Annals of Global Analysis and Geometry **28:** 371–394 (2005)

Calibrated Subbundles in Noncompact Manifolds of Special Holonomy

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(Received: 21 April 2005; accepted: 9 August 2005)

Abstract. This paper is a continuation of *Math. Res. Lett.* **12** (2005), 493–512. We first construct special Lagrangian submanifolds of the Ricci-flat Stenzel metric (of holonomy SU(n)) on the cotangent bundle of S^n by looking at the conormal bundle of appropriate submanifolds of S^n. We find that the condition for the conormal bundle to be special Lagrangian is the same as that discovered by Harvey–Lawson for submanifolds in $\mathbb{R}^n$ in their pioneering paper, *Acta Math.* 148 (1982), 47–157. We also construct calibrated submanifolds in complete metrics with special holonomy G_2 and Spin(7) discovered by Bryant and Salamon (*Duke Math. J.* 58 (1989), 829–850) on the total spaces of appropriate bundles over self-dual Einstein four manifolds. The submanifolds are constructed as certain subbundles over immersed surfaces. We show that this construction requires the surface to be minimal in the associative and Cayley cases, and to be (properly oriented) real isotropic in the coassociative case. We also make some remarks about using these constructions as a possible local model for the intersection of compact calibrated submanifolds in a compact manifold with special holonomy.

Mathematics Subject Classification (2000): 53-XX, 58-XX.

Key words: Stenzel metric, Bryant–Salamon metric, Calabi metric, special Lagrangian, associative, coassociative, Cayley, calibrated submanifolds.

1. Introduction

The study of calibrated geometries begun in the paper [19] of Harvey and Lawson. Calibrated submanifolds (in particular special Lagrangian submanifolds) are believed to play a crucial role in mirror symmetry [40] and M-theory, and hence they have recently received much attention. There has been extensive research done on special Lagrangian submanifolds of $\mathbb{C}^n$, most notably by Joyce but see also [23] and the many references contained therein. Much less progress has been made in studying associative, coassociative, and Cayley submanifolds even in flat space. The earliest explicit nonflat examples of special holonomy metrics were constructed on vector bundles. These explicit metrics are all cohomogeneity one examples and are obtained by reducing the conditions for special holonomy to an exactly solvable ordinary differential equation. Explicit Calabi–Yau metrics were found on the cotangent bundle of spheres, initially discovered by Eguchi–Hanson for S^2 and Candelas and others for S^3, but see Stenzel [39] for the general case.

Similarly, Calabi discovered hyper-Kähler metrics on the cotangent bundle of the complex projective space [8] and Bryant and Salamon [7] found explicit examples of metrics of full holonomy G_2 and Spin(7) on the bundles of anti-self-dual 2-forms and negative chirality spinors over specific four manifolds (see Remark 4.7 for a note on orientation conventions). These bundles, although noncompact, also serve as local models for a general metric of special holonomy and they have also received a lot of attention from mathematical physicists, who have generalized these metrics and studied them in detail [2, 9, 10, 15, 16].

In our first paper [21], along with Marianty Ionel, we generalized a bundle construction of Harvey and Lawson for special Lagrangian submanifolds in $\mathbb{C}^n$ to analogous constructions of coassociative, associative, and Cayley submanifolds in $\mathbb{R}^7$ and $\mathbb{R}^8$. In this paper, we further generalize this construction to the case of several explicit, nonflat, noncompact manifolds with complete metrics of special holonomy which are vector bundles over a compact base. The authors recommend that readers first consult [21], as many of the calculations, especially in Section 4, are very similar and are covered in more detail in [21]. In particular, without further mention, all of our local calculations are done using normal coordinates.

In Section 2, we briefly review the relevant facts from calibrated geometry that we will use, and set up some notation. In particular, it should be noted that in Propositions 2.3 and 2.5 we present alternative characterizations of the associative and Cayley conditions. These characterizations are entirely in terms of the calibrating forms and the associated cross-products and metrics (which are all derivable from the forms). This is similar to the special Lagrangian and coassociative conditions. In [21] our proofs in the associative and Cayley cases relied on a choice of identification of the tangent spaces with octonions or purely imaginary octonions and was perhaps not as satisfying. At least the invariant description of the Cayley condition seems not to have appeared in the literature before.

In Section 3, we describe the Stenzel Calabi–Yau metrics on $T^*(S^n)$ and show that the conormal bundle over an immersed submanifold X in S^n is special Lagrangian with respect to some phase (which depends on the codimension of X in S^n) if and only if X is *austere* in S^n. This is the same result as Harvey and Lawson found [19] for $\mathbb{C}^n$ but it is perhaps surprising, especially since the complex structure on $T^*(S^n)$ is obtained in an extremely different way from that of $\mathbb{C}^n = T^*(\mathbb{R}^n)$, namely by identifying it with a complex quadric hypersurface in $\mathbb{C}^{n+1}$.

In Section 4, we construct coassociative and Cayley submanifolds in $\wedge^2_-(S^4)$ and $\wedge^2_-(\mathbb{CP}^2)$ by taking vector subbundles over an immersed surface Σ in the base. As in [21], the associative construction requires Σ to be minimal, while the coassociative case needs Σ to be (properly oriented) isotropic. (Sometimes also called superminimal.) In this case, it is perhaps not so surprising that the results are the same as in the flat case, since the calculations are extremely similar, differing bascially only by the presence of some conformal scaling factors. This is entirely due to the fact that these cohomogeneity one metrics have a high degree of symmetry. We also construct Cayley submanifolds in the negative spinor bundle

$\mathcal{S}_-(S^4)$ over S^4 by taking rank 2 vector bundles over a minimal surface Σ in S^4. The result is again the same as the flat case of $\mathbb{R}^8$ found in [21], although this time the calculation is done in a very different way. It should also be noted that in the case of $\mathbb{R}^8$, we obtained degenerate examples. That is, they were products of lower order constructions. However, this time in the case of $\mathcal{S}_-(S^4)$ the Cayley examples are not degenerate.

Finally, in Section 5, we make some remarks about how these constructions might be used as local models for the intersections of compact calibrated submanifolds of a compact manifold with special holonomy. We hope to expand on this topic further in a subsequent paper.

Remark. Similar although different statements to some of the results of Section 4 appeared, without proof, in an unpublished preprint by S. H. Wang [42] back in 2001. As remarked in [21], the original statement appeared in the preprint was incorrect, but the authors were recently notified by Robert Bryant that a corrected version of Wang's paper will appear soon.

2. Review of Calibrated Geometries

In this section, we review the necessary facts about the calibrated geometries that we study in this paper, and set up notation. Some references are [19, 23, 24]. Calibrated submanifolds are a distinguished class of submanifolds of a Riemannian manifold (M, g) which are absolutely volume minimizing in their homology class. Being minimal is a second order differential condition, but being calibrated is a *first order* differential condition.

DEFINITION 2.1. A closed k-form α on M is called a calibration if it satisfies $\alpha(e_1, \ldots, e_k) \leqslant 1$ for any choice of k orthonormal tangent vectors $e_1, \ldots, e_k$ at any point $p \in M$. A calibrated subspace of $T_p(M)$ is an oriented k-dimensional subspace V_p for which $\alpha(V_p) = 1$. Then a calibrated submanifold L of M is a k-dimensional oriented submanifold for which each tangent space is a calibrated subspace. Equivalently, L^k is calibrated if

$$\alpha|_L = \mathrm{vol}_L$$

where vol_L is the volume form of L associated to the induced Riemannian metric from M and the choice of orientation.

Here are the four main examples of calibrated geometries. (More will be said later about G_2 and Spin(7) structures.)

(i) Complex submanifolds L^{2k} (of complex dimension k) of a *Kähler* manifold M where the calibration is given by $\alpha = \omega^k/k!$, and ω is the Kähler form on M. Kähler manifolds are characterized by having Riemannian holonomy contained in U(n), where n is the complex dimension of M. These submanifolds come in all even real dimensions.

(ii) Special Lagrangian submanifolds L^n *with phase* $e^{i\theta}$ of a *Calabi–Yau* manifold M where the calibration is given by $\mathrm{Re}(e^{i\theta}\Omega)$, where Ω is the holomorphic $(n, 0)$ volume form on M. Calabi–Yau manifolds have Riemannian holonomy contained in $\mathrm{SU}(n)$. Special Lagrangian submanifolds are always half-dimensional, but there is an S^1 family of these calibrations for each M, corresponding to the $\mathrm{e}^{i\theta}$ freedom of choosing Ω. Note that Calabi–Yau manifolds, being Kähler, also possess the Kähler calibration.

(iii) Associative submanifolds L^3 and coassociative submanifolds L^4 of a G_2 manifold M^7. Here the calibrations are given by the 3-form φ and the 4-form $*\varphi$, respectively, where φ is the fundamental 3-form corresponding to the G_2-structure. G_2 manifolds have Riemannian holonomy contained in G_2. These calibrated submanifolds only come in dimensions 3 and 4.

(iv) Cayley submanifolds L^4 of a Spin(7) manifold M^8. Here the calibration is given by the 4-form Φ which is the fundamental 4-form corresponding to the Spin(7) structure. Spin(7) manifolds have Riemannian holonomy contained in Spin(7). These calibrated submanifolds only come in dimension 4.

Remark 2.2. If M^{4n} is a *hyper-Kähler manifold*, which means its Riemannian holonomy is contained in $\mathrm{Sp}(n)$, then it has an S^2 family of Kähler structures and each one is Calabi–Yau. There is thus a wealth of calibrated submanifolds in the hyper-Kähler case. Also, a Calabi–Yau manifold M^8 of complex dimension 4 is always a Spin(7) manifold, and thus contains special Lagrangian, complex, and Cayley submanifolds.

In practice, it is not easy to check if $\alpha|_L = \mathrm{vol}_L$ but there are alternative, equivalent conditions for a submanifold to be calibrated which we now describe.

(i) Complex submanifolds L of a Kähler manifold M are characterized by the fact that their tangent spaces are invariant under the action of the complex structure J on M.

(ii) Harvey and Lawson showed in [19] that, up to a possible change of orientation, L is special Lagrangian of phase $e^{i\theta}$ if and only if

$$\omega|_L = 0 \tag{1}$$

and

$$\mathrm{Im}(\mathrm{e}^{i\theta}\Omega)|_L = 0. \tag{2}$$

Condition (1) say that L is Lagrangian, while (2) is the *special* condition.

(iii) A Riemannian manifold M^7 which possesses a G_2 structure has a globally defined, two-fold vector cross-product

$$\begin{aligned} \times\colon\ & T(M) \times T(M) \to T(M) \\ & (v, w) \mapsto v \times w \end{aligned}$$

which satisfies

$$v \times w = -w \times v \qquad \times \text{ is alternating,}$$
$$\langle v \times w, v\rangle = 0 \qquad \forall v, w \text{ (orthogonal to its arguments),}$$
$$|v \times w|^2 = |v \wedge w|^2 \qquad \forall v, w,$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on M and $|\cdot|$ is its associated norm. The metric, cross-product, and fundamental 3-form φ are related by

$$\varphi(u, v, w) = \langle u \times v, w\rangle \tag{3}$$

from which it follows that

$$(u \times v)^\flat = v \lrcorner u \lrcorner \varphi, \tag{4}$$

where $^\flat$ is the isomorphism from vector fields to 1-forms induced by the Riemannian metric. It is shown in [19] that a three-dimensional submanifold L^3 is associative if and only if its tangent space is preserved by the cross product $\times$. Similarly, a four-dimensional submanifold L^4 is coassociative if and only if $u \times v$ is a normal vector for every pair of vectors u, v tangent to L^4. There exist vector valued alternating 3- and 4-forms on M called the associator and coassociator which vanish on associative and coassociative submanifolds, respectively, but these are difficult to work with directly as they are related to octonion algebra. In [19] Harvey and Lawson showed that the coassociative condition is equivalent (up to a change of orientation), to the vanishing of the 3-form φ:

$$\varphi|_{L^4} = 0. \tag{5}$$

This reformulation should be compared to (1) and (2).

We now present an alternative characterization of the associative condition. Let u, v, w be a linearly independent set of tangent vectors at a point $p \in M$. We want to check when the three-dimensional subspace that they span is an associative subspace. Now if we have chosen an identification of T_pM with Im $\mathbb{O}$, then we need to check the vanishing of the associator:

$$[u, v, w] = u(vw) - (uv)w.$$

When u and v are imaginary octonions, their product is $uv = -\langle u, v\rangle + u \times v$, in terms of the inner product and the cross-product. Thus, we have

$$\begin{aligned}[u, v, w] &= u(-\langle v, w\rangle + v \times w) - (-\langle u, v\rangle + u \times v)w \\ &= -\langle v, w\rangle u - \langle u, v \times w\rangle + u \times (v \times w) + \\ &\quad + \langle u, v\rangle w + \langle u \times v, w\rangle - (u \times v) \times w \\ &= \langle u, v\rangle w - \langle v, w\rangle u + u \times (v \times w) - (u \times v) \times w\end{aligned}$$

where we have used (3) to cancel two of the terms. Now from Lemma 2.4.3 in [27] we have the formula

$$u \times (v \times w) = -\langle u, v\rangle w + \langle u, w\rangle v - (u \lrcorner v \lrcorner w \lrcorner * \varphi)^\#.$$

Substituting this into the above expression for the associator and simplifying, we obtain

$$[u, v, w] = -2(u \lrcorner v \lrcorner w \lrcorner * \varphi)^{\#}$$

Thus, we have proved the following proposition:

PROPOSITION 2.3. *The subspace spanned by the tangent vectors u, v, w is an associative subspace if and only if*

$$u \lrcorner v \lrcorner w \lrcorner * \varphi = 0. \tag{6}$$

Remark 2.4. The left-hand side of (6) is (using the metric isomorphism) a vector valued 3-form which is invariant under the action of G_2. Therefore, representation theory arguments say it must be the associator, and here we show this directly.

(iv) A Riemannian manifold M^8 which possesses a Spin(7) structure has a globally defined, three-fold vector cross-product

$$\begin{aligned} X\colon T(M) \times T(M) \times T(M) &\to T(M) \\ (u, v, w) &\mapsto X(u, v, w) \end{aligned}$$

which satisfies

$$\begin{array}{ll} X(u, v, w) & \text{is totally skew-symmetric,} \\ \langle X(u, v, w), u\rangle = 0 & \forall u, v, w \text{ (orthogonal to its arguments),} \\ |X(u, v, w)|^2 = |u \wedge v \wedge w|^2 & \forall u, v, w. \end{array}$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on M and $| \cdot |$ is its associated norm. As in the G_2 case, the metric, cross product, and fundamental 4-form Φ are related by

$$\Phi(u, v, w, y) = \langle X(u, v, w), y\rangle \tag{7}$$

from which it follows that

$$X(u, v, w)^{\flat} = w \lrcorner v \lrcorner u \lrcorner \Phi. \tag{8}$$

It is shown in [19] that a four-dimensional submanifold L^4 is Cayley if and only if its tangent space is preserved by the cross product X. As in the G_2 case, there exists a rank 7 bundle valued 4-form η on M that vanishes on Cayley submanifolds. This form η is defined in terms of octonion multiplication. Let u, v, w, y be a linearly independent set of tangent vectors at a point $p \in M$. We want to check when the four-dimensional subspace that they span is a Cayley subspace. Assuming an explicit identification of T_pM with $\mathbb{O}$, the form η is:

$$\eta = \tfrac{1}{4}\mathrm{Im}(\bar{u}X(v, w, y) + \bar{v}X(w, u, y) + \bar{w}X(u, v, y) + \bar{y}X(v, u, w)).$$

We now describe a characterization of the Cayley condition which is analogous to (6), that does not seem to have explicitly appeared in the literature before. The fact we use is the following. The space of 2-forms on M splits as $\wedge^2 = \wedge^2_7 \oplus \wedge^2_{21}$, where at each point $\wedge^2_k$ is k-dimensional (see [22, 27]). One can check by explicit computation that if u and v are tangent vectors, identified as octonions, then

$$\mathrm{Im}(\bar{u}v) \cong \pi_7(u^\flat \wedge v^\flat),$$

where π_7 is projection onto $\wedge^2_7$. Thus, up to isomorphism, the expression for the form η becomes

$$\eta = \pi_7(u^\flat \wedge X(v,w,y)^\flat + v^\flat \wedge X(w,u,y)^\flat + w^\flat \wedge X(u,v,y)^\flat + \\ + y^\flat \wedge X(v,u,w)^\flat).$$

We have an explicit formula for the projection π_7 in terms of the 4-form Φ. (See [27], for example, although we differ by a sign here because of the opposite choice of orientation.) This formula is

$$\pi_7(u^\flat \wedge v^\flat) = \tfrac{1}{4}(u^\flat \wedge v^\flat + u \lrcorner v \lrcorner \Phi).$$

Combining these expressions, we have proved the following proposition.

PROPOSITION 2.5. *The subspace spanned by the tangent vectors u, v, w, y is a Cayley subspace if and only if the $\wedge^2_7$ valued 2-form η vanishes:*

$$\eta = u^\flat \wedge X(v,w,y)^\flat + u \lrcorner X(v,w,y) \lrcorner \Phi + v^\flat \wedge X(w,u,y)^\flat + \\ + v \lrcorner X(w,u,y) \lrcorner \Phi + w^\flat \wedge X(u,v,y)^\flat + w \lrcorner X(u,v,y) \lrcorner \Phi + \\ + y^\flat \wedge X(v,u,w)^\flat + y \lrcorner X(v,u,w) \lrcorner \Phi = 0.$$

Remark 2.6. It should be evident that calibrated submanifolds seem to fall into two different categories. There are those whose tangent spaces are preserved by a cross-product operation. These are the complex, associative, and Cayley submanifolds, whose tangent spaces are preserved by J, $\times$, and X, respectively. These are called *instantons*. There are also those which are determined by the vanishing of differential forms, namely the special Lagrangian and coassociative submanifolds, and these are called *branes*. Branes have a nice, unobstructed deformation theory, which was first studied by McLean [35]. Instantons, on the other hand, are generally obstructed and are more complicated to study. See [29] for more details on the differences between branes and instantons.

3. Special Lagrangians in $T^*(S^n)$ with the Stenzel Metric

In this section, we construct special Lagrangian submanifolds in $T^*(S^n)$ with the Calabi–Yau metric discovered by Stenzel [39] and discussed in detail in [9].

It is a classical fact that if X^p is a p-dimensional submanifold of $\mathbb{R}^n$, then the *conormal bundle* $N^*(X)$ is a *Lagrangian* submanifold of the symplectic manifold $T^*(\mathbb{R}^n)$, with its canonical symplectic structure. Harvey and Lawson found conditions ([19, Theorem III.3.11]) on the immersion $X \subset \mathbb{R}^n$ that makes $N^*(X)$ a special Lagrangian submanifold of $T^*(\mathbb{R}^n) \cong \mathbb{C}^n$, in terms of the second fundamental form of the immersion. We generalize this construction to the case of the Calabi–Yau metric on $T^*(S^n)$, which we now describe.

Following Szöke [41], we can map the space

$$T^*(S^n) = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \| x| = 1, \langle x, \xi\rangle = 0\}$$

diffeomorphically and equivariantly with respect to $\mathrm{SO}(n; \mathbb{R}) \subset \mathrm{O}(n; \mathbb{C})$ onto the complex quadric

$$Q = \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \,\middle|\, \sum z_k^2 = 1 \right\}$$

in $\mathbb{C}^{n+1}$ by

$$\begin{aligned} \Psi : T^*S^n &\to Q \\ (x, \xi) &\mapsto x \cosh|\xi| + i \frac{\xi}{|\xi|} \sinh(|\xi|). \end{aligned} \tag{9}$$

In this way $Q \cong T^*(S^n)$ inherits a complex structure, since it is a complex hypersurface of $\mathbb{C}^{n+1}$. It also posseses a holomorphic $(n, 0)$ form Ω which is defined by

$$\Omega(v_1, \dots, v_n) = (\mathrm{d}z_0 \wedge \mathrm{d}z_1 \cdots \wedge \mathrm{d}z_n)(Z, v_1, \dots, v_n), \tag{10}$$

where

$$Z = z_0 \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_1} + \cdots + z_n \frac{\partial}{\partial z_n}$$

is the holomorphic radial vector field on $\mathbb{C}^{n+1}$. With respect to this complex structure, Stenzel showed ([39]) that there exists a Ricci-flat Kähler metric on $T^*(S^n)$, thought of as the quadic Q, whose Kähler form ω_{St} in a neighbourhood of a point where $z_0 \neq 0$, is given by

$$\omega_{st} = \frac{i}{2} \sum_{j,k=1}^{n} a_{jk} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k, \tag{11}$$

where we have (see also Anciaux [1] for more details) that

$$a_{jk} = \left(\delta_{jk} + \frac{z_j \bar{z}_k}{|z_0|^2} \right) u' + 2\mathrm{Re}\left(\bar{z}_j z_k - \frac{\bar{z}_0}{z_0} z_j z_k \right) u''. \tag{12}$$

Here u is a function of the radial variable $r = |z|$ and satisfies a certain ordinary differential equation that makes the metric Ricci-flat. The precise form of u depends on the dimension n but it will not concern us, since our results depend only on the fact that u is a function of r. We note that $r^2 = \cosh^2|\xi| + \sinh^2|\xi|$. It is easy to check from (11) and (12) that when restricted to the zero section, this gives the standard round metric on S^n. In dimension $n = 2$ this metric coincides with the well-known Eguchi–Hanson and Calabi metrics on $T^*(S^2)$ ([8–10, 13]).

Now let X be a p-dimensional submanifold of the standard round sphere S^n with the induced metric. The conormal bundle of X^p in S^n will be denoted by $L = N^*(X) \subset T^*(S^n)$. Then L is a submanifold of dimension n and can be locally parametrized as

$$(s, t) \mapsto \big(x(s), \Sigma t_k \nu^k\big) \quad s = (s_1, \dots, s_p), \quad t = (t_{p+1}, \dots, t_n),$$

where $x = (x_0, \dots, x_n) \in X \subset S^n$ and $\nu = (\nu^{p+1}, \dots, \nu^n) \in \mathbb{R}^{n+1}$ are orthonormal conormal vectors in $N^*(X)$. Let $e_1, \dots, e_p$ be an orthonormal base of tangent vectors to X. Then $(e_0 = x(s), e_1, \dots, e_p, \nu^{p+1}, \dots, \nu^n)$ form an adapted orthonormal moving frame of $\mathbb{R}^{n+1}$ along the submanifold X.

We restrict the map in (9) to the subbundle $L = N^*(X)$:

$$\Psi(x(s), \Sigma t_k \nu^k) = x(s) \cosh |t| + i\,\hat{\nu}(s, t) \sinh |t|,$$

where $|t|^2 = t_{p+1}^2 + \cdots + t_n^2$, and $\hat{\nu} = \Sigma t_k \nu^k / |t|$ is a unit conormal vector. Note that $\hat{\nu}$ is homogeneous as a function of t. That is, $\hat{\nu}(s, \lambda t) = \hat{\nu}(s, t)$ for all $\lambda \neq 0$ and $\hat{\nu}(s, t) \sinh |t|$ is well defined for $t = 0$.

THEOREM 3.1. *The conormal bundle L of a submanifold $X \subset S^n$ is special Lagrangian in $T^*(S^n)$ equipped with the Ricci-flat Stenzel metric if and only if X is* austere *in S^n.*

Proof. We show that the tangent space of L at each point is a special Lagrangian subspace. Fix a point $(x, \xi) \in L$. By the equivariance of the embedding we can choose an orthonormal basis $(e_0, \dots, e_n)$ of $\mathbb{R}^{n+1}$ so that at the point (x, ξ) the moving frame is given by these vectors and so the point has coordinates $(x(0) = e_0, \Sigma t_k \nu^k)$ with $\nu^k(0) = e_k$, for $k = p+1, \dots, n$. In fact, since we still have the freedom of rotating the conormal vectors, we can assume that $\hat{\nu} = \nu^{p+1} = e_{p+1}$. In other words, we can rotate so that the point we are considering has t coordinates $t_{p+1} = |t| = t \geqslant 0$ and $t_k = 0$ for $k = p+2, \dots, n$.

Now we compute a basis for the tangent space at this point $\Psi(x, \xi) = e_0 \cosh |t| + i e_{p+1} \sinh |t|$. We differentiate the immersion with respect to the s and t coordinates and evaluate at the point. From $s_1, \dots, s_p$ we have

$$E_j = \cosh |t| e_j + i \sinh |t| A^{\hat{\nu}}(e_j) \quad j = 1, \dots, p, \tag{13}$$

where $A^{\hat{\nu}}$ is the second fundamental form in the direction of the unit normal vector $\hat{\nu}$ of the submanifold X in S^n. That is, $A^{\hat{\nu}}(u) = \bar{\nabla}_u \hat{\nu}$, where $\bar{\nabla}$ is the Levi-Civita connection for the standard round metric on S^n. When we differentiate with respect to t_k we get

$$F_k = x(s) \frac{\sinh |t|}{|t|} t_k + i \left(\nu^k \frac{\sinh |t|}{|t|} + \left(\sum_t t_l \nu^l \right) \left(\frac{|t| \cosh |t| - \sinh |t|}{|t|^3} t_k \right) \right).$$

Now we evaluate at our fixed point by putting $s = 0$, $t_k = 0$ for $k \neq p+1$, and $t_{p+1} = |t|$ to obtain

$$F_{p+1} = \sinh|t|e_0 + i\cosh|t|e_{p+1},$$
$$F_k = i\frac{\sinh|t|}{|t|}e_k \quad k = p+2, \dots, n. \tag{14}$$

At the point $e_0\cosh|t| + ie_{p+1}\sinh|t|$, $z_0 = \cosh|t| \neq 0$, $z_{p+1} = i\sinh|t|$ and all the other coordinates $z_1, \dots, z_p, z_{p+2}, \dots, z_n$ are zero. This simplifies (and in fact diagonalizes) the Stenzel metric in (11) and (12) and, at that point, we have

$$a_{jk} = u', \qquad j, k \neq p+1,$$
$$a_{p+1,p+1} = (1 + \tanh^2|t|)u' + 4\sinh^2|t|u''.$$

and so

$$\omega_{St} = u'\frac{i}{2}\sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k + \frac{i}{2}(u'\tanh^2|t|u' + 4u''\sinh^2|t|)\mathrm{d}z_{p+1} \wedge \mathrm{d}\bar{z}_{p+1}.$$

Since from (13) the E_j's have a zero component in the e_{p+1}-direction, $\mathrm{d}z_{p+1} \wedge \mathrm{d}\bar{z}_{p+1}$ vanishes on $E_j \wedge E_k$ for all j, k and we have

$$\omega_{St}(E_j, E_k) = u'\sinh|t|\cosh|t|(\langle A^{\hat{\nu}}(e_j, e_k)\rangle - \langle A^{\hat{\nu}}(e_k, e_j)\rangle)$$
$$= 0$$

since the second fundamental form is symmetric. From (13) and (14) we see that E_j has nonzero components only in the $z_1, \dots, z_p$ directions and F_k for $k = p+2, \dots, n$ has a nonzero component only in the z_k-direction. Hence,

$$\omega_{St}(E_j, F_k) = 0 \quad j = 1, \dots, p \quad \text{and} \quad k = p+2, \dots, n.$$

Similarly, F_{p+1} has nonzero components only in the direction of z_0 and z_{p+1}. Thus,

$$\omega_{St}(E_j, F_{p+1}) = 0,$$
$$\omega_{St}(F_k, F_{p+1}) = 0.$$

Thus, we have shown that that $L = N^*(X)$ is always Lagrangian with respect to the symplectic form associated to the Stenzel metric for any submanifold X of S^n.

In order to find the conditions for L to be special Lagrangian, we have to evaluate the holomorphic $(n, 0)$-form Ω on the tangent vectors E_j and F_k of our submanifold. In a neighbourhood of a point where $z_0 \neq 0$, it follows from (10) that

$$\Omega = \frac{1}{2z_0}\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n. \tag{15}$$

This calculation is very similar to the original calculation done by Harvey and Lawson [19], except that we have factors involving the function u and the hyperbolic trigonometric functions of the radial variable $|t|$.

We can choose $e_1, \dots, e_p$ to diagonalize the second fundamental in the direction $\hat{\nu}$ at the point under consideration. Let λ_j be the corresponding eigenvalues (principal curvatures). Then we have

$$E_j = \cosh|t|e_j + i\lambda_j \sinh|t|e_j \quad j = 1, \dots, p,$$
$$F_{p+1} = \sinh|t|e_0 + i\cosh|t|e_{p+1},$$
$$F_k = i\frac{\sinh|t|}{|t|}e_k, \quad k = p+2, \dots, n$$

and, hence, plugging into (15),

$$\begin{aligned}\Omega(E_1 \wedge \cdots \wedge E_p \wedge F_{p+1} \wedge \cdots \wedge F_n) \\ = \frac{1}{2\cosh|t|}\cosh|t|\left(\frac{\sinh|t|}{|t|}\right)^{n-p-1} i^{n-p}\prod_{j=1}^{p}(\cosh|t| + i\lambda_j \sinh|t|) \\ = (***)i^{n-p}\prod_{j=1}^{p}(1 + i\lambda_j \tanh|t|),\end{aligned}$$

where $(***)$ denotes an always positive factor. Hence from (2) we see that L will be special Lagrangian with phase i^{p-n} if the product on the right-hand side in the earlier equation vanishes for all t. This happens if and only if all odd symmetric polynomials in the eigenvalues λ_j have to be zero, or equivalently if all eigenvalues occur in pairs of opposite signs. This has to be true in all normal directons ν and so the submanifold must be *austere* as defined by Harvey and Lawson [19]. This completes the proof. □

Remark 3.2. The first symmetric polynomial is the trace, so the submanifold M^p is necessarily minimal. If $p = 1, 2$ this is the only condition, but for $p \geqslant 3$ the austere condition is much stronger than minimal.

Remark 3.3. It is interesting to note that we cannot construct special Lagrangian submanifolds in this way of arbitrary phase. The factor of i^{p-n} means that the allowed phase (up to orientation) depends on the codimension $n - p$ of the immersion. We will say more about this in Section 5.

Austere submanifolds have been studied, for example, in [6, 11]. A particularly simple (and in some sense trivial) example comes from equators: a sphere S^p immersed in S^n as an equator is totally geodesic and, hence, the conormal bundle $N^*(S^p)$ is a special Lagrangian submanifold of $T^*(S^n)$ with respect to the Stenzel metric. (Of phase i^{n-p}.)

4. Calibrated Submanifolds for the Bryant–Salamon Metrics

In this section, we will construct calibrated submanifolds as subbundles inside the Bryant–Salamon metrics [7] of exceptional holonomy G_2 or Spin(7) which are themselves defined on appropriate bundles over four manifolds with a self-dual Einstein metric. The subbundles are defined exactly in the same way as in [21], except that the ambient manifold, instead of being flat $\mathbb{R}^7$ or $\mathbb{R}^8$ is the total space of a vector bundle over a four manifold X^4.

4.1. CALIBRATED SUBMANIFOLDS OF $\wedge^2_-(X^4)$

Let (X^4, g) be an oriented self-dual Einstein manifold. The examples for which Bryant and Salamon obtained complete G_2 metrics are those with positive scalar curvature $\mathbb{CP}^2$ and S^4. Let $M^7 = \wedge^2_-(T^*X^4)$ be the bundle of anti-self-dual 2-forms on X^4. This vector bundle has a connection induced by the Levi-Civita connection of (X, g). The tangent space $T_\omega M$ of M at a point $\omega \in \wedge^2_-$, has therefore a canonical splitting $T_\omega M \cong \mathcal{H}_\omega \oplus \mathcal{V}_\omega$ into horizontal and vertical subspaces.

The projection map is a submersion and maps the horizontal space isometrically onto the tangent space of the base manifold at that point. The metric g on the base X^4 has a unique lift to the horizontal space $g_{\mathcal{H}}$. The vertical space $\mathcal{V}_\omega$, which can be identified with the vector space (the fibre) $\wedge^2_-(T^*_xX)$ also has a natural metric $g_{\mathcal{V}}$ induced by g.

THEOREM 4.1 (Bryant–Salamon [7]). *There exist positive functions u and v, depending only on the radial coordinate in the vertical fibres and satisfying a certain set of ordinary differential equations such that the metric*

$$g_{M^7} = u^2 g_{\mathcal{H}} \oplus v^2 g_{\mathcal{V}} \tag{16}$$

on the total space $M^7 = \wedge^2_-(T^*X^4)$ of a self-dual Einstein 4-manifold has G_2-holonomy with fundamental 3-form φ given by

$$\varphi = v^3 \operatorname{vol}_{\mathcal{V}} + u^2 v \mathrm{d}\theta,$$

where θ is the canonical (soldering) 2-form on $\wedge^2_-(T^*X^4)$ and $\mathrm{vol}_{\mathcal{V}}$ is the volume 3-form of $g_{\mathcal{V}}$ on the vertical fibres.

Remark 4.2. The canonical p-form θ on $\wedge^p(T^*X)$ for any manifold X is defined to be $\theta(u_1 \wedge \cdots \wedge u_p)_\omega = \omega(\pi_* u_1 \wedge \cdots \wedge_* u_p)$, at the point ω where π is the projection onto the base manifold. For $p = 1$ this is the usual canonical 1-form on $T^*(X)$.

Let e^0, e^1, e^2, e^3 be an orthonormal coframe for $T^*(X)$ and f^1, f^2, f^3 be an (orthonormal) basis of anti-self-dual 2-forms in the vertical fibres defined by $f^i = e^0 \wedge e^i - e^j \wedge e^k$ with i, j, k forming a cyclic permutation of 1, 2, 3. We denote horizontal lifts of tangent vectors e_i on the base to $\mathcal{H}$ by $\bar{e}_i$, with dual horizontal 1-forms $\bar{e}^i$. Similarly, we think of the anti-self dual two forms f^i as being vertical tangent vectors $\check{f}^i$ in $\mathcal{V}$ on the total space with dual vertical 1-forms $\check{f}_i$. Then, locally, the fundamental three form φ is given by

$$\varphi = v^3(\check{f}_1 \wedge \check{f}_2 \wedge \check{f}_3) + u^2 v \check{f}_1 \wedge (\bar{e}^0 \wedge \bar{e}^1 - \bar{e}^2 \wedge \bar{e}^3) + \\ + u^2 v \check{f}_2 \wedge (\bar{e}^0 \wedge \bar{e}^2 - \bar{e}^3 \wedge \bar{e}^1) + u^2 v \check{f}_3 \wedge (\bar{e}^0 \wedge \bar{e}^3 - \bar{e}^1 \wedge \bar{e}^2). \tag{17}$$

On this basis, the dual 4-form is given by

$$^*\varphi = u^4(\bar{e}^0 \wedge \bar{e}^1 \wedge \bar{e}^2 \wedge \bar{e}^3) - u^2 v^2 \check{f}_2 \wedge \check{f}_3 \wedge (\bar{e}^0 \wedge \bar{e}^1 - \bar{e}^2 \wedge \bar{e}^3) - \\ - u^2 v^2 \check{f}_3 \wedge \check{f}_1 \wedge (\bar{e}^0 \wedge \bar{e}^2 - \bar{e}^3 \wedge \bar{e}^1) - u^2 v^2 \check{f}_1 \\ \wedge \check{f}_2 \wedge (\bar{e}^0 \wedge \bar{e}^3 - \bar{e}^1 \wedge \bar{e}^2). \tag{18}$$

It was proved in [7] that the functions u and v are globally defined and the Bryant–Salamon metric is complete only in the cases where X is either S^4 or $\mathbb{CP}^2$ with the standard metrics (round metric on S^n and Fubini–Study metric on $\mathbb{CP}^2$). In other cases, like for hyperbolic space, the functions are not globally defined and we only obtain an incomplete metric defined near the zero section of the vector bundle $\wedge^2_-(T^*X^4)$. Our later constructions of associative and coassociative submanifolds are of a general nature and hence works in both cases (complete or incomplete).

An oriented surface $\Sigma^2 \subset X^4$ equipped with the induced metric defines a canonical lift

$$f^1_\Sigma \colon \Sigma^2 \longrightarrow M^7 = \wedge^2_-(X^4)$$

locally defined by the anti-self-dual 2-form $f^1 = e^1 \wedge e^2 - \nu^1 \wedge \nu^2$, where e^1, e^2 are orthonormal co-tangent vectors and ν^1, ν^2 are orthonormal conormal vectors to the surface Σ. That is, (e^1, e^2, ν^1, ν^2) is an oriented adapted co-frame along the surface. It is easily seen that f^1_Σ is globally well defined and is independent of the local frame. More invariantly we can define it by

$$f^1_\Sigma = \mathrm{vol}_\Sigma - *\mathrm{vol}_\Sigma,$$

where vol_Σ is the induced volume form on Σ and $*$ is the Hodge star operator on X^4. The span of f^1 defines a line bundle $L^3 \subset M^7 = \wedge^2_-(X)$. We also define $L^\perp = \{\omega \in \wedge^2_- \mid \omega \perp \omega^1\}$ to be the (real) two-dimensional subbundle orthogonal to L with respect to the Bryant–Salamon metric. Locally, $L^\perp$ is spanned by the two anti-self-dual 2-forms

$$f^2 = e^1 \wedge \nu^1 - \nu^2 \wedge e^2 \quad f^3 = e^1 \wedge \nu^2 - e^2 \wedge \nu^1.$$

We want to determine necessary and sufficient conditions on the second fundamental form of Σ for L to be associative and $L^\perp$ to be coassociative with respect to the Bryant–Salamon G_2-structure on M^7.

THEOREM 4.3. *The bundle L defined earlier which is canonically associated to a surface Σ in a four-dimensional self-dual Einstein manifold (X^4, g) is associative in $M^7 = \wedge^2_-(T^*X)$ equipped with the G_2 metric of Bryant and Salamon if and only if Σ is a minimal surface in X^4. The bundle $L^\perp$ is coassociative if and only if Σ is a (properly oriented) real isotropic surface in X^4.*

Proof. We check that at each point, the tangent space is a calibrated subspace. We begin with the associative case. At a point $t_1 f^1 \in L$, the following three vectors form a basis of the tangent space $T_{t_1 f^1}$ of L. (We denote the dual vectors with a lower index.)

$$\begin{aligned} E_i &= \bar{e}_i + t_1 \alpha(e_i, f^1) \quad i = 1, 2, \\ F_1 &= \check{f}^1, \end{aligned} \tag{19}$$

where the bar denotes the horizontal lift and $\alpha(e_i, f^1) = (\bar{\nabla}_{e_i} f^1)_{\mathcal{V}}$ is a vertical vector, which can be expressed (locally) in terms of the second fundamental form of the submanifold as follows:

$$\alpha(e_i, f^1) = (-A^{\nu_1}(e_i, e_1) - A^{\nu_2}(e_i, e_2))\check{f}_3 + (-A^{\nu_1}(e_i, e_2) + A^{\nu_2}(e_i, e_1))\check{f}_2,$$

where we use the notation $A^{\nu}(u, v) = \langle \bar{\nabla}_u u, v \rangle = -\langle \bar{\nabla}_u u, \nu, \rangle$ for $u, v \in T(X)$ and $\nu \perp T(X)$.

From Proposition 2.3 we have that the subbundle L is associative if and only if the 1-form $E_1 \lrcorner E_2 \lrcorner F_1 \lrcorner * \varphi$ vanishes at all points of L. Using (19) and (18) we compute:

$$F_1 \lrcorner * \varphi = -u^2 v(\check{f}_2 \wedge (\bar{e}^1 \wedge \bar{\nu}^2 - \bar{e}^2 \wedge \bar{\nu}^1) - \check{f}_3 \wedge (\bar{e}^1 \wedge \bar{\nu}^1 - \bar{\nu}^2 \wedge \bar{e}^2)),$$

where u, v are just functions. Using the symmetry of the second fundamental form A and the index notation $A^i_{jk} = A^{\nu_i}(e_j, e_k)$, we continue to compute:

$$\begin{aligned} & E_2 \lrcorner F_1 \lrcorner * \varphi \\ &\quad = -u^2 v\big(\check{f}_2 \wedge \bar{\nu}^1 + \check{f}_3 \wedge \bar{\nu}^2 + t_1\big(A^1_{12} + A^2_{22}\big)(e^{-1} \wedge \bar{\nu}^1 - \bar{\nu}^2 \wedge \bar{e}^2)\big) - \\ &\qquad - u^2 v\big(t_1\big(- A^1_{22} + A^2_{12}\big)(\bar{e}^1 \wedge \bar{\nu}^2 - \bar{e}^2 \wedge \bar{\nu}^1)\big) \end{aligned}$$

and further

$$\begin{aligned} & E_1 \lrcorner E_2 \lrcorner F_1 \lrcorner * \varphi \\ &\quad = -u^2 v\big(t_1\big(A^1_{12} + A^2_{22}\big)\bar{\nu}^1 + t_1\big(-A^1_{22} + A^2_{12}\big)\bar{\nu}^2\big) - \\ &\qquad - u^2 v\big(t_1\big(-A^1_{11} - A^2_{12}\big)\bar{\nu}^2 + t_1\big(-A^1_{12} + A^2_{11}\big)\bar{\nu}^1\big) \\ &\quad = -t_1 u^2 v\big(\big(A^2_{11} + A^2_{22}\big)\bar{\nu}^1 - \big(A^1_{11} + A^1_{22}\big)\bar{\nu}^2\big). \end{aligned}$$

Since *u, v*, are positive functions and since this expression must vanish at all points on L (that is, for all t_1), we must have $A^1_{11} + A^1_{22} = 0$ and $A^2_{11} + A^2_{22} = 0$. Thus, L is associative if and only if Σ is a minimal surface in X^4, proving the first half of the theorem.

We now move on to the coassociative case. For the subbundle $L^{\perp}$ we have the following description of a basis of four tangent vectors at a given point $f = t_2 f^2 + t_3 f^3$:

$$\begin{aligned} E_i &= \bar{e}_i + t_2 \alpha(e_i, f^2) + t_3 \alpha(e_i, f^3), \quad i = 1, 2, \\ F_j &= \check{f}^j, \quad j = 2, 3. \end{aligned}$$

Here the vertical correction terms are given by:

$$\alpha(e_i, f^2) = \big(\bar{\nabla}_{e_i} f^2\big)_{\mathcal{V}} = (A^{\nu_1}(e_i, e_2) - A^{\nu_2}(e_i, e_1))\check{f}^1,$$
$$\alpha(e_i, f^3) = \big(\bar{\nabla}_{e_i} f^3\big)_{\mathcal{V}} = (A^{\nu_2}(e_i, e_2) + A^{\nu_1}(e_i, e_1))\check{f}^1.$$

In order to check coassociativity, by (5) we need to check that $\varphi|_{L^\perp} = 0$. As in [21] we define $\nu = t_2\nu_1 + t_3\nu_2$ and $\nu^\perp = -t_3\nu_1 + t_2\nu_2$ and thus

$$E_1 = \bar{e}_1 + \big(A^{\nu}_{12} - A^{\nu^\perp}_{11}\big)\check{f}^1,$$
$$E_2 = \bar{e}_2 + \big(A^{\nu}_{22} - A^{\nu^\perp}_{12}\big)\check{f}^1.$$

It is easy to compute that

$$\begin{aligned}\varphi(E_1, E_2, \cdot) &= E_2 \lrcorner E_1 \lrcorner \varphi \\ &= u^2 v(\check{f}_1 + (\cdots)\bar{e}^1 + (\cdots)\bar{e}^2)\end{aligned}$$

and, hence, since $F_j = \check{f}^j$ we see that $\varphi(E_1, E_2, F_2) = \varphi(E_1, E_2, F_3) = 0$ always. It remains to check when $\varphi(F_2, F_3, E_j) = 0$ for $j = 1, 2$. Since $\varphi(F_2, F_3, \cdot) = v^3\check{f}_1$ and v is always positive, these become the conditions

$$A^{\nu}_{12} - A^{\nu^\perp}_{11} = 0, \quad A^{\nu}_{22} + A^{\nu^\perp}_{12} = 0. \tag{20}$$

for the tangent space at $(\mathbf{x}_0, t_2, t_3)$ to be coassociative. We get two more conditions that must be satisfied by demanding that the tangent space at $(\mathbf{x}_0, -t_3, t_2)$ also be coassociative. This corresponds to changing $t_2 \mapsto -t_3$ and $t_3 \mapsto t_2$ in the earlier equations, which is equivalent to $\nu \mapsto \nu^\perp$ and $\nu^\perp \mapsto -\nu$. This gives

$$A^{\nu^\perp}_{12} + A^{\nu}_{11} = 0, \quad A^{\nu^\perp}_{22} + A^{\nu}_{12} = 0. \tag{21}$$

Conditions (20) and (21) are exactly the same as those obtained in the case of $\mathbb{R}^7$ in [21]. These surfaces are called isotropic (with negative orientation) or superminimal surfaces. These surfaces are necessarily minimal, but the condition is in fact stronger (and overdetermined). See [3, 14, 21, 37] and the references contained therein for more details. □

Remark 4.4. Although the associative case is computed using a different method from that of [21], the calculations here and in Section 4.2 are very similar to [21], basically differing by the presence of certain conformal scaling factors. This is due to the high degree of symmetry in the cohomogeneity one metrics.

4.2. CAYLEY SUBMANIFOLDS OF $\mathcal{S}_-(S^4)$

In order to construct Cayley submanifolds, we now look at the Bryant–Salamon construction on the negative spin bundle of four manifolds. Let (X^4, g) be an oriented self-dual Einstein *spin* manifold of positive scalar curvature. The only example now is S^4, since $\mathbb{CP}^2$ is not spin. Let $M^8 = \mathcal{S}_-(X^4) \longrightarrow X$ be the complex two-dimensional vector bundle of negative chirality spinors on S^4. This is in fact the quaternionic Hopf bundle of the quaternionic projective line $\mathbb{HP}^1 \cong S^4$. Its unit sphere bundle $S^7 \longrightarrow S^4$, can be viewed as the associated principal $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$-bundle. Note that $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. This vector bundle has a natural Hermitian inner product and a connection induced by the Levi-Civita connection of the standard metric on S^4. The tangent space T_sM of M at a point $s \in \mathcal{S}_-$ has therefore a canonical splitting $T_sM \cong \mathcal{H}_s \oplus \mathcal{V}_s$ into horizontal and vertical subspaces. It is well known that this connection defines the standard SU(2)-instanton on S^4 with (anti-) self-dual curvature. The horizontal space of the connection is orthogonal to the vertical space with respect to the standard metric on S^7 and the curvature, which is the Lie bracket of horizontal vector fields identifies the anti-self-dual 2-forms on the base with the vertical fibres which form the Lie algebra $\mathrm{su}(2) \cong \mathbb{R}^3$. The projection map is a submersion and maps the horizontal space isometrically onto $T(S^4)$. The vertical space $\mathcal{V}_s$ also has a natural induced metric $g_{\mathcal{V}}$ and the connection form is an isomorphism between anti-self-dual 2-forms and the Lie algebra of SU(2).

THEOREM 4.5 (Bryant–Salamon [7]). *There exist positive functions u and v, depending only on the radial coordinate in the vertical fibres and satisfying a certain set of ordinary differential equations such that the metric*

$$g_{M^8} = u^2 g_{\mathcal{H}} \oplus v^2 g_{\mathcal{V}} \tag{22}$$

on the total space $M^8 = \mathcal{S}_-(S^4)$ has Spin*(7)-holonomy with self-dual fundamental 4-form Φ given by*

$$\Phi = u^4 \mathrm{vol}_{\mathcal{H}} + u^2 v^2 \beta + v^4 \mathrm{vol}_{\mathcal{V}},$$

where $\mathrm{vol}_{\mathcal{H}}$, $vol_{\mathcal{V}}$ are the volume 4-forms of $g_{\mathcal{H}}$, $g_{\mathcal{V}}$ on the horizontal and vertical spaces respectively and β is the 4-form defined as follows:

$$\beta = \sum_{k=1}^{3} \omega_k \wedge \sigma^k,$$

where ω_k is an orthonormal basis for anti-self-dual 2*-forms on the horizontal space and σ^k is the corresponding orthonormal basis for anti-self-dual* 2*-forms on the vertical space.*

Remark 4.6. Given an orthonormal basis of three anti-self-dual 2-forms, we get the corresponding vertical vectors at a spinor s by Clifford multiplication, since the curvature of the connection is anti-self-dual.

Remark 4.7. A note on orientations. With our chosen convention for the Spin(7) 4-form Φ, the natural local model for this structure is the negative spinor bundle over $\mathbb{R}^4$. With the opposite choice of orientation, we would be working with the positive spinor bundle. See [28] for more about sign conventions and orientations. As we are working only on S^4 in this paper, it does not make a difference.

Let e_1, e_2, e_3, e_4 be an oriented orthonormal frame for S^4 with horizontal lifts to the total space $\mathcal{S}_-(S^4)$ denoted by $\bar{e}_i$ with dual 1-forms $\bar{e}^i$. Let $\check{f}^1, \check{f}^2, \check{f}^3, \check{f}^4$ be the corresponding oriented orthonormal basis for the fibres. Then (dropping the wedge product symbols for clarity), the form Φ can be written as

$$\begin{aligned}\Phi = {} & u^4\bar{e}^1\bar{e}^2\bar{e}^3\bar{e}^4 + u^2v^2(\bar{e}^1\bar{e}^2 - \bar{e}^3\bar{e}^4)(\check{f}^1\check{f}^2 - \check{f}^3\check{f}^4) + \\ & + u^2v^2(\bar{e}^1\bar{e}^3 - \bar{e}^4\bar{e}^2)(\check{f}^1\check{f}^3 - \check{f}^4\check{f}^2) \\ & u^2v^2(\bar{e}^1\bar{e}^4 - \bar{e}^2\bar{e}^3)(\check{f}^1\check{f}^4 - \check{f}^2\check{f}^3) + v^4\check{f}^1\check{f}^2\check{f}^3\check{f}^4\end{aligned} \tag{23}$$

Now let $\Sigma^2 \subset S^4$ be an oriented surface equipped with the induced metric and let (e_1, e_2, ν_1, ν_2) be an oriented adapted frame along the surface. That is, (e_1, e_2) are orthonormal tangent vectors and (ν_1, ν_2) are orthonormal normal vectors to the surface. We are interested in the operator

$$\Gamma = \gamma(e^1 \wedge e^2) = \pm\gamma(\nu^1 \wedge \nu^2) \quad \text{on } \mathcal{S}_\pm$$

acting on spinors. The operator Γ leaves $\mathcal{S}_\pm$ invariant and it is easily seen that Γ is well defined globally and is independent of the local frame. Moreover, Γ is a skew-Hermitian operator satisfying $\Gamma^2 = -1$. The eigenspace decomposition of $\mathcal{S}_-$ with respect to Γ defines a natural splitting of the spinor bundle $\mathcal{S}_-$ restricted to the surface: $\mathcal{S}_-|_\Sigma \cong \mathcal{S}_-^+ \oplus \mathcal{S}_-^-$, where

$$\mathcal{S}_-^\pm = \{s \in \mathcal{S}_- \mid \Gamma(\Sigma) = \gamma(e^1 \wedge e^2)s = \pm i\, s\}.$$

The two bundles $\mathcal{S}_-^+$ and $\mathcal{S}_-^-$ are complex line bundles and are orthogonal to each other. We want to determine necessary and sufficient conditions on the second fundamental form of Σ for the total space of these bundles to be Cayley submanifolds with respect to the Bryant–Salamon Spin(7)-structure on M^8.

THEOREM 4.8. *The total space of either rank 2 bundle $\mathcal{S}_-^\pm$ over Σ is a Cayley submanifold of $\mathcal{S}_-(S^4)$ if and only the immersion $\Sigma \subset S^4$ is minimal.*

Proof. We show every tangent space to the total space of $\mathcal{S}_-^+$ is a Cayley subspace of the corresponding tangent space to $\mathcal{S}_-(S^4)$. The proof for $\mathcal{S}_-^-$ is identical.

Let $\dot{\Gamma}$ denote the covariant derivative of the operator Γ along the surface. Since $\Gamma^2 = -1$, we have $\Gamma\dot{\Gamma} + \dot{\Gamma}\Gamma = 0$, so Γ and $\dot{\Gamma}$ anti-commute and $\dot{\Gamma}$ interchanges the two eigenspaces of Γ. Differentiating the eigenvalue equation $\Gamma_s = is$, we get $(\Gamma - i)\dot{s} = -\dot{\Gamma}_s$ and hence

$$\dot{s} = -\frac{1}{2} i \dot{\Gamma}_s.$$

Now at a fixed point on S^4 let s_1 be a unit spinor in the fibre $\mathcal{S}_-^+$. Then $s_2 = \Gamma s_1 = i\, s_1$ is another unit spinor in $\mathcal{S}_-^+$ orthogonal to s_1. Therefore, the fibres of the negative spinor bundle at a point are given by $t_1 s_1 + t_2 s_2$ where $t_1, t_2 \in \mathbb{R}$.

Thus, the following four vectors form a basis of the tangent space at $t_1 s_1 + t_2 s_2$ of $\mathcal{S}^+_-$:

$$
\begin{aligned}
E_1 &= \bar{e}_1 - \frac{i}{2} t_1 \nabla_{e_1}(\Gamma)(s_1) - \frac{i}{2} t_2 \nabla_{e_1}(\Gamma)(s_2),\\
E_2 &= \bar{e}_2 - \frac{i}{2} t_1 \nabla_{e_2}(\Gamma)(s_1) - \frac{i}{2} t_2 \nabla_{e_2}(\Gamma)(s_2),\\
E_1 &= s_1,\\
F_2 &= s_2 = i s_1,
\end{aligned}
$$

where the bar denotes the horizontal lift and the $\nabla_{e_i}(\Gamma)(s_j)$ are vertical vectors which can be expressed in terms of the second fundamental form of the submanifold as we now describe.

Using the adapted frame (e_1, e_2, ν_1, ν_2), we have at a given point (recall we are always using normal coordinates)

$$
\begin{aligned}
\nabla_{e_k}\Gamma &= (\gamma(\nabla_{e_k} e^1)\gamma(e^2) + \gamma(e^1)\gamma(\nabla_{ek} e^2))\\
&= -A^1_{k\,1}\gamma(\nu^1 \wedge e^2) - A^2_{k\,1}\gamma(\nu^2 \wedge e^2) - A^1_{k\,2}\gamma(e^1 \wedge \nu^1) - A^2_{k\,2}\gamma(e^1 \wedge \nu^2),
\end{aligned}
$$

where we have used the notation $A^l_{k\,j} = \langle \nabla_{e_k} e_j, \nu_l \rangle$. Note that the operators $\gamma(e^j \wedge \nu^l)$ all anti-commute with $\Gamma = \gamma(e^1 \wedge e^2)$ as expected and hence they permute the two subbundles $\mathcal{S}^\pm_-$. Let $\check{f}^1$ be the 1-form dual to the vertical tangent vector $\check{f}_1$ which corresponds to the spinor s_1. Then one can check easily that $\check{f}^2$, $\check{f}^3$, $\check{f}^4$ correspond to the spinors

$$
s_2 = \frac{\omega_1}{2} \cdot s_1, \qquad s_3 = \frac{\omega_2}{2} \cdot s_1, \qquad s_4 = \frac{\omega_3}{2} \cdot s_1,
$$

respectively. It can also be checked that

$$
\gamma(e^1)\gamma(\nu^1) = \gamma(e^2)\gamma(\nu^2) \quad \text{and} \quad \gamma(e^1)\gamma(\nu^2) = -\gamma(e^2)\gamma(\nu^1),
$$

since we are on the negative spinor bundle so Clifford multiplication by $-\gamma(e^1 e^2 \nu^1 \nu^2)$ is equal to -1. Using all these facts, the tangent vectors can be expressed as

$$
\begin{aligned}
E_1 &= \bar{e}_1 + \frac{t_1}{2}\big(\big(-A^1_{11} - A^2_{12}\big)\check{f}_3 + \big(-A^2_{11} + A^1_{12}\big)\check{f}_4\big) +\\
&\quad + \frac{t_2}{2}\big(\big(A^2_{11} - A^1_{12}\big)\check{f}_3 + \big(-A^1_{11} - A^2_{12}\big)\check{f}_4\big),\\
E_2 &= \bar{e}_2 + \frac{t_1}{2}\big(\big(-A^1_{12} - A^2_{22}\big)\check{f}_3 + \big(-A^2_{12} + A^1_{22}\big)\check{f}_4\big) +\\
&\quad + \frac{t_2}{2}\big(\big(A^2_{12} - A^1_{22}\big)\check{f}_3\big(-A^1_{12} - A^2_{22}\big)\check{f}_4\big),\\
F_1 &= \check{f}_1,\\
F_2 &= \check{f}_2.
\end{aligned}
$$

In order to check that the space spanned by E_1, E_2, F_1, F_2 is Cayley, we need to check the vanishing of the $\wedge^2_7$ form η from Proposition 2.5 using the explicit form of Φ in (23). Recall that from (22) we have that $e_k^{-b} = u^2\bar{e}^k$ and $\check{f}_k^{\flat} = v^2\check{f}^k$. Then (again omitting the wedge product symbols), one can tediously compute that

$$\begin{aligned}\eta = {} & 2u^2v^2\big(t_1\big(A^1_{11} + A^1_{22}\big) - t_2\big(A^2_{11} + A^2_{22}\big)\big) \times \\ & \times \big(\bar{e}^1\check{f}^3 - \bar{e}^2\check{f}^4 - \bar{e}^3\check{f}^1 + \bar{e}^4\check{f}^2\big) \\ & + 2u^2v^2\big(t_2\big(A^1_{11} + A^1_{22}\big) + t_1\big(A^2_{11} + A^2_{22}\big)\big) \times \\ & \times \big(\bar{e}^1\check{f}^4 + \bar{e}^2\check{f}^3 - \bar{e}^3\check{f}^2 - \bar{e}^4\check{f}^1\big)\end{aligned}$$

which clearly vanishes for all t_1, t_2 if and only if Σ is minimal in S^4. □

An obvious example again in this case is to take an equatorial S^2 sitting inside S^4, which is totally geodesic. Then there exist two different real rank 2 vector bundles over this S^2 which are Cayley with respect to the Bryant–Salamon metric on $\mathcal{S}_-(S^4)$. In fact, by the results of Bryant [3], any genus Riemann surface may be immersed in S^4 as a minimal surface and, hence, we can find Cayley submanifolds of $\mathcal{S}_-$ (S^4) which are rank 2 bundles over any possible compact surface.

5. Local Intersections of Calibrated Submanifolds

In this section, we make some remarks about possible uses of these constructions to study the local intersections of compact calibrated submanifolds in a compact manifold with special holonomy. In [35] McLean studied the local moduli spaces of compact calibrated submanifolds. One of his observations was the following.

THEOREM 5.1 (McLean [35]). *Let X be a compact calibrated submanifold of a manifold M with special holonomy. A small neighbourhood of X in M is naturally isomorphic to a small neighbourhood of the zero section of the normal bundle $N(X)$ of X in M. We also have the following explicit identifications of $N(X)$ for the various cases of calibrations:*

Calibration	*Normal bundle N(X) is isomorphic to*
Special Lagrangian	*Cotangent bundle $T^*(X)$ (intrinsic)*
Coassociative	*Bundle of anti-self-dual 2-forms $\wedge^2_-(X)$ (intrinsic)*
Associative	*Twisted spinor bundle $\mathcal{S}\otimes_{\mathbb{H}} E$ over X (nonintrinsic)*
Cayley	*Twisted negative spinor bundle $\mathcal{S}\otimes_{\mathbb{H}}$ F over X (nonintrinsic)*

where E and F are some explicitly described quaternionic line bundles.

Now in all the explicit noncompact manifolds with complete metrics of special holonomy that we have been discussing in this paper, the base of the bundle (the zero section), is an example of a calibrated submanifold. (In fact, the zero section is always rigid with respect to deformations through calibrated submanifolds by the results of McLean [35].) Explicitly, S^n is special Lagrangian in $T^*(S^n)$ with respect to the Stenzel metric, $\mathbb{CP}^2$ is coassociative in $\wedge^2_-(\mathbb{CP}^2)$ with respect to the Bryant–Salamon metric, and so on. The ambient manifolds in all cases are complete versions of the local neighbourhoods described in Theorem 5.1. This is immediate for the special Lagrangian and coassociative cases. In the case of S^4, McLean shows that the quaternionic line bundle F is trivial in this case so the normal bundle is isomorphic to $\mathcal{S}_-(S^4)$, which is the ambient space of the complete Bryant–Salamon Spin(7) metric. Finally, there is also a complete G_2 metric on $\mathcal{S}(S^3)$ that was discovered by Bryant and Salamon [7]. We do not discuss this metric in the current paper because the calculations are almost identical to the $\mathcal{S}_-(S^4)$ case, but see [21] for some brief remarks on this metric. The zero section S^3 is associative in $\mathcal{S}(S^3)$, and the quaternionic line bundle E mentioned in Theorem 5.1 is again trivial in this case.

Hence, we see that these noncompact manifolds (at least near the zero section) are good local models for a small neighbourhood of a rigid, compact calibrated submanifold. Furthermore, one can check that in all these cases the fibres of the vector bundle total space are also calibrated submanifolds. The fibres are examples of calibrated submanifolds which intersect the base calibrated submanifold in only a point. However, the calibrated submanifolds which we constructed in Sections 3 and 4 were defined as subbundles of the total space restricted to a submanifold of the base. These calibrated submanifolds interesect the base calibrated submanifold in a surface in the exceptional cases, and in submanifolds of many different possible dimensions in the special Lagrangian case.

From the characterizations of calibrated submanifolds in terms of cross-product structures and calibrating forms in Section 2, one can deduce that (nonsingular) calibrated submanifolds can only intersect in submanifolds of certain allowable dimensions. For instance, since an associative 3-plane is closed under the cross product, two associative 3-planes can only intersect in 0, 1, or 3 dimensions. This is because if they intersect in two dimensions spanned by orthogonal vectors e_1 and e_2, the fact that they are both associative means that must also both contain the third direction $e_1 \times e_2$. Now because co-associative 4-planes are orthogonal complements to associative 3-planes, one can use a similar argument to show that two co-associative submanifolds can only intersect in 0, 2, or 4 dimensions. Similarly, since Cayley 4-planes are closed under the triple cross-product X, it is easy to deduce that they too can only intersect in 0, 2, or 4 dimensions. Finally, consider the local model of $\mathbb{R}^n \subset \mathbb{C}^n$ of a special Lagrangian of phase 0 in $\mathbb{C}^n$, with coordinates $z^j = x^j + iy^j$. Then the real n-plane with coordinates $(x^1, \dots, x^p, iy^{p+1}, \dots, iy^n)$ is a $U(n)$ rotation of $\mathbb{R}^n$ with determinant i^{n-p} and, hence, is special Lagrangian in $\mathbb{C}^n$ with phase i^{n-p}, and intersects $\mathbb{R}^n$ in p dimensions. Thus, we have essentially shown the following.

PROPOSITION 5.2. *Let X_1 and X_2 be two nonsingular calibrated submanifolds of a manifold M with special holonomy. Suppose that X_1 and X_2 intersect at some* *point x, and that in a neighbourhood U of x the intersection $X_1 \cap X_2$ is not just the point* x *and not all of $X_1 \cap U$ (and equivalently not all of $X_2 \cap U$). Then we must have*

Calibration	*Intersection of X_1 and X_2 near* x *must be*
Special Lagrangian	*p-dimensional, when phases of X_1, X_2 differ by i^{n-p}*
Coassociative	*a surface (two-dimensional)*
Associative	*a curve (one-dimensional)*
Cayley	*a surface (two-dimensional)*

The constructed calibrated submanifolds in this paper all intersect the base (zero section) calibrated submanifold in precisely the dimensions expected by Proposition 5.2. (Compare Remark 3.3.) Furthermore, our constructions required strong conditions on the intersection with the base, thought of as an isometrically immersed submanifold of the base. Based on this evidence, it is natural to ask the following question.

QUESTION 5.3. Let X_1 and X_2 be two compact calibrated submanifolds of a compact manifold M with special holonomy. Recall that both X_1 and X_2 inherit induced Riemannian metrics g_1 and g_2 from M, respectively. Suppose that X_1 and X_2 intersect at some point x, and that in a neighbourhood U of **x** the intersection $X_1 \cap X_2$ is not just the point x and not all of $X_1 \cap U$ (and equivalently not all of $X_2 \cap U$). Then is it true that we must have the following:

- If X_1 and X_2 are special Lagrangian, with phases differing by i^{n-p}, then the local intersection of X_1 and X_2 near **x** is a p-dimensional submanifold, which is an austere immersion with respect to (X_1, g_1) or (X_2, g_2).
- If X_1 and X_2 are coassociative, then the local intersection of X_1 and X_2 near **x** is a two-dimensional surface, which is a properly oriented isotropic (that is, negative superminimal) immersion with respect to (X_1, g_1) or (X_2, g_2).
- If X_1 and X_2 are associative, then the local intersection of X_1 and X_2 near **x** is a one-dimensional curve, which is a geodesic (minimal) immersion with respect to (X_1, g_1) or (X_2, g_2).
- if X_1 and X_2 are Cayley, then the local intersection of X_1 and X_2 near **x** is a two-dimensional surface, which is a minimal immersion with respect to (X_1, g_1) or (X_2, g_2).

We are currently investigating this question. A related problem is the following. In symplectic geometry, a neighbourhood of a Lagrangian submanifold X in a symplectic manifold M is naturally identified with a neigbourhood of the zero section in $T^*(X)$. It would be useful to have similar neighbourhood theorems in the case of calibrated submanifolds, describing the Ricci-flat metric on the ambient space to a certain order of approximation. Topologically, this was done by McLean [35].

It would also be useful to discover to what extent these bundle constructions of calibrated submanifolds generalize to other explicitly known metrics. There is a wealth of new explicit examples of G_2 and Spin(7) metrics, for example, that have been recently discovered by physicists. (See [9, 10], and the references therein.)

6. Conclusion

Besides the possible applications to the study of intersections of calibrated submanifolds discussed in Section 5, there are several other future directions to explore. It would be interesting to study the possible singularities that can occur in such examples. It should be noted that even when the submanifold over which we build our calibrated subbundle is only immersed in the base, with self-intersections, the resulting calibrated submanifold which we construct is in fact embedded. It is also worth studying how these calibrated submanifolds can be deformed. This would require extending the work of McLean [35] to the case of noncompact calibrated submanifolds. Some study has been made of deformations of noncompact asymptotically conical [25, 33, 34, 36] or asymptotically cylindrical [26] calibrated submanifolds. This, of course, is closely related to the possible nonexistence of other kinds of calibrated submanifolds built as bundles over the same submanifold, discussed at the end of Section 5. It may be that the only way to deform our constructed calibrated submanifolds through calibrated submanifolds would be to deform the base of the subbundle. For example, the moduli space of associative three-folds near a fixed associative submanifold L which is a rank 1 line bundle over a minimal surface Σ as constructed in Section 4 may be just those which arise via the same construction by deforming the minimal surface inside the base, through minimal surfaces. These moduli of course always exist as possible deformations, the only question being whether or not there are any others.

Acknowledgements

The authors would like to thank Marianty Ionel and Nai-Chung Conan Leung for helpful discussions. Spiro Karigiannis was partially supported by NSERC. Maung Min-Oo was partially supported by NSERC.

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