

Poisson fiber bundles and coupling Dirac structures

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Abstract Poisson fiber bundles are studied. We give sufficient conditions for the existence of a Dirac structure on the total space of a Poisson fiber bundle endowed with a compatible connection. We also provide some examples.

Keywords Poisson structure · Fiber bundle · Connections · Dirac structure

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1 Introduction

Several constructions of symplectic forms on the total space of a symplectic fiber bundle have appeared in the literature, among others, Thurston's construction (see [10, 12]). One also has the method of coupling forms developed by Guillemin, Lerman and Sternberg (see [5, 6, 11]). Moreover in Ref. [7], Gotay, Lashof, Śniatycki and Weinstein gave necessary and sufficient conditions for the existence of a pre-symplectic form on the total space of a symplectic fiber bundle which restricts to the symplectic structure along its fibers. Symplectic fiber bundles have been extensively studied in recent years. They have various applications in gauge theories.

Unlike symplectic fiber bundles, generic Poisson fiber bundles are not well understood. Although, they were considered by Marsden, Montgomery and Ratiu in connection with the study of moving systems (see [9]). Based on Cartan's theory of classical space-times, they introduced the notion of a *Cartan-Hannay-Berry connection*, which is an important tool for the study of moving systems such as the ball in a rotating hoop. Various examples of systems having the Cartan connection as underlying geometric structure can be found in Ref. [9].

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It turns out that Cartan and Cartan-Hannay-Berry connections give rise to coupling Dirac structures in the sense of Vaisman [13] (see Sect. 4 below). This suggests that a natural framework for the study of certain moving systems is the setting of coupling Dirac structures and, partly, motivates our study of the problem of finding conditions under which the Poisson structure along the fibers of a Poisson fiber bundle endowed with a Poisson–Ehresmann connection can be extended to a (non-vertical) Dirac structure. Our aim is to investigate that problem. Our main results are Theorems 3.2 and 3.4. In Ref. [3], we give another construction of a coupling structure on the total space of a Poisson fiber bundle extending the Sternberg–Weinstein phase space of particles in a Yang-Mills field to the setting of coupling Dirac structures.

Here is an outline of the article. Section 2 provides the tools that will be used to prove the main results. In Sect. 3, we establish Theorems 3.2 and 3.4. In Sect. 4, we show that Cartan and Cartan-Hannay-Berry connections induce coupling Dirac structures.

Note: After submitting this work for publication, an interesting paper by Olivier Brahic and Rui Loja Fernandes appeared Ref. [1]. It addresses some problems closely related to our work.

2 Basic definitions and results

All manifolds are assumed to be paracompact, Hausdorff, smooth and connected. We also assume that all maps between manifolds are smooth.

2.1 Poisson fiber bundles

Let (F, \mathcal{V}_F) be a finite-dimensional Poisson manifold. A *Poisson fiber bundle* is a fiber bundle $F \rightarrow E \xrightarrow{\pi} B$ whose structure group preserves the Poisson structure on F . In other words, there is an open cover (U_i) of B and diffeomorphisms $\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times F$ satisfying the properties:

1. The following diagram commutes

$$\begin{array}{ccc}
 \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times F \\
 \pi \searrow & & \swarrow \text{pr} \\
 & U_i &
 \end{array}$$

2. If $b \in U_i \cap U_j$ then the transition map $\phi_{ij}(b) = \phi_i(b) \circ \phi_j(b)^{-1}$ is a Poisson isomorphism of (F, \mathcal{V}_F) .

Notice that the Poisson tensor on each fiber E_b is given by $\mathcal{V}_b = (\phi_i(b))_*^{-1} \mathcal{V}_F$. It is independent of the local trivialization map ϕ_i . Consider the vertical sub-bundle

$$\text{Vert} = \text{Ker}(T\pi) \subset TE.$$

There is a vertical Poisson bivector field $\mathcal{V} \in \mathfrak{X}_{\text{Vert}}^2(E)$ which coincides with the Poisson structure along the fibers, i.e. $(i_b)_* \mathcal{V}_b = \mathcal{V}$, where $i_b : E_b \rightarrow E$ is the injection map.

2.2 Ehresmann connections and integrable geometric data

Let $E \xrightarrow{\pi} B$ be a fiber bundle. An *Ehresmann connection* on E is a smooth sub-bundle $\text{Hor} \subset TE$ such that $TE = \text{Hor} \oplus \text{Vert}$ and the following condition is satisfied: given any smooth path

$$c : [0, 1] \rightarrow B$$

$$t \mapsto c(t)$$

from b_0 to b_1 and for any $x_0 \in \pi^{-1}(b_0)$, there exists a horizontal lift $\gamma : t \mapsto \gamma(t)$ in E starting from x_0 with $c(t) = \pi \circ \gamma(t)$. There is an associated bundle projection map $\Gamma : TE \rightarrow \text{Vert}$, i.e. $\Gamma_e^2 = \Gamma_e$ for every $e \in E$. One has $\text{Hor} = \ker \Gamma$.

Definition 2.1 [9] Let $\pi : E \rightarrow B$ be a Poisson fiber bundle together with its associated vertical Poisson bivector field $\mathcal{V} \in \mathfrak{X}_{\text{Vert}}^2(E)$. An Ehresmann connection Γ on E is *Poisson* if \mathcal{V} is preserved by parallel transport. i.e.,

$$\mathcal{L}_{\text{hor}_\Gamma(X)} \mathcal{V} = 0, \quad \text{for all } X \in \mathfrak{X}(B),$$

where $\text{hor}_\Gamma(X)$ is the Γ -horizontal lift of X .

Definition 2.2 [14] Let $\pi : E \rightarrow B$ be a fiber bundle. A triple $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ formed by a vertical bivector field $\mathcal{V} \in \mathfrak{X}_{\text{Vert}}^2(E)$, an Ehresmann connection Γ , and a horizontal 2-form $\overline{\mathbb{F}} \in \Omega^2(E)$ is called *geometric data*. It is said to be *integrable* if the following properties are satisfied:

- \mathcal{V} is a Poisson tensor, i.e. $[\mathcal{V}, \mathcal{V}] = 0$.
- Γ is a Poisson–Ehresmann connection with respect to \mathcal{V} .
- The curvature 2-form of Γ is a Hamiltonian vector field given by:

$$\text{Curv}_\Gamma(X, Y) = \mathcal{V}^\sharp \left(d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y))) \right), \quad \forall X, Y \in \mathfrak{X}(B).$$

- The 2-form $\overline{\mathbb{F}}$ is horizontally closed.

Remark

(a) Define the operator $\partial_\Gamma : \Omega^k(B) \otimes C^\infty(E) \rightarrow \Omega^{k+1}(B) \otimes C^\infty(E)$ by setting

$$\begin{aligned} \partial_\Gamma \alpha(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i \mathcal{L}_{\text{hor}_\Gamma(X)}(\alpha(X_0, \dots, \widehat{X}_i, \dots, X_k)) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k). \end{aligned}$$

The fact that $\overline{\mathbb{F}}$ is horizontally closed can be alternatively expressed by the following equation (see [14])

$$\partial_\Gamma \overline{\mathbb{F}} = 0,$$

where \mathbb{F} is the 2-form defined by

$$\mathbb{F}(X, Y) = \overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)). \tag{1}$$

- (b) Let $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ be integrable geometric data. In general $\partial_\Gamma^2 \neq 0$, but its restriction to the Casimir valued k -forms, denoted by $\partial_\mathcal{V} : \Omega^k(S) \otimes \text{Casim}(E, \mathcal{V}) \rightarrow \Omega^{k+1}(S) \otimes \text{Casim}(E, \mathcal{V})$, satisfies $\partial_\mathcal{V}^2 = 0$.
- (c) Let $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ be integrable geometric data on $E \rightarrow B$. Every $\Phi \in \Omega^1(B) \otimes \text{Casim}(E, \mathcal{V})$ induces new integrable geometric data $(\mathcal{V}, \Gamma, \overline{\mathbb{F}}')$, where the new horizontal 2-form is defined by

$$\overline{\mathbb{F}}'(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)) = \overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)) + (\partial_\Gamma \Phi)(X, Y),$$

for any $X, Y \in \mathfrak{X}(B)$. In this case, we say these geometric data are equivalent. This defines an equivalence relation among the set of all integrable geometric data with a fixed vertical Poisson structure and a fixed Poisson–Ehresmann connection.

2.3 Coupling Dirac structures

2.3.1 Dirac structures

Let N be a finite-dimensional manifold. Consider the canonical symmetric pairing $\langle \cdot, \cdot \rangle_+$ on the vector bundle $TN \oplus T^*N$ defined by

$$\langle (X_1, \xi_1), (X_2, \xi_2) \rangle_+ = \frac{1}{2} (\xi_1(X_2) + \xi_2(X_1)).$$

The space of smooth sections of $TN \oplus T^*N$ is endowed with a bilinear operation, called the Courant bracket, which is an extension of the Lie bracket of vector fields to $TN \oplus T^*N$ defined by

$$[(X_1, \xi_1), (X_2, \xi_2)] = ([X_1, X_2], \mathcal{L}_{X_1}\xi_2 - i_{X_2}d\xi_1),$$

for all $(X_1, \xi_1), (X_2, \xi_2)$ smooth sections of $TN \oplus T^*N$.

Definition 2.3 [2] An *almost Dirac structure* on a manifold N is a sub-bundle L of $TN \oplus T^*N \rightarrow N$ which is maximally isotropic with respect to the symmetric pairing $\langle \cdot, \cdot \rangle_+$. If, in addition, the space of sections L is closed under the Courant bracket then L is called a *Dirac structure* on M .

Basic examples of Dirac structures are regular foliations, Poisson and pre-symplectic structures (see [2]).

2.3.2 Induced Dirac structures on sub-manifolds

Let L be a Dirac structure on a manifold N , Q a sub-manifold of N . At every point $q \in Q$, one has a maximal isotropic vector space

$$(L_Q)_q = \frac{L_q \cap (T_q Q \oplus T_q^* N)}{L_q \cap (\{0\} \oplus \text{Ann}(T_q Q))}.$$

Using the map $(L_Q)_q \rightarrow T_q Q \oplus T_q^* Q$ defined by $(u, v) \mapsto (u, v|_{T_q Q})$, one can identify $(L_Q)_q$ with a sub-space of $T_q Q \oplus T_q^* Q$. Moreover, L_Q defines a smooth sub-bundle of $TQ \oplus T^*Q$ if and only if $L_q \cap (T_q Q \oplus T_q^* N)$ has constant dimension. The following result was proved in Ref. [2].

Proposition 2.4 [2] *If $L_q \cap (T_q Q \oplus T_q^* N)$ has constant dimension then L_Q defines a Dirac structure on Q .*

Definition 2.5 A Poisson fiber bundle $F \rightarrow E \xrightarrow{\pi} B$ is *coherent* if there exists a Dirac structure L on E whose restriction to the fibers coincides with the Poisson structure along the fibers and such that $L \cap (\text{Vert} \oplus \text{Ann}(\text{Vert})) = \{0\}$.

We have the following result:

Proposition 2.6 *Every coherent Poisson fiber bundle $\pi: E \rightarrow B$ admits a Poisson–Ehresmann connection.*

Proof Suppose $F \rightarrow E \xrightarrow{\pi} B$ is a coherent Poisson fiber bundle. Let L be a Dirac structure on E that coincides with the Poisson structure on the fibers and such that $L \cap (\text{Vert} \oplus \text{Ann}(\text{Vert})) = \{0\}$. Then $L_x \cap (\text{Vert}_x \oplus T_x^* E)$ has constant dimension $n = \dim F$. In fact, $L_x \cap (\text{Vert}_x \oplus T_x^* E)$

is isomorphic to $T_x^*E_{\pi(x)}$ since the restriction of L to $E_{\pi(x)}$ is the graph of the Poisson bivector field $\mathcal{V}_{\pi(x)}$. Set

$$H_x(L) = \{Y_x \in T_x E \mid \exists \beta_x \in \text{Ann}(\text{Vert}_x) \text{ such that } (Y_x, \beta_x) \in L_x\}.$$

We have

$$H_x(L) \cong \text{Ann}\left(\text{pr}_2(L_x \cap (\text{Vert}_x \oplus T_x^*E))\right).$$

It follows that

$$\dim H_x(L) = \dim T_x E - \dim(T_x^*E_{\pi(x)}).$$

Hence

$$T_x E = \text{Vert}_x \oplus H_x(L),$$

for all $x \in E$. This shows that the distribution $\text{Hor}(L)$ defined by the sub-spaces $H_x(L) \subset T_x E$ is normal to the sub-bundle Vert . We will prove that $\text{Hor}(L)$ is smooth. Fix a neighborhood U of a point $x \in E$ and let $(Z_i, \eta_i), (X_j, \alpha_j)$ be local bases on U for L and $L \cap (\text{Vert} \oplus T^*E)$, respectively. A vector Y tangent to the distribution $\text{Hor}(L)$ has the form $Y = \sum_i f_i Z_i$ with $\langle Y, \alpha_j \rangle = 0$, for all j . The existence of smooth solutions for such a system of equations implies the smoothness of $\text{Hor}(L)$. Consequently, there is an Ehresmann Γ_L connection associated with $\text{Hor}(L)$. The fact that Γ_L is Poisson is an immediate consequence of the integrability of L , i.e. the sections of L are closed under the Courant bracket. \square

Definition 2.7 Suppose the geometric data $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ defined on the fiber bundle $E \rightarrow B$ is integrable. Set

$$L = \left\{ (X, i_X \overline{\mathbb{F}}) + (\mathcal{V}^\sharp \alpha, \alpha) \mid X \in \text{Hor}_\Gamma, \alpha \in \text{Ann}(\text{Hor}_\Gamma) \right\}. \tag{2}$$

Then L is called a *coupling Dirac structure*.

We refer the reader to Ref. [13] for a more general definition of a coupling Dirac structure on a foliated manifold. Coupling Dirac structures naturally appeared in Ref. [4] when we considered the transverse Poisson structure at a pre-symplectic of a Dirac manifold.

Remark 2.8 (a) The Dirac structure L defined in (2) satisfies $L \cap (TE \oplus \{0\}) = \{0\}$ if and only if $\overline{\mathbb{F}}$ is non-degenerate. In other words, L is the graph of a Poisson bivector field if and only if $\overline{\mathbb{F}}$ is non-degenerate.

(b) The distribution \mathcal{D} given by the set of all horizontal vector fields X satisfying $i_X \overline{\mathbb{F}} = 0$ is integrable. It defines a foliation \mathcal{F} , called the characteristic foliation or the null foliation of L . Moreover, E/\mathcal{F} is a Poisson manifold when L is reducible (see [8]). \square

We have the following result:

Theorem 2.9 *Let $E \rightarrow B$ be a fiber bundle. The integrability of geometric data $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ is equivalent to the fact that the space of smooth sections of the corresponding sub-bundle $L \subset TE \oplus T^*E$ (defined as in Eq. 2) is closed under the Courant bracket.*

Proof Consider geometric data $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ and define its corresponding almost Dirac structure as in Eq. 2. Set

$$e_X = \left(\text{hor}_\Gamma(X), i_{\text{hor}_\Gamma(X)} \overline{\mathbb{F}} \right) \quad \text{and} \quad e_\alpha = (\mathcal{V}^\sharp(\alpha), \alpha),$$

for all $X \in \mathfrak{X}(B)$ and for all $\alpha \in \text{Ann}(\text{Hor}_\Gamma)$. Since

$$\text{Curv}_\Gamma(X, Y) = \text{hor}_\Gamma([X, Y]) - [\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)],$$

we get

$$\begin{aligned} [e_X, e_Y] &= \left(\text{hor}_\Gamma([X, Y]) - \text{Curv}_\Gamma(X, Y), \mathcal{L}_{\text{hor}_\Gamma(X)}(i_{\text{hor}_\Gamma(Y)}\overline{\mathbb{F}}) - i_{\text{hor}_\Gamma(Y)}d(i_{\text{hor}_\Gamma(X)}\overline{\mathbb{F}}) \right) \\ &= \left(\text{hor}_\Gamma([X, Y]) - \text{Curv}_\Gamma(X, Y), i_{[\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)]}\overline{\mathbb{F}} - d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y))) \right). \end{aligned}$$

There follows

$$\langle [e_X, e_Y], e_\alpha \rangle_+ = \frac{1}{2} \left\langle \mathcal{V}^\sharp \left(d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y))) \right) - \text{Curv}_\Gamma(X, Y), \alpha \right\rangle_+,$$

for any $X, Y \in \mathfrak{X}(B)$ and for any $\alpha \in \text{Ann}(\text{Hor}_\Gamma)$. Hence

$$\langle [e_X, e_Y], e_\alpha \rangle_+ = 0, \forall e_\alpha \iff \text{Curv}_\Gamma(X, Y) = \mathcal{V}^\sharp \left(d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y))) \right). \tag{3}$$

Moreover, we have

$$\langle [e_X, e_Y], e_Z \rangle = 0 \iff d\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y), \text{hor}_\Gamma(Z)) = 0 \tag{4}$$

for all $X, Y, Z \in \mathfrak{X}(B)$. We also have

$$\begin{aligned} [e_\alpha, e_\beta] &= \left([\mathcal{V}^\sharp(\alpha), \mathcal{V}^\sharp(\beta)], \mathcal{L}_{\mathcal{V}^\sharp(\alpha)}\beta - i_{\mathcal{V}^\sharp(\beta)}d\alpha \right) \\ &= \left(\mathcal{V}^\sharp(\mathcal{L}_{\mathcal{V}^\sharp(\alpha)}\beta - i_{\mathcal{V}^\sharp(\beta)}d\alpha) + [\mathcal{V}, \mathcal{V}](\alpha, \beta, \cdot), \mathcal{L}_{\mathcal{V}^\sharp(\alpha)}\beta - i_{\mathcal{V}^\sharp(\beta)}d\alpha \right) \end{aligned}$$

for all $\alpha, \beta \in \text{Ann}(\text{Hor}_\Gamma)$. Therefore, $[e_\alpha, e_\beta]$ is a smooth section of L if and only if $[\mathcal{V}, \mathcal{V}](\alpha, \beta, \cdot) = 0$ for all vertical 1-forms α, β . Since the trivector field $[\mathcal{V}, \mathcal{V}]$ is vertical this is equivalent to say that all brackets $[e_\alpha, e_\beta]$ are smooth sections of L if and only if

$$[\mathcal{V}, \mathcal{V}] = 0. \tag{5}$$

Furthermore, we have

$$[e_X, e_\alpha] = \left([\text{hor}_\Gamma(X), \mathcal{V}^\sharp(\alpha)], \mathcal{L}_{\text{hor}_\Gamma(X)}\alpha - i_{\mathcal{V}^\sharp(\alpha)}d(i_{\text{hor}_\Gamma(X)}\overline{\mathbb{F}}) \right),$$

for any $X \in \mathfrak{X}(B), \alpha \in \text{Ann}(\text{Hor}_\Gamma)$. Using the fact that

$$[\text{hor}_\Gamma(X), \mathcal{V}^\sharp(\alpha)] = \left(\mathcal{L}_{\text{hor}_\Gamma(X)}\mathcal{V} \right)(\alpha, \cdot) + \mathcal{V}^\sharp(\mathcal{L}_{\text{hor}_\Gamma(X)}\alpha),$$

one gets

$$\langle [e_X, e_\alpha], e_\beta \rangle = 0, \forall \alpha, \beta \in \text{Ann}(\text{Hor}_\Gamma) \iff \mathcal{L}_{\text{hor}_\Gamma(X)}\mathcal{V} = 0. \tag{6}$$

Relations (3)–(6) show that if L is a Dirac structure then $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ is integrable. The converse is true because of (3)–(6) and the fact that

$$\begin{aligned}
 \langle [e_X, e_\alpha], e_Y \rangle_+ &= i_{\text{hor}_\Gamma(Y)} i_{\text{hor}_\Gamma(X)} d\alpha - \mathcal{V}^\sharp \left(\alpha, d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y))) \right) \\
 &= - \left\langle [\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)] + \mathcal{V}^\sharp \left(d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y))), \alpha \right), \alpha \right\rangle_+ \\
 &= \left\langle \text{Curv}_\Gamma(X, Y) - \mathcal{V}^\sharp(d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y))), \alpha \right\rangle_+.
 \end{aligned}$$

This completes the proof of Theorem 2.9. □

3 Dirac extensions of Poisson fiber bundles

In this section, we give constructions of Dirac structures on the total space of certain Poisson fiber bundles. First, we recall from Ref. [16] the following definition:

Definition 3.1 A classical Yang-Mills-Higgs setup is a triple (G, P, F) formed by a finite-dimensional Lie group G , a principal G -bundle P , and a Hamiltonian Poisson G -space F .

3.1 Dirac structures and principal bundles

We have the following result:

Theorem 3.2 Let (G, P, F) be a classical Yang-Mills-Higgs setup. Then every connection Θ on P gives rise to a coupling Dirac structure on the associated bundle $E = P \times_G F$.

Proof Let $\pi_{P \times F} : P \times F \rightarrow P \times_G F$ denote the canonical projection. Define the vertical bivector field \mathcal{V} on E as follows

$$\mathcal{V} = (\pi_{P \times F})_* \mathcal{V}_F.$$

It satisfies $[\mathcal{V}, \mathcal{V}] = 0$ since \mathcal{V}_F is Poisson. Moreover, every connection Θ on P induces a connection Γ on E . The Γ -horizontal lift of $X \in \mathfrak{X}(B)$ is given by

$$(\text{hor}_\Gamma(X))([p, m]) = T_{(p,f)} \pi_{P \times F}(\overline{X}_p, \mathbf{0}_f), \quad \forall [p, m] \in P \times_G F, \tag{7}$$

where $\mathbf{0}_f$ is the zero tangent vector at $f \in F$ and $\overline{X}_p \in T_p P$ is the Θ -horizontal lift of X at p . Consequently, one gets

$$\mathcal{L}_{\text{hor}_\Gamma(X)} \mathcal{V} = 0,$$

for all $X \in \mathfrak{X}(B)$. Recall that the curvature of Θ is a vertical \mathfrak{g} -valued 2-form. Moreover, for all $X, Y \in \mathfrak{X}(B)$, we have

$$(\text{Curv}_\Gamma(X, Y))([p, f]) = T_{(p,f)} \pi_{P \times F} \left(\mathbf{0}_p, (\varrho_F \circ \text{Curv}_\Theta(\overline{X}_p, \overline{Y}_p)) \right), \tag{8}$$

where $\varrho_F : \mathfrak{g} \rightarrow \mathfrak{X}(F)$ is the infinitesimal action associated to the G -action on F . Let $J : F \rightarrow \mathfrak{g}^*$ be the momentum map associated with the G -action on F . Now, define the horizontal 2-form $\overline{\mathbb{F}}$ as follows

$$\left(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)) \right)([p, f]) = \left\langle J(f), \text{Curv}_\Theta(\overline{X}_p, \overline{Y}_p) \right\rangle, \tag{9}$$

for all $X, Y \in \mathfrak{X}(B)$, and for all $[p, f] \in E$. Using Relations (8) and (9) and the fact that the G -action on F is Hamiltonian, one obtains

$$\text{Curv}_\Gamma(X, Y) = \mathcal{V}^\sharp \left(d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y))) \right), \quad \text{for all } X, Y \in \mathfrak{X}(B).$$

To check that $\overline{\mathbb{F}}$ is horizontally closed, it is enough to notice that if we set

$$\Phi_{[p, f]}(\widehat{\mathbf{A}}_p, \widehat{\mathbf{B}}_f) = \left\langle J(f), \Theta_p(\mathbf{A}_p) \right\rangle,$$

for all tangent vectors

$$(\widehat{\mathbf{A}}_p, \widehat{\mathbf{B}}_f) = T_{(p, f)}\pi_{P \times F}(\mathbf{A}_p, \mathbf{B}_f)$$

then we get

$$\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)) = d\Phi(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)),$$

for all $X, Y \in \mathfrak{X}(B)$. The fact that $\overline{\mathbb{F}}$ is horizontally closed follows from $d^2\Phi = 0$. We have constructed integrable geometric data $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$. Finally, we can apply Theorem 2.9 which gives the result we sought. \square

3.2 Fat bundles

Let $P \rightarrow B$ be a (left) principal G -bundle, \mathfrak{g}^* the dual of the Lie algebra of G , and S a sub-set of \mathfrak{g}^* . A connection Θ on P is *S-fat* [15] if for every $\mu \in S$, $\mu \circ \text{Curv}_\Theta$ is non-degenerate.

Proposition 3.3 *Let (G, P, F) be a classical Yang-Mills-Higgs setup. Then every $J(F)$ -fat connection Θ on P gives rise to a Poisson structure on the associated bundle $E = P \times_G F \rightarrow B$.*

The proof of Proposition 3.3 is similar to that of Theorem 3.2. Precisely, one can notice that the 2-form defined by Eq. 9 is nondegenerate when the given connection Θ is $J(F)$ -fat connection. So using Remark 2.8, we conclude that the Dirac structure obtained (as in the proof of Theorem 3.2) is the graph of a Poisson bivector field on E . \square

3.3 Another construction of a Dirac extension of a Poisson fiber bundle

Theorem 3.4 *Let (F, \mathcal{V}_F) be a compact Poisson manifold whose first Poisson cohomology group $H^1_{\mathcal{V}_F}(F)$ vanishes. Let $F \rightarrow E \xrightarrow{\pi} B$ be a Poisson fiber bundle. Then every Poisson–Ehresmann connection on E gives rise to an equivalence class of coupling Dirac structures on E such that each representative restricts to the Poisson structure on the fibers.*

Proof Consider the structure group $G = \text{Iso}(F, \mathcal{V}_F)$ which consists of all Poisson isomorphisms of (F, \mathcal{V}_F) . The *Poisson frame bundle*, denoted by P , is the principal G -bundle whose fiber over b is the set of all Poisson isomorphisms $\varphi_b: (F, \mathcal{V}_F) \rightarrow (E_b, \mathcal{V}_b)$. We can identify E with $P \times_G F$. The vertical Poisson vector field \mathcal{V} (defining the Poisson fiber bundle structure) can be viewed as the push-forward of \mathcal{V}_F by the projection map $\pi_{P \times F}: P \times F \rightarrow P \times_G F$. Moreover, every Poisson–Ehresmann connection Γ on E induces a connection Θ on P . These connections are related as in Eq. 7.

Consider the \mathbb{R} -linear map J from the Lie algebra of G into $C^\infty(F)/\{\text{Casimir functions}\}$ such that $J(Z) = g_Z$ is the unique function on F (up to Casimir functions) whose Hamiltonian vector field equals Z . Notice that the Lie algebra of G coincides with the space of Hamiltonian vector fields of (F, \mathcal{V}_F) because of the hypothesis $H^1_{\mathcal{V}_F}(F) = \{0\}$. Using this map J and the connection 1-form $\Theta(p): T_p P \rightarrow \text{Ham}(F, \mathcal{V}_F)$, we define a class of 1-forms Ψ on E as follows:

$$(\Psi(Y))(e) = \left(J \circ (\Theta(p)(Y_p^1)) \right)(f),$$

for every $e = [p, f] \in E$ and for all $Y \in \mathfrak{X}(E)$ defined by

$$Y([p, f]) = T_{(p,f)}\pi_{P \times F}(Y_p^1, Y_f^2).$$

Define the class of horizontal 2-forms

$$\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)) = d\Psi(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)),$$

which is determined up to elements of the form $\overline{\partial}_\Gamma\Phi$, where $\Phi \in \Omega^1(B) \otimes \text{Casim}(F, \mathcal{V}_F)$ and

$$\overline{\partial}_\Gamma\Phi(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)) = (\partial_\Gamma\Phi)(X, Y),$$

for $X, Y \in \mathfrak{X}(B)$. By construction, each representative element, also denoted by $\overline{\mathbb{F}}$, is horizontally closed. Furthermore, by arguments similar to those used in the proof of Theorem 3.2, one gets

$$\text{Curv}_\Gamma(X, Y) = \mathcal{V}^\sharp(d(\overline{\mathbb{F}}(\text{hor}_\Gamma(X), \text{hor}_\Gamma(Y)))) \quad \text{for all } X, Y \in \mathfrak{X}(B).$$

There follows Theorem 3.4. □

4 The Cartan-Hannay-Berry connection

In this section, our goal is to show that the notion of a Cartan-Hannay-Berry connection provides specific examples of coupling Dirac structures. We will use the following lemma.

Lemma 4.1 *Let $\pi : F \times B \rightarrow B$ be a Poisson fiber bundle together with its associated vertical Poisson bivector field \mathcal{V} . Consider an Ehresmann connection Γ on E such that*

$$\Gamma(0, X) = \mathcal{V}^\sharp(d\Phi(X)), \quad \forall X \in \mathfrak{X}(B),$$

for some $\Phi \in \Omega^1(B) \otimes C^\infty(E)$. Set

$$\mathbb{F}(X, Y) = d\Phi(X, Y) - \{\Phi(X), \Phi(Y)\}_{\mathcal{V}}, \quad \forall X, Y \in \mathfrak{X}(B).$$

Then the curvature of Γ is given by

$$\text{Curv}_\Gamma(X, Y) = \left(\mathcal{V}^\sharp(d(\mathbb{F}(X, Y))), 0 \right),$$

for any $X, Y \in \mathfrak{X}(B)$. Moreover, the associated horizontal 2-form $\overline{\mathbb{F}}$ (defined as in Eq. 1) is horizontally closed.

The proof of this lemma is straightforward. It is left to the reader. Now, we recall from Ref. [9] the definition and properties of a Cartan connection. Let S be a Riemannian manifold, Q the configuration space of a given mechanical system, and B a finite-dimensional space of embeddings of Q into S . Given a vector field $U \in \mathfrak{X}(B)$ and a point $b \in B$, the tangent vector $U_b \in T_bB$ is a map $U_b : Q \rightarrow TS$ with $U_b(q) \in T_{b(q)}S$. There is a canonical vector field $\mathcal{U}_b \in \mathfrak{X}(Q)$ associated with U_b . It is defined as follows: let $U_b^\perp(q)$ be the orthogonal projection of $U_b(q)$ to $T_{b(q)}b(Q)$ then

$$\mathcal{U}_b(q) = (Tb)^{-1}(U_b^\perp(q)).$$

The Cartan connection γ_0 on the trivial fiber bundle $Q \times B \rightarrow B$ is given by

$$(\gamma_0(V, U))(q, b) = (V_b + \mathcal{U}_b(q), 0_b).$$

Consider the Poisson fiber bundle $T^*Q \times B \rightarrow B$ with typical fiber $(T^*Q, \omega_{\text{can}})$. Denote by $\mathcal{V} \in \mathfrak{X}_{\text{Vert}}^2(T^*Q \times B)$ the vertical Poisson structure determined by the Poisson structure on the fibers.

Definition 4.2 [9] The induced Cartan connection on $E = T^*Q \times B$ is the map $\Gamma_0 : TE \rightarrow \text{Vert}$ defined by

$$\Gamma_0(W, U) = (W + X_{\mathcal{P}(U)}, 0),$$

where $\mathcal{P} \in \Omega(B) \otimes C^\infty(T^*Q \times B)$ is the 1-form defining the momentum function of U . Precisely, we have

$$(\mathcal{P}(U))(\alpha_q, b) = \langle \alpha_q, \mathcal{U}_b(q) \rangle \quad \forall \alpha_q \in T_q^*Q, \forall b \in B,$$

and $X_{\mathcal{P}(U)}$ is the Hamiltonian vector field of $\mathcal{P}(U)$ relative to \mathcal{V} . Moreover,

$$(\text{hor}_{\Gamma_0} U) = (0, U) + (-X_{\mathcal{P}(U)}, 0), \quad \text{for every } U \in \mathfrak{X}(B).$$

Let G be a compact Lie group. Given a left action of G on T^*Q with equivariant momentum map $J : T^*Q \rightarrow \mathfrak{g}^*$, we denote by $\langle \cdot, \cdot \rangle_G$ the averaging operation (see [9]).

Definition 4.3 [9] The Cartan-Hannay-Berry connection on $T^*Q \times B$ is the vertical valued 1-form Γ defined as follows:

$$\Gamma(W, U) = (W + X_{\langle \mathcal{P}(U) \rangle_G}, 0),$$

for any $W \in \mathfrak{X}(T^*Q)$, $U \in \mathfrak{X}(B)$. In other words, the horizontal lift for $U \in \mathfrak{X}(B)$ is given by

$$(\text{hor}_\Gamma U)(\alpha_q, b) = (-X_{\langle \mathcal{P}(U) \rangle_G}(\alpha_q, b), U(b)).$$

Now we are going to define the integrable data associated with the Cartan-Hannay-Berry connection. We set

$$\mathbb{F}_0(U_1, U_2) = U_1 \cdot \mathcal{P}(U_2) - U_2 \cdot \mathcal{P}(U_1) - \mathcal{P}([U_1, U_2]) - \{\mathcal{P}(U_1), \mathcal{P}(U_2)\}_{\mathcal{V}},$$

and

$$\mathbb{F}(U_1, U_2) = \langle \mathbb{F}_0(U_1, U_2) \rangle_G.$$

for all $U_1, U_2 \in \mathfrak{X}(B)$. Lemma 4.1 implies that $(\mathcal{V}, \Gamma_0, \mathbb{F}_0)$ and $(\mathcal{V}, \Gamma, \mathbb{F})$ are integrable geometric data on $T^*Q \times B$. Their associated coupling Dirac structures are defined as in Eq. 1. In other words, every Cartan (resp. Cartan-Hannay-Berry) connection gives rise to a coupling Dirac structure. □

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