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Surfaces in $\mathbb{S}^2 \times \mathbb{R}$ with a canonical principal direction

Franki Dillen · Johan Fastenakels · Joeri Van der Veken

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Abstract We show a way to choose nice coordinates on a surface in $\mathbb{S}^2 \times \mathbb{R}$ and use this to study minimal surfaces. We show that only open parts of cylinders over a geodesic in $\mathbb{S}^2$ are both minimal and flat. We also show that the condition that the projection of the direction tangent to $\mathbb{R}$ onto the tangent space of the surface is a principal direction, is equivalent to the condition that the surface is normally flat in $\mathbb{E}^4$. We present classification theorems under the extra assumption of minimality or flatness.

Keywords Minimal surfaces · Flat · Product manifold

Mathematics Subject Classification (2000) 53B25

1 Introduction

In recent years, a lot of research has been done about surfaces in a three-dimensional Riemannian product of a surface $\mathbb{M}^2$ and $\mathbb{R}$. This was motivated by the study of minimal surfaces. In particular H. Rosenberg and W. Meeks initiated this in [8,9]. This work inspired other geometers, for example, in [1–3,5–7].

In this article, we consider a special case of a $\mathbb{M}^2 \times \mathbb{R}$, namely, we take $\mathbb{M}^2$ to be the unit 2-sphere $\mathbb{S}^2$. We first show how we can take local coordinates on a surface in $\mathbb{S}^2 \times \mathbb{R}$ that are

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F. Dillen · J. Fastenakels · J. Van der Veken (✉)
Departement Wiskunde, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3001 Leuven, Belgium
e-mail: joeri.vanderveken@wis.kuleuven.be

F. Dillen
e-mail: franki.dillen@wis.kuleuven.be

J. Fastenakels
e-mail: johan.fastenakels@wis.kuleuven.be

adapted to the structure of $\mathbb{S}^2 \times \mathbb{R}$. Next we show that we can take easier coordinates when the surface is minimal. Furthermore, we prove that all flat and minimal surfaces are open parts of vertical cylinders on a geodesic in $\mathbb{S}^2$, which means surfaces for which the angle between the unit normal and the $\mathbb{R}$-direction is everywhere equal to $\frac{\pi}{2}$ and for which the intersection with $\mathbb{S}^2$ is a great circle.

In the Sect. 5 we investigate the condition that the projection of $\frac{\partial}{\partial t}$, i.e. the canonical unit vector tangent to the $\mathbb{R}$-direction, onto the tangent space of an immersed surface, is a principal direction. We show that this is equivalent to the condition that the surface is normally flat if we look at a surface in $\mathbb{S}^2 \times \mathbb{R}$ as a codimension 2 immersion of a surface in $\mathbb{E}^4$. Moreover, we give a characterization of these surfaces and classification theorems under the additional assumption of minimality or flatness.

2 Preliminaries

Let $\mathbb{S}^2 \times \mathbb{R}$ be the product of the 2-sphere $\mathbb{S}^2(1)$ and $\mathbb{R}$ with the Riemannian product metric $\langle\ ,\ \rangle$ and Levi-Civita connection $\widetilde{\nabla}$. We denote by $\frac{\partial}{\partial t}$ a unit vector field in the tangent bundle $T(\mathbb{S}^2 \times \mathbb{R})$ that is tangent to the $\mathbb{R}$-direction.

For $p \in \mathbb{S}^2 \times \mathbb{R}$, the Riemann–Christoffel curvature tensor $\widetilde{R}$ of $\mathbb{S}^2 \times \mathbb{R}$ is given by

$$\langle \widetilde{R}(X, Y)Z, W\rangle = \langle X_{\mathbb{S}^2}, W_{\mathbb{S}^2}\rangle\langle Y_{\mathbb{S}^2}, Z_{\mathbb{S}^2}\rangle - \langle X_{\mathbb{S}^2}, Z_{\mathbb{S}^2}\rangle\langle Y_{\mathbb{S}^2}, W_{\mathbb{S}^2}\rangle,$$

where $X, Y, Z, W \in T_p(\mathbb{S}^2 \times \mathbb{R})$ and $X_{\mathbb{S}^2} = X - \langle X, \frac{\partial}{\partial t}\rangle \frac{\partial}{\partial t}$ is the projection of X to the tangent space of $\mathbb{S}^2$.

Let us consider $F : M \to \widetilde{M}$, an isometric immersion of a submanifold M into a Riemannian manifold $\widetilde{M}$ with Levi-Civita connection $\widetilde{\nabla}$. Then we have the formulas of Gauss and Weingarten which state that for every X and Y tangent to M and for every N normal to M there holds that

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$
$$\widetilde{\nabla}_X N = -S_N X + \nabla^{\perp}_X N, \tag{2}$$

with ∇ the Levi-Civita connection of the submanifold. Here h is a symmetric $(1, 2)$-tensor field, taking values in the normal bundle, called the second fundamental form of the submanifold, S_N is a symmetric $(1, 1)$-tensor field, called the shape operator associated to N and $\nabla^{\perp}$ is a connection in the normal bundle.

Now consider a surface M in $\mathbb{S}^2 \times \mathbb{R}$. Let us denote by ξ a unit normal to M with associated shape operator S. Then we can decompose $\frac{\partial}{\partial t}$ at every point p of M as

$$\frac{\partial}{\partial t} = T + \cos(\theta(p))\, \xi, \tag{3}$$

where T is the projection of $\frac{\partial}{\partial t}$ on the tangent space of M and θ is the angle function defined by

$$\cos(\theta(p)) = \left\langle \frac{\partial}{\partial t}, \xi \right\rangle. \tag{4}$$

If we denote by R the curvature tensor of M, then with the previous notations, the equations of Gauss and Codazzi are given by

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \langle SY,Z\rangle\langle SX,W\rangle - \langle SX,Z\rangle\langle SY,W\rangle \\ + \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle \\ + \langle Y,T\rangle\langle W,T\rangle\langle X,Z\rangle + \langle X,T\rangle\langle Z,T\rangle\langle Y,W\rangle \\ - \langle X,T\rangle\langle W,T\rangle\langle Y,Z\rangle - \langle Y,T\rangle\langle Z,T\rangle\langle X,W\rangle,\end{aligned} \tag{5}$$

$$\nabla_X SY - \nabla_Y SX - S[X,Y] = \cos(\theta)\,(\langle Y,T\rangle X - \langle X,T\rangle Y). \tag{6}$$

Note that Eq. 5 is equivalent to

$$K = \det S + \cos^2(\theta), \tag{7}$$

where K is the Gaussian curvature of M.

Furthermore, we have the following proposition.

Proposition 1 *For every $X \in T(M)$, we have that*

$$\nabla_X T = \cos(\theta)\, SX, \tag{8}$$

$$X[\cos(\theta)] = -\langle SX, T\rangle. \tag{9}$$

We can prove this by using the fact that $\frac{\partial}{\partial t}$ is a parallel vector field in $\mathbb{S}^2 \times \mathbb{R}$ and the decomposition (3).

The Eqs. 5–6, 8–9 are called the compatibility equations for $\mathbb{S}^2 \times \mathbb{R}$.

In [5] the following theorem was proven.

Theorem 1 (B. Daniel) *Let M be a simply connected Riemannian surface, g its metric and ∇ its Levi-Civita connection. Let S be a field of symmetric operators $S_p : T_p(M) \to T_p(M)$, T a vector field on M and θ a smooth function on M such that $\|T\|^2 = \sin^2(\theta)$.*
Assume that (g, S, T, θ) satisfies the compatibility equations for $\mathbb{S}^2 \times \mathbb{R}$. Then there exists an isometric immersion $F : M \to \mathbb{S}^2 \times \mathbb{R}$ such that the shape operator with respect to the unit normal ξ is given by S and such that

$$\frac{\partial}{\partial t} = T + \cos(\theta)\,\xi.$$

Moreover the immersion is unique up to global isometries of $\mathbb{S}^2 \times \mathbb{R}$ preserving the orientations of both $\mathbb{S}^2$ and $\mathbb{R}$.

In the next sections, we will use the notation f_x for the partial derivative of a function f with respect to x.

3 Surfaces in $\mathbb{S}^2 \times \mathbb{R}$

In this section, we consider arbitrary surfaces in $\mathbb{S}^2 \times \mathbb{R}$. The following proposition gives a nice way to choose local coordinates adapted to the structure of $\mathbb{S}^2 \times \mathbb{R}$.

Proposition 2 *If M is an immersed surface in $\mathbb{S}^2 \times \mathbb{R}$ and p a point of M for which $\theta(p) \neq 0$ and $\theta(p) \neq \frac{\pi}{2}$, then we can choose local coordinates (x, y) in a neighborhood of p such that $\frac{\partial}{\partial x}$ is in the direction of T, the metric g has the form*

$$g = \frac{1}{\sin^2(\theta)} dx^2 + \beta^2(x,y) dy^2, \tag{10}$$

and the shape operator S with respect to the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ *is given by*

$$S = \begin{pmatrix} \theta_x \sin(\theta) & \theta_y \sin(\theta) \\ \frac{\theta_y}{\sin(\theta)\beta^2} & \frac{\sin^2(\theta)\beta_x}{\cos(\theta)\beta} \end{pmatrix}. \tag{11}$$

Moreover the functions θ *and* β *are related by the PDE*

$$\begin{aligned} &\frac{\sin(\theta)}{\cos^2(\theta)}\theta_x\frac{\beta_x}{\beta} + \frac{\sin^2(\theta)}{\cos(\theta)}\frac{\beta_{xx}}{\beta} + 2\frac{\cos(\theta)}{\sin^2(\theta)}\theta_y^2\frac{1}{\beta^2} \\ &\quad -\frac{1}{\sin(\theta)}\theta_{yy}\frac{1}{\beta^2} + \frac{1}{\sin(\theta)}\theta_y\frac{\beta_y}{\beta^3} + \cos(\theta) = 0. \end{aligned} \tag{12}$$

Proof Take an arbitrary point p in M such that the angle function $\theta(p) \notin \{0, \frac{\pi}{2}\}$. Then we can take local coordinates (x, y) on M such that $\frac{\partial}{\partial x}$ is in the direction of T and the metric g has the form

$$g = \alpha^2(x, y)dx^2 + \beta^2(x, y)dy^2, \tag{13}$$

where α and β are functions on M.

By computing the Levi-Civita connection of the metric (13) and using (8) and (9) with $T = \frac{\sin(\theta)}{\alpha}\frac{\partial}{\partial x}$, we find that the shape operator S takes the form

$$S = \begin{pmatrix} \frac{\theta_x}{\alpha} & \frac{\theta_y}{\alpha} \\ \frac{\alpha\theta_y}{\beta^2} & \frac{\tan(\theta)\beta_x}{\alpha\beta} \end{pmatrix},$$

with respect to the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ and that α satisfies $\frac{\partial}{\partial y}(\alpha \sin(\theta)) = 0$, since $\theta(p) \neq \frac{\pi}{2}$. Hence we obtain $\alpha = \frac{\phi(x)}{\sin(\theta)}$ for some function ϕ on M only depending on x. By changing the x-coordinate, we can thus assume that $\alpha = \frac{1}{\sin(\theta)}$.

The equations of Gauss and Codazzi, (5) and (6), give the PDE relating the functions θ and β. This concludes the proof. □

Remark 1 Combining Proposition 2 with Theorem 1 we see that for every two functions θ and β on a simply connected Riemannian surface with metric given by (10), which satisfy (12), we can construct an immersion into $\mathbb{S}^2 \times \mathbb{R}$ with shape operator (11).

4 Minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$

In this section, we look at minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$. We will use Proposition 2 to choose nice local coordinates.

Proposition 3 *Let M be an immersed surface in* $\mathbb{S}^2 \times \mathbb{R}$ *and p a point of M for which* $\theta(p) \notin \{0, \frac{\pi}{2}\}$. *If M is minimal, then we can choose coordinates* (x, y) *in a neighborhood of p such that* $\frac{\partial}{\partial x}$ *is in the direction of T, the metric g has the form*

$$g = \frac{1}{\sin^2(\theta)}\left(dx^2 + dy^2\right), \tag{14}$$

and the shape operator S with respect to the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ *is given by*

$$S = \sin(\theta)\begin{pmatrix} \theta_x & \theta_y \\ \theta_y & -\theta_x \end{pmatrix}. \tag{15}$$

Moreover the angle function θ *must satisfy the PDE*

$$\Delta \ln\left(\tan\left(\frac{\theta}{2}\right)\right) = \cos(\theta), \tag{16}$$

where $\Delta = \sin^2(\theta)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ *is the Laplacian of* M.

Proof Take an arbitrary point p in M such that the angle function θ is not zero or $\frac{\pi}{2}$ at p. Using Proposition 2 we find local coordinates (x, y) on M such that $\frac{\partial}{\partial x}$ is in the direction of T, the metric g has the form (10) and the shape operator S with respect to the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is given by (11). Since the surface is minimal, we must have $\mathrm{Tr}(S) = 0$. This means

$$\theta_x \sin(\theta) + \frac{\sin^2(\theta)\beta_x}{\cos(\theta)\beta} = 0, \tag{17}$$

which is equivalent to

$$(\beta \sin(\theta))_x = 0. \tag{18}$$

Thus $\beta = \frac{\phi(y)}{\sin(\theta)}$ for some function ϕ on M depending only on y. After changing the y-coordinate, we can assume $\beta = \frac{1}{\sin(\theta)}$. This gives us (14) and (15). From Eq. 12, we also find that

$$\cos(\theta)(\theta_x^2 + \theta_y^2 + 1) - \sin(\theta)(\theta_{xx} + \theta_{yy}) = 0, \tag{19}$$

which is equivalent to (16). □

To give non-trivial examples of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ we must find solutions θ of the PDE (16). Suppose that there is a constant k such that $\theta_x = k\theta_y$. Then we can find other coordinates u and v such that

$$\begin{cases} \frac{\partial}{\partial u} = \frac{1}{\sqrt{1+k^2}}(k\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \\ \frac{\partial}{\partial v} = \frac{1}{\sqrt{1+k^2}}(\frac{\partial}{\partial x} - k\frac{\partial}{\partial y}). \end{cases}$$

With these coordinates (16) becomes

$$\left(\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\right)_{uu} = \frac{\cos(\theta)}{\sin^2(\theta)}, \tag{20}$$

$$\theta_v = 0. \tag{21}$$

By making the substitution $\theta(u) = \arctan(\frac{c}{\cos(\rho(u))})$ with $c \in \mathbb{R}$, (20) becomes

$$\frac{\partial^2\rho(u)}{\partial u^2}\sin(\rho(u))(\cos^2(\rho(u)) + c^2) + \left(\frac{\partial\rho(u)}{\partial u}\right)^2 \cos(\rho(u))(1 + c^2) = \frac{\cos(\rho(u)}{c^2}(\cos^2(\rho(u) + c^2)^2. \tag{22}$$

Equation 22 is satisfied if $\frac{\partial\rho(u)}{\partial u} = \frac{\sqrt{\cos^2(\rho(u))+c^2}}{c}$. Thus we find that

$$\theta = \arctan\left(\frac{c}{\cos\left(\mathrm{am}\left(\frac{u\sqrt{1+c^2}}{c}, \frac{1}{\sqrt{1+c^2}}\right)\right)}\right) \tag{23}$$

is a solution for this system of PDE's. Here am denotes the inverse function of the normal elliptic integral of the first kind. More details on elliptic functions can be found in [4]. In view of Theorem 1, we thus have an example of a minimal surface, but to construct this explicitly the calculations get very difficult.

We show that the only flat and minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ are also totally geodesic.

Theorem 2 *The only surfaces in $\mathbb{S}^2 \times \mathbb{R}$ which are both flat and minimal are vertical cylinders on a geodesic in $\mathbb{S}^2$, i.e. open parts of surfaces of type $\mathbb{S}^1 \times \mathbb{R}$.*

Proof Consider first the case that the angle function θ is constant. Then we know from (7) and (9) that the Gaussian curvature is also constant and equal to $\cos^2(\theta)$. However, since we have a flat surface, we immediately find that $\theta = \frac{\pi}{2}$. Suppose that the surface is given by $F(s,t) = (\gamma(s), t)$, where γ is a curve in $\mathbb{S}^2$ parametrized by arc length. A straightforward calculation shows that the shape operator is given by

$$\begin{pmatrix} \kappa_\gamma & 0 \\ 0 & 0 \end{pmatrix},$$

where κ_γ is the geodesic curvature of γ. So if we assume the surface to be minimal, γ has to be a geodesic of $\mathbb{S}^2$.

Now consider the case that θ is not constant. Then there exists a point p on M for which $\theta(p) \neq 0$ and $\theta(p) \neq \frac{\pi}{2}$. So we can choose local coordinates $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ in a neighborhood of p as in Proposition 3. Computing the Gaussian curvature from the metric (14), respectively from (7) and (15), we obtain

$$\theta_{xx} + \theta_{yy} = \frac{\cos(\theta)}{\sin^3(\theta)}, \tag{24}$$

$$\theta_x^2 + \theta_y^2 = \cot^2(\theta), \tag{25}$$

since the surface is flat.

By deriving (25) with respect to x and substituting $\frac{\cos(\theta)}{\sin^3(\theta)}$ from (24) we get

$$2\theta_x\theta_{xx} + \theta_y\theta_{xy} + \theta_x\theta_{yy} = 0. \tag{26}$$

In an analogous way, by deriving with respect to y, we get

$$2\theta_y\theta_{yy} + \theta_x\theta_{xy} + \theta_y\theta_{xx} = 0. \tag{27}$$

Combining (26) and (27) with (24) and (25), we see that θ must satisfy

$$\cos(\theta)\sin(\theta)\theta_{xx} + 3\theta_x^2 = 2\cot^2(\theta). \tag{28}$$

Now by making the substitution $f(x, y) = \ln(\cos(\theta(x, y)))$, Eq. 28 reduces to

$$2f_x^2 - (1 - e^{2f})f_{xx} = 2. \tag{29}$$

By deriving (29) with respect to x we see that

$$2\frac{2 + e^{2f}}{1 - e^{2f}} f_x = \frac{f_{xxx}}{f_{xx}}. \tag{30}$$

Integrating (30) we get

$$f_{xx} = \phi(y)\frac{e^{4f}}{(1 - e^{2f})^3}, \tag{31}$$

for a function ϕ on M. Substituting (31) in (29) gives

$$f_x^2 = 1 + \frac{\phi(y)}{2}\left(\frac{e^{2f}}{1-e^{2f}}\right)^2. \tag{32}$$

In an analogous way one has that

$$f_{yy} = \psi(x)\frac{e^{4f}}{(1-e^{2f})^3}, \tag{33}$$

$$f_y^2 = 1 + \frac{\psi(x)}{2}\left(\frac{e^{2f}}{1-e^{2f}}\right)^2, \tag{34}$$

for a function ψ on M.

Substituting (31) and (33) in (24) gives

$$\phi(y) + \psi(x) = -2\left(\frac{1-e^{2f}}{e^{2f}}\right)^2. \tag{35}$$

By substituting (32), (34) in (25), we also find Eq. 35.

Combining (32) and (34) with (35), we get

$$f_x^2 = \frac{\psi(x)}{\psi(x)+\phi(y)}, \tag{36}$$

$$f_y^2 = \frac{\phi(y)}{\psi(x)+\phi(y)}. \tag{37}$$

From the integrability of this system, we see that

$$\psi(x) = -(\alpha x+\beta)^2, \tag{38}$$

$$\phi(y) = -(\alpha x+\delta)^2. \tag{39}$$

Note that ψ and ϕ are negative because of (35), (36) and (37).

Thus, (36) and (37) now become

$$f_x = \pm\frac{\alpha x+\beta}{\sqrt{(\alpha x+\beta)^2+(\alpha y+\delta)^2}}, \tag{40}$$

$$f_y = \pm\frac{\alpha y+\delta}{\sqrt{(\alpha x+\beta)^2+(\alpha y+\delta)^2}}. \tag{41}$$

We can see from the integrability condition that in fact f_x and f_y must have the same sign. Solving this system, we then find

$$f(x,y) = \pm\frac{1}{c}\sqrt{(\alpha x+\beta)^2+(\alpha y+\delta)^2}. \tag{42}$$

But (42) gives a contradiction with (35), (38) and (39). So we can conclude that this case does not occur, which proves the theorem. □

5 Surfaces for which *T* is a principal direction

In [8] surfaces in $\mathbb{S}^2 \times \mathbb{R}$ for which the angle function θ is constant, were studied. These constant angle surfaces were characterized by the fact that the projection T of $\frac{\partial}{\partial t}$ on the tangent space of the surface, is a principal direction with principal curvature 0. A natural generalization of constant angle surfaces is thus the study of surfaces in $\mathbb{S}^2 \times \mathbb{R}$ for which T is a principal direction, but the principal curvature doesn't need to be zero. Note that we can consider a surface in $\mathbb{S}^2 \times \mathbb{R}$ also as a codimension 2 immersion of a surface in $\mathbb{E}^4$. The condition that T is a principal direction is equivalent to the condition of the vanishing of the normal curvature of the surface in $\mathbb{E}^4$. This is again an indication that this condition is very natural in $\mathbb{S}^2 \times \mathbb{R}$.

Theorem 3 *Let M be an immersed surface in $\mathbb{S}^2 \times \mathbb{R}$ and p a point of M for which $\theta(p) \notin \{0, \frac{\pi}{2}\}$. Then T is a principal direction if and only if M considered as a surface in $\mathbb{E}^4$ is normally flat.*

Proof Take an arbitrary point p in M such that the angle function θ is not zero or $\frac{\pi}{2}$ at p. Choose local coordinates as in Proposition 2. Now let us consider the surface M as a codimension 2 immersed surface in $\mathbb{E}^4$ and denote by $F = (F_1, F_2, F_3, F_4)$ the immersion, with D the Euclidean connection and with $\nabla^\perp$ the normal connection. Then we have two unit normals: $\xi = (\xi_1, \xi_2, \xi_3, \cos(\theta))$ tangent to $\mathbb{S}^2 \times \mathbb{R}$ and $\widetilde{\xi} = (F_1, F_2, F_3, 0)$ normal to $\mathbb{S}^2 \times \mathbb{R}$ with shape operator S respectively $\widetilde{S}$. We have for every $X = (X_1, X_2, X_3, X_4) \in T_p(M)$

$$\begin{aligned} \nabla^\perp_X \widetilde{\xi} &= \langle D_X \widetilde{\xi}, \xi \rangle \, \xi \\ &= \langle (X_1, X_2, X_3, 0), \xi \rangle \, \xi \\ &= -\cos(\theta) \langle X, T \rangle \, \xi, \end{aligned} \tag{43}$$

and hence

$$\nabla^\perp_X \xi = \cos(\theta) \langle X, T \rangle \, \widetilde{\xi}. \tag{44}$$

Choose coordinates (x, y) as in Proposition 2. From (43) and (44), we obtain

$$\left\langle R^\perp \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \xi, \widetilde{\xi} \right\rangle = \sin(\theta) \theta_y. \tag{45}$$

Hence, $R^\perp = 0$ if and only if $\theta_y = 0$. From (11) in Proposition 2, we obtain the result. □

We have the following propositions.

Proposition 4 *Let M be an immersed surface in $\mathbb{S}^2 \times \mathbb{R}$ and p a point of M for which $\theta(p) \notin \{0, \frac{\pi}{2}\}$. If T is a principal direction, then we can choose coordinates (x, y) in a neighborhood of p such that $\frac{\partial}{\partial x}$ is in the direction of T, the metric g has the form*

$$g = dx^2 + \beta^2(x, y) dy^2, \tag{46}$$

and the shape operator S with respect to the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is given by

$$S = \begin{pmatrix} \theta_x & 0 \\ 0 & \tan(\theta) \frac{\beta_x}{\beta} \end{pmatrix}. \tag{47}$$

Moreover the functions θ and β are related by the PDE

$$\beta_{xx} + \beta_x \tan(\theta) \theta_x + \beta \cos^2(\theta) = 0, \tag{48}$$

and $\theta_y = 0$.

Proof Take an arbitrary point p in M such that the angle funtion θ is not zero or $\frac{\pi}{2}$ at p. From Proposition 2 and the assumption that T is a principal direction, we see that $\theta_y = 0$. This means that by changing the x-coordinate we can assume that the metric is given by (46). Now we can find (47) and (48) in an analogous way as in Proposition 2. □

Proposition 5 *A surface M immersed in $\mathbb{S}^2 \times \mathbb{R}$ is a surface for which T is a principal direction if and only if the immersion F is (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$) in the neighborhood of a point p where $\theta(p) \notin \{0, \frac{\pi}{2}\}$ given by*

$$F : M \to \mathbb{S}^2 \times \mathbb{R} : (x, y) \mapsto (F_1(x, y), F_2(x, y), F_3(x, y), F_4(x)) \tag{49}$$

with

$$F_j(x, y) = \int_{y_0}^{y} \alpha_j(v) \sin(\psi(x) + \phi(v))\, dv \tag{50}$$

for $j = 1, 2, 3$ where $\psi'(x) = \cos(\theta(x))$, $F_4'(x) = \sin(\theta(x))$, $(\alpha_1, \alpha_2, \alpha_3)$ is a curve in $\mathbb{S}^2$ and $F_1^2 + F_2^2 + F_3^2 = 1$. Moreover $\alpha_1, \alpha_2, \alpha_3, \psi$ and ϕ are functions on M related by

$$\alpha_j'(y) = -\cos(\psi(x) + \phi(y)) \int_{y_0}^{y} \alpha_j(v) \cos(\psi(x) + \phi(v))\, dv$$
$$- \sin(\psi(x) + \phi(y)) \int_{y_0}^{y} \alpha_j(v) \sin(\psi(x) + \phi(v))\, dv. \tag{51}$$

Proof Take an arbitrary point p in M such that the angle function θ is not zero or $\frac{\pi}{2}$ at p. We take coordinates as in Proposition 4.

From (48) we find that β satisfies $\frac{\beta_x^2}{\cos^2(\theta)} + \beta^2 = k(y)^2$ for some function k on M and hence $\beta(x, y) = k(y)\sin(\psi(x) + \phi(y))$ for some function ϕ on M and a primitive function ψ of $\cos(\theta)$. By changing the y-coordinate we can assume that we have

$$g = dx^2 + \sin^2(\psi(x) + \phi(y))dy^2, \tag{52}$$

and thus the Levi-Civita connection is given by

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \tag{53}$$
$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \cot(\psi(x) + \phi(y)) \cos(\theta) \frac{\partial}{\partial y}, \tag{54}$$
$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\sin(\psi(x) + \phi(y)) \cos(\psi(x) + \phi(y)) \cos(\theta) \frac{\partial}{\partial x}$$
$$+ \cot(\psi(x) + \phi(y))\phi'(y) \frac{\partial}{\partial y}. \tag{55}$$

Denote the two normals by ξ and $\widetilde{\xi}$. The normal $\xi = (\xi_1, \xi_2, \xi_3, \cos(\theta))$, with shape operator S, is tangent to $\mathbb{S}^2 \times \mathbb{R}$ and the normal $\widetilde{\xi} = (F_1, F_2, F_3, 0)$ with shape operator $\widetilde{S}$, is normal to $\mathbb{S}^2 \times \mathbb{R}$. S and $\widetilde{S}$ are given by

$$S = \begin{pmatrix} \theta_x & 0 \\ 0 & \cot(\psi(x) + \phi(y)) \sin(\theta) \end{pmatrix}, \quad \widetilde{S} = \begin{pmatrix} -\cos^2(\theta) & 0 \\ 0 & -1 \end{pmatrix}. \tag{56}$$

From the form of $\widetilde{S}$ and (45) in the proof of Theorem 3 we also have that

$$\xi_j = -\tan(\theta)(F_j)_x, \tag{57}$$

for $j = 1, 2, 3$.

For the first three components of F the formula of Gauss (1) together with (53), (54), (55) and (56) gives the following system of PDE's

$$(F_j)_{xx} = -\tan(\theta)\theta_x(F_j)_x - \cos^2(\theta)F_j, \tag{58}$$

$$(F_j)_{xy} = \cot(\psi(x)+\phi(y))\cos(\theta)(F_j)_y, \tag{59}$$

$$\begin{aligned}(F_j)_{yy} = &-\frac{1}{2}\sin(2(\psi(x)+\phi(y)))\frac{1}{\cos(\theta)}(F_j)_x \\ &+\cot(\psi(x)+\phi(y))\phi'(y)(F_j)_y \\ &-\sin^2(\psi(x)+\phi(y))F_j.\end{aligned} \tag{60}$$

From (59) we find that

$$F_j(x,y) = \int_{y_0}^{y} \alpha_j(v)\sin\left(\psi(x)+\phi(v)\right)\, dv, \tag{61}$$

where α_j is a function on M for $j = 1, 2, 3$.

By substituting (61) into (60) we find that α_j must satisfy

$$\begin{aligned}\alpha_j'(y) = &-\cos\left(\psi(x)+\phi(y)\right)\int_{y_0}^{y}\alpha_j(v)\cos\left(\psi(x)+\phi(v)\right)\, dv \\ &-\sin(\psi(x)+\phi(y))\int_{y_0}^{y}\alpha_j(v)\sin\left(\psi(x)+\phi(v)\right)\, dv.\end{aligned} \tag{62}$$

Since we must have that $(F_1, F_2, F_3) \in \mathbb{S}^2$ we also need that $F_1^2 + F_2^2 + F_3^2 = 1$ and from $\langle F_y, F_y\rangle = \sin^2(\psi(x)+\phi(y))$ we find that $(\alpha_1, \alpha_2, \alpha_3)$ is a curve in $\mathbb{S}^2$.

For the fourth component of the immersion note that

$$(F_4)_x = \left\langle F_x, \frac{\partial}{\partial t}\right\rangle = \sin(\theta), \tag{63}$$

$$(F_4)_y = \left\langle F_y, \frac{\partial}{\partial t}\right\rangle = 0. \tag{64}$$

Thus we see that F_4 only depends on x and is a primitive function of $\sin(\theta)$. This proves the proposition. □

We can give many examples of these type of surfaces. The constant angle surfaces, as defined in [8], and the rotation surfaces, as defined in [7], all satisfy this condition. Since we know from Theorem 3 that the condition that T is a principal direction is equivalent to normal flatness in $\mathbb{E}^4$ and from [6] that all rotation surfaces are normally flat, one could think that every surface for which T is a principal direction must be a rotation surface. This is however not true as can be seen from the following example:

$$F : M \to \mathbb{S}^2 \times \mathbb{R} : (x, y) \mapsto (\cos x \cos y,\ \cos x \sin y,\ \sin x,\ f(x, y)) \tag{65}$$

with $f(x, y) = \int_0^x \sqrt{c - \tan^2 u}\, du + y$ where $c \in \mathbb{R}$ is a constant such that on a neighborhood of 0 we have that $c - \tan^2 u \geq 0$.

We can give classification theorems for surfaces for which T is a principal direction under an additional assumption.

Theorem 4 *A surface M immersed in* $\mathbb{S}^2 \times \mathbb{R}$ *is a minimal surface with T a principal direction if and only if the immersion F is* (*up to isometries of* $\mathbb{S}^2 \times \mathbb{R}$) *in the neighborhood of a point p where* $\theta(p) \notin \{0, \frac{\pi}{2}\}$ *given by*

$$F : M \to \mathbb{S}^2 \times \mathbb{R} :$$
$$(x, y) \mapsto \left(\frac{\sin x}{\sqrt{1+c^2}}, \frac{\sqrt{\cos^2 x + c^2}\cos y}{\sqrt{1+c^2}}, \frac{\sqrt{\cos^2 x + c^2}\sin y}{\sqrt{1+c^2}}, F_4(x) \right). \tag{66}$$

with

$$F_4(x) = \int_0^x \frac{c}{\sqrt{\cos^2(u) + c^2}} du. \tag{67}$$

Proof Take an arbitrary point p in M such that the angle function θ is not zero or $\frac{\pi}{2}$ at p. Choose local coordinates as in Proposition 4. Since now M is minimal, it follows from (47) that we can take $\beta = \frac{1}{\sin(\theta)}$. Summarizing this, we obtain the metric g

$$g = dx^2 + \frac{1}{\sin^2(\theta)} dy^2, \tag{68}$$

and the shape operator S

$$S = \begin{pmatrix} \theta_x & 0 \\ 0 & -\theta_x \end{pmatrix}. \tag{69}$$

Since $\theta_y = 0$ if T is a principal direction and $\beta = \frac{1}{\sin(\theta)}$, Eq. 48 reduces to an ordinary differential equation in θ

$$\theta_{xx} - 2\cot(\theta)\theta_x^2 + \cos(\theta)\sin(\theta) = 0. \tag{70}$$

Making the substitution $\theta(x) = \arctan(f(x))$, we get

$$\frac{\left(\frac{f'}{f}\right)'}{1 + \left(\frac{f'}{f}\right)^2} = 1. \tag{71}$$

Integrating (71) we find $\frac{f'}{f} = \tan(x + \tilde{d})$ for some constant $\tilde{d}$. By changing the x-coordinate if necessary, we can assume $\tilde{d} = 0$. Integrating this last equation, we find $f(x) = \frac{c}{\cos(x)}$ for a constant c and thus

$$\theta = \arctan\left(\frac{c}{\cos(x)}\right). \tag{72}$$

Now we know from Theorem 1 that this function will give the surface we are looking for. To get an explicit parametrization we will integrate the formula of Gauss (1). In order to do so, we consider the surface as a surface in $\mathbb{E}^4$. From the metric (68) we find that the Levi-Civita connection ∇ is given by

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \tag{73}$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = -\cot(\theta)\theta_x \frac{\partial}{\partial y}, \tag{74}$$

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{\cos(\theta)}{\sin^3(\theta)} \frac{\partial}{\partial x}. \tag{75}$$

As in the proof of Proposition 5 we take two unit normals ξ and $\widetilde{\xi}$ with shape operators S and $\widetilde{S}$ given by

$$S = \begin{pmatrix} \theta_x & 0 \\ 0 & -\theta_x \end{pmatrix}, \quad \widetilde{S} = \begin{pmatrix} -\cos^2(\theta) & 0 \\ 0 & -1 \end{pmatrix}. \tag{76}$$

with

$$\xi_j = -\tan(\theta)(F_j)_x. \tag{77}$$

Combining (73), (74), (75), (76) and (77) with (1) we find for $j = 1, 2, 3$ that

$$(F_j)_{xx} = -\tan(\theta)\theta_x(F_j)_x - \cos^2(\theta)F_j, \tag{78}$$
$$(F_j)_{xy} = -\cot(\theta)\theta_x(F_j)_y, \tag{79}$$
$$(F_j)_{yy} = \frac{\theta_x}{\cos(\theta)\sin^3(\theta)}(F_j)_x - \frac{1}{\sin^2(\theta)}F_j. \tag{80}$$

From (79) we find that F_j is given by

$$F_j = \frac{\phi_j(y)}{\sin(\theta)} + \psi_j(x), \tag{81}$$

where ϕ_j and ψ_j are functions on M for $j = 1, 2, 3$ and θ is given by (72).

Substituting (72) and (81) in (78) gives ordinary differential equations for the functions ψ_j:

$$\psi_j''(x)\cos^3(x) + \psi_j''(x)c^2\cos(x) + \psi_j'(x)c^2\sin(x) + \psi_j(x)\cos^3(x) = 0. \tag{82}$$

Substituting (72) and (81) in (80) gives

$$\phi_j''(y) + \frac{c^2+1}{c^2}\phi_j(y) = \epsilon d_j, \tag{83}$$

$$\begin{aligned} &\sin(x)\cos^2(x)\psi_j'(x) + c^2\sin(x)\psi_j'(x) - \cos^3(x)\psi_j(x) \\ &\quad - c^2\cos(x)\psi_j(x) = -d_j c\cos(x)\sqrt{\cos^2(x)+c^2}, \end{aligned} \tag{84}$$

where $\epsilon = 1$ if $\cos(x) \geq 0$ in a neighborhood of p and $\epsilon = -1$ in the other case and d_j is a constant.

Note that Eq. 84 implies Eq. 82, since we get (82) by deriving (84) with respect to x. Eq. 84 is equivalent to

$$\sin(x)\psi_j'(x) - \cos(x)\psi_j(x) = \frac{d_j c\cos(x)}{\sqrt{\cos^2(x)+c^2}}. \tag{85}$$

From (85) we see that ψ_j is given by

$$\psi_j(x) = A_j\sin(x) - \frac{d_j c\sqrt{\cos^2(x)+c^2}}{c^2+1}, \tag{86}$$

for a constant A_j.

And by solving (83), we have

$$\phi_j(y) = \epsilon\left(B_j\cos\left(y\sqrt{\frac{c^2+1}{c^2}}\right) + C_j\sin\left(y\sqrt{\frac{c^2+1}{c^2}}\right) + \frac{d_j c^2}{c^2+1}\right), \tag{87}$$

with B_j and C_j constants.

Combining (81), (86), and (87) we find

$$F_j = \frac{\sqrt{\cos^2(x)+c^2}}{c}\left(B_j \cos\left(y\sqrt{\frac{c^2+1}{c^2}}\right) + C_j \sin\left(y\sqrt{\frac{c^2+1}{c^2}}\right)\right) + A_j \sin(x), \tag{88}$$

for $j = 1, 2, 3$.

Since we must have that $\langle (F_1, F_2, F_3), (F_1, F_2, F_3)\rangle = 1$ we find that $\langle A, B\rangle = \langle A, C\rangle = \langle B, C\rangle = 0$ and $c^2\|A\|^2 = \|B\|^2 = \|C\|^2 = \frac{c^2}{1+c^2}$ where $A = (A_1, A_2, A_3)$, $B = (B_1, B_2, B_3)$ and $C = (C_1, C_2, C_3)$. Thus we can assume that $A = (\frac{1}{\sqrt{1+c^2}}, 0, 0)$, $B = (0, \frac{1}{\sqrt{1+c^2}}, 0)$ and $C = (0, 0, \frac{c}{\sqrt{1+c^2}})$.

From the proof of Proposition 5, we know that $F_4(x, y) = \int_{x_0}^{x} \sin(\theta(u))du$. Together with (72), we see that

$$F_4 = \int_0^x \frac{c}{\sqrt{\cos^2(u)+c^2}}du. \tag{89}$$

By changing the y-coordinate, equations (88) and (89) give the immersion (66). □

Remark 2 We can rewrite F_4 in the following way:

$$\begin{aligned} F_4 &= \frac{c}{\sqrt{1+c^2}}\int_0^x \frac{du}{\sqrt{1-\frac{1}{1+c^2}\sin^2(u)}} \\ &= \frac{c}{\sqrt{1+c^2}} F\left(x, \sqrt{\frac{1}{1+c^2}}\right), \end{aligned}$$

where F is the normal elliptic integral of the first kind. More details on elliptic functions can be found in [4].

Theorem 5 *A surface M immersed in $\mathbb{S}^2 \times \mathbb{R}$ is a flat surface with T a principal direction if and only if the immersion F is (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$) in the neighborhood of a point p where $\theta(p) \notin \{0, \frac{\pi}{2}\}$ given by*

$$F : M \to \mathbb{S}^2 \times \mathbb{R} : (x, y) \mapsto \left(\frac{\sqrt{1+d-x^2}}{\sqrt{1+d}}, \frac{x\cos y}{\sqrt{1+d}}, \frac{x \sin y}{\sqrt{1+d}}, F_4(x)\right) \tag{90}$$

with

$$F_4(x) = \int_0^x \frac{\sqrt{d-u^2}}{\sqrt{1+d-u^2}}du. \tag{91}$$

Proof Take an arbitrary point p in M such that the angle function θ is not zero or $\frac{\pi}{2}$ at p. Choose local coordinates as in Proposition 4. From the metric (46) and the fact that M is flat we know that $\beta_{xx} = 0$. Thus we have $\beta = a(y)x + b(y)$ for some functions a and b on M. Substituting this in (48) we get

$$\frac{a(y)}{b(y)} = -\frac{\cos^2(\theta)}{\tan(\theta)\theta_x + x\cos^2(\theta)}. \tag{92}$$

Since the left hand side of (92) depends only on y and the right hand side only on x, they must be constant. So there is a constant c such that

$$b(y) = c\,a(y) \tag{93}$$
$$-c\cos^2(\theta) = \tan(\theta)\theta_x + x\cos^2(\theta). \tag{94}$$

From (93) we see that $\beta = a(y)(x+c)$. So after changing the x-coordinate we can assume that $c = 0$ and that the metric g is given by

$$g = dx^2 + x^2 dy^2. \tag{95}$$

We can rewrite (94) as $\left(\frac{1}{2}\tan^2(\theta)\right)_x = -x$. This means that

$$\theta = \arctan\left(\sqrt{d - x^2}\right), \tag{96}$$

for some positive constant d.

From (95) we find the Levi-Civita connection

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \tag{97}$$
$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \frac{1}{x}\frac{\partial}{\partial y}, \tag{98}$$
$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -x\frac{\partial}{\partial x}. \tag{99}$$

As before, take two unit normals $\xi = (\xi_1, \xi_2, \xi_3, \cos(\theta))$ with shape operator

$$S = \begin{pmatrix} \theta_x & 0 \\ 0 & \frac{\tan(\theta)}{x} \end{pmatrix}, \tag{100}$$

where $\xi_j = -\tan(\theta)(F_j)_x$ and $\widetilde{\xi} = (F_1, F_2, F_3, 0)$ with shape operator

$$\widetilde{S} = \begin{pmatrix} -\cos^2\theta & 0 \\ 0 & -1 \end{pmatrix}. \tag{101}$$

Combining all this with the formula of Gauss (1) we find the following system for $j = 1, 2, 3$:

$$(F_j)_{xx} = -\tan(\theta)\theta_x(F_j)_x - \cos^2(\theta)F_j, \tag{102}$$
$$(F_j)_{xy} = \frac{1}{x}(F_j)_y, \tag{103}$$
$$(F_j)_{yy} = \frac{-x}{\cos^2(\theta)}(F_j)_x - x^2F_j. \tag{104}$$

From (103) we conclude that

$$F_j = \phi_j(y)x + \psi_j(x), \tag{105}$$

for some functions ϕ_j and ψ_j on M.

Substituting (96) and (105) in (102) gives us

$$(x^2 - d - 1)\psi_j''(x) + x\psi_j'(x) - \psi_j(x) = 0, \tag{106}$$

and substituting this in (104) gives

$$\phi_j''(y) + (1+d)\phi_j(y) = k_j, \tag{107}$$

$$(1+d)\psi_j'(x) - x^2\psi_j'(x) + x\psi_j(x) = -k_j, \tag{108}$$

for a constant k_j. Note that Eq. 108 implies Eq. 106, since we get (106) by deriving (108) with respect to x. Solving (108), we find

$$\psi_j(x) = A_j\sqrt{1+d-x^2} - \frac{k_j}{1+d}x, \tag{109}$$

for a constant A_j. And by solving (107), we get

$$\phi_j(y) = B_j \cos\left(y\sqrt{1+d}\right) + C_j \sin\left(y\sqrt{1+d}\right) + \frac{k_j}{1+d}, \tag{110}$$

with B_j and C_j constants, since we know that d is a positive constant.

From (109) and (110) we conclude that

$$F_j(x,y) = B_j x \cos\left(y\sqrt{1+d}\right) + C_j x \sin\left(y\sqrt{1+d}\right) + A_j\sqrt{1+d-x^2} \tag{111}$$

for $j = 1, 2, 3$.

Since we must have that $\langle (F_1, F_2, F_3), (F_1, F_2, F_3)\rangle = 1$ we find that $\langle A, B\rangle = \langle A, C\rangle = \langle B, C\rangle = 0$ and $\|A\|^2 = \|B\|^2 = \|C\|^2 = \frac{1}{1+d}$ where $A = (A_1, A_2, A_3)$, $B = (B_1, B_2, B_3)$ and $C = (C_1, C_2, C_3)$.

From the proof of Proposition 5, we know that $F_4(x, y) = \int_{x_0}^{x} \sin(\theta(u))du$. Using (96) we find

$$F_4(x,y) = \int_0^x \frac{\sqrt{d-u^2}}{\sqrt{1+d-u^2}}du. \tag{112}$$

By changing the y-coordinate, Eq. 111 and 112 give us the immersion (90). □

Remark 3 Again we can rewrite F_4, by making the substitution $t = \frac{u}{1+d}$, in the following way,

$$\begin{aligned} F_4(x,y) &= \frac{1}{\sqrt{1+d}} \int_0^x \frac{\sqrt{d-u^2}}{\sqrt{1-\left(\frac{u}{1+d}\right)^2}}du \\ &= \sqrt{d(1+d)} \int_0^{\frac{x}{1+d}} \frac{\sqrt{1-\frac{(1+d)^2}{d}t^2}}{\sqrt{1-t^2}}dt \\ &= \sqrt{d(1+d)}E\left(\frac{x}{1+d}, \frac{1+d}{\sqrt{d}}\right), \end{aligned}$$

where E is the normal elliptic integral of the second kind.

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