Annals of Global Analysis and Geometry **28:** 107–122 (2005)
DOI: 10.1007/s10455-005-0039-5

On Stability of Cones in $\boldsymbol{R}^{n+1}$ with Zero Scalar Curvature

J. L. M. BARBOSA[1] and M. P. DO CARMO[2]
[1]*Universidade Federal do Ceará (UFC), Departamento de Matemática, Campus do Pici, 60455-760 Fortaleza-Ce, Brazil. e-mail: lucas@mat.ufc.br*
[2]*Instituto Nacional de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, 22460-320 Rio de Janeiro-RJ, Brazil. e-mail: manfredo@impa.br*

(Received: 22 March 2004; accepted: 7 December 2004)

Abstract. In this work we generalize the case of scalar curvature zero the results of Simmons (*Ann. Math.* **88** (1968), 62–105) for minimal cones in R^{n+1}. If M^{n-1} is a compact hypersurface of the sphere $S^n(1)$ we represent by $C(M)_\varepsilon$ the truncated cone based on M with center at the origin. It is easy to see that M has zero scalar curvature if and only if the cone base on M also has zero scalar curvature. Hounie and Leite (*J. Differential Geom.* **41** (1995), 247–258) recently gave the conditions for the ellipticity of the partial differential equation of the scalar curvature. To show that, we have to assume $n \geqslant 4$ and the three-curvature of M to be different from zero. For such cones, we prove that, for $n \leqslant 7$ there is an ε for which the truncate cone $C(M)_\varepsilon$ is not stable. We also show that for $n \geqslant 8$ there exist compact, orientable hypersurfaces M^{n-1} of the sphere with zero scalar curvature and S_3 different from zero, for which all truncated cones based on M are stable.

Mathematics Subject Classifications (2000): 53C42, 53C40, 49F10, 57R70.

Key words: stability, r-curvature, cone, scalar curvature.

1. Introduction

A natural generalization of minimal hypersurfaces in Euclidean spaces was known to Reilly since 1973. Reilly considered the elementary symmetric functions S_r, $r = 0, 1, \dots, n$, of the principal curvatures $k_1, \dots, k_n$ of an orientable hypersurface $x\colon M^n \to R^{n+1}$ given by

$$S_0 = 1, \quad S_r = \sum_{i_1<\cdots<i_r} k_{i_1}\dots k_{i_r}.$$

Here, $k_{i_1}, \dots, k_{i_n}$ are the eigenvalues of $A = -\mathrm{d}g$, where $g\colon M^n \to S^n(1)$ is the Gauss map of the hypersurface. Reilly showed in [8] that orientable hypersurfaces with $S_{r+1} = 0$ are critical points of the functional

$$\mathcal{A}_r = \int_M S_r \mathrm{d}M$$

for variations of M with compact support. Thus, such hypersurfaces generalize the fact that minimal hypersurfaces are critical points of the area functional $A_0 = \int_M S_0 \mathrm{d}M$ for compactly supported variations.

A breakthrough in the study of these hypersurfaces occurred in 1995 when Hounie and Leite [6, 7] found conditions for the linearization of the partial differential equation $S_{r+1} = 0$ to be an elliptic equation. This linearization involves a second order differential operator L_r (see the definition of $L_r 2$ in Section 2) and the Hounie–Leite conditions read as follows:

$$L_r \text{ is elliptic} \iff \operatorname{rank}(A) > r + 1 \iff S_{r+2} \neq 0 \text{ everywhere.}$$

In this paper, we will be interested in the case $S_2 = 0$. For this situation, since rank(A) cannot be two, the ellipticity condition is equivalent to rank $(A) \geqslant 3$.

In Alencar et al. [2], a general notion of stability was introduced for bounded domains of hypersurfaces of Euclidean spaces with $S_{r+1} = 0$. In the case we are interested, namely $S_2 = 0$, it can be shown that if we assume that L_1 is elliptic, an orientation can be chosen so that a bounded domain $D \subset M$ is stable if

$$\left.\frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2}\right|_{t=0} > 0 \quad \text{for all variations with support in (the open set) } D.$$

In what follows, we denote by $B_r(0)$ the ball of radius r centered at the origin 0 of R^{n+1}. Let M^{n-1} be a smooth hypersurface of the sphere $S^n(1)$. A cone $\mathcal{C}(M)$ in R^{n+1} is the union of half-lines starting at 0 and passing through the points of M. It is clear that $\mathcal{C}(M) \cap S^n(1) = M$. It is easy to show that $\mathcal{C}(M) - \{0\}$ is a smooth n-dimensional hypersurface of R^{n+1}. The manifold $\mathcal{C}(M)$ is referred to as the *cone based on* M^{n-1}. The part of the cone contained in the closure of the ring $B_1(0)\backslash B_\varepsilon(0)$, $0 < \varepsilon < 1$, is called a *truncated cone* and is denoted by $\mathcal{C}(M)_\varepsilon$.

In this work we will prove the following two theorems which provide a nice description of the stability of truncated cones in R^{n+1} based on compact, orientable hypersurfaces of $S^n(1)$, with $S_2 = 0$ and $S_3 \neq 0$ everywhere.

THEOREM 1. *Let M^{n-1}, $n \geqslant 4$, be an orientable, compact, hypersurface of $S^n(1)$ with $S_2 = 0$ and $S_3 \neq 0$ everywhere. Then, if $n \leqslant 7$, there exists an $\varepsilon > 0$ so that the truncated cone $\mathcal{C}(M)_\varepsilon$ is not stable.*

THEOREM 2. *For $n \geqslant 8$, there exist compact, orientable hypersurfaces M^{n-1} of the sphere $S^n(1)$, with $S_2 = 0$ and $S_3 \neq 0$ everywhere, so that, for all $\varepsilon > 0$, $\mathcal{C}(M)_\varepsilon$ is stable.*

Although Theorems 1 and 2 are interesting in their own right, a further motivation to prove these theorems is that, for the minimal case, they provide the geometric basis to prove the generalized Bernstein theorem, namely, that a complete minimal graph $y = f(x_1, \ldots, x_{n-1})$ in R^n, $n \leqslant 8$, is a linear function (see Simons [9], Theorems 6.1.1, 6.1.2, 6.2.1, 6.2.2).

For elliptic graphs in R^n with vanishing scalar curvature, the question appears in a natural way. Of course, since we want to consider graphs with $S_2 = 0$ and S_3 that are never zero, we must start with $n \geqslant 4$, and the solution cannot be a hyperplane.

Thus the question is whether there exists an elliptic graph in R^n, $n \geqslant 4$, with vanishing scalar curvature.

So far, the arguments leading to the above quoted Theorems 6.2.1 and 6.2.2 of [9], which depend on geometric measure theory, have not been extended to the case of hypersurfaces of R^n with $S_2 = 0$ and S_3 nowhere zero. To the best of our knowledge, one has not been able to solve a Plateau problem for the above situation, even for the simplest case of $n = 4$.

2. Preliminaries

Given a manifold $\bar{M}$ and an immersion $Y\colon \bar{M}^n \to R^{n+1}$, we represent by A the second fundamental form of Y. The elementary symmetric functions S_r of A are defined by the identity

$$\det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r}$$

and the r-curvatures H_r by

$$H_r = \binom{n}{r}^{-1} S_r.$$

The functions S_r can be considered as homogeneous polynomials of the principal curvatures $k_1, k_2, \ldots, k_n$ given by

$$S_r = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} k_{i_1} k_{i_2} \ldots k_{i_r}.$$

It is well known that the scalar curvature K of the immersion Y is equal to H_2 and its mean curvature is H_1.

We are interested in studying the immersions with $K = 0$, or, in other words, with $H_2 = 0$. Such immersions are critical points to the functional

$$\mathcal{A}_1 = \int_{\bar{M}} H_1 \mathrm{d}\bar{M}$$

with respect to variations of compact support. This variational problem has been studied by Reilly [8], Hounie and Leite [6, 7], Alencar *et al.* [2] and various others. To express its second variation formula, one has to consider the Newton Transformations P_r, that are inductively given by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r\, I - A P_{r-1}, \end{aligned} \tag{1}$$

and then define the differential operator L_1 by

$$L_r f = \operatorname{trace}\{P_r \operatorname{Hess} f\}. \tag{2}$$

It turns out that L_r is self-adjoint and that $L_r f = \operatorname{div}(P_r \operatorname{grad} f)$.

The second variation formula for the mentioned variational problem is, up to a positive constant, given by the functional

$$I(f) = -\int_{\bar{M}} f(L_1 f - 3S_3 f)\,\mathrm{d}\bar{M} \tag{3}$$

for test functions f of compact support in $\bar{M}$. For variational problems involving the integral of S_r, Alencar *et al.* [2] have established a definition of stability. In our case, if we assume that L_1 is elliptic, it turns out that we can choose an orientation so that a bounded domain $D \subset M$ is stable if $I(f) > 0$ for all f supported in D.

Consider now a compact orientable $(n-1)$-dimensional manifold M immersed as a hypersurface of the unit sphere $S^n(1)$ of the Euclidean space R^{n+1}. The cone $\mathcal{C}(M)$ based on M is the immersed hypersurface of R^{n+1} described by

$$\begin{aligned} M \times (0, \infty) &\to R^{n+1} \\ (m, t) &\to tm \end{aligned} \tag{4}$$

Given a positive number ε the truncated cone $\mathcal{C}(M)_\varepsilon$ is the same application restricted to $M \times [\varepsilon, 1]$. The truncated cone is a compact hypersurface with boundary of the Euclidean Space.

Of course, the geometry of $\mathcal{C}(M)$ is closely related to the one of M. Let X describe the immersion of M into the sphere. Then, $Y = tX$ describes parametrically $\mathcal{C}(M)$. In fact, if the metric of M is given by $\mathrm{d}s^2$, the metric of $\mathcal{C}(M)$ is

$$\mathrm{d}\sigma^2 = \mathrm{d}t^2 + t^2\mathrm{d}s^2; \tag{5}$$

and if $N(m)$ is the unit normal vector of M at the point m, then

$$N(m, t) = N(m) \tag{6}$$

is a unit normal vector to $\mathcal{C}(M)$ at the point (m, t). Let $X = e_0, e_1, \ldots, e_{n-1}, e_n = N$ be a local frame field, adapted to X in $S^n(1)$. Let θ_i be the form dual to e_i, $1 \leqslant i \leqslant n-1$. Represent by θ_{ij} the connection forms. Then, the structural equations of the immersion X are

$$\mathrm{d}\theta_i = \sum_{i=1}^{n-1} \theta_{ij} \wedge \theta_j, \tag{7}$$

$$\begin{aligned} \Omega_{ij} &= \mathrm{d}\theta_{ij} - \sum_{k=1}^{n-1} \theta_{ik} \wedge \theta_{kj} \\ &= -\theta_i \wedge \theta_j - \sum h_{ik} h_{jm} \theta_k \wedge \theta_m, \end{aligned} \tag{8}$$

where $A = (h_{ij})$ is the matrix of the second fundamental form of X with respect to the frame field.

Now, translating this frame field along the lines through the origin we may define a frame field in the cone by

$$e_0 = \frac{Y}{|Y|} = X, e_1, e_2, \ldots, e_{n-1}, e_n = N.$$

Let ω_i represent the dual forms and ω_{ij} represent the connection forms for this frame. We have the structural equations

$$\mathrm{d}\omega_i = \sum_{j=0}^{n-1} \omega_{ij} \wedge \omega_j, \tag{9}$$

$$\bar{\Omega}_{ij} = \mathrm{d}\omega_{ij} - \sum_{k=0}^{n-1} \omega_{ik} \wedge \omega_{kj}. \tag{10}$$

Since $\partial Y/\partial t = X$ we have that

$$\mathrm{d}Y = \frac{\partial Y}{\partial t}\mathrm{d}t + t\mathrm{d}X = X\mathrm{d}t + t\sum_{i=1}^{n-1} \theta_i e_i = \sum_{i=0}^{n-1} \omega_i e_i \tag{11}$$

It follows that

$$\omega_0 = \mathrm{d}t, \qquad \omega_i = t\theta_i. \tag{12}$$

From where one deduces

$$0 = d(\mathrm{d}t) = \mathrm{d}\omega_0 = \sum \omega_{0j} \wedge \omega_j$$

and, for $i > 0$,

$$\begin{aligned} \mathrm{d}\omega_i &= \mathrm{d}t \wedge \theta_i + t\sum_{j=1}^{n-1} \theta_{ij} \wedge \theta_j \\ &= -\theta_i \wedge \omega_0 + \sum_{j=1}^{n-1} \theta_{ij} \wedge \theta_j. \end{aligned}$$

It follows that

$$\omega_{i0} = -\theta_i, \tag{13}$$

$$\omega_{ij} = \theta_{ij}. \tag{14}$$

To compute the second fundamental form $\bar{A}$ of $\mathcal{C}(M)$ in terms of the second fundamental form A of M, we proceed as follows. Since

$$\mathrm{d}N = \mathrm{d}e_n = \sum_{j=0}^{n-1} \omega_{nj} e_j = \sum_{j=1}^{n-1} \theta_{nj} e_j,$$

we have

$$\omega_{n0} = 0, \tag{15}$$

$$\omega_{ni} = \theta_{ni} = \sum_{j=1}^{n-1} h_{ij}\theta_j = \frac{1}{t}\sum_{j=1}^{n-1} h_{ij}\omega_j. \tag{16}$$

Hence, if we set $\omega_{ni} = \sum_{j=0}^{n-1} \bar{h}_{ij}\omega_j$, we obtain

$$\begin{aligned} \bar{h}_{0j} &= 0 && \text{for} \quad 0 \leqslant j \leqslant n-1, \\ \bar{h}_{ij} &= \tfrac{1}{t} h_{ij} && \text{for} \quad 1 \leqslant i, j \leqslant n-1. \end{aligned}$$

PROPOSITION 2.1. *If $\bar{S}_r$ represents the elementary symmetric function of order r of $\mathcal{C}(M)$ and $\bar{P}_r$ its Newton transformations, then*

(a) $\bar{S}_r = (1/t^r)S_r$,
(b) $\bar{S}_r = 0$ *if and only if* $S_r = 0$,
(c) $|\bar{A}| = (1/t)|A|$,
(d) $\bar{P}_r = (1/t^r)\left[\begin{array}{c|c} S_r & 0 \\ \hline 0 & P_r \end{array}\right]$.

Proof. The proof is obvious except for the last item. But this can be done using finite induction and the definition of $\bar{P}_r$. For that, we may assume the matrix of A and $\bar{A}$ are diagonalized. Then, for $r = 1$ we have

$$\begin{aligned}
\bar{P}_1 &= \bar{S}_1\bar{I} - \bar{A} = (1/t)S_1\bar{I} - \bar{A} \\
&= (1/t)\begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_1 \end{bmatrix} - (1/t)\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array}\right] \\
&= (1/t)\left[\begin{array}{c|c} S_1 & 0 \\ \hline 0 & S_1 I - A \end{array}\right] = (1/t)\left[\begin{array}{c|c} S_1 & 0 \\ \hline 0 & P_1 \end{array}\right].
\end{aligned}$$

Assume now the result is true for $\bar{P}_r$ and let us prove it for $\bar{P}_{r+1}$.

$$\begin{aligned}
\bar{P}_{r+1} &= \bar{S}_{r+1}\bar{I} - \bar{A}\bar{P}_r = (1/t^{r+1})S_{r+1}\bar{I} - \bar{A}\bar{P}_r \\
&= (1/t^{r+1})\begin{bmatrix} S_{r+1} & & \\ & \ddots & \\ & & S_{r+1} \end{bmatrix} - (1/t^{r+1})\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & AP_r \end{array}\right]
\end{aligned}$$

$$= (1/t^{r+1}) \left[\begin{array}{c|c} S_{r+1} & 0 \\ \hline 0 & S_{r+1}I - AP_r \end{array} \right]$$

$$= (1/t^{r+1}) \left[\begin{array}{c|c} S_{r+1} & 0 \\ \hline 0 & P_{r+1} \end{array} \right]. \tag{17}$$

This concludes the proof. □

Let $F\colon \mathcal{C}(M) \to R$ be a C^2 function. For each $t > 0$ define $\tilde{F}_t\colon M \to R$ by $\tilde{F}_t(m) = F(m,t)$.

PROPOSITION 2.2. *With the above notation we have*

$$\bar{L}_r F = \frac{1}{t^r} S_r \frac{\partial^2 F}{\partial t^2} + \frac{n-r-1}{t^{r+1}} S_r \frac{\partial F}{\partial t} + \frac{1}{t^{r+1}} L_r(\tilde{F}_t).$$

Proof. Since $\bar{L}_1 F = \operatorname{trace}(\bar{P}_1 \operatorname{Hess}(F))$, we start by computing Hess (F). In the following computation we use the frame field and equations deduced earlier and represent by d_M the differential of functions on M. First of all, observe that

$$\mathrm{d}F = \sum_{i=0}^{n-1} F_i \omega_i = \frac{\partial F}{\partial t}\mathrm{d}t + \sum_{i=1}^{n-1} F_i \omega_i,$$

$$\mathrm{d}_M \tilde{F}_t = \sum_{i=1}^{n-1} (\tilde{F}_t)_i \theta_i = \sum_{i=1}^{n-1} F_i \omega_i.$$

It follows that

$$\begin{aligned} tF_i &= (\tilde{F}_t)_i, \quad \text{for } i > 0, \\ F_0 &= \partial F/\partial t. \end{aligned} \tag{18}$$

Now we compute the diagonal of the Hessian of F.

$$\begin{aligned} DF_0 &= \frac{\partial^2 F}{\partial t^2}\mathrm{d}t + \sum \left(\frac{\partial F}{\partial t}\right)_i \omega_i + \sum F_j \omega_{j0} \\ &= \frac{\partial^2 F}{\partial t^2}\mathrm{d}t + \sum \left(\frac{\partial F}{\partial t}\right)_i \omega_i - \frac{1}{t}\sum F_j \omega_j. \end{aligned}$$

Since $DF_0 = \sum F_{0k}\omega_k$, the above equality tells us that

$$F_{00} = \partial^2 F/\partial t^2. \tag{19}$$

For $i > 0$ we have

$$DF_i = \frac{\partial}{\partial t}\left(\frac{1}{t}(\tilde{F}_t)_i\right)\mathrm{d}t + \frac{1}{t}\mathrm{d}_M((\tilde{F}_t)_i) + \frac{1}{t}\sum(\tilde{F}_t)_j\omega_{ji} + \frac{\partial F}{\partial t}\omega_{0i}$$
$$= -\frac{1}{t^2}(\tilde{F}_t)_i\mathrm{d}t + \frac{1}{t}\sum(\tilde{F}_t)_{ij}\theta_j + \frac{\partial F}{\partial t}\theta_i.$$

Since $DF_i = \sum F_{ij}\omega_j$, we conclude that

$$F_{ij} = \frac{1}{t^2}(\tilde{F}_t)_{ij} + \frac{1}{t}\frac{\partial F}{\partial t}\delta_{ij}. \tag{20}$$

Then we have

$$\bar{P}_r\mathrm{Hess}(F) = \left[\begin{array}{c|c} \frac{1}{t^r}S_r & 0 \\ \hline 0 & \frac{1}{t^r}P_r \end{array}\right]\left[\begin{array}{c|c} \frac{\partial^2 F}{\partial t^2} & (*) \\ \hline (*) & \left(\frac{1}{t^2}(\tilde{F}_t)_{ij} + \frac{1}{t}\frac{\partial F}{\partial t}\delta_{ij}\right) \end{array}\right]$$
$$= \left[\begin{array}{c|c} \frac{1}{t^r}S_r\frac{\partial^2 F}{\partial t^2} & 0 \\ \hline 0 & \left(\frac{1}{t^{r+2}}P_r(\tilde{F}_t)_{ij} + \frac{1}{t^{r+1}}P_r\frac{\partial F}{\partial t}\delta_{ij}\right) \end{array}\right].$$

From there we compute

$$\bar{L}_rF = \mathrm{trace}\,(\bar{P}_r\mathrm{Hess}(F))$$
$$= \frac{1}{t^r}S_r\frac{\partial^2 F}{\partial t^2} + \frac{1}{t^{r+1}}\mathrm{trace}\,(P_r)\frac{\partial F}{\partial t} + \frac{1}{t^{r+2}}\mathrm{trace}(P_r\mathrm{Hess}(\tilde{F}_t)).$$

Since the dimension of M is $n-1$, $\mathrm{trace}(P_r) = (n-1-r)S_r$ (see Barbosa and Colares [4]). Using this plus the fact that $\mathrm{trace}(P_r\mathrm{Hess}(\tilde{F}_t)) = L_r(\tilde{F}_t)$ we conclude the proof of the Proposition. □

3. The Main Theorem

In this section we are going to prove the following theorem:

THEOREM 3.1. *Let M^{n-1} be a compact, orientable, immersed hypersurface of $S^n(1)$. Assume that $n \geqslant 4$, that the immersion has S_2 identically zero and that $S_3 \neq 0$ for all points of M. Then, if $n \leqslant 7$, for some ε, $0 < \varepsilon < 1$, the truncated cone $\mathcal{C}(M)_\varepsilon$ is not stable.*

Proof. First of all let us observe that, according to Proposition 2.1, our hypotheses imply that, for the cone $\mathcal{C}(M)$, we have $\bar{S}_2 \equiv 0$ and $\bar{S}_3$ never zero. It was proved by Hounie and Leite [7] that, for a hypersurface of R^{n+1} with $\bar{S}_r \equiv 0$, $2 \leqslant r < n$, the operator $\bar{L}_{r-1}$ is elliptic if and only if $\bar{S}_{r+1}$ is never zero. □

LEMMA 3.2. *Under the same hypothesis of the theorem and choosing properly the normal vector to M, we have that*

(a) *S_1 and $\bar{S}_1$ are positive,*
(b) *L_1 and $\bar{L}_1$ are elliptic.*

Proof. Elementary computation tell us that

$$(S_1)^2 = |A|^2 + 2S_2 = |A|^2 \geqslant 0, \tag{21}$$

where the last equality is a consequence of the hypothesis $S_2 \equiv 0$. Therefore, if there is a point where S_1 is zero, then, at this point, all the entries of the matrix A are zero and, consequently, each S_r is zero at this same point. Since this contradicts our hypothesis that S_3 is never zero, we conclude that $(S_1)^2 > 0$. By properly choosing the normal vector we may assume, from now on, that $S_1 > 0$. Using Proposition 2.1 this implies that $\bar{S}_1 > 0$. (b) follows immediately from Proposition (2.1) and the above result of Hounie and Leite.

To prove the theorem, we are going to show the existence of a truncated cone $\mathcal{C}(M)_\epsilon$ for which the second variation formula attains negative values. Hence, from now on we are going to work on a truncated cone, with test functions f that have a support contained in the interior of the truncated cone. As we did before, for each test function $f\colon \mathcal{C}(M)_\epsilon \to R$ and each fixed t we define $\tilde{f}_t\colon M \to R$ by $\tilde{f}_t(m) = f(m,t)$. From Proposition 2.2 we have that

$$\bar{L}_1 f = \frac{1}{t} S_1 \frac{\partial^2 f}{\partial t^2} + \frac{n-2}{t^2} S_1 \frac{\partial f}{\partial t} + \frac{1}{t^3} L_1(\tilde{f}_t). \tag{22}$$

From (5), the volume element of $\mathcal{C}(M)$ is given by

$$\mathrm{d}\bar{M} = t^{n-1}\mathrm{d}t \wedge \mathrm{d}M. \tag{23}$$

Hence, using (3), (22) and the expression of the volume, the second variation formula on f becomes

$$\begin{aligned} I(f) &= -\int_{M\times[\epsilon,1]} f(\bar{L}_1 f - 3\bar{S}_3 f) t^{n-1}\mathrm{d}t \wedge \mathrm{d}M \\ &= -\int_{M\times[\epsilon,1]} (\tilde{f}_t L_1(\tilde{f}_t) - 3S_3(\tilde{f}_t)^2) t^{n-4}\mathrm{d}t \wedge \mathrm{d}M - \\ &\quad -\int_{M\times[\epsilon,1]} \left(t^2 f \frac{\partial^2 f}{\partial t^2} + (n-2) t f \frac{\partial f}{\partial t} \right) t^{n-4} S_1 \mathrm{d}t \wedge \mathrm{d}M. \end{aligned} \tag{24}$$

Since $S_1 > 0$ according to Lemma 3.2, then $t^{n-4}S_1\mathrm{d}t \wedge \mathrm{d}M$ is a volume element in $\mathcal{C}(M)$, in particular in $\mathcal{C}(M)_\epsilon$. We will represent it by $\mathrm{d}S$. In fact, $\mathrm{d}S$ is a product of two measures. The first one on the real line: $\mathrm{d}\xi = t^{n-4}\mathrm{d}t$; the second, on M, given by $\mathrm{d}\mu = S_1 \mathrm{d}M$. So, $\mathrm{d}S = \mathrm{d}\xi \wedge \mathrm{d}\mu$. We can then rewrite the second variation formula on f as

$$I(f) = -\int_{M\times[\epsilon,1]} \frac{1}{S_1}(\tilde{f}_t L_1(\tilde{f}_t) - 3S_3(\tilde{f}_t)^2)\,\mathrm{d}\xi \wedge \mathrm{d}\mu - \\ - \int_{M\times[\epsilon,1]} \left(t^2 f \frac{\partial^2 f}{\partial t^2} + (n-2)tf\frac{\partial f}{\partial t}\right)\mathrm{d}\xi \wedge \mathrm{d}\mu. \tag{25}$$

Define, now, the following two operators:

$$\begin{aligned} &\mathcal{L}_1 \colon C^\infty(M) \to C^\infty(M) && \text{by} \quad \mathcal{L}_1 f = -(1/S_1)L_1 f + 3(S_3/S_1)f. \\ &\mathcal{L}_2 \colon C^\infty[\epsilon, 1] \to C^\infty[\epsilon, 1] && \text{by} \quad \mathcal{L}_2 g = -t^2 g'' - (n-2)tg'. \end{aligned} \tag{26}$$

Observe that we are considering the space $C^\infty(M)$ with the inner product

$$\langle\langle f_1, f_2\rangle\rangle = \int_M f_1 f_2 \,\mathrm{d}\mu \tag{27}$$

and $C^\infty[\epsilon, 1]$ with the inner product

$$\langle g_1, g_2\rangle = \int_\epsilon^1 g_1 g_2 \mathrm{d}\xi. \tag{28}$$

Since L_1 is elliptic and M is compact then L_1, and so $\mathcal{L}_1$, is strongly elliptic. The same is true for the operator $\mathcal{L}_2$. Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \nearrow \infty$ be the eigenvalues of $\mathcal{L}_1$ and $\delta_1 < \delta_2 < \cdots \nearrow \infty$ be the eigenvalues of $\mathcal{L}_2$. □

LEMMA 3.3. *For any test function f we have*

$$I(f) \geqslant (\lambda_1 + \delta_1) \int_{M\times[\epsilon,1]} f^2 \mathrm{d}\xi \wedge \mathrm{d}\mu.$$

There exists a test function f such that $I(f) < 0$ if and only if $\lambda_1 + \delta_1 < 0$

Proof. Let $\{f_i(m); 1 \leqslant i < \infty\}$ and $\{g_j(t); 1 \leqslant j < \infty\}$ be orthonormal bases of proper functions for $C^\infty(M)$ and $C^\infty[\epsilon, 1]$ respectively, chosen in such way that, for each i and j, f_i corresponds to eigenvalue λ_i and g_j to the eigenvalue δ_j. A test function $f \colon \mathcal{C}(M)_\epsilon \to R$ can now be expressed as

$$f(m, t) = \sum a_{ij} f_i(m) g_j(t). \tag{29}$$

Using (25) we then have

$$\begin{aligned}
I(f) &= \int_{M\times[\epsilon,1]} f(\mathcal{L}_1 f + \mathcal{L}_2 f)\mathrm{d}\xi \wedge \mathrm{d}\mu \\
&= \int_{M\times[\epsilon,1]} \left(\sum a_{ij} g_j \mathcal{L}_1 f_i + \sum a_{ij} f_i \mathcal{L}_2 g_j\right) \sum a_{km} f_k g_m \mathrm{d}\xi \wedge \mathrm{d}\mu \\
&= \sum a_{km} a_{ij} \lambda_i \int_M f_i f_k \mathrm{d}\mu \int_\epsilon^1 g_j g_m \mathrm{d}\xi + \\
&\quad + \sum a_{km} a_{ij} \delta_j \int_M f_i f_k \mathrm{d}\mu \int_\epsilon^1 g_j g_m \mathrm{d}\xi \\
&= \sum a_{ij}^2 \lambda_i + \sum a_{ij}^2 \delta_j \\
&\geqslant (\lambda_1 + \delta_1) \sum a_{ij}^2 = (\lambda_1 + \delta_1) \int_{M\times[\epsilon,1]} f^2 \mathrm{d}\xi \wedge \mathrm{d}\mu. \qquad (30)
\end{aligned}$$

Hence, if $I(f) < 0$ then $\lambda_1 + \delta_1 < 0$. On the other hand, if $\lambda_1 + \delta_1 < 0$ we choose $f(m, t) = f_1(m) g_1(t)$, what gives $I(f) = \lambda_1 + \delta_1 < 0$. This completes the proof of the lemma. □

LEMMA 3.4. *The operator $\mathcal{L}_2$ has eigenvalues*

$$\delta_k = \left(\frac{n-3}{2}\right)^2 + \left(\frac{k\pi}{\log \epsilon}\right)^2, \qquad (31)$$

where $1 \leqslant k < \infty$.

Proof. One search for solutions in the form $g(t) = t^\alpha \sin \varphi(t)$. Then one computes

$$\begin{aligned}
g'(t) &= \alpha t^{\alpha-1} \sin \varphi(t) + t^\alpha \varphi'(t) \cos \varphi(t), \\
g''(t) &= \alpha(\alpha - 1) t^{\alpha-2} \sin \varphi(t) + 2\alpha t^{\alpha-1} \varphi'(t) \cos \varphi(t) + \\
&\quad + t^\alpha \varphi''(t) \cos \varphi(t) - t^\alpha (\varphi'(t))^2 \sin \varphi(t).
\end{aligned}$$

Substitution of this values in the equation $-t^2 g'' - (n-2) t g' = \delta g$ yields

$$\begin{aligned}
&(\alpha(\alpha - 1) t^\alpha - t^{\alpha+2} (\varphi')^2 + (n-2)\alpha t^\alpha + \delta t^\alpha) \sin \varphi + \\
&\quad + (2\alpha t^{\alpha+1} \varphi' + t^{\alpha+2} \varphi'' + (n-2) t^{\alpha+1} \varphi') \cos \varphi \equiv 0.
\end{aligned}$$

Since $\sin \varphi$ and $\cos \varphi$ are linearly independent, each one of the terms in parentheses are zero. It follows that $\varphi'(t) = c/t$, where c is a constant and, consequently,

$$\begin{aligned}
&\alpha(\alpha - 1) - c^2 + \alpha(n-2) + \delta = 0, \\
&2\alpha - 1 + (n-2) = 0.
\end{aligned}$$

It follows that $\alpha = -(n-3)/2$ and $\delta = c^2 + (n-3)^2/4$ and so $g(t) = t^{(n-3)/2} \sin \varphi(t)$, being $\varphi(t) = c \log t$. Since g must be zero in the boundary of $[\epsilon, 1]$, then $c \log \epsilon = k\pi$ for $k = 1, 2, 3, \ldots$. Therefore, the functions $g_k = t^{(n-3)/2} \sin(k\pi \log t / \log \epsilon)$ are eigenfunctions corresponding to the eigenvalues $\delta_k = ((n-3)/2)^2 + (k\pi/\log \epsilon)^2$. It is now a simple matter to verify that the functions g_k and g_m, $k \neq m$, are orthogonal with respect to the inner product defined in (28). This proves the lemma. □

LEMMA 3.5. *Let M^{n-1} be a compact, orientable, immersed hypersurface of $S^n(1)$ with $S_2 \equiv 0$ e S_3 never zero. Suppose that $n \geqslant 4$. The first eigenvalue of the operator $\mathcal{L}_1$ in M satisfy $\lambda_1 \leqslant -(n-2)$.*

Proof. According to recent work of Alencar *et al.* [1] (Lemmas (3.7) and (4.1))(see also [3]), we have

$$L_1 S_1 = |\nabla A|^2 - |\nabla S_1|^2 + (n-1)|A|^2 - S_1^2 + 3S_1 S_3 \tag{32}$$

and

$$1\nabla A|^2 \geqslant |\nabla S_1|^2. \tag{33}$$

Using this and (21), we deduce that

$$\mathcal{L}_1 S_1 \leqslant -(n-2)S_1. \tag{34}$$

Hence

$$\int_M S_1 \mathcal{L}_1 S_1 d\mu \leqslant -(n-2) \int_M S_1^2 \mathrm{d}\mu. \tag{35}$$

But

$$\lambda_1 = \min_f \frac{\int_M S_1 \mathcal{L}_1 S_1 \mathrm{d}\mu}{\int_M S_1^2 \mathrm{d}\mu} \leqslant -(n-2). \tag{36}$$

This concludes the proof of the lemma. □

LEMMA 3.6. *Let M^{n-1} be a compact, orientable, immersed hypersurface of $S^n(1)$ with $S_2 \equiv 0$, S_3 never zero and $n \geqslant 4$.*

If $n \leqslant 7$ then there exists $\epsilon > 0$ such that the truncated cone $\mathcal{C}M_\epsilon$ is not stable.

We observe that the lemma completes the proof of the theorem.

Proof of the lemma. From Lemmas 3.4 and 3.5 we have

$$\lambda_1 + \delta_1 \leqslant -(n-2) + \left(\frac{n-3}{2}\right)^2 + \left(\frac{\pi}{\log \epsilon}\right)^2.$$

It is trivial to verify that the sum of the first two terms of the right-hand side of this inequality is a quadratic polynomial, with positive second-order term, whose roots are approximately 2.2 and 7.8. Hence, it is strictly negative for values of $n \in \{4, 5, 6, 7\}$, in fact, it is less than or equal to -1. Hence,

$$\lambda_1 + \delta_1 \leq -1 + \left(\frac{\pi}{\log \epsilon}\right)^2$$

Choosing ϵ sufficiently small we can guarantee that the right-hand side is negative. Now, by Lemma 3.3 we see that $\mathcal{C}M_\epsilon$ is not stable. This proves the lemma and completes the proof of the theorem. □

4. Existence of Stable Cones

In this section we will prove the following theorem:

THEOREM 4.1. *If $n \geqslant 8$ there exist compact, orientable hypersurfaces of $S^n(1)$ with $S_2 \equiv 0$ and S_3 never zero whose cone $\mathcal{C}(M)_\varepsilon$ for all ε, $0 < \varepsilon < 1$, is stable as a hypersurface of R^{n+1}.*

Proof. The following example has been considered in [3] for another purpose. Consider $R^{p+2} = R^{r+1} \oplus R^{s+1}$, $r + s = p$. Write down the vectors of R^{p+2} as $\xi_1 + \xi_2$, with $\xi_1 \in R^{r+1}$ and $\xi_2 \in R^{s+1}$. If ξ_1 describes $S^r(1) \subset R^{r+1}$ and ξ_2 describes $S^s(1) \subset R^{s+1}$ and if $a_1^2 + a_2^2 = 1$, where a_1 and a_2 are positive numbers, then

$$X = a_1\xi_1 + a_2\xi_2 \tag{37}$$

describes a submanifold M of dimension $p = r + s$ of the sphere $S^{p+1}(1)$ of R^{p+2}. The manifold M is diffeomorphic to $S^r(1) \times S^s(1)$ and so it is compact and orientable. It is clearly embedded as a hypersurface of the unit sphere of R^{p+2}. We are going to show that it is possible to choose values for a_1 and a_2 so that $\mathcal{C}(M)$ is stable as a hypersurface of R^{n+1} when $r + s + 1 = 8$. □

A normal vector field for M is given by

$$N = -a_2\xi_1 + a_1\xi_2 \tag{38}$$

Then we have

$$\begin{aligned} \mathrm{d}X &= a_1\mathrm{d}\xi_1 + a_2\mathrm{d}\xi_2, \\ \mathrm{d}N &= -a_2\mathrm{d}\xi_1 + a_1\mathrm{d}\xi_2. \end{aligned}$$

Hence, if $\mathrm{d}\sigma_1^2$ is the metric of $S^r(1)$ and $\mathrm{d}\sigma_2^2$ is the metric of $S^s(1)$, the first fundamental form of M is

$$\mathrm{d}s^2 = a_1^2\mathrm{d}\sigma_1^2 + a_2^2\mathrm{d}\sigma_2^2 \tag{39}$$

and its second fundamental form is

$$II = a_1a_2\mathrm{d}\sigma_1^2 - a_1a_2\mathrm{d}\sigma_2^2 \tag{40}$$

whose matrix is

$$A = \left(\begin{array}{c|c} (a_2/a_1)I_r & 0 \\ \hline 0 & -(a_1/a_2)I_s \end{array} \right),$$

where I_r and I_s are the identity matrices of R^r and R^s respectively. Since A is already diagonalized, we conclude that its eigenvalues are

$$\underbrace{\frac{a_2}{a_1}, \dots, \frac{a_2}{a_1}}_{r}, \underbrace{-\frac{a_1}{a_2}, \dots, -\frac{a_1}{a_2}}_{s}. \tag{41}$$

Since this eigenvalues are constant, then its r-mean curvature are constant for any value of r. It is clear that

$$S_1 = r(a_2/a_1) - s(a_1/a_2)$$

and

$$|A|^2 = r(a_2/a_1)^2 + s(a_1/a_2)^2.$$

Thus, using (21), we obtain

$$2S_2 = |A|^2 - S_1^2 = r(r-1)\left(\frac{a_2}{a_1}\right)^2 - 2rs + s(s-1)\left(\frac{a_1}{a_2}\right)^2. \tag{42}$$

Hence, $S_2 \equiv 0$ if and only if

$$r(r-1)a_2^4 - 2rsa_1^2a_2^2 + s(s-1)a_1^4 \equiv 0. \tag{43}$$

To find values of a_1 and a_2 that solve this equation one may transform it into a quadratic equation in $(a_2/a_1)^2$ by simply dividing it by a_1^4. Solving this equation and discarding the negative root, we obtain

$$\left(\frac{a_2}{a_1}\right)^2 = \frac{rs + \sqrt{rs(p-1)}}{r(r-1)}. \tag{44}$$

Since $a_1^2 + a_2^2 = 1$, we may solve this to obtain

$$a_1^2 = r(r-1)/(r(p-1) + \sqrt{rs(p-1)}),$$
$$a_2^2 = (rs + \sqrt{rs(p-1)})/(r(p-1) + \sqrt{rs(p-1)}).$$

When $r + s = p = 7$, which corresponds to $n = 8$, we have five distinct solutions which correspond to the pairs $(r, s) \in \{(2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$. For all of them we have $S_2 \equiv 0$ and $S_1 > 0$, in fact equal to $(a_1/a_2)[s + \sqrt{rs(p-1)}]/(p-1)$.

LEMMA 4.2. *For each one of these cones* $S_3 = -(p-1)S_1/3$.

Proof. It is well known that $\text{trace}(A^2 P_1) = S_1 S_2 - 3S_3$. Since $S_2 = 0$ then we have

$$S_3 = -(1/3)\text{trace}(A^2 P_1) = -(1/3)\text{trace}\{A^2(S_1 I - A)\}. \tag{45}$$

Since A is diagonal and the entries of its diagonal are given in (41), we obtain

$$\begin{aligned}
&\text{trace}\{A^2(S_1 I - A)\}\\
&= r\left(\frac{a_2}{a_1}\right)^2\left((r-1)\frac{a_2}{a_1} - s\frac{a_1}{a_2}\right) + s\left(\frac{a_1}{a^2}\right)^2\left(r\frac{a_2}{a_1} - (s-1)\frac{a_1}{a_2}\right)\\
&= r(r-1)\left(\frac{a_2}{a_1}\right)^3 - rs\frac{a_2}{a_1} + sr\frac{a_1}{a_2} - s(s-1)\left(\frac{a_1}{a_2}\right)^3\\
&= \frac{a_1}{a_2}\left\{r(r-1)\left(\frac{a_2}{a_1}\right)^4 - rs\left(\frac{a_2}{a_1}\right)^2\right\} - \frac{a_2}{a_1}\left\{s(s-1)\left(\frac{a_1}{a_2}\right)^4 - sr\left(\frac{a_1}{a_2}\right)^2\right\}.
\end{aligned}$$

Using (43) to substitute the terms inside braces, we obtain

$$\begin{aligned}
&\text{trace}\{A^2(S_1 I - A)\}\\
&= \left(-s(s-1) + rs\left(\frac{a_2}{a_1}\right)^2\right)\frac{a_1}{a_2} - \left(-r(r-1) + rs\left(\frac{a_1}{a_2}\right)^2\right)\frac{a_2}{a_1}\\
&= -s(s-1)\frac{a_1}{a_2} + rs\frac{a_2}{a_1} + r(r-1)\frac{a_2}{a_1} - rs\frac{a_1}{a_2}\\
&= \frac{a_2}{a_1}r(s+r-1) - \frac{a_1}{a_2}s(r+s-1)\\
&= (p-1)S_1
\end{aligned}$$

This proves the lemma. □

A corollary of this lemma is that, for the surfaces we have been studying, S_3 is zero if and only if S_1 is zero. Since we already know that $S_1 > 0$, then we conclude that S_3 is never zero.

Now observe that the L_1 operator in M is given by

$$\begin{aligned}
L_1 f &= \sum_{i=1}^{p}(S_1 - k_i) f_{ii}\\
&= \left[(r-1)\frac{a_2}{a_1} - s\frac{a_1}{a_2}\right]\sum_{i=1}^{r} f_{ii} + \left[r\frac{a_2}{a_1} - (s-1)\frac{a_1}{a_2}\right]\sum_{i=r+1}^{p} f_{ii}\\
&= \left[(r-1)\frac{a_2}{a_1} - s\frac{a_1}{a_2}\right]\triangle^r f + \left[r\frac{a_2}{a_1} - (s-1)\frac{a_1}{a_2}\right]\triangle^s f,
\end{aligned}$$

where $\triangle^r$ and $\triangle^s$ represent the Laplace operator in the Euclidean spheres $S^r(a_1)$ and $S^s(a_2)$ respectively. Since the metric on M is that of the product of these two spheres and the first nonzero eigenvalue of the Laplace operator on a sphere $S^k(b)$ is known to be k/b^2, then the first nonzero eigenvalue of L_1 will be

$$\tilde{\lambda}_1 = \min\left\{\left[(r-1)\frac{a_2}{a_1} - s\frac{a_1}{a_2}\right]\frac{r}{a_1^2}, \left[r\frac{a_2}{a_1} - (s-1)\frac{a_1}{a_2}\right]\frac{s}{a_2^2}\right\}. \tag{46}$$

It then follows that, for the operator

$$\mathcal{L}_1 = -\frac{1}{S_1}L_1 + 3\frac{S_3}{S_1}$$

the first eigenvalue is going to correspond to the constant functions, for which the corresponding eigenvalue is simply

$$\lambda_1 = 3S_3/S_1 = -(p-1), \tag{47}$$

where the last equality comes from Lemma 4.2.

For our manifold M we have been able to effectively compute the value of λ_1. The value of δ_1 was already computed in Lemma 3.4. Observe that, in our case $n = p + 1$. So, using Lemma 3.4 and (47) we obtain

$$\lambda_1 + \delta_1 = -(n-2) + \left(\frac{n-3}{2}\right)^2 + \left(\frac{\pi}{\log\epsilon}\right)^2 \tag{48}$$

Taking $n = 8$, the sum of the first two terms on the right-hand side becomes $1/4$. Hence, we have $\lambda_1 + \delta_1 > 0$ for any choice of ϵ. This completes the proof of the theorem.

Acknowledgement

Work partially supported by PRONEX, and CNPq.

References

1. Alencar, H., do Carmo, M. P. and Colares, A. G.: Stable hypersurfaces with constant scalar curvature, *Math Z.* **213** (1993), 117–131.
2. Alencar, H., do Carmo, M. P. and Elbert, M. F.: Stability of hypersurfaces with vanishing r-mean curvatures in euclidean spaces, *J. Reine Angew. Math.* **554** (2003), 201–216.
3. Alencar, H., do Carmo, M. P. and Santos, W.: A gap theorem for hypersurfaces of the sphere with constant scalar curvature one, *Commun. Math. Helv.* **77** (2002), 549–562.
4. Barbosa, J. L. M. and Colares, A. G.: Stability of hypersurfaces with constant r-mean curvature, *Ann. Global Anal. Geom.* **15** (1997), 277–297.
5. Chern, S. S.: *Minimal Submanifolds in a Riemannian Manifold*, University of Kansas, Lawrence, Kansas, 1968.
6. Hounie, J. and Leite, M. L.: The maximum principle for hypersurfaces with vanishing curvature functions, *J. Differential Geom.* **41** (1995), 247–258.
7. Hounie, J. and Leite, M. L.: Two-ended hypersurfaces with zero scalar curvature, *Indiana Univ. Math. J.* **48** (1999), 867–882.
8. Reilly, R.: Variational properties of functions of the mean curvatures for hypersurfaces in space forms, *J. Differential Geom.* **8** (1973), 465–477.
9. Simmons, J.: Minimal varieties in Riemannian manifolds, *Ann. Math.* **88** (1968), 62–105.