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ORIGINAL PAPER

Stability of area-preserving variations in space forms

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Abstract In this article, we deal with compact hypersurfaces without boundary immersed in space forms with $\frac{S_{r+1}}{S_1} =$ constant. They are critical points for an area-preserving variational problem. We show that they are r-stable if and only if they are totally umbilical hypersurfaces.

Keywords rth mean curvatures · r-stability · Area-preserving variation

Mathematics Subject Classification (2000) Primary 53C42 · 53A30 · Secondary 53B25

1 Introduction

Let $\tilde{M}(c)$ be $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, an open hemisphere of the unit sphere $S^{n+1}(1)$ or the hyperbolic space $H^{n+1}(-1)$ according to $c = 0$, 1 or -1, respectively. Let $x: M \to \tilde{M}(c)$ be a smooth immersion of a compact and oriented hypersurface without boundary.

Volume preserving variational problem has been studied by many authors, see [1–8]. It is well known that immersions with constant mean curvature are critical points for the variational problem of minimizing the area functional keeping the balance of volume zero. A local solution for this variational problem is said to be stable. This concept was introduced by Barbosa, do Carmo and Eschenburg in [8].

For immersions of hypersurfaces with constant $(r+1)$th mean curvature in space forms, Alencar, do Carmo and Rosenberg studied the case of $\mathbb{R}^{n+1}$ in [3], Barbosa and Colares studied the case of an open hemisphere of $S^{n+1}(1)$ or the hyperbolic space $H^{n+1}(-1)$ in [5].

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These hypersurfaces are critical points for a variational problem of minimizing a curvature integral of the type

$$\mathscr{A}_r = \int_M F_r(S_1, \ldots, S_r) dM,$$

keeping the balance of volume zero, where F_r is a suitable function. For this problem, they introduced the concept of r-stability of hypersurfaces, generalized the one introduced in [8]. Other variational problems for hypersurfaces involving functions of $S_1, \ldots, S_r$ can be found in [18].

In this article, we consider hypersurfaces in $\tilde{M}(c)$ with positive mean curvature and constant ratio of $(r+1)$th mean curvature and mean curvature, where $c = 0$ or r is even when $c \neq 0$, which are critical points for a variational problem of minimizing the functional $\mathscr{A}_r$ in [5] keeping the area of the hypersurfaces. We introduce the concept of r-stability similar to [5]. We prove that totally umbilical hypersurfaces are the only r-stable immersed compact oriented hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$, an open hemisphere of the unit sphere $S^{n+1}(1)$ or the hyperbolic space $H^{n+1}(-1)$(see Theorems 5.4 and 5.6).

2 Preliminaries

Let $\tilde{M}(c)$ be an $(n+1)$-dimensional space form with constant sectional curvature c, where $c = 0, 1$, or -1 and respectively $\tilde{M}(c)$ is either the Euclidean space $\mathbb{R}^{n+1}$, the unit sphere $S^{n+1}(1)$ or the hyperbolic space $H^{n+1}(-1)$. We represent $\langle \cdot, \cdot \rangle$ the Riemannian structure of $\tilde{M}(c)$. Let $x: M \to \tilde{M}(c)$ be a smooth immersion of a compact, connected, oriented hypersurface without boundary. Let N be a globally defined unit normal vector field along M.

The shape operator B of x associated to N is defined by $B(Y) = -\tilde{\nabla}_Y N$, where Y is any tangent vector field on M, $\tilde{\nabla}$ is the Levi Civita connection on $\tilde{M}(c)$. Its eigenvalues, the principal curvatures are represented by $k_1, k_2, \ldots, k_n$. Using the characteristic polynomial of B, the elementary symmetric function S_r is defined by

$$\det(tI - B) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r}. \tag{1}$$

the rth mean curvature H_r is defined by $H_r = S_r / C_n^r$. Clearly H_1 is the mean curvature H.

In this article, we assume the mean curvature H of M is positive and the ratio of $(r+1)$th mean curvature and mean curvature H_{r+1}/H is constant where $c = 0$, or $c \neq 0$ and r is even, $1 \leq r \leq n-1$.

The classical Newton transformation T_r are inductively defined by

$$\begin{aligned} T_0 &= I \\ T_r &= S_r I - T_{r-1} B. \end{aligned} \tag{2}$$

Let $e_1, e_2, \ldots, e_n$ be orthonormal eigenvectors of B corresponding respectively to the eigenvalues $k_1, k_2, \ldots, k_n$. Represent by B_i the restriction of B to the subspace normal to e_i, and by $S_r(B_i)$ the rth symmetric function associated to B_i. Then, it is obvious that

$$S_{r+1} = k_i S_r(B_i) + S_{r+1}(B_i) \quad \text{for each } 1 \leq r \leq n-1 \text{ and } 1 \leq i \leq n. \tag{3}$$

We state the following properties of T_r which can be found in [5] or [16]:

Lemma 2.1 ([5,16]) *For each* $1 \leq r \leq n-1$

1. $T_r(e_i) = S_r(B_i)e_i$ for each $1 \leq r \leq n$;
2. $trace(T_r) = (n-r)S_r$;
3. $trace(BT_r) = (r+1)S_{r+1}$;
4. $trace(B^2T_r) = S_1 S_{r+1} - (r+2)S_{r+2}$.

3 L_r operator

Let $\{e_1, \ldots, e_n, N\}$ be a local orthonormal frame field along hypersurface M in $\tilde{M}(c)$ where N is a normal vector field, and $\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}$ its dual coframe field. We have the structure equations (see [10,13,14])

$$\begin{cases} dx = \sum_{i=1}^{n} \omega_i e_i, \\ de_i = \sum_{j=1}^{n} \omega_{ij} e_j + \sum_{j=1}^{n} h_{ij}\omega_j N - cx\omega_i, \quad 1 \leq i \leq n, \\ dN = -\sum_{i,j=1}^{n} h_{ij}\omega_j e_i, \end{cases} \tag{4}$$

where $h_{ij} = h_{ji}$, $Be_i = \sum_j h_{ij} e_j$.

For any smooth function f on M, we define f_i and f_{ij} by (see [13,14])

$$df = \sum_{i=1}^{n} f_i \omega_i, \tag{5}$$

$$df_i + \sum_{j=1}^{n} f_j \omega_{ji} = \sum_{j=1}^{n} f_{ij}\omega_j, \tag{6}$$

where $f_{ij} = f_{ji}$.

Then the gradient ∇f and Hessian Hess(f) of f are defined by

$$\nabla f = \sum_{i=1}^{n} f_i e_i, \tag{7}$$

and

$$\text{Hess}(f)e_i = \sum_{j=1}^{n} f_{ij} e_j, \quad 1 \leq i \leq n, \tag{8}$$

respectively.

In [10], Cheng and Yau introduced an operator $\square \colon C^\infty(M) \to C^\infty(M)$, $\square f = \text{trace}(\Phi \text{Hess}(f))$, where $\Phi = \sum_{ij} \phi_{ij}\omega_i\omega_j$ is a symmetric tensor. They also have shown that $\square$ is self-adjoint if and only if $\sum_j \phi_{ijj} = 0$ for all i. It is a simple consequence of their computation that under the above condition,

$$\square f = \text{div}(\Phi \nabla f),$$

where div stands for the divergence operator on M.

For each T_r defined by (2), we have a second order differential operator L_r defined by

$$L_r f = \text{trace}(T_r \text{Hess}(f)).$$

From the Codazzi equation, and Cheng-Yau's result above we have $L_r = \text{div}(T_r \nabla f)$. A proof of this fact was done by Reilly [15] (see also Rosenberg in [16]). Thus, we have the following lemma by Stokes theorem:

Lemma 3.1 (see [5]) *For any function f, g on M, we have*

$$\int_M L_r(f)\text{d}M = 0, \tag{9}$$

and

$$\int_M f L_r(g)\text{d}M = -\int_M \langle T_r \nabla f, \nabla g\rangle \text{d}M. \tag{10}$$

We need the following theorem:

Theorem 3.2 (see [5,9]) *Let $x: M \to \tilde{M}(c)$ be a hypersurface with unit normal vector field N. Then we have*

$$L_r x = (r+1)S_{r+1}N - (n-r)cS_r x, \tag{11}$$

$$L_r N = -\nabla S_{r+1} - (S_1 S_{r+1} - (r+2)S_{r+2})N + c(r+1)S_{r+1}x. \tag{12}$$

For a hypersurface M in $\mathbb{R}^{n+1}$, taking $f = \langle x, N\rangle$ and $g = \frac{1}{2}|x|^2$ in (10), we can obtain the following lemma:

Lemma 3.3 *Let $x: M \to \mathbb{R}^{n+1}$ be a hypersurface with unit normal vector field N. Then we have*

$$\int_M \langle x, N\rangle\{(n-r)S_r + (r+1)S_{r+1}\langle x, N\rangle\}\text{d}M = \int_M \langle T_r B x^T, x^T\rangle \text{d}M, \tag{13}$$

where x^T denotes the tangent component of x.

Proof Through a direct calculation, we have

$$g_{ij} = \delta_{ij} + h_{ij}\langle x, N\rangle,$$
$$\nabla f = -\sum_{i,j=1}^{n} h_{ij}\langle x, e_j\rangle e_i = -Bx^T,$$
$$\nabla g = \sum_{i=1}^{n}\langle x, e_i\rangle e_i = x^T,$$

and using Lemma 2.1, we have

$$\begin{aligned} L_r g &= \sum_{i,j=1}^{n} (T_r)_{ij} g_{ij} \\ &= \text{trace}(T_r) + \text{trace}(T_r B)\langle x, N\rangle \\ &= (n-r)S_r + (r+1)S_{r+1}\langle x, N\rangle. \end{aligned}$$

From these formulas and (10) of Lemma 3.1, we have (13). □

We define an operator $\tilde{L}_r$ by

$$\tilde{L}_r(f) = L_r f - \frac{S_{r+1}}{S_1}\Delta f. \tag{14}$$

We have the following proposition:

Proposition 3.4 *Let M be an n-dimensional connected, compact without boundary and oriented Riemannian manifold, let $\tilde{M}(c)$ be $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, an open hemisphere of the unit sphere $S^{n+1}(1)$ or the hyperbolic space $H^{n+1}(-1)$ according to $c = 0, 1$ or -1, respectively. Let $x: M \to \tilde{M}(c)$ be an isometric immersion. If S_1 and S_{r+1} are all positive, then for $1 \le j \le r$,*

(1) *both the operators L_j and $\tilde{L}_j$ are elliptic;*
(2) *each jth mean curvature H_j is positive.*

Proof The ellipticity of L_j and the positiveness of H_j were proved in [5]. We note for odd r, the positiveness of S_{r+1} can not derive the positiveness of H_j for $1 \le j \le r$ unless we choose the unit normal vector field N such that all the principal curvatures of x are positive at a point p. Hence, we add the condition that S_1 is positive. Thus, we only need to prove $\tilde{L}_j$ is elliptic. But $\tilde{L}_j(f) = \mathrm{trace}[(T_j - \frac{S_{j+1}}{S_1}I)\mathrm{Hess}(f)]$, and S_1 is positive, so it is equivalent to the positiveness of the eigenvalues of $S_1T_j - S_{j+1}I$.

From (1) of Lemma 2.1 and (3), the eigenvalues of $S_1T_j - S_{j+1}I$ are:

$$S_1S_j(B_i) - S_{j+1} = (S_1(B_i) + k_i)S_j(B_i) - S_{j+1} = S_1(B_i)S_j(B_i) - S_{j+1}(B_i).$$

We define $H_j(B_i) = S_j(B_i)/C_{n-1}^j$, $1 \le j \le n-1$, then we have

$$S_1(B_i)S_j(B_i) - S_{j+1}(B_i) = (n-1)C_{n-1}^j H_1(B_i)H_j(B_i) - C_{n-1}^{j+1}H_{j+1}(B_i)$$

From the ellipticity of L_j and (1) of Lemma 2.1, $H_j(B_i) > 0$ for each $1 \le j \le r$, so we have $H_1(B_i)H_j(B_i) \ge H_{j+1}(B_i)$ (see [11] and [17]). So we have

$$\begin{aligned} S_1(B_i)S_j(B_i) - S_{j+1}(B_i) &\ge ((n-1)C_{n-1}^j - C_{n-1}^{j+1})H_1(B_i)H_j(B_i) \\ &= jC_n^{j+1}H_1(B_i)H_j(B_i) > 0. \end{aligned} \tag{15}$$

□

Corollary 3.5 *Under the same assumptions of Proposition 3.4, we have*

$$(r+2)S_1S_{r+2} - 2S_2S_{r+1} < 0. \tag{16}$$

Proof From (see [11] and [17])

$$H_i^2 - H_{i-1}H_{i+1} \ge 0,$$

and the positiveness of $H_1, \ldots, H_{r+1}$, we have,

$$\frac{H_2}{H_1} \ge \frac{H_3}{H_2} \ge \cdots \ge \frac{H_{r+2}}{H_{r+1}}.$$

So $H_1H_{r+2} \le H_2H_{r+1}$. Thus, we have

$$\begin{aligned} (r+2)S_1S_{r+2} - 2S_2S_{r+1} &= (r+2)nC_n^{r+2}H_1H_{r+2} - n(n-1)C_n^{r+1}H_2H_{r+1} \\ &\le ((r+2)nC_n^{r+2} - n(n-1)C_n^{r+1})H_2H_{r+1} \\ &= -nrC_n^{r+1}H_2H_{r+1} \\ &< 0. \end{aligned}$$

□

4 The area-preserving variational problem

We set for each r, $0 \le r \le n$,

$$\mathscr{A}_r = \int_M F_r \mathrm{d}M, \tag{17}$$

where the functions F_r are defined inductively by

$$\begin{aligned} F_0 &= 1\\ F_1 &= S_1\\ F_r &= S_r + \frac{c(n-r+1)}{r-1} F_{r-2}, \quad \text{for} \quad 2 \le r \le n-1. \end{aligned} \tag{18}$$

Clearly $\mathscr{A} \stackrel{def}{=} \mathscr{A}_0$ is the area of M.

Let $X\colon M \times (-\epsilon, \epsilon) \to \tilde{M}(c)$ be a variation of x. That is, for each $t \in (-\epsilon, \epsilon)$, $x_t(p) = X(p,t)$, $p \in M$ is an immersion, $x_0 = x$. We denote the unit normal vector field of immersion x_t by N_t. Variation X is said to be *area-preserving* if for any $t \in (-\epsilon, \epsilon)$, $\mathscr{A}(t) \equiv \mathscr{A}$.

We consider the variational problem of minimizing $\mathscr{A}_r$ keeping the area of M, where $r \ge 1$. By a standard argument involving Lagrange multipliers, this means that we are considering critical points of the functional

$$J_r(t) = \mathscr{A}_r(t) + \lambda \mathscr{A}(t). \tag{19}$$

Let the variational vector field $\partial X/\partial t$ be decomposed to

$$\frac{\partial X}{\partial t} = \xi + fN, \tag{20}$$

where ξ is tangent to M. then we have the following lemmas (see [5] and [15]):

Lemma 4.1 $S'_{r+1} = L_r(f) + (S_1 S_{r+1} - (r+2)S_{r+2})f + c(n-r)S_r f + \langle \nabla S_{r+1}, \xi \rangle$.

Lemma 4.2 $(\partial/\partial t)(\mathrm{d}M_t) = (-S_1 f + \mathrm{div}\xi)\mathrm{d}M_t$.

Lemma 4.3 $\mathscr{A}'(t) = -\int_M S_1 f(t)\mathrm{d}M_t$.

Lemma 4.4 *If* $c = 0$, *then* $\mathscr{A}'_r(t) = -(r+1)\int_M S_{r+1} f(t)\mathrm{d}M_t$.

Proof In this case, we have $\mathscr{A}_r = \int_M S_r \mathrm{d}M$, so from Lemmas 4.1, 4.2 and (9), we have the conclusion. □

Lemma 4.5 *If* $c \ne 0$ *and* r *is even*, $\mathscr{A}'_r(t) = -(r+1)\int_M S_{r+1} f(t)\mathrm{d}M_t$.

Proof We prove Lemma 4.5 inductively. For $r = 0$, it is Lemma 4.3. Suppose Lemma 4.5 is true for $r-2$, then we have

$$\begin{aligned} \mathscr{A}'_r(t) &= \int_M S'_r \mathrm{d}M_t + \int_M S_r(\partial/\partial t)(\mathrm{d}M_t) + \frac{c(n-r+1)}{r-1}\mathscr{A}'_{r-2}(t)\\ &= \int_M (L_{r-1}(f) - (r+1)S_{r+1}f + \langle \nabla S_r, \xi \rangle + S_r \mathrm{div}\xi)\mathrm{d}M_t\\ &= -(r+1)\int_M S_{r+1} f(t)\mathrm{d}M_t. \end{aligned}$$

□

From Lemmas 4.4 and 4.5, we immediately get the following variational formula:

Proposition 4.6 (the First Variational Formula) *Suppose* $c = 0$*, or* $c \neq 0$ *and* r *is even,* $1 \leq r \leq n-1$*, then for any variation of* x*, we have*

$$J_r'(t) = -\int_M \{(r+1)S_{r+1} + \lambda S_1\} f(t) \mathrm{d}M_t.$$

From Proposition 4.6 we know, the critical points of the above variational problem are the immersion x for which

$$S_{r+1}/S_1 = -\frac{\lambda}{r+1} = \text{constant}.$$

In order to decide if x is or not a local minimum, we restrict ourselves to area-preserving variations and compute the second derivative of $\mathscr{A}_r(t)$ at $t = 0$. As $\mathscr{A}(t) \equiv \mathscr{A}$, we have $\mathscr{A}_r''(0) = J_r''(0)$. So we can get the following proposition by a direct calculation using Lemma 4.1:

Proposition 4.7 (the Second Variational Formula) *Let* $x : M \to \tilde{M}(c)$ *be a hypersurface for which* S_1 *is positive,* $S_{r+1}/S_1 = $ *constant, where* $c = 0$*, or* $c \neq 0$ *and* r *is even,* $1 \leq r \leq n-1$*. For area-preserving variations, the second derivative of* $\mathscr{A}_r$ *at* $t = 0$ *is given by*

$$\begin{aligned}\mathscr{A}_r''(0) &= -(r+1)\int_M \frac{\partial}{\partial t}[(r+1)S_{r+1} + \lambda S_1]|_{t=0} f \mathrm{d}M \\ &= -(r+1)\int_M f\left\{L_r f - \frac{S_{r+1}}{S_1}\Delta f + f\left[\frac{2S_2S_{r+1}}{S_1} - (r+2)S_{r+2}\right.\right. \\ &\quad \left.\left. + c(n-r)S_r - \frac{cnS_{r+1}}{S_1}\right]\right\}\mathrm{d}M.\end{aligned}$$

5 Stability of hypersurfaces in $\tilde{M}(c)$

A variation X of the immersion x is called a normal variation if the variational vector field is parallel to N. We have the following lemma:

Lemma 5.1 *For any function* $f : M \to \mathbb{R}$ *that satisfying*

$$\int_M f S_1 \mathrm{d}M = 0, \tag{21}$$

there exists an area-preserving normal variation X *of the immersion* x *such that the variational vector field is* fN*.*

Proof Let $g : M \to \mathbb{R}$ be a smooth function such that $\int_M gS_1 \mathrm{d}M \neq 0$. We consider the two parameter variation

$$X(t, \bar{t}) \overset{def}{=} \exp_x\{(tf + \bar{t}g)N\}, \tag{22}$$

where exp is the exponential map on $\tilde{M}(c)$. Denote the area of M under the induced metric from immersion $X(t, \bar{t})$ by $\mathscr{A}(t, \bar{t})$, and consider the following equation:

$$\mathscr{A}(t, \bar{t}) = \text{constant}. \tag{23}$$

From the property of exponential map we have

$$\frac{\partial X}{\partial t}|_{t=\bar{t}=0} = fN, \tag{24}$$

$$\frac{\partial X}{\partial \bar{t}}|_{t=\bar{t}=0} = gN. \tag{25}$$

Thus, from Lemma 4.4 we have

$$\frac{\partial \mathscr{A}(t,\bar{t})}{\partial t}|_{t=\bar{t}=0} = -\int_M f S_1 \mathrm{d}M = 0. \tag{26}$$

$$\frac{\partial \mathscr{A}(t,\bar{t})}{\partial \bar{t}}|_{t=\bar{t}=0} = -\int_M g S_1 \mathrm{d}M \neq 0. \tag{27}$$

Hence, from implicit function theorem, in a neighborhood of $(t,\bar{t}) = (0,0)$, we can get a solution $\bar{t} = s(t)$ of Eq. 23 satisfies $s(0) = 0$. Thus we obtain an area-preserving variation

$$X(t) = \exp_x\{(tf + s(t)g)N\}. \tag{28}$$

Observe that

$$s'(0) = -\left\{\frac{\partial \mathscr{A}(t,\bar{t})}{\partial t} / \frac{\partial \mathscr{A}(t,\bar{t})}{\partial \bar{t}}\right\}|_{t=\bar{t}=0} = -\int_M f S_1 \mathrm{d}M / \int_M g S_1 \mathrm{d}M = 0,$$

we obtain that the variational vector field of $X(t)$ is

$$\frac{\partial X(t)}{\partial t}|_{t=0} = (f + s'(0)g)N = fN.$$

□

From Lemma 5.1 and Proposition 4.7, the expression of $\mathscr{A}_r''(0)$ depends only on the immersion x and on the function f which can be any function satisfies (21).

So, we fix the following notation:

$$I_r(f) = -\int_M f\left\{L_r f - \frac{S_{r+1}}{S_1}\Delta f + f\left[\frac{2S_2 S_{r+1}}{S_1} - (r+2)S_{r+2} + c(n-r)S_r - \frac{cnS_{r+1}}{S_1}\right]\right\}\mathrm{d}M. \tag{29}$$

Definition 5.2 We say that a hypersurface $x\colon M \to \tilde{M}(c)$ with S_1 positive and $S_{r+1}/S_1 =$ constant is r-stable if $I_r(f) \geq 0$ for any function $f\colon M \to \mathbb{R}$ that satisfies (21), where $c = 0$, or $c \neq 0$ and r is even, $1 \leq r \leq n-1$.

Proposition 5.3 *Totally umbilical hypersurfaces of $\tilde{M}(c)$ which are not totally geodesic are r-stable, where $c = 0$, or $c \neq 0$ and r is even, $1 \leq r \leq n-1$.*

Proof Let Σ be a totally umbilical hypersurfaces of $\tilde{M}(c)$, and suppose Σ is not totally geodesic. We choose normal vector such the principal curvatures of Σ are equal to $k > 0$. Then we have

$$S_j = C_n^j k^j, \quad S_j(B_i) = C_{n-1}^j k^j,$$

and

$$L_r(f) = C_{n-1}^r k^r \Delta f.$$

Hence, Σ is a hypersurface with $S_1 > 0$ and S_{r+1}/S_1 is a constant, and from (29) we have

$$\begin{aligned} I_r(f) &= -\int_\Sigma \Big\{(C_{n-1}^r - \frac{1}{n}C_n^{r+1})k^r f\Delta f + (\frac{2}{n}C_n^{r+1}C_n^2 - (r+2)C_n^{r+2})k^{r+2}f^2 \\ &\quad + c((n-r)C_n^r - C_n^{r+1})k^r\Big\}\mathrm{d}M \\ &= -\frac{r}{n}C_n^{r+1}k^r\int_\Sigma\{f\Delta f + n(k^2+c)f^2\}\mathrm{d}M \\ &\geq \frac{r}{n}C_n^{r+1}k^r\int_\Sigma\{\lambda_1 - n(k^2+c)\}f^2\mathrm{d}M \\ &= 0, \end{aligned}$$

where λ_1 stands for the first eigenvalue of the Laplacian Δ of Σ. The last equality is because Σ is isometric to an Euclidean n-sphere with constant curvature k^2+c. Hence $\lambda_1 = n(k^2+c)$. Therefore, Σ is r-stable. □

Now we state our main theorems

Theorem 5.4 *Let M be an n-dimensional connected, compact without boundary and oriented Riemannian manifold, $1 \leq r \leq n-1$. An isometric immersion $x\colon M \to \mathbb{R}^{n+1}$ for which S_1 is positive and S_{r+1}/S_1 is a constant is r-stable if and only if M is a sphere and x is its inclusion as a totally umbilical hypersurface.*

Proof From Proposition 5.3, the condition is sufficient. Now we prove that it is also necessary. By Proposition 3.4, the operator $\tilde{L}_r$ is elliptic.

Let $\int_M xS_1\mathrm{d}M = C$, constant vector in $\mathbb{R}^{n+1}$, then

$$\tilde{x} = x - \frac{1}{\int_M S_1\mathrm{d}M}C$$

satisfies $\int_M \tilde{x}S_1\mathrm{d}M = 0$. Because the qualities of (29) are same for x and $\tilde{x}$, so without loss of generality, we can assume that

$$\int_M xS_1\mathrm{d}M = 0.$$

Take an orthonormal basis $E_1, E_2, \ldots, E_{n+1}$ of $\mathbb{R}^{n+1}$ and define functions f_A, g_A by

$$f_A = \langle N, E_A\rangle, \qquad g_A = \langle x, E_A\rangle. \tag{30}$$

The hypothesis of r-stability implies that $I(g_A) \geq 0$ for each A, $1 \leq A \leq n+1$. Hence, using (29) and Theorem 3.2 in the case $c = 0$, we obtain

$$\begin{aligned} 0 \leq I_r(g_A) &= -\int_M g_A\Big\{L_r g_A - \frac{S_{r+1}}{S_1}\Delta g_A + g_A\Big[\frac{2S_2S_{r+1}}{S_1} - (r+2)S_{r+2}\Big]\Big\}\mathrm{d}M \\ &= \int_M\Big\{-\Big[\frac{2S_2S_{r+1}}{S_1} - (r+2)S_{r+2}\Big]g_A^2 - rS_{r+1}f_Ag_A\Big\}\mathrm{d}M. \end{aligned} \tag{31}$$

Adding these equations for $1 \leq A \leq n+1$ and noting $\sum_{A=1}^{n+1}\langle X, E_A\rangle\langle E_A, Y\rangle = \langle X, Y\rangle$, we conclude that

$$\begin{aligned} 0 &\leq \int_M\{-rS_{r+1}\langle x, N\rangle - \Big(\frac{2S_2S_{r+1}}{S_1} - (r+2)S_{r+2}\Big)|x|^2\}\mathrm{d}M \\ &= \int_M\Big\{-rS_{r+1}\langle x, N\rangle - \Big(\frac{2S_2S_{r+1}}{S_1} - (r+2)S_{r+2}\Big)\Big(|x^T|^2 + \langle x, N\rangle^2\Big)\Big\}\mathrm{d}M. \end{aligned} \tag{32}$$

Notice that S_{r+1}/S_1 is a constant, and using Lemma 3.3, through a direct calculation, we derive from (13),

$$\int_M \left\langle \left(T_{r+1} - \frac{S_{r+1}}{S_1} T_1 \right) Bx^T, x^T \right\rangle \mathrm{d}M$$
$$= -\int_M \left[-rS_{r+1} < x, N > -\left(\frac{2S_2S_{r+1}}{S_1} - (r+2)S_{r+2} \right) < x, N >^2 \right] \mathrm{d}M. \quad (33)$$

Combining (32) with (33), we get

$$0 \le \int_M \left\{ \left((r+2)S_{r+2} - \frac{2S_2S_{r+1}}{S_1} \right) |x^T|^2 + \left\langle \left(T_{r+1} - \frac{S_{r+1}}{S_1} T_1 \right) Bx^T, x^T \right\rangle \right\} \mathrm{d}M. \quad (34)$$

Let $e_1, e_2, \ldots, e_n$ be orthonormal principal vectors corresponding to the principal curvatures $k_1, k_2, \ldots, k_n$ respectively. Then we have by use of (1) of Lemma 2.1 and (3)

$$\left((r+2)S_{r+2} - \frac{2S_2S_{r+1}}{S_1} \right) |x^T|^2 + \left\langle \left(T_{r+1} - \frac{S_{r+1}}{S_1} T_1 \right) Bx^T, x^T \right\rangle$$
$$= \sum_{i=1}^n \{(r+2)S_{r+2} - \frac{2S_2S_{r+1}}{S_1} + k_iS_{r+1}(B_i) - \frac{S_{r+1}}{S_1} k_iS_1(B_i)\} \langle x, e_i \rangle^2$$
$$= \frac{1}{S_1} \sum_{i=1}^n \{(r+2)S_1S_{r+2} - 2S_2S_{r+1} + k_i^2(S_{r+1}(B_i) - S_1(B_i)S_r(B_i))\} \langle x, e_i \rangle^2. \quad (35)$$

But, from (15) and (16),

$$(r+2)S_1S_{r+2} - 2S_2S_{r+1} + k_i^2(S_{r+1}(B_i) - S_1(B_i)S_r(B_i)) < 0.$$

From (34) and (35), we must have $x^T = 0$, this means $x = kN$ for some function k. But then we have

$$d|x|^2 = 2\langle dx, x \rangle = 2k\langle dx, N \rangle = 0.$$

This means $|x|^2$ is a constant, i.e. M is a sphere. This completes the proof of Theorem 5.4.□

Remark 5.5 When $r = 1$, Theorem 5.4 has been proved in [12].

In the case of M is a hypersurface of an open hemisphere of the unit sphere $S^{n+1}(1)$ or the hyperbolic space $H^{n+1}(-1)$, we have the following theorem:

Theorem 5.6 *Let M be an n-dimensional connected, compact without boundary and oriented Riemannian manifold, let $\tilde{M}(c)$ be an open hemisphere of the unit sphere $S^{n+1}(1)$ or the hyperbolic space $H^{n+1}(-1)$ according to $c = 1$ or -1 respectively. Let r be even, $1 \le r \le n-1$. An isometric immersion $x : M \to \tilde{M}(c)$ for which S_1 is positive and S_{r+1}/S_1 is a constant is r-stable if and only if M is a sphere and x is its inclusion as a non totally geodesic, totally umbilical hypersurface.*

Proof From Proposition 5.3, the condition is sufficient. Now we prove that it is also necessary. By Proposition 3.4, the operator $\tilde{L}_r$ is elliptic. We consider separately the two cases.

Case 1 Suppose $\tilde{M}(c) =$ open hemisphere of $S^{n+1}(1) \subset \mathbb{R}^{n+2}$.

Set $\bar{N} = \int_M N S_1 \mathrm{d}M$. Assume $\bar{N} = 0$. Take an orthonormal basis $E_0, E_1, \ldots, E_{n+1}$ of $\mathbb{R}^{n+2}$ and define f_A, g_A as in (30). The hypothesis of r-stability implies that $I(f_A) \geq 0$ for each A, $0 \leq A \leq n+1$. Hence, using theorem 3.2, we obtain

$$\begin{aligned} 0 &\leq -\int_M f_A \Big\{ L_r f_A - \frac{S_{r+1}}{S_1} \Delta f_A \\ &\quad + f_A \Big[\frac{2S_2 S_{r+1}}{S_1} - (r+2) S_{r+2} + (n-r) S_r - \frac{n S_{r+1}}{S_1} \Big] \Big\} \mathrm{d}M \\ &= \int_M \Big\{ -r S_{r+1} f_A g_A - \Big[(n-r) S_r - \frac{n S_{r+1}}{S_1} \Big] f_A^2 \Big\} \mathrm{d}M. \end{aligned} \tag{36}$$

Adding these equations for all A, using $\langle x, N \rangle = 0$ and $|N|^2 = 1$, we obtain

$$\begin{aligned} 0 &\leq \int_M \Big\{ -r S_{r+1} \langle x, N \rangle - \Big[(n-r) S_r - \frac{n S_{r+1}}{S_1} \Big] |N|^2 \Big\} \mathrm{d}M \\ &= -\int_M \Big\{ \frac{1}{S_1} \big[(n-r) S_1 S_r - n S_{r+1} \big] R \Big\} \mathrm{d}M \\ &= -\int_M \Big\{ \frac{1}{S_1} \big[n(n-r) C_n^r H_1 H_r - n C_n^{r+1} H_{r+1} \big] \Big\} \mathrm{d}M \\ &\leq -\int_M \Big\{ \frac{1}{S_1} \big[n(n-r) C_n^r H_{r+1} - n C_n^{r+1} H_{r+1} \big] \Big\} \mathrm{d}M \\ &= -\int_M \frac{n r S_{r+1}}{S_1} \mathrm{d}M \\ &< 0. \end{aligned} \tag{37}$$

So, $\bar{N}$ cannot be zero.

Thus, we assume that $\bar{N} \neq 0$. Let $E_0, E_1, \ldots, E_{n+1}$ be orthonormal basis of $\mathbb{R}^{n+2}$ such that $E_0 = \bar{N}/|\bar{N}|$, and define f_A and g_A as in (30). Now we have

$$\int_M f_A S_1 \mathrm{d}M = 0, \quad \text{for} \quad 1 \leq A \leq n+1.$$

For these functions r-stability implies the validity of (36). We may add them from 1 to $n+1$ to obtain

$$\begin{aligned} 0 &\leq \int_M \Big\{ -r S_{r+1} \sum_{A=1}^{n+1} f_A g_A - \Big[(n-r) S_r - \frac{n S_{r+1}}{S_1} \Big] \sum_{A=1}^{n+1} f_A^2 \Big\} \mathrm{d}M \\ &= \int_M \{ r S_{r+1} f_0 g_0 - \Big[(n-r) S_r - \frac{n S_{r+1}}{S_1} \Big] (1 - f_0^2) \} \mathrm{d}M. \end{aligned} \tag{38}$$

On the other hand, we have

$$1 = |E_0|^2 \geq \langle E_0, x \rangle^2 + \langle E_0, N \rangle^2 = g_0^2 + f_0^2.$$

Therefore, using Theorem 3.2 and the ellipticity of $\tilde{L}_r$, we can get

$$\begin{aligned} 0 &\le \int_M \left\{ rS_{r+1} f_0 g_0 - \left[(n-r)S_r - \frac{nS_{r+1}}{S_1} \right] g_0^2 \right\} \mathrm{d}M \\ &= \int_M \left\{ g_0 \tilde{L}_r g_0 \right\} \mathrm{d}M \\ &= -\int_M \left\langle \left(T_r - \frac{S_{r+1}}{S_1} I \right) \nabla g_0, \nabla g_0 \right\rangle \mathrm{d}M \\ &\le 0. \end{aligned} \tag{39}$$

Hence all the inequalities must be equalities. In particular, we obtain that $\nabla g_0 = 0$, so g_0 is a constant. Thus, the image of x is contained in the intersection of a hyperplane and the sphere $S^{n+1}(1)$. Therefore, $x(M)$ is a non-geodesic hypersphere of $S^{n+1}(1)$.

Case 2 Suppose $\tilde{M}(c) = H^{n+1}(-1) \subset \mathbb{R}_1^{n+2}$, where $\mathbb{R}_1^{n+2}$ is the Lorentz space.

We define $\bar{x} = \int_M x S_1 \mathrm{d}M$. From

$$\langle x, x \rangle = -x_0^2 + \sum_{A=1}^{n+1} x_A^2 = -1,$$

and $x_0 > 0$, we have

$$x_0 = \sqrt{1 + \sum_{A=1}^{n+1} x_A^2}.$$

Hence, from the Minkowski integral formula, we have

$$\left(\int_M S_1 \mathrm{d}M \right)^2 + \sum_{A+1}^{n+1} \left(\int_M x_A S_1 \mathrm{d}M \right)^2 \le \left(\int_M \sqrt{1 + \sum_{A=1}^{n+1} x_A^2} S_1 \mathrm{d}M \right)^2 = \left(\int_M x_0 S_1 \mathrm{d}M \right)^2.$$

Thus, we have

$$\langle \bar{x}, \bar{x} \rangle = -\left(\int_M x_0 S_1 \mathrm{d}M \right)^2 + \sum_{A+1}^{n+1} \left(\int_M x_A S_1 \mathrm{d}M \right)^2 \le -\left(\int_M S_1 \mathrm{d}M \right)^2 < 0.$$

We choose $E_0 = \bar{x}/|\bar{x}|$ and complete it into an orthonormal basis of $\mathbb{R}_1^{n+2}$. For such a basis, define f_A and g_A as in (30). It is clear that $\int_M g_A S_1 \mathrm{d}M = 0$ for $1 \le A \le n+1$. Thus, for such g_A, r-stability implies:

$$\begin{aligned} 0 \le I_r(g_A) &= -\int_M g_A \left\{ L_r g_A - \frac{S_{r+1}}{S_1} \Delta g_A \right. \\ &\quad \left. + g_A \left[\frac{2S_2 S_{r+1}}{S_1} - (r+2)S_{r+2} - (n-r)S_r + \frac{nS_{r+1}}{S_1} \right] \right\} \mathrm{d}M \\ &= \int_M \left\{ \left[(r+2)S_{r+2} - \frac{2S_2 S_{r+1}}{S_1} \right] g_A^2 - rS_{r+1} f_A g_A \right\} \mathrm{d}M. \end{aligned} \tag{40}$$

Adding these equations for $1 \le A \le n+1$ and using the fact that

$$-f_0 g_0 + \sum_{A=1}^{n+1} f_A g_A = \langle x, N \rangle = 0$$

and

$$-g_0^2+\sum_{A=1}^{n+1} g_A^2=\langle x, x\rangle=-1,$$

we conclude that

$$0 \leq \int_M\left\{-r S_{r+1} f_0 g_0+\left[\frac{2 S_2 S_{r+1}}{S_1}-(r+2) S_{r+2}\right](1-g_0^2)\right\} \mathrm{d} M. \tag{41}$$

On the other hand, we have

$$-1=\langle E_0, E_0\rangle \geq-\langle E_0, x\rangle^2+\langle E_0, N\rangle^2=-g_0^2+f_0^2.$$

Therefore, using Theorem 3.2 and the ellipticity of $\tilde{L}_r$, we can get

$$\begin{aligned}
0 & \leq \int_M\left\{-r S_{r+1} f_0 g_0-\left[\frac{2 S_2 S_{r+1}}{S_1}-(r+2) S_{r+2}\right] f_0^2\right\} \mathrm{d} M \\
& =\int_M\left\{f_0 \tilde{L}_r f_0\right\} \mathrm{d} M \\
& =-\int_M\left\langle\left(T_r-\frac{S_{r+1}}{S_1} I\right) \nabla f_0, \nabla f_0\right\rangle \mathrm{d} M \\
& \leq 0.
\end{aligned} \tag{42}$$

This implies that f_0 is constant and $-g_0^2+f_0^2=1$. Hence g_0 is also a constant and the proof can be concluded as in Case 2. This completes the proof of Theorem 5.6. □

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