamma - \sigma), \\ R_{2323} = e_2(\psi) - e_3(\nu) - \nu^2 + \psi^2 + \kappa\tau - \mu\sigma - \gamma(\mu + \sigma), \\ R_{1213} = e_1(\mu) - e_2(\beta) + \alpha(\beta - \nu) + \gamma(\kappa - \tau) + \mu(\kappa + \tau), \\ R_{1323} = e_1(\psi) - e_3(\gamma) - \gamma(\beta + \nu) - \sigma(\beta - \nu) + \tau(\alpha + \psi), \\ R_{1223} = e_3(\kappa) - e_2(\sigma) + \alpha(\mu + \sigma) + \nu(\kappa - \tau) + \psi(\mu - \sigma). \end{cases} \tag{2.9}$$

We now use (2.9) in (2.1) to calculate the Ricci components in function of (b^i_{jk}). We get

$$\begin{cases} \varrho_{11} = e_2(\alpha) + e_3(\beta) - e_1(\kappa) - e_1(\tau) - \alpha^2 + \beta^2 - \kappa^2 - \tau^2 + \alpha\psi + \beta\nu - 2\mu\sigma, \\ \varrho_{22} = e_2(\alpha) - e_1(\kappa) + e_3(\nu) - e_2(\psi) - \alpha^2 - \kappa^2 + \nu^2 - \psi^2 + \beta\nu - \kappa\tau + 2\gamma\sigma, \\ \varrho_{33} = e_1(\tau) - e_3(\beta) - e_3(\nu) + e_2(\psi) - \beta^2 + \tau^2 - \nu^2 + \psi^2 - \alpha\psi + \kappa\tau - 2\gamma\mu, \\ \varrho_{12} = e_3(\gamma) - e_1(\psi) + \gamma(\beta + \nu) + \sigma(\beta - \nu) - \tau(\alpha + \psi), \\ \varrho_{13} = e_2(\sigma) - e_3(\kappa) - \alpha(\mu + \sigma) - \nu(\kappa - \tau) - \psi(\mu - \sigma), \\ \varrho_{23} = e_1(\mu) - e_2(\beta) + \alpha(\beta - \nu) + \gamma(\kappa - \tau) + \mu(\kappa + \tau). \end{cases} \tag{2.10}$$

Before specializing our study to the different Segre types, we remark that functions α, .., ψ are not all independent. In fact, from (2.2) and the constancy of ϱ_{ij} it easily follows

$$\nabla_i \varrho_{jk} = -\sum_t \Big(\varepsilon_j b^i_{jt} \varrho_{tk} + \varepsilon_k b^i_{kt} \varrho_{tj} \Big), \tag{2.11}$$

for all indeces i, j, k. Since (M, g) is curvature homogeneous, its scalar curvature $r = \mathrm{tr}\varrho$ is constant. The well-known *divergence formula* $dr = 2\mathrm{div}\varrho$ [10] then implies

$$\sum_j \varepsilon_j \nabla_j \varrho_{ij} = 0 \quad \text{for all } i, \tag{2.12}$$

which, taking into account (2.11), gives some restrictions for the connection functions (2.5). Explicitly, from (2.12) we get

$$\begin{cases} \varrho_{11}(\kappa + \tau) - \varrho_{22}\kappa + \varrho_{33}\tau - \varrho_{12}(2\alpha - \psi) - \varrho_{13}(2\beta + \nu) - \varrho_{23}(\mu - \sigma) = 0, \\ \varrho_{11}\alpha - \varrho_{22}(\alpha - \psi) + \varrho_{33}\psi + \varrho_{12}(2\kappa + \tau) - \varrho_{13}(\gamma + \sigma) - \varrho_{23}(\beta + 2\nu) = 0, \\ \varrho_{11}\beta + \varrho_{22}\nu + \varrho_{33}(\beta + \nu) - \varrho_{12}(\gamma + \mu) - \varrho_{13}(\kappa + 2\tau) + \varrho_{23}(\alpha - 2\psi) = 0, \end{cases} \tag{2.13}$$

which may be used, for example, to express ν, τ, ψ in function of $\alpha, \beta, \gamma, \kappa, \mu, \sigma$.

Summarizing, curvature homogeneous Lorentzian three-manifolds (M, g) are characterized by Eqs. 2.10 and 2.13. According to (2.6) (equivalently, to (2.7)), functions $\alpha, \beta, \gamma, \kappa, \mu, \sigma$, appearing in (2.10) and (2.13) completely determine the Levi Civita connection of (M, g). In this way, we proved the following.

Theorem 2.1 *Let (M, g) be a three-dimensional Lorentzian manifold. (M, g) is curvature homogeneous if and only if there exist (at least, locally) a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ and six functions $\alpha, \beta, \gamma, \kappa, \mu, \sigma$, such that* (2.7)*,* (2.10) *and* (2.13) *hold for six constants $\varrho_{11}, \ldots, \varrho_{23}$.*

From now on, we shall focus on curvature homogeneous Lorentzian three-spaces (M, g) whose Ricci operator is neither of diagonal Segre type $\{11, 1\}$ nor of degenerate Segre type $\{21\}$, because non-homogeneous examples have already been studied in these cases. Under this restriction, we can now give a simple criterion to recognize locally homogeneous Lorentzian three-spaces among all solutions of (2.10) and (2.13).

As the author proved in [5], a three-dimensional locally homogeneous Lorentzian three-manifold is either locally symmetric or locally isometric to a Lie group, equipped with a left-invariant Lorentzian metric. Moreover, he also proved that Lorentzian symmetric three-spaces only occur when the Ricci operator is of degenerate Segre type either $\{11, 1\}$ or $\{21\}$.

Henceforth, under the assumption above, (M, g) can not be locally symmetric and we can easily prove the following

Theorem 2.2 *Let (M, g) be a curvature homogeneous Lorentzian three-space, whose Ricci operator Q is neither diagonal nor of degenerate Segre type $\{21\}$. Let $\{e_1, e_2, e_3\}$ be a (local) pseudo-orthonormal frame field on (M, g) for which (2.7), (2.10) and (2.13) hold. Then, (M, g) is locally homogeneous if and only if all functions $\alpha, \beta, \gamma, \kappa, \mu, \sigma$ are constant.*

Proof If (2.7) holds for some constants $\alpha, \beta, \gamma, \kappa, \mu, \sigma$, then (2.8) yields that all b^i_{jk} are constant (at least, locally). Since the components of the Ricci tensor and of its derivatives of any order with respect to $\{e_i\}$ depend on b^i_{jk}, we have that (M, g) is curvature homogeneous up to any order k and so, is locally homogeneous.

Conversely, assume now that (M, g) is locally homogeneous. Because of the Segre type of its Ricci operator, (M, g) is not locally symmetric. Hence, the main result of [5] implies that (M, g) is locally isometric to a three-dimensional Lie group G, equipped with a left-invariant Lorentzian metric. The Lie algebra $\mathfrak{g}$ of G admits a pseudo-orthonormal basis $\{e'_1, e'_2, e'_3\}$, such that

$$\left[e'_1, e'_2\right] = k_1 e'_1 + k_2 e'_2 + k_3 e'_3, \quad \left[e'_1, e'_3\right] = k_4 e'_1 + k_5 e'_2 + k_6 e'_3,$$
$$\left[e'_2, e'_3\right] = k_7 e'_1 + k_8 e'_2 + k_9 e'_3,$$

for some real constants $k_1, \ldots, k_9$, and the conclusion follows by comparing formulas above with formulas (2.7) □

Remark 2.3 Conditions for local homogeneity given in Theorem 2.2 appear rather restrictive compared with equations (2.10) and (2.13), which express curvature homogeneity when all ϱ_{ij} are constant. In particular, it suffices to have *one* non-constant connection function between $\alpha, \beta, \gamma, \kappa, \mu, \sigma$, to have a non-homogeneous solution of (2.10) and (2.13). In the rest of this article, we shall construct explicitly such solutions.

3 Examples with Ricci operator of either Segre type {3} or nondegenerate Segre type {21}

The existence of a parallel line field has strong and interesting consequences on the geometry of a manifold. If a Riemannian manifold (M, g) admits such a line field, then (M, g) is locally reducible. The same property remains true for a pseudo-Riemannian manifold admitting a parallel non-degenerate line field. However, in the pseudo-Riemannian framework, a peculiar phenomenon arises: it can exist a parallel degenerate line field, that is, one generated by a null vector field.

The geometry of Lorentzian three-manifolds admitting a parallel degenerate line field has been studied in [8]. These manifolds are described in terms of a suitable system of local coordinates (t, x, y) and form a large class, depending on an arbitrary three-variables function $f(t, x, y)$. We briefly report here the decription of these manifolds, which we shall denote by (M, g_f), referring to [8] for more details.

A three-dimensional Lorentzian manifold (M, g_f) admitting a parallel degenerate line field has local coordinates (t, x, y) such that with respect to the local frame fields $\{\partial_t, \partial_x, \partial_y\}$, the Lorentzian metric is expressed by

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t, x, y) \end{pmatrix} \tag{3.1}$$

for some function $f(t, x, y)$, where $\varepsilon = \pm 1$. Here we fix $\varepsilon = 1$, so that the Lorentzian metric tensor will have signature $(+, +, -)$. The parallel degenerate line field is given by $\bar{\mathcal{D}}$ =Span(∂_t). When $U = \partial_t$ is a parallel null vector field, then $f = f(x, y)$.

With respect to the coordinate basis $\{\partial_t, \partial_x, \partial_y\}$, the Levi Civita connection and curvature tensor of (M, g_f) are determined by the following formulas:

$$\begin{aligned} \nabla_{\partial_t}\partial_y &= \tfrac{1}{2} f'_t \partial_t, \\ \nabla_{\partial_x}\partial_y &= \tfrac{1}{2} f'_x \partial_t, \\ \nabla_{\partial_y}\partial_y &= \tfrac{1}{2}(f f'_t + f'_y)\partial_t - \tfrac{1}{2} f'_x \partial_x - \tfrac{1}{2} f'_t \partial_y, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} R(\partial_t, \partial_y)\partial_t &= -\tfrac{1}{2} f''_{tt}\partial_t \\ R(\partial_t, \partial_y)\partial_x &= -\tfrac{1}{2} f''_{tx}\partial_t \\ R(\partial_t, \partial_y)\partial_y &= -\tfrac{1}{2} f f''_{tt}\partial_t + \tfrac{1}{2} f''_{tx}\partial_x + \tfrac{1}{2} f''_{tt}\partial_y \\ R(\partial_x, \partial_y)\partial_t &= -\tfrac{1}{2} f''_{tx}\partial_t \\ R(\partial_x, \partial_y)\partial_x &= -\tfrac{1}{2} f''_{xx}\partial_t, \\ R(\partial_x, \partial_y)\partial_y &= -\tfrac{1}{2} f f''_{tx}\partial_t + \tfrac{1}{2} f''_{xx}\partial_x + \tfrac{1}{2} f''_{tx}\partial_y. \end{aligned} \tag{3.3}$$

In particular, with respect to $\{\partial_t, \partial_x, \partial_y\}$, the Ricci operator Q and the Ricci tensor ϱ of (M, g_f) are respectively given by:

$$Q = \begin{pmatrix} \frac{1}{2} f''_{tt} & \frac{1}{2} f''_{tx} & -\frac{1}{2} f''_{xx} \\ 0 & 0 & \frac{1}{2} f''_{tx} \\ 0 & 0 & \frac{1}{2} f''_{tt} \end{pmatrix} \quad \text{and} \quad \varrho = \begin{pmatrix} 0 & 0 & \frac{1}{2} f''_{tt} \\ 0 & 0 & \frac{1}{2} f''_{tx} \\ \frac{1}{2} f''_{tt} & \frac{1}{2} f''_{tx} & \frac{1}{2}(f f''_{tt} - f''_{xx}) \end{pmatrix}. \tag{3.4}$$

The eigenvalues of the Ricci operator Q are $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \frac{1}{2} f''_{tt}$. Henceforth, the constancy of f''_{tt} is a necessary condition for the curvature homogeneity of (M, g_f). Note that all Ricci eigenvalues are real and so, Q is never of Segre type $\{1z\bar{z}\}$.

We now describe (M, g_f) in terms of a (local) pseudo-orthonormal frame field. Consider local coordinates (t, x, y) for which (3.1) holds. Then, it is easy to check that

$$e_1 = \partial_x, \quad e_2 = \frac{2 - f}{2\sqrt{2}}\partial_t + \frac{1}{\sqrt{2}}\partial_y, \quad e_3 = \frac{2 + f}{2\sqrt{2}}\partial_t + \frac{1}{\sqrt{2}}\partial_y \tag{3.5}$$

is a local pseudo-orthonormal frame field on (M, g_f), with e_3 timelike. Using (3.2) and (3.5), we easily find that with respect to $\{e_1, e_2, e_3\}$ the Levi Civita connection is completely determined by

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_1 = \tfrac{1}{4} f'_x (e_2 + e_3), \qquad \nabla_{e_3} e_1 = -\tfrac{1}{4} f'_x (e_2 + e_3), \\ &\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -\tfrac{1}{4} f'_x e_1 + \tfrac{1}{2\sqrt{2}} f'_t e_3, \quad \nabla_{e_3} e_2 = \tfrac{1}{4} f'_x e_1 - \tfrac{1}{2\sqrt{2}} f'_t e_3, \\ &\nabla_{e_1} e_3 = 0, \quad \nabla_{e_2} e_3 = \tfrac{1}{4} f'_x e_1 + \tfrac{1}{2\sqrt{2}} f'_t e_2, \quad \nabla_{e_3} e_3 = -\tfrac{1}{4} f'_x e_1 - \tfrac{1}{2\sqrt{2}} f'_t e_2, \end{aligned} \tag{3.6}$$

while (3.4) and (3.5) yield that the Ricci operator with respect to $\{e_1, e_2, e_3\}$ is given by

$$Q = \begin{pmatrix} 0 & \frac{1}{2\sqrt{2}} f''_{tx} & -\frac{1}{2\sqrt{2}} f''_{tx} \\ \frac{1}{2\sqrt{2}} f''_{tx} & \frac{1}{4}(2f''_{tt} - f''_{xx}) & \frac{1}{4} f''_{xx} \\ \frac{1}{2\sqrt{2}} f''_{tx} & -\frac{1}{4} f''_{xx} & \frac{1}{4}(2f''_{tt} + f''_{xx}) \end{pmatrix}, \tag{3.7}$$

and for the Ricci components (ϱ_{ij}) it suffices to change the signs in the last row of (3.7).

We now find conditions ensuring that (M, g_f) is curvature homogeneous and has Ricci operator either of Segre type $\{3\}$ or of nondegenerate Segre type $\{21\}$.

Segre type $\{3\}$ In order to be of Segre type $\{3\}$, Q must admit a triple eigenvalue $\lambda_1 = \lambda_2 = \lambda_3$. Therefore, we necessarily have $f''_{tt} = 0$. In this case, by either (3.4) or (3.7) it easily follows that Q is of Segre type $\{3\}$ if and only if $f''_{tx} \neq 0$. In fact, if $f''_{tx} \neq 0$, then the associated eigenspace is one-dimensional, while $f''_{tx} = 0$ implies that the eigenspace is at least two-dimensional and so, Q is not of Segre type $\{3\}$. In particular, if the defining function f satisfies

$$\begin{cases} f''_{tt} = 0, \\ f''_{tx} = a_1, \\ f''_{xx} = a_2, \end{cases} \tag{3.8}$$

where $a_1 \neq 0$ and a_2 are two real constants, then (3.7) becomes

$$Q = \begin{pmatrix} 0 & \frac{1}{2\sqrt{2}} a_1 & -\frac{1}{2\sqrt{2}} a_1 \\ \frac{1}{2\sqrt{2}} a_1 & -\frac{1}{4} a_2 & \frac{1}{4} a_2 \\ \frac{1}{2\sqrt{2}} a_1 & -\frac{1}{4} a_2 & \frac{1}{4} a_2 \end{pmatrix} \tag{3.9}$$

and so, (M, g_f) is curvature homogeneous. Integrating (3.8), we find

$$f(t, x, y) = a_1 x t + \frac{a_2}{2} x^2 + p(y)t + q(y)x + s(y), \tag{3.10}$$

where p, q, s are the three arbitrary one-variable functions. Note that when (3.10) holds, (M, g_f) is neither locally symmetric nor locally homogeneous. This follows both from Theorem 2.2 and by direct calculation. In fact, if f satisfies (3.10), then the Ricci components (3.9) with respect to $\{e_1, e_2, e_3\}$ are constants, but a straightforward calculation gives

$$\nabla_2 \varrho_{22} = -2\varrho(\nabla_{e_2} e_2, e_2) = \frac{1}{4\sqrt{2}}(a_1 t^2 + a_1 q(y) - a_2 p(y)),$$

which is not constant. In this way, we proved the following.

Theorem 3.1 *For any defining function f satisfying* (3.10)*, the Lorentzian manifold (M, g_f) with metric tensor* (3.1) *is curvature homogeneous and has Ricci operator of Segre type* $\{3\}$*.*

Hence, (3.10) *and* (3.1) *determine explicitly a family of (nonhomogeneous) curvature homogeneous Lorentzian metrics on $\mathbb{R}^3[t, x, y]$, with Ricci operator of Segre type* $\{3\}$*, depending on three arbitrary functions of one variable.*

We end the study of this case by discussing when two of the Lorentzian metrics described in Theorem 3.1 are locally isometric. Let (M, g) and (M', g') be curvature homogeneous Lorentzian three-manifolds admitting (at least, locally) pseudo-orthonormal frame fields $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$ respectively, for which (3.9) hold. Then, we have the following

Theorem 3.2 *A differentiable mapping $\varphi : M \to M'$ is an isometry if and only if*

$$\varphi_*(e_i) = \varepsilon e_i' = \pm e_i'.$$

Proof If φ satisfies the condition above, then clearly φ is an isometry. Conversely, suppose that φ is an isometry. By (3.9) it easily follows that the one-dimensional Ricci eigenspace of (M, g) is spanned by the null vector $u = \frac{e_2+e_3}{\sqrt{2}}$. For this reason, we pass from the pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ to the null frame field $\{e_1, u, v\}$, where $v = \frac{e_2-e_3}{\sqrt{2}}$. Taking into account (3.9), the Lorentzian metric and the Ricci tensor with respect to $\{e_1, u, v\}$ are completely determined by conditions

$$g(e_1, e_1) = g(u, v) = 1, \quad g(e_1, u) = g(e_1, v) = g(u, u) = g(v, v) = 0 \tag{3.11}$$

and

$$\varrho(e_1, e_1) = \varrho(e_1, u) = \varrho(u, u) = \varrho(u, v) = 0, \quad \varrho(e_1, v) = \frac{a_1}{2}, \quad \varrho(v, v) = -a_2 \tag{3.12}$$

The null frame field $\{e_1', u', v'\}$ is defined in the same way starting from the pseudo-orthonormal frame field $\{e_1', e_2', e_3'\}$, and (3.11), (3.12) hold for g' and ϱ'.

Since φ preserves the Ricci eigenspace, we have $\mathrm{Span}(u') = \mathrm{Span}(\varphi_*(u))$. Hence, there exists a smooth function $r \neq 0$ such that $\varphi_*(u) = ru'$, and

$$\begin{cases} \varphi_*(e_1) = h_1 e_1' + h_2 u' + h_3 v', \\ \varphi_*(u) = ru', \\ \varphi_*(v) = s_1 e_1' + s_2 u' + s_3 v'. \end{cases} \tag{3.13}$$

for some smooth functions $h_1, h_2, h_3, s_1, s_2, s_3$. Since φ both preserves the Lorentzian metric and the Ricci tensor, (3.11) and (3.12) easily imply $h_1 = r = s_3 = \varepsilon$ and $h_2 = h_3 = s_1 = s_2 = 0$. Then, (3.13) reduces to $f_*(e_1) = \varepsilon e_1'$, $f_*(u) = \varepsilon u'$, $f_*(v) = \varepsilon v'$, from which the conclusion follows at once, since $e_2 = \frac{u+v}{\sqrt{2}}$ and $e_3 = \frac{u-v}{\sqrt{2}}$. □

By Theorem 3.2, two locally isometric curvature homogeneous pseudo-Riemannian three-manifolds (M, g) and (M', g') having the same Ricci operator (3.9), with respect to the suitable pseudo-orthonormal frames $\{e_i\}$ and $\{e_i'\}$, necessarily have the same connection functions (at most, up to sign). Therefore, (3.6) yields that two of the solutions given in Theorem 3.1, constructed starting by two defining functions f and $\bar{f}$ satisfying (3.10) but such that either $f_x' \neq \bar{f}_x'$ or $f_t' \neq \bar{f}_t'$, is not (locally) isometric. So, Theorems 3.1 and 3.2 ensure that *there are infinitely many curvature homogeneous Lorentzian metrics on* $\mathbb{R}^3[w, x, y]$*, all having the same Ricci operator* (3.9) *of Segre type* $\{3\}$*, not locally isometric to one another.*

Nondegenerate Segre type $\{21\}$ It is easy to check that if $f_{tx}'' = 0$ and $f_{xx}'' \neq 0 \neq f_{tt}''$, then the Ricci operator Q of (M, g_f) is of nondegenerate Segre type $\{21\}$. In fact, by either (3.4) or (3.7), it follows that the Ricci eigenvalues are $\lambda_1 = 0 \neq \frac{1}{2} f_{tt}'' = \lambda_2 = \lambda_3$, and the eigenspace associated to the eigenvalue $\lambda_2 = \lambda_3$ is one-dimensional.

In particular, (3.7) easily implies that (M, g_f) is curvature homogeneous and has Ricci operator of nondegenerate Segre type $\{21\}$ if the defining function f satisfies the following system of partial differential equations:

$$\begin{cases} f_{tt}'' = b_1, \\ f_{tx}'' = 0, \\ f_{xx}'' = b_2, \end{cases} \tag{3.14}$$

for two real constants $b_1 \neq 0$ and $b_2 \neq 0$. In this case, (3.7) becomes

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4}(2b_1 - b_2) & \frac{1}{4}b_2 \\ 0 & -\frac{1}{4}b_2 & \frac{1}{4}(2b_1 + b_2) \end{pmatrix} \tag{3.15}$$

and the Ricci eigenvalues are $\lambda_1 = 0 \neq \frac{b_1}{2} = \lambda_2 = \lambda_3$. We integrate (3.14) and we find

$$f(t, x, y) = \frac{b_1}{2}t^2 + \frac{b_2}{2}x^2 + \bar{p}(y)t + \bar{q}(y)x + \bar{s}(y), \tag{3.16}$$

for three arbitrary one-variable functions $\bar{p}, \bar{q}, \bar{s}$. If f satisfies (3.16), then the Ricci components with respect to $\{e_1, e_2, e_3\}$ are constant, but for example

$$\nabla_2 \varrho_{23} = -\varrho(\nabla_{e_2} e_2, e_3) - \varrho(e_2, \nabla_{e_2} e_3) = \frac{b_1}{4\sqrt{2}}(b_1 t + \bar{p}(y))$$

is not a constant and so, (M, g_f) is neither locally symmetric nor locally homogeneous. Thus, we proved

Theorem 3.3 *For any defining function f satisfying* (3.16)*, the Lorentzian manifold (M, g_f) with metric tensor* (3.1) *is curvature homogeneous and has Ricci operator of nondegenerate Segre type* $\{21\}$.

Hence, (3.16) *and* (3.1) *determine explicitly a family of (nonhomogeneous) curvature homogeneous Lorentzian metrics on $\mathbb{R}^3[t, x, y]$, with Ricci operator of nondegenerate Segre type* $\{21\}$*, depending on three arbitrary functions of one variable.*

Suppose now that (M, g) and (M', g') are curvature homogeneous Lorentzian three-manifolds admitting (at least, locally) pseudo-orthonormal frame fields $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$ respectively, for which (3.15) hold. Then, we can prove the following

Theorem 3.4 *A differentiable mapping $\varphi : M \to M'$ is an isometry if and only if*

$$\varphi_*(e_1) = \varepsilon_1 e'_1, \quad \varphi_*(e_2) = \varepsilon_2 e'_2, \quad \varphi_*(e_3) = \varepsilon_2 e'_3,$$

where $\varepsilon_i = \pm 1$ for $i = 1, 2$.

Proof The "if" part is trivial. As regards the "only if" part, suppose that φ is an isometry. Then, φ preserves the Ricci eigenspaces. In particular, $\text{Span}(\varphi_* e_1) = \text{Span}(e'_1)$ and so, $\varphi_*(e_1) = \varepsilon_1 e'_1$. Moreover, since φ is an isometry and $\varphi_*(e_1) = \varepsilon_1 e'_1$, it is easy to check that $\varphi_* e_2$ and $\varphi_* e_3$ are both orthogonal to e'_1. Therefore, $\text{Span}(\varphi_* e_2, \varphi_* e_3) = \text{Span}(e'_2, e'_3)$ and so, there exists a real-valued function θ, such that

$$\varphi_*(e_2) = \varepsilon_2(\cosh\theta e'_2 + \sinh\theta e'_3), \quad \varphi_*(e_3) = \varepsilon_3(\sinh\theta e'_2 + \cosh\theta e'_3), \tag{3.17}$$

where $\varepsilon_i = \pm 1$ for $i = 1, 2$. Since φ preserves the Ricci tensor, using (3.15) and (3.17) to calculate ϱ'_{22} we easily get

$$(\cosh\theta - \sinh\theta)^2 = 1$$

which admits $\theta = 0$ as the unique solution. Then, (3.17) gives $\varphi_*(e_2) = \varepsilon_2 e'_2$, $\varphi_*(e_3) = \varepsilon_3 e'_3$.

Finally, since φ preserves the one-dimensional eigenspace associated to the Ricci eigenvector $\lambda_2 = \lambda_3 = \frac{b_1}{2}$, we have that $\varphi_*(e_2) + \varphi_*(e_3)$ and $e'_2 + e'_3$ are collinear. Hence, $\varepsilon_2 = \varepsilon_3$ and this ends the proof. □

The same argument used in the previous case leads here to conclude that, by Theorems 3.3 and 3.4, *there are infinitely many curvature homogeneous Lorentzian metrics on* $\mathbb{R}^3[w, x, y]$, *all having the same Ricci operator* (3.15) *of nondegenerate Segre type* $\{21\}$, *not locally isometric to one another*.

4 Examples with Ricci operator of Segre type $\{1z\bar{z}\}$

In order to complete the proof of our Main Theorem, we must exhibit a family of non-homogeneous curvature homogeneous Lorentzian three-spaces whose Ricci operator is of Segre type $\{1z\bar{z}\}$. Let (M, g) be a three-dimensional curvature homogeneous Lorentzian manifold. Adapting to the Lorentzian case the technique used in [9] (see also [6]), we shall express conditions (2.10) and (2.13) through a system of partial differential equations for some three-variables functions, whose solutions permit to build explicitly Lorentzian metrics on $\mathbb{R}^3$ with the curvature properties of (M, g). We refer to [9] for a more detailed explanation of how the corresponding equations for the connection and the curvature are obtained.

We fix a point $p \in M$ and consider a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ as in Theorem 2.1. We then choose a surface S through p transversal to the lines generated by e_3, a local coordinates system (w, x) on S and a neighborhood U_p of p, sufficiently small that each $q \in U_p$ is situated on exactly one line generated by e_3 and passing through one point $\bar{q} \in S$.

Choose an orientation of S and define the coordinate function y in U_p as the oriented distance of the point q from S along the corresponding line, that is, $y(q) = \mathrm{dist}(q, \pi(q))$, where $\pi : U_p \to S$ is the corresponding projection. We also define $w(q) = w(\pi(q))$ and $x(q) = x(\pi(q))$.

In this way, a local coordinate system (w, x, y) is introduced in U_p. Notice that $e_3 = \frac{\partial}{\partial y}$ and the coframe $\{\omega^1, \omega^2, \omega^3\}$ of $\{e_1, e_2, e_3\}$ takes the form

$$\omega^1 = A dw + B dx, \quad \omega^2 = C dw + D dx, \quad \omega^1 = G dw + H dx + dy, \tag{4.1}$$

for some functions A, B, C, D, G, H. Next, we introduce the connection forms on (M, g), putting

$$\omega^i_j = \sum_k \varepsilon_j b^i_{jk} \omega^k. \tag{4.2}$$

Connection forms completely determine the Levi Civita connection, because

$$\nabla_{e_i} e_j = \sum_k \omega^k_j(e_i) e_k,$$

for all i, j. Note also that from (2.4) we easily get

$$\omega^i_j + \varepsilon_i \varepsilon_j \omega^j_i = 0 \tag{4.3}$$

for all i, j. In particular, $\omega^i_i = 0$ for all i. The *structure equations* for ω^i_j give

$$d\omega^i + \sum_j \omega^i_j \wedge \omega^j = 0, \tag{4.4}$$

for all indices i. As regards the curvature forms $\Omega^i_j(X, Y) = \omega^i(R(X, Y)e_j)$, they are completely determined by the standard formulas

$$-\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j. \tag{4.5}$$

Using (2.1) and taking into account (4.3), we obtain that (4.5) is equivalent to

$$\begin{aligned} d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 &= R_{1212}\,\omega^1 \wedge \omega^2 + \varrho_{23}\,\omega^1 \wedge \omega^3 - \varrho_{13}\,\omega^2 \wedge \omega^3, \\ d\omega^1_3 + \omega^1_2 \wedge \omega^2_2 &= \varrho_{23}\,\omega^1 \wedge \omega^2 + R_{1313}\,\omega^1 \wedge \omega^3 - \varrho_{12}\,\omega^2 \wedge \omega^3, \\ d\omega^2_3 - \omega^1_2 \wedge \omega^1_3 &= -\varrho_{13}\,\omega^1 \wedge \omega^2 - \varrho_{12}\,\omega^1 \wedge \omega^3 + R_{2323}\,\omega^2 \wedge \omega^3. \end{aligned} \tag{4.6}$$

We now use (4.1) in (4.4). After some long but standard calculations, we obtain that (4.4) is equivalent to the following system of nine partial differential equations:

$$\begin{aligned} &A'_x - B'_w = -\alpha\mathcal{D} + \beta\mathcal{E} + (\mu + \sigma)\mathcal{F}, \\ &A'_y = \beta A + (\mu + \sigma)C, \quad B'_y = \beta B + (\mu + \sigma)D, \\ &C'_x - D'_w = -\kappa\mathcal{D} + (\gamma - \sigma)\mathcal{E} + \nu\mathcal{F}, \\ &C'_y = (\gamma - \sigma)A + \nu C, \quad D'_y = (\gamma - \sigma)B + \nu D, \\ &G'_x - H'_w = (\gamma - \mu)\mathcal{D} - \tau\mathcal{E} - \psi\mathcal{F}, \\ &G'_y = -\tau A - \psi C, \quad H'_y = -\tau B - \psi D, \end{aligned} \tag{4.7}$$

where $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ are auxiliary functions, defined by

$$\mathcal{D} = AD - BC, \quad \mathcal{E} = AH - BG, \quad \mathcal{F} = CH - DG. \tag{4.8}$$

Note that, because of (4.1), $\mathcal{D} \neq 0$ is a necessary and sufficient condition for linear independence of ω^i. Starting from the connection functions b^i_{jk} of (Mg), system (4.7) determines the functions $A, .., H$ and so, gives explicit Lorentzian metrics on $\mathbb{R}^3$, with the same Levi Civita connection of (M, g). Notice that, conversely, if $A, \ldots, H$ are known, then by (4.7) we can determine uniquely the connection functions b^i_{jk}.

Next, we use (4.1) to express curvature conditions (4.6). As in [9], we shall restrict ourselves to the case when *all connection functions* b^i_{jk} *are independent of the variable* y. Notice that, by Theorem 2.2, this condition is also satisfied when (M, g) is locally homogeneous.

A long but straightforward calculation leads to conclude that, under this assumption, (4.6) is equivalent to the following system of nine differential equations:

$$
\begin{aligned}
&\sigma'_w = -(V_3+\varrho_{23})A-(W_3-\varrho_{13})C, \qquad \sigma'_x = -(V_3+\varrho_{23})B-(W_3-\varrho_{13})D,\\
&\tau'_w = -(V_2-R_{1313})A-(W_2+\varrho_{12})C, \quad \tau'_x = -(V_2-R_{1313})B-(W_2+\varrho_{12})D,\\
&\psi'_w = -(V_1+\varrho_{12})A-(W_1-R_{2323})C, \quad \psi'_x = -(V_1+\varrho_{12})B-(W_1-R_{2323})D,\\
&A\alpha'_x-B\alpha'_w+C\kappa'_x-D\kappa'_w+G\sigma'_x-H\sigma'_w = \mathcal{D}(U_3+R_{1212})+\mathcal{E}(V_3+\varrho_{23})+\mathcal{F}(W_3-\varrho_{13}),\\
&A\beta'_x-B\beta'_w+C\mu'_x-D\mu'_w+G\tau'_x-H\tau'_w = \mathcal{D}(U_2-\varrho_{23})+\mathcal{E}(V_2-R_{1313})+\mathcal{F}(W_2+\varrho_{12}),\\
&A\gamma'_x-B\gamma'_w+C\nu'_x-D\nu'_w+G\psi'_x-H\psi'_w = \mathcal{D}(U_1+\varrho_{13})+\mathcal{E}(V_1+\varrho_{12})+\mathcal{F}(W_1-R_{2323}),
\end{aligned}
\tag{4.9}
$$

where we put

$$
\begin{aligned}
U_1 &= \alpha(\gamma+\mu)-\kappa(\beta-\nu)-\psi(\gamma-\mu),\\
V_1 &= -\beta(\gamma+\sigma)-\nu(\gamma-\sigma)+\tau(\alpha+\psi),\\
W_1 &= -\nu^2+\psi^2-\gamma(\mu+\sigma)+\kappa\tau-\mu\sigma,\\
U_2 &= \alpha(\beta-\nu)+\kappa(\gamma+\mu)-\tau(\gamma-\mu),\\
V_2 &= -\beta^2+\tau^2-\mu(\gamma-\sigma)-\alpha\psi+\gamma\sigma,\\
W_2 &= -\beta(\mu+\sigma)-\nu(\mu-\sigma)-\psi(\kappa-\tau),\\
U_3 &= \alpha^2+\kappa^2-\sigma(\gamma-\mu)-\beta\nu+\gamma\mu,\\
V_3 &= -\beta(\alpha+\psi)-\kappa(\gamma-\sigma)+\tau(\gamma+\sigma),\\
W_3 &= -\alpha(\mu+\sigma)-\nu(\kappa-\tau)-\psi(\mu-\sigma).
\end{aligned}
\tag{4.10}
$$

In this way, we proved the following.

Proposition 4.1 *Let A, B, C, D, G, H be smooth functions on the three variables w, x, y, satisfying partial differential equations (4.7) and (4.9). Then, (4.1) describes a curvature homogeneous Lorentzian metric g on $\mathbb{R}^3$, whose Ricci tensor has constant (local) components (ϱ_{ij}).*

Note that Proposition 4.1 is valid for any Segre type of the Ricci operator. We shall now apply this result to describe some nonhomogeneous curvature homogeneous Lorentzian metrics on $\mathbb{R}^3$ with Ricci operator of Segre type $\{1z\bar{z}\}$. More precisely, we shall construct examples whose Ricci operator is of the form

$$
Q = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & c\\ 0 & -c & 0 \end{pmatrix}, \tag{4.11}
$$

for any real constant $c \neq 0$, that is, having $\lambda_1 = 0$ as the sole real Ricci eigenvalue. We choose the connection functions in the following way:

$$\alpha = \beta = \kappa = \mu = \nu = \sigma = \psi = 0, \quad \gamma = -\frac{c}{\tau}, \tag{4.12}$$

with $\tau \neq 0$. By (4.12), Eq. 2.10 reduce to

$$\varrho_{11} = -\varrho_{33} = -e_1(\tau) - \tau^2, \quad \varrho_{22} = \varrho_{13} = 0, \quad \varrho_{12} = \frac{c}{\tau^2} e_3(\tau), \quad \varrho_{23} = c.$$

Hence, the Ricci operator Q assumes the form (4.11) for any smooth function τ satisfying

$$e_1(\tau) + \tau^2 = 0, \quad e_3(\tau) = 0.$$

Notice that (4.11) and (4.12) imply that all Eq. 2.13 are automatically satisfied.

Next, we consider the curvature Eq. 4.9. Using (4.11) and (4.12) in (4.10), we easily get

$$U_1 = U_3 = V_1 = W_1 = W_2 = W_3 = 0, \quad U_2 = -V_3 = c, \quad V_2 = \tau^2$$

and so, the curvature equations (4.9) reduce to

$$A = -\frac{\tau'_w}{\tau^2}, \quad B = -\frac{\tau'_x}{\tau^2} \tag{4.13}$$

and

$$G\tau'_x - H\tau'_w = \tau^2 \mathcal{E}, \quad \frac{c}{\tau^2} A\tau'_x - \frac{c}{\tau^2} B\tau'_w = 0. \tag{4.14}$$

But equations (4.14) follow at once from (4.13) and $\mathcal{E} = AH - BG$. Hence, when (4.11) and (4.12) hold, system (4.9) just reduces to (4.13).

Next, as regards connection Eq. 4.7, because of (4.12) they reduce to

$$\begin{cases} A'_y = 0, & B'_y = 0, & A'_x - B'_w = 0, \\ C'_y = -\frac{c}{\tau} A, & D'_y = -\frac{c}{\tau} B, & C'_x - D'_w = -\frac{c}{\tau} \mathcal{E} \\ G'_y = -\tau A, & H'_y = -\tau B, & G'_x - H'_w = -\frac{c}{\tau} \mathcal{D} - \tau \mathcal{E}. \end{cases} \tag{4.15}$$

It is easily seen that the equations in the first row of (4.15) follow at once from (4.13) and the fact that τ does not depend on y. Summarizing, when (4.11) and (4.12) hold, all connection and curvature Eqs. 4.7, 4.9 reduce to the following system of partial differential equations:

$$\begin{cases} A = -\frac{\tau'_w}{\tau^2} & B = -\frac{\tau'_x}{\tau^2}, \\ C'_y = \frac{c\tau'_w}{\tau^3}, & D'_y = \frac{c\tau'_x}{\tau^3}, & C'_x - D'_w = -\frac{c}{\tau} \mathcal{E}, \\ G'_y = \frac{\tau'_w}{\tau}, & H'_y = \frac{\tau'_x}{\tau}, & G'_x - H'_w = -\frac{c}{\tau} \mathcal{D} - \tau \mathcal{E}. \end{cases} \tag{4.16}$$

Since τ does not depend on y, integrating the first two equations in the second and third row of (4.16) we obtain at once

$$C = \frac{c\tau'_w}{\tau^3} y + C_0(w, x), \quad D = \frac{c\tau'_x}{\tau^3} y + D_0(w, x), \quad G = \frac{\tau'_w}{\tau} y + G_0(w, x),$$
$$H = \frac{c\tau'_x}{\tau} y + H_0(w, x), \tag{4.17}$$

where C_0, D_0, G_0, H_0 are two-variables functions. Finally, we use (4.17) to rewrite the last equations in the second and third row of (4.16). Taking into account the definition of $\mathcal{D}$ and $\mathcal{E}$, we get

$$\begin{cases} (C_0)'_x - (D_0)'_w = -\frac{c}{\tau}(AH_0 - BG_0), \\ (G_0)'_x - (H_0)'_w = -\frac{c}{\tau}(AD_0 - BC_0) - \tau(AH_0 - BG_0). \end{cases} \tag{4.18}$$

Therefore, all solutions of (4.16) are determined by A, B given by (4.13) and C, D, G, H of the form (4.17), where C_0, D_0, G_0, H_0 satisfy (4.18). If for example we choose D_0 and H_0 as arbitrary smooth functions on $\mathbb{R}^2[w, x]$, then (4.18) is a system of two linear ordinary differential equations of the first order for C_0, G_0, with w as a parameter (see also [9]). The standard existence theorem ensures that this system can be solved. The solution (C_0, G_0) exists on the whole of $\mathbb{R}^2[w, x]$ and involves two arbitrary functions of the variable w. Moreover, $\mathcal{D} = AD_0 - BC_0 \neq 0$ in a dense open subset of $\mathbb{R}^2[w, x]$. Therefore, A, B, C_0, D_0, G_0, H_0 determine a Lorentzian metric on a dense open subset of $\mathbb{R}^3[w, x, y]$, with Ricci operator (4.11) of Segre type $\{1z\bar{z}\}$. Thus, we proved the following.

Theorem 4.2 *For any real constant $c \neq 0$, let Q be the linear operator of Segre type $\{1z\bar{z}\}$ described by (4.11) and $\tau \neq 0$ an arbitrary smooth function on $\mathbb{R}^2[w, x]$, satisfying either $\tau'_w \neq 0$ or $\tau'_x \neq 0$. Then, (4.1) determines a family of curvature homogeneous Lorentzian metrics on $\mathbb{R}^3[w, x, y]$ having Q as the Ricci operator at any point, where functions A, B, C, D, G, H are described by (4.13), (4.17) and (4.18). They depend on two arbitray functions of two variables and two arbitrary functions of one variable.*

All the solutions given in Theorem 4.2 are nonhomogeneous. In fact, either $\tau'_w \neq 0$ or $\tau'_x \neq 0$. Hence, the connection function τ is not constant and Theorem 2.2 implies that the corresponding metric is not locally homogeneous.

In order to decide whether two of such solutions are locally isometric or not, consider more generally two curvature homogeneous Lorentzian three-manifolds (M, g), (M', g'), both with Ricci operator of Segre type $\{1z\bar{z}\}$, admitting (at least, locally) pseudo-orthonormal frame fields $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$ respectively, for which (2.7), (2.10) and (2.13) hold. Then, we have the following

Theorem 4.3 *A differentiable mapping $\varphi : M \to M'$ is an isometry if and only if*

$$\varphi_*(e_i) = \varepsilon_i e'_i,$$

where $\varepsilon_i = \pm 1$ for all $i = 1, 2, 3$.

Proof Conditions above imply that φ is an isometry. Conversely, if φ is an isometry, then it preserves the eigenspace of the real Ricci eigenvalue. Therefore, Span(e'_1)=Span($\varphi_*(e_1)$) and so, $\varphi_*(e_1) = \varepsilon_1 e'_1 = \pm e'_1$. Again because φ is an isometry, it preserves the orthogonal complement of the eigenspace, that is, Span(e'_2, e'_3)=Span($\varphi_*(e_2), \varphi_*(e_3)$). Hence, there exists a real-valued function θ, such that

$$\varphi_*(e_2) = \varepsilon_2(\cosh\theta e'_2 + \sinh\theta e'_3), \quad \varphi_*(e_3) = \varepsilon_3(\sinh\theta e'_2 + \cosh\theta e'_3). \tag{4.19}$$

Next, φ preserves the Ricci components. Thus, by (1.1) and (4.19) we have

$$b = \varrho_{22} = \varrho'_{22} = b + 2c\sinh\theta\cosh\theta,$$

which implies $\sinh\theta = 0$, because $c\cosh\theta \neq 0$. Then, (4.19) reduces to $\varphi_*(e_2) = \varepsilon_2 e'_2$, $\varphi_*(e_3) = \varepsilon_3 e'_3$ and this ends the proof. □

Theorem 4.3 ensures that two locally isometric curvature homogeneous pseudo-Riemannian three-manifolds (M, g) and (M', g') having the same Ricci operator of Segre type $\{1z\bar{z}\}$, with respect to the suitable frames $\{e_i\}$ and $\{e_i'\}$, must have the same connection functions (at most, up to sign). Therefore, two of the solutions given in Theorem 4.2, constructed starting by two different functions $\tau = b_{13}^3$, are never (locally) isometric. In other words, Theorems 4.2 and 4.3 ensure that *there are infinitely many curvature homogeneous Lorentzian metrics on* $\mathbb{R}^3[w, x, y]$*, all having the same Ricci operator* (4.11)*, not locally isometric to one another.*

We end this section presenting some explicit solutions of (4.16), that is, some explicit curvature homogeneous Lorentzian metrics with Ricci operator (4.11). To construct them, we assume $\tau = \tau(w)$ and in (4.17) we choose $C_0 = H_0 = 0$. Then, it is easily seen that functions

$$A = -\frac{\tau'}{\tau^2} \quad B = 0, \quad C = \frac{c\tau_w'}{\tau^3}y, \quad D = D_0, \quad G = \frac{\tau'}{\tau}y + G_0, \quad H = 0$$

are a solution of (4.16) whenever

$$(D_0)_w' = 0, \quad (G_0)_x' = \frac{c\tau'}{\tau^3}D_0. \tag{4.20}$$

Integrating (4.20), we get at once

$$D_0 = p(x), \quad G_0 = \frac{c\tau'}{\tau^3}\int p(x)\mathrm{d}x + q(w),$$

where $p \neq 0$ and q are arbitrary one-variable functions. In this way, we proved the following

Corollary 4.4 *Consider an arbitrary one-variable function* $\tau = \tau(w) \neq 0$ *with* $\tau' \neq 0$*. Then, the following functions*

$$\begin{array}{lll} A = -\frac{\tau'}{\tau^2} & B = 0, & C = \frac{c\tau_w'}{\tau^3}y \\ D = p(x), & G = \frac{\tau'}{\tau}y + \frac{c\tau'}{\tau^3}\int p(x)\mathrm{d}x + q(w), & H = 0, \end{array} \tag{4.21}$$

where $p \neq 0$ *and* q *are arbitrary one-variable functions, are solutions of* (4.16)*. So,* (4.1) *and* (4.21) *determine explicit nonhomogeneous curvature homogeneous Lorentzian metrics on* $\mathbb{R}^3[w, x, y]$ *having the Ricci operator of the form* (4.11).

Remark 4.5 In all the examples we constructed in Theorems 3.1, 3.3, and 4.2 of nonhomogeneous curvature homogeneous Lorentzian three-manifolds, with Ricci operator of Segre type $\{3\}$, nondegenerate $\{21\}$, and $\{1z\bar{z}\}$ respectively, $\lambda_1 = 0$ occurs as Ricci eigenvalue. We do not know whether the nullity of a Ricci eigenvalue is a necessary condition for the existence of nonhomogeneous examples with the Ricci operator of these Segre types, or there exist examples with non-vanishing Ricci eigenvalues.

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