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On relaxed inertial projection and contraction algorithms for solving monotone inclusion problems

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Abstract
We present three novel algorithms based on the forward-backward splitting technique for the solution of monotone inclusion problems in real Hilbert spaces. The proposed algorithms work adaptively in the absence of the Lipschitz constant of the single-valued operator involved thanks to the fact that there is a non-monotonic step size criterion used. The weak and strong convergence and the R-linear convergence of the developed algorithms are investigated under some appropriate assumptions. Finally, our algorithms are put into practice to address the restoration problem in the signal and image fields, and they are compared to some pertinent algorithms in the literature.

Keywords Monotone inclusion · Inclusion problem · Forward-backward method · Projection and contraction method · Convergence rate

Mathematics Subject Classification (2010) 47J20 · 49J40 · 65K15 · 68W10 · 90C33

1 Introduction

In this paper, we aim to present some accelerated algorithms based on the forward-backward technique [1, 2] to solve an inclusion problem in the framework of real Hilbert spaces. The inclusion problem refers to the problem of determining whether one mathematical object, such as a set or a solution to an equation, is entirely contained

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within another object. Recall that the inclusion problem of the sum of two operators is described as follows:

$$\text{find } x^* \in \mathcal{H} \text{ such that } 0 \in (A + B)x^*, \tag{1.1}$$

where $\mathcal{H}$ denotes a real Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and induced norm $\|\cdot\|$, $A : \mathcal{H} \to \mathcal{H}$ is a single-valued operator, and $B : \mathcal{H} \to 2^{\mathcal{H}}$ is a multi-valued operator. The solution set of problem (1.1) is denoted by Ω throughout this paper.

The inclusion problem is a fundamental problem in mathematics, with applications in various areas including set theory, topology, geometry, and numerical analysis. Inclusion problems are closely connected to optimization problems, such as split feasibility problems, variational inequalities, and convex minimization problems (see, e.g., [3, Section 1]). In conclusion, the inclusion problem is a fundamental concept in mathematics with numerous applications in a variety of fields and continues to be an active area of research and development. Many numerical methods have been developed to find solutions of problem (1.1) over the last decade. Our interest in this paper is in the `forward-backward splitting algorithm`, which was proposed by Lions and Mercier [1] and Passty [2]. The method is updated by the following iterative process:

$$s_{n+1} = (I + \chi_n B)^{-1} (I - \chi_n A) s_n, \quad s_0 \in \mathcal{H}, \forall n \geq 0, \tag{1.2}$$

where $(\cdot)^{-1}$ stands for the inverse of $(\cdot)$, and $I : \mathcal{H} \to \mathcal{H}$ represents the identity operator, $\chi_n > 0$ for all $n \geq 0$. It can be seen that each step of iteration (1.2) involves only A as a forward step and B as a backward step. Forward-backward splitting algorithms offer a range of solutions to large-scale optimization projects where structures that favor decomposition can be exploited. Recently, the forward-backward algorithm and its variants have been introduced and further developed for applications in sparse signal recovery [4], image processing [5], and machine learning [6].

In order to weaken the restriction on the weak convergence of iterative scheme (1.2) (i.e., requiring the inverse of A to be strongly monotone), Tseng [7] proposed a new iterative procedure known as the `forward-backward-forward splitting algorithm` (also known as the `Tseng splitting algorithm`) for solving problem (1.1). This solution introduces a new forward step based on iterative (1.2), described as follows:

$$\begin{cases} t_n = (I + \chi_n B)^{-1} (I - \chi_n A) s_n, \\ s_{n+1} = t_n - \chi_n (A t_n - A u_n), \quad s_0 \in \mathcal{H}, \forall n \geq 0. \end{cases} \tag{1.3}$$

Both (1.2) and (1.3) have received a great deal of attention from the optimization community since they were proposed. Many variants have been introduced by scholars for solving optimization tasks such as inclusion problems and variational inequalities (see, e.g., [3, 5, 8–14] and the references therein). We next present some of these results, which inspire us to develop new efficient algorithms. In 2018, Gibali and Thong [3] introduced a Mann-type Tseng algorithm to solve problem (1.1), where A

is Lipschitz continuous and monotone and B is maximally monotone. Their iterative scheme is demonstrated as Algorithm 1.

Algorithm 1.1

Initialization: Give $\chi_1 > 0$ and $\kappa \in (0, 1)$. Let $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (a, b) \subset (0, 1 - \alpha_n)$ for some $a > 0, b > 0$. Let $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $s_0 \in \mathcal{H}$ and set $n := 0$.
Iterative Steps: Given the iterates s_n, s_{n-1}, perform the following steps.
Step 1. Compute $t_n = (I + \chi_n B)^{-1} (I - \chi_n A) s_n$. If $t_n = s_n$ then stop and $t_n \in \Omega$. Otherwise, go to *Step 2*.
Step 2. Compute $s_{n+1} = (1 - \alpha_n - \beta_n)s_n + \beta_n (t_n - \chi_n(At_n - Au_n))$. Update χ_{n+1} by

$$\chi_{n+1} = \begin{cases} \min\left\{\dfrac{\kappa \|u_n - t_n\|}{\|Au_n - At_n\|}, \chi_n\right\} & \text{if } Au_n - At_n \neq 0; \\ \chi_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to *Step 1*.

The advantage of Algorithm 1 is that it can work adaptively without the prior information on the Lipschitz constant of the operator A, and strong convergence is established in real Hilbert spaces by means of the Mann-type method. Subsequently, Gibali et al. [10] combined the inertial method, the projection and contraction method [15], and the forward-backward algorithm and proposed a new iterative procedure (see Algorithm 2 below) to solve monotone inclusion problem (1.1). The weak convergence of Algorithm 2 is confirmed in the case that A satisfies L-Lipschitz continuity and monotonicity and B meets maximal monotonicity.

Algorithm 1.2

Initialization: Give $\delta \in (1, 2)$, $\{\zeta_n\} \subset [0, 1)$, and $\{\chi_n\} \subset [a, b] \subset \left(0, \frac{1}{L}\right)$. Select starting points $s_0, s_1 \in \mathcal{H}$ and set $n := 1$.
Iterative Steps: Given the iterates s_n, s_{n-1}, perform the following steps.
Step 1. Compute $u_n = s_n + \zeta_n \left(s_n - s_{n-1}\right)$.
Step 2. Compute $t_n = (I + \chi_n B)^{-1} (I - \chi_n A) u_n$. If $t_n = u_n$ then stop and $t_n \in \Omega$. Otherwise, go to *Step 3*.
Step 3. Compute $s_{n+1} = u_n - \delta\theta_n g_n$, where

$$g_n := u_n - t_n - \chi_n (Au_n - At_n), \quad \theta_n := \frac{\langle u_n - t_n, g_n\rangle}{\|g_n\|^2}.$$

Set $n := n + 1$ and go to *Step 1*.

It is worth noting that Algorithm 2 also adds an extra forward step making the convergence condition weaker than that of (1.2). The technique incorporating this extra forward step is known as the `projection and contraction method`, introduced by He [15], and this method is now widely used by researchers, also in this paper. On the other hand, Algorithm 1 uses an adaptive non-increasing step size scheme while Algorithm 2 employs a fixed step size approach limited by the Lipschitz constant, and the use of these step sizes will affect the convergence speed of the algorithms. Recently, Thong et al. [11] proposed a modified Tseng algorithm incorporating inertial extrapolation steps and relaxation effects for finding the solutions to problem (1.1).

In their algorithm, a non-monotonic step size scheme is used in order to improve the computational efficiency of Algorithms 1 and 2. More precisely, the iterative solution method of Thong et al. [11] is presented in Algorithm 1.3.

Algorithm 1.3

Initialization: Give $\chi_1 > 0$, $\zeta \in (0,1)$, $\kappa \in (0,1)$, and $\varphi \in [0, \frac{1}{2})$. Let $\{\tau_n\}$ be a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$. Select starting points $s_0, s_1 \in \mathcal{H}$ and set $n := 1$.
Iterative Steps: Given the iterates s_n, s_{n-1}, perform the following steps.
Step 1. Compute $u_n = s_n + \zeta\left(s_n - s_{n-1}\right)$.
Step 2. Compute $t_n = (I + \chi_n B)^{-1}(I - \chi_n A)u_n$. If $t_n = u_n$ then stop and $t_n \in \Omega$. Otherwise, go to *Step 3*.
Step 3. Compute $s_{n+1} = (1-\varphi)s_n + \varphi(t_n - \chi_n(At_n - Au_n))$. Update χ_{n+1} by

$$\chi_{n+1} = \begin{cases} \min\left\{\dfrac{\kappa\|u_n - t_n\|}{\|Au_n - At_n\|}, \chi_n + \tau_n\right\} & \text{if } Au_n - At_n \neq 0; \\ \chi_n + \tau_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to *Step 1*.

The weak convergence and R-linear convergence of Algorithm 3 are established under the condition that the parameters and operators satisfy some suitable conditions.

Inspired and motivated by the results above, we introduce in this paper three new iterative algorithms based on the projection and contraction technique to solve monotone inclusion problem (1.1). Our contributions are listed below.

(i) The inertial method and the relaxation method are utilized in our algorithms to accelerate the convergence speed of the algorithms. In addition, a different relaxation technique is used than in Algorithm 3. Indeed, we compute s_{n+1} in Algorithm 4 using the information of u_n instead of s_n. This modification can also improve the computational speed of the algorithms (see the numerical experiments in Sect. 5).
(ii) To weaken the convergence conditions of our algorithms, we use the projection and contraction technique instead of the Tseng algorithm. Numerical experiments in this paper demonstrate that our algorithms converge faster than Tseng-type algorithms [3, 11].
(iii) A more general non-monotonic step size criterion (cf. (3.3)) is adopted and designed to overcome the difficulty when the Lipschitz constant of the operator is unknown, i.e., our algorithms can work adaptively.
(iv) The weak and strong convergence of the proposed algorithms in the framework of real Hilbert spaces is proved under some mild conditions. Moreover, we establish the R-linear convergence of Algorithms 4 and 5 in the case that operator B satisfies strong monotonicity.
(v) The proposed algorithms are applied to signal processing problems and image denoising problems and demonstrate good computational performance.

The remainder of the paper is organized as follows. In the next section, some basic definitions and lemmas are provided for the purpose of the subsequent convergence analysis. In Sect. 3, we present three forward-backward algorithms with non-monotonic adaptive step sizes and inertial effects to solve the monotone inclusion problem (1.1) and analyze their convergence. In Sect. 4, the R-linear convergence of the suggested Algorithms 4 and 5 is established under the condition that the multi-valued operator B satisfies strong monotonicity. In Sect. 5, we apply the proposed algorithms to the signal recovery problem and image processing problem and compare them with some related ones. Finally, we conclude the paper with a brief summary in Sect. 6.

2 Preliminaries

In this section, we state some basic concepts and lemmas for subsequent use. Let $\mathbb{C}$ be a nonempty, closed, and convex subset of real Hilbert space $\mathcal{H}$. The strong and weak convergence of $\{s_n\}_{n=1}^{\infty}$ converges to x (as $n \to \infty$) is denoted by $s_n \to x$ and $s_n \rightharpoonup x$, respectively. The following two relations will be used several times in the proofs in Sect. 3.

(i) $\|\zeta x + (1-\zeta)y\|^2 = \zeta\|x\|^2 + (1-\zeta)\|y\|^2 - \zeta(1-\zeta)\|x-y\|^2, \quad \forall x, y \in \mathcal{H}, \zeta \in [0, 1]$.
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in \mathcal{H}$.

Definition 2.1 Let $A : \mathcal{H} \to \mathcal{H}$ denote a single-valued operator and $B : \mathcal{H} \to 2^{\mathcal{H}}$ a multi-valued operator.

(i) The operator A is said to be L-`Lipschitz continuous` with $L > 0$ if

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

(ii) The operator A is said to be `monotone` if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}.$$

(iii) The operator B is said to be `monotone` if

$$\langle u - v, x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}, u \in Bx, v \in By.$$

(iv) The operator B is said to be ν-`strongly monotone` if there exists a number $\nu > 0$ such that

$$\langle u - v, x - y\rangle \geq \nu\|x - y\|^2, \quad \forall x, y \in \mathcal{H}, u \in Bx, v \in By.$$

(v) The operator B is said to be $B : H \to 2^H$ is called `maximal monotone`, if it is monotone and if for any $(x, u) \in \mathcal{H} \times \mathcal{H}$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in \text{Graph}(B)$ (the graph of operator B) implies that $u \in Bx$.

Definition 2.2 For all $x \in \mathcal{H}$, there exists a unique nearest point in $\mathbb{C}$, denoted by $P_{\mathbb{C}}(x)$, such that

$$\|x - P_{\mathbb{C}}(x)\| \leq \|x - y\|, \quad \forall y \in \mathbb{C},$$

where $P_{\mathbb{C}}$ is the `metric projection` of $\mathcal{H}$ onto $\mathbb{C}$.

Remark 2.1 Let $\mathbb{C}$ be a nonempty, convex, and closed subset of $\mathcal{H}$. The projection $P_{\mathbb{C}}(x)$ of a point $x \in \mathcal{H}$ onto $\mathbb{C}$ is characterized by (see, e.g., [16, p. 535, Eq. (29.1)])

$$\langle x - P_{\mathbb{C}}(x), y - P_{\mathbb{C}}(x)\rangle \leq 0, \quad \forall x \in \mathcal{H}, y \in \mathbb{C}. \tag{2.1}$$

Let $J_\chi^B : \mathcal{H} \to \mathcal{H}$ denote the resolvent operator of the multi-valued operator $B : \mathcal{H} \to 2^{\mathcal{H}}$, defined as

$$J_\chi^B(x) := (I + \chi B)^{-1}(x), \quad \forall x \in \mathcal{H}, \chi > 0,$$

where I stands for the identity operator on $\mathcal{H}$.

Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone and let $\chi > 0$. Then, $\operatorname{Dom}\left(J_\chi^B\right) = \mathcal{H}$ and $J_\chi^B : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive (see [16, Corollary 23.11]).

Lemma 2.1 ([13]) *Let $A : \mathcal{H} \to \mathcal{H}$ be an operator on $\mathcal{H}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. Define $T_\chi := (I + \chi B)^{-1}(I - \chi A)$, $\chi > 0$. Then, $\operatorname{Fix}\left(T_\chi\right) = (A + B)^{-1}(0)$, where $\operatorname{Fix}\left(T_\chi\right)$ represents the fixed point set of T_χ.*

Lemma 2.2 ([17]) *Let $A : \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous and monotone and $B : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone. Then, $A + B$ is maximal monotone.*

The following Lemmas 2.3–2.5 are used to prove the weak convergence of our Algorithms 4 and 5.

Lemma 2.3 ([18]) *Let $\{\varphi_n\}$, $\{\nu_n\}$, and $\{\chi_n\}$ be nonnegative sequences such that*

$$\varphi_{n+1} \leq \varphi_n + \chi_n (\varphi_n - \varphi_{n-1}) + \nu_n, \quad \forall n \geq 1.$$

If there exists a real number ζ with $0 \leq \chi_n \leq \zeta < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \nu_n < +\infty$, then the following hold:

(i) $\sum_{n=1}^{\infty} \left[\varphi_n - \varphi_{n-1}\right]_+ < +\infty$, where $[t]_+ := \max\{t, 0\}$;
(ii) there exists $\varphi^* \in [0, +\infty)$ such that $\lim_{n\to+\infty} \varphi_n = \varphi^*$.

Lemma 2.4 ([19]) *Let $\{s_n\}$ be a sequence in $\mathcal{H}$ and $\mathbb{C}$ be a nonempty set of $\mathcal{H}$. If the following two conditions hold: (i) for every $x \in \mathbb{C}$, $\lim_{n\to\infty} \|s_n - x\|$ exists; (ii) every sequential weak cluster point of $\{s_n\}$ is in $\mathbb{C}$. Then, $\{s_n\}$ converges weakly to a point in $\mathbb{C}$.*

Lemma 2.5 *Let $\{s_n\}$ be a nonnegative sequence and p be a fixed point. Let $\alpha \in (0, 1)$, $\beta \in [0, +\infty)$, and $\gamma \in (0, +\infty)$. Denote by*

$$\Gamma_n := \|s_n - p\|^2 - \alpha \|s_{n-1} - p\|^2 + \beta \|s_n - s_{n-1}\|^2 .$$

If the following inequality holds

$$\Gamma_{n+1} - \Gamma_n \leq -\gamma \|s_n - s_{n-1}\|^2 , \quad \forall n \geq 1,$$

then $\sum_{n=1}^{\infty} \|s_n - s_{n-1}\|^2 < +\infty$.

Proof Based on the assumptions, it is easy to see that $\Gamma_{n+1} - \Gamma_n \le 0,\ \forall n \ge 1$. Hence, the sequence $\{\Gamma_n\}$ is non-increasing. From the definition of Γ_n (noting that $\beta \ge 0$), one has

$$\Gamma_n \ge \|s_n - p\|^2 - \alpha \|s_{n-1} - p\|^2 .$$

This yields that

$$\begin{aligned} \|s_n - p\|^2 &\le \alpha \|s_{n-1} - p\|^2 + \Gamma_n \\ &\le \alpha \|s_{n-1} - p\|^2 + \Gamma_1 \\ &\le \cdots \le \alpha^n \|s_0 - p\|^2 + \Gamma_1 \left(\alpha^{n-1} + \cdots + \alpha + 1\right) \\ &\le \alpha^n \|s_0 - p\|^2 + \frac{\Gamma_1}{1-\alpha}. \end{aligned} \tag{2.2}$$

By the definition of Γ_{n+1}, we find that

$$\begin{aligned} \Gamma_{n+1} &= \|s_{n+1} - p\|^2 - \alpha \|s_n - p\|^2 + \beta \|s_{n+1} - s_n\|^2 \\ &\ge -\alpha \|s_n - p\|^2 . \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we observe that

$$-\Gamma_{n+1} \le \alpha \|s_n - p\|^2 \le \alpha^{n+1} \|s_0 - p\|^2 + \frac{\alpha \Gamma_1}{1-\alpha}.$$

According to assumption $\Gamma_{n+1} - \Gamma_n \le -\gamma \|s_n - s_{n-1}\|^2$ and noting that $\alpha \in (0, 1)$, one has

$$\begin{aligned} \gamma \sum_{n=1}^{k} \|s_n - s_{n-1}\|^2 \le \Gamma_1 - \Gamma_{k+1} &\le \alpha^{k+1} \|s_0 - p\|^2 + \frac{\Gamma_1}{1-\alpha} \\ &\le \|s_0 - p\|^2 + \frac{\Gamma_1}{1-\alpha}. \end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} \|s_n - s_{n-1}\|^2 < +\infty.$$

This completes the proof of the lemma. □

The Lemma 2.6 listed below is essential for the strong convergence analysis of the proposed Algorithm 6.

Lemma 2.6 ([20]) *Let $\{s_n\}$ be a nonnegative sequence, $\{\sigma_n\}$ be a sequence of real numbers, and $\{\zeta_n\} \subset (0, 1)$ be a sequence such that $\sum_{n=1}^{\infty} \zeta_n = \infty$. Assume that*

$$s_{n+1} \le (1 - \zeta_n) s_n + \zeta_n \sigma_n, \quad \forall n \ge 1.$$

If $\limsup_{k\to\infty} \sigma_{n_k} \le 0$ *for every subsequence* $\{s_{n_k}\}$ *of* $\{s_n\}$ *satisfying* $\liminf_{k\to\infty} \left(s_{n_k+1} - s_{n_k}\right) \ge 0$, *then* $\lim_{n\to\infty} s_n = 0$.

3 Weak and strong convergence

In this section, we provide three modified forward-backward algorithms for solving monotone inclusion problem (1.1). These iterative schemes are inspired by the projection and contraction algorithm, the inertial method, the relaxation method, and the forward-backward algorithm. The first two algorithms yield weak convergence results in real Hilbert spaces, while the third achieves strong convergence results. Now, we assume that Algorithm 4 satisfies the following Conditions (C1)–(C3).

(C1) The solution set of inclusion problem (1.1) is nonempty, i.e., $\Omega := (A+B)^{-1}(0) \neq \emptyset$.

(C2) $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous and monotone, and $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone.

(C3) Let $\chi_1 > 0$, $\kappa \in (0, 1)$, $\zeta \in (0, 1)$, $\varphi \in (0, 1)$, and $\delta \in (0, 2)$. Choose $\{\xi_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(\xi_n - 1) < \infty$, and $\{\tau_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \tau_n < \infty$. Let the parameters φ and ζ satisfy the following condition:

$$\frac{1}{\varphi} - 1 - \varphi\zeta(1+\zeta) > 0. \tag{3.1}$$

Remark 3.1 By solving equation (3.1), we can obtain an upper bound for the parameter φ as

$$\overline{\varphi} = \frac{\sqrt{1+4\zeta(1+\zeta)} - 1}{2\zeta(1+\zeta)}.$$

That is, the range of φ is $\varphi \in (0, \overline{\varphi})$ when the parameter $\zeta \in (0, 1)$ is fixed. The variation between ζ and $\overline{\varphi}$ is demonstrated in Fig. 1.

Now, we are in a position to introduce the suggested Algorithm 4.

The following lemmas are crucial for the convergence analysis of Algorithm 4.

Lemma 3.1 ([22]) *Suppose that Condition (C3) holds. Then, the sequence* $\{\chi_n\}$ *generated by* (3.3) *is well defined and* $\lim_{n\to\infty} \chi_n$ *exists.*

Lemma 3.2 *If* $t_n = u_n$ *or* $g_n = 0$ *in Algorithm 4, then* $t_n \in \Omega$.

Proof From the definition of g_n and (3.3), one has

$$\begin{aligned} \|g_n\| &\ge \|u_n - t_n\| - \chi_n\|Au_n - At_n\| \\ &\ge \left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)\|u_n - t_n\|. \end{aligned} \tag{3.4}$$

It can be easily proved that $\|g_n\| \le \left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)\|u_n - t_n\|$. Hence,

$$\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)\|u_n - t_n\| \le \|g_n\| \le \left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)\|u_n - t_n\|. \tag{3.5}$$

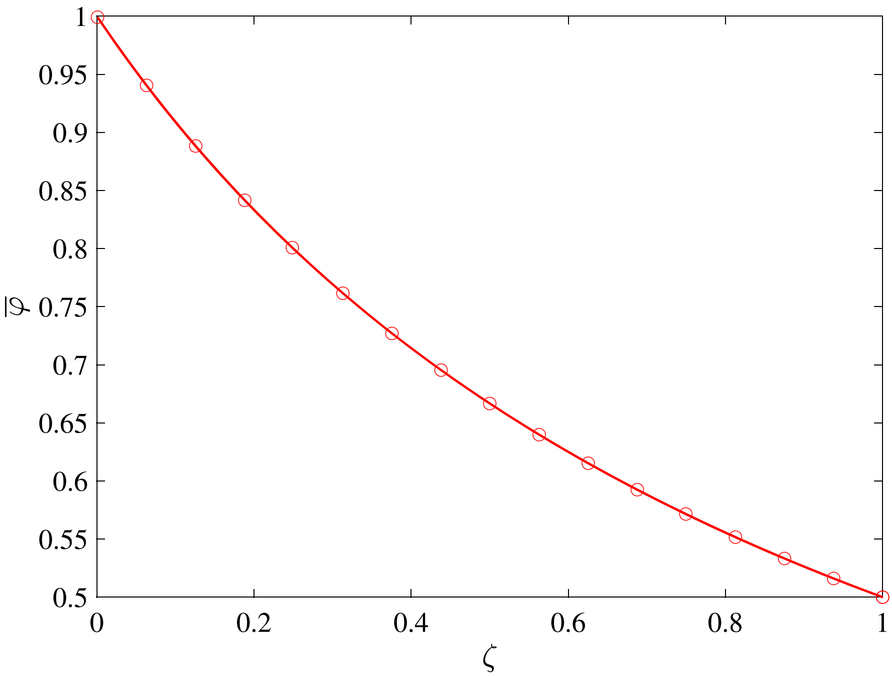


Fig. 1 The relationship between parameters ζ and $\overline{\varphi}$

From (3.5), one can check that $u_n = t_n$ if and only if $g_n = 0$. Therefore, if $u_n = t_n$ or $g_n = 0$, then $t_n \in \Omega$ according to *Step 2* and Lemma 2.1. □

Algorithm 3.1

Initialization: Give $\chi_1 > 0$, $\kappa \in (0, 1)$, $\zeta \in (0, 1)$, $\varphi \in (0, 1)$, and $\delta \in (0, 2)$. Choose φ and ζ satisfy (3.1). Let $\{\xi_n\}$ and $\{\tau_n\}$ meet Condition (C3). Select initial points $s_0, s_1 \in \mathcal{H}$ and set $n := 1$.

Iterative Steps: Given the iterates s_n, s_{n-1}, perform the following steps.

Step 1. Compute $u_n = s_n + \zeta \left(s_n - s_{n-1}\right)$.

Step 2. Compute

$$t_n = (I + \chi_n B)^{-1} (I - \chi_n A) u_n.$$

If $t_n = u_n$ then stop and $t_n \in \Omega$. Otherwise, go to *Step 3*.

Step 3. Compute $q_n = u_n - \delta\theta_n g_n$, where

$$g_n := u_n - t_n - \chi_n (A u_n - A t_n), \quad \theta_n := \frac{\langle u_n - t_n, g_n\rangle}{\|g_n\|^2}. \tag{3.2}$$

Step 4. Compute

$$s_{n+1} = (1 - \varphi)s_n + \varphi q_n.$$

Update

$$\chi_{n+1} = \begin{cases} \min\left\{\dfrac{\kappa \|u_n - t_n\|}{\|A u_n - A t_n\|}, \xi_n \chi_n + \tau_n\right\} & \text{if } A u_n - A t_n \neq 0; \\ \xi_n \chi_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n + 1$ and go to *Step 1*.

Lemma 3.3 ([13]) *Let $\{u_n\}$ and $\{t_n\}$ be two sequences generated by Algorithm 4. If the subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges weakly to some $p \in \mathcal{H}$ and $\lim_{n\to\infty} \|u_n - t_n\| = 0$, then $p \in \Omega$.*

Lemma 3.4 *Let $\{u_n\}$, $\{t_n\}$, and $\{q_n\}$ be three sequences formed by Algorithm 4. Then,*

$$\|q_n - p\|^2 \le \|u_n - p\|^2 - \frac{2-\delta}{\delta} \|q_n - u_n\|^2, \quad \forall p \in \Omega, \tag{3.6}$$

and

$$\|u_n - t_n\|^2 \le \frac{\left(1 + \frac{\kappa \chi_n}{\chi_{n+1}}\right)^2}{\left[(1 - \frac{\kappa \chi_n}{\chi_{n+1}})\delta\right]^2} \|u_n - q_n\|^2. \tag{3.7}$$

Proof It follows from the definition of q_n that

$$\begin{aligned} \|q_n - p\|^2 &= \|u_n - \delta\theta_n g_n - p\|^2 \\ &= \|u_n - p\|^2 - 2\delta\theta_n \langle u_n - p, g_n\rangle + \delta^2\theta_n^2 \|g_n\|^2 . \end{aligned} \tag{3.8}$$

By the definition of g_n, one obtains

$$\begin{aligned} \langle u_n - p, g_n\rangle &= \langle u_n - t_n, g_n\rangle + \langle t_n - p, g_n\rangle \\ &= \langle u_n - t_n, g_n\rangle + \langle t_n - p, u_n - t_n - \chi_n (Au_n - At_n)\rangle . \end{aligned} \tag{3.9}$$

According to the definition of t_n, one has $(I - \chi_n A) u_n \in (I + \chi_n B) t_n$. Since B is maximal monotone, there exists $v_n \in Bt_n$ satisfying $(I - \chi_n A) u_n = t_n + \chi_n v_n$. This implies that

$$v_n = \chi_n^{-1} (u_n - t_n - \chi_n Au_n) . \tag{3.10}$$

We have that $(A + B)$ is maximal monotone by means of Lemma 2.2. From $At_n + v_n \in (A + B)t_n$ and $0 \in (A + B)p$, we obtain

$$\langle At_n + v_n, t_n - p\rangle \ge 0. \tag{3.11}$$

Combining (3.10) and (3.11), we deduce that

$$\langle u_n - t_n - \chi_n (Au_n - At_n), t_n - p\rangle \ge 0. \tag{3.12}$$

By using (3.8), (3.9), (3.12), and the definitions of θ_n and q_n, we have

$$\begin{aligned} \|q_n - p\|^2 &\le \|u_n - p\|^2 - 2\delta\theta_n \langle u_n - t_n, g_n\rangle + \delta^2\theta_n^2 \|g_n\|^2 \\ &= \|u_n - p\|^2 - 2\delta\theta_n^2 \|g_n\|^2 + \delta^2\theta_n^2 \|g_n\|^2 \\ &= \|u_n - p\|^2 - \frac{2-\delta}{\delta} \|\delta\theta_n g_n\|^2 \\ &= \|u_n - p\|^2 - \frac{2-\delta}{\delta} \|q_n - u_n\|^2 . \end{aligned}$$

It follows from (3.3) that

$$\|Au_n - At_n\| \le \frac{\kappa}{\chi_{n+1}}\|u_n - t_n\|, \quad \forall n \ge 1,$$

which combining with the definition of θ_n yields that

$$\begin{aligned}\theta_n \|g_n\|^2 = \langle g_n, u_n - t_n\rangle &\ge \|u_n - t_n\|^2 - \chi_n \|Au_n - At_n\| \|u_n - t_n\| \\ &\ge \left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)\|u_n - t_n\|^2 .\end{aligned} \tag{3.13}$$

Combining (3.5) and (3.13), one sees that

$$\theta_n^2\|g_n\|^2 \ge \left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2 \frac{\|u_n - t_n\|^4}{\|g_n\|^2} \ge \frac{\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}\|u_n - t_n\|^2.$$

This together with the definition of q_n yields that

$$\|q_n - u_n\|^2 = \delta^2\theta_n^2\|g_n\|^2 \ge \delta^2\frac{\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}\|u_n - t_n\|^2.$$

Thus, we obtain

$$\|u_n - t_n\|^2 \le \frac{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}{\left[(1 - \frac{\kappa\chi_n}{\chi_{n+1}})\delta\right]^2}\|u_n - q_n\|^2.$$

The proof is completed. □

Theorem 3.1 *Let $\{s_n\}$ be any sequence generated by Algorithm 4 and Conditions (C1)–(C3) hold. Then, $\{s_n\}$ converges weakly to an $p \in \Omega$.*

Proof It follows from Lemma 3.4 that

$$\|q_n - p\|^2 \le \|u_n - p\|^2, \quad \forall n \ge 1. \tag{3.14}$$

Using (3.14) and the definition of s_{n+1}, we have

$$\begin{aligned}\|s_{n+1} - p\|^2 &= \|(1-\varphi)(s_n - p) + \varphi(q_n - p)\|^2 \\ &= (1-\varphi)\|s_n - p\|^2 + \varphi\|q_n - p\|^2 - \varphi(1-\varphi)\|q_n - s_n\|^2 \\ &\le (1-\varphi)\|s_n - p\|^2 + \varphi\|u_n - p\|^2 - \varphi(1-\varphi)\|q_n - s_n\|^2 .\end{aligned} \tag{3.15}$$

By the definition of s_{n+1}, one sees that $\|q_n - s_n\| = \frac{1}{\varphi}\|(s_{n+1} - s_n)\|$. This together with (3.15) implies that

$$\|s_{n+1} - p\|^2 \le (1-\varphi)\|s_n - p\|^2 + \varphi\|u_n - p\|^2 - \frac{1}{\varphi}(1-\varphi)\|s_{n+1} - s_n\|^2 . \tag{3.16}$$

Using the definition of u_n, one has

$$\begin{aligned}\|u_n - p\|^2 &= \|s_n + \zeta(s_n - s_{n-1}) - p\|^2 \\ &= \|(1+\zeta)(s_n - p) - \zeta(s_{n-1} - p)\|^2 \\ &= (1+\zeta)\|s_n - p\|^2 - \zeta\|s_{n-1} - p\|^2 + \zeta(1+\zeta)\|s_n - s_{n-1}\|^2 .\end{aligned} \tag{3.17}$$

Substituting (3.17) into (3.16), we deduce that

$$\begin{aligned}\|s_{n+1} - p\|^2 \le{}& (1-\varphi)\|s_n - p\|^2 + \varphi(1+\zeta)\|s_n - p\|^2 - \varphi\zeta\|s_{n-1} - p\|^2 \\ &+ \varphi\zeta(1+\zeta)\|s_n - s_{n-1}\|^2 - \frac{1}{\varphi}(1-\varphi)\|s_{n+1} - s_n\|^2 \\ ={}& (1+\varphi\zeta)\|s_n - p\|^2 - \varphi\zeta\|s_{n-1} - p\|^2 + \varphi\zeta(1+\zeta)\|s_n - s_{n-1}\|^2 \\ &- \frac{1}{\varphi}(1-\varphi)\|s_{n+1} - s_n\|^2 .\end{aligned} \tag{3.18}$$

This follows that

$$\begin{aligned}&\|s_{n+1} - p\|^2 - \varphi\zeta\|s_n - p\|^2 + \frac{1}{\varphi}(1-\varphi)\|s_{n+1} - s_n\|^2 \\ &\le \|s_n - p\|^2 - \varphi\zeta\|s_{n-1} - p\|^2 + \frac{1}{\varphi}(1-\varphi)\|s_n - s_{n-1}\|^2 \\ &\quad - \left(\frac{1}{\varphi}(1-\varphi) - \varphi\zeta(1+\zeta)\right)\|s_n - s_{n-1}\|^2 .\end{aligned} \tag{3.19}$$

Denote by

$$\Gamma_n := \|s_n - p\|^2 - \varphi\zeta\|s_{n-1} - p\|^2 + \frac{1}{\varphi}(1-\varphi)\|s_n - s_{n-1}\|^2 .$$

It follows from (3.19) that

$$\Gamma_{n+1} - \Gamma_n \le -\left(\frac{1}{\varphi}(1-\varphi) - \varphi\zeta(1+\zeta)\right)\|s_n - s_{n-1}\|^2 . \tag{3.20}$$

By using (3.1), one sees that $\varphi\zeta \in (0,1)$, $\frac{1}{\varphi}(1-\varphi) > 0$, and $\frac{1}{\varphi}(1-\varphi) - \varphi\zeta(1+\zeta) > 0$. Combining (3.20) and Lemma 2.5, we have

$$\sum_{n=1}^{\infty}\|s_n - s_{n-1}\|^2 < +\infty. \tag{3.21}$$

From (3.18), we have

$$\begin{aligned}\|s_{n+1} - p\|^2 \le{}& \|s_n - p\|^2 + \varphi\zeta\left(\|s_n - p\|^2 - \|s_{n-1} - p\|^2\right) \\ &+ \varphi\zeta(1+\zeta)\|s_n - s_{n-1}\|^2 .\end{aligned} \tag{3.22}$$

Combining (3.21), (3.22), and Lemma 2.3, we arrive at

$$\lim_{n\to\infty} \|s_n - p\|^2 = l. \tag{3.23}$$

This implies that the sequence $\{s_n\}$ is bounded and thus the sequences $\{u_n\}$ and $\{t_n\}$ are also bounded. Moreover, by (3.21), one can see that

$$\lim_{n\to\infty} \|s_{n+1} - s_n\| = 0.$$

Hence, we have $\lim_{n\to\infty} \|q_n - s_n\| = \lim_{n\to\infty} \frac{1}{\varphi} \|s_{n+1} - s_n\| = 0$, and

$$\lim_{n\to\infty} \|s_n - u_n\| = \lim_{n\to\infty} \zeta \|s_n - s_{n-1}\| = 0. \tag{3.24}$$

So $\lim_{n\to\infty} \|q_n - u_n\| = 0$. This together with (3.7) yields that

$$\lim_{n\to\infty} \|u_n - t_n\| = 0. \tag{3.25}$$

Since $\{s_n\}$ is bounded, there exists a subsequence $\left\{s_{n_k}\right\}$ of $\{s_n\}$ and $z \in \mathcal{H}$ such that $s_{n_k} \rightharpoonup z$. It follows from (3.24) that $u_{n_k} \rightharpoonup z$. Combining (3.25) and Lemma 3.3, we obtain $z \in \Omega$. This together with (3.23) and Lemma 2.4 infers that the sequence $\{s_n\}$ converges weakly to $p \in \Omega$. □

Notice that *step 4* in the suggested Algorithm 4 does not use the information of u_n when computing s_{n+1}. Next, we present an improved version of Algorithm 4 in which we use u_n instead of s_n in the relaxation step of the new algorithm. Indeed, our second iterative scheme is shown in Algorithm 5 below.

Algorithm 3.2

Initialization: Given $\chi_1 > 0$, $\kappa \in (0, 1)$, $\zeta \in (0, 1)$, $\varphi \in (0, 1)$, and $\delta \in (0, 2)$. Choose φ, δ, and ζ such that (3.26) holds. Let $\{\xi_n\}$ and $\{\tau_n\}$ meet Condition (C3). Select initial points $s_0, s_1 \in \mathcal{H}$ and set $n := 1$.
Iterative Steps: Given the iterates s_n, s_{n-1}, perform the following steps.
Step 1. Compute

$$u_n = s_n + \zeta \left(s_n - s_{n-1}\right).$$

Step 2. Compute

$$t_n = (I + \chi_n B)^{-1} (I - \chi_n A) u_n.$$

If $t_n = u_n$ then stop and $t_n \in \Omega$. Otherwise, go to *Step 3.*
Step 3. Compute

$$q_n = u_n - \delta\theta_n g_n,$$

where g_n and θ_n are defined in (3.2).
Step 4. Compute

$$s_{n+1} = (1 - \varphi)u_n + \varphi q_n.$$

Update χ_{n+1} by (3.3).
Set $n := n + 1$ and go to *Step 1.*

Before starting the convergence analysis of Algorithm 5, we need the following condition to hold.

(C4) Let $\chi_1 > 0$, $\kappa \in (0,1)$, $\zeta \in (0,1)$, $\varphi \in (0,1)$, and $\delta \in (0,2)$. Choose $\{\xi_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(\xi_n - 1) < \infty$, and $\{\tau_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \tau_n < \infty$. Choose the parameters φ, δ, and ζ such that

$$\varphi \frac{2-\delta}{\delta}(1-\zeta)^2 - (1+\zeta)\zeta > 0. \tag{3.26}$$

Remark 3.2 By solving (3.26), we can easily derive the lower bound of φ as

$$\underline{\varphi} = \frac{\delta\zeta(1+\zeta)}{(2-\delta)(1-\zeta)^2}.$$

That is, when $\delta \in (0,2)$ and $\zeta \in (0,1)$ are fixed, we can choose $\varphi \in (\underline{\varphi}, 1]$. To better describe the relationship between $\underline{\varphi}$, δ, and ζ, we draw the variation of $\underline{\varphi}$ when δ and ζ are fixed in Fig. 2.

We are now in a position to prove the weak convergence of Algorithm 5.

Theorem 3.2 *Assume that $\{s_n\}$ is generated by Algorithm 5 and that Conditions (C1), (C2), and (C4) hold. Then, $\{s_n\}$ converges weakly to an element p in the Ω.*

Proof From the definition of s_{n+1}, one has

$$\begin{aligned} \|s_{n+1} - p\|^2 &= \|(1-\varphi)(u_n - p) + \varphi(q_n - p)\|^2 \\ &\le (1-\varphi)\|u_n - p\|^2 + \varphi\|q_n - p\|^2. \end{aligned} \tag{3.27}$$

Combining (3.6) and (3.27), we have

$$\|s_{n+1} - p\|^2 \le \|u_n - p\|^2 - \varphi\frac{2-\delta}{\delta}\|q_n - u_n\|^2, \quad \forall p \in \Omega. \tag{3.28}$$

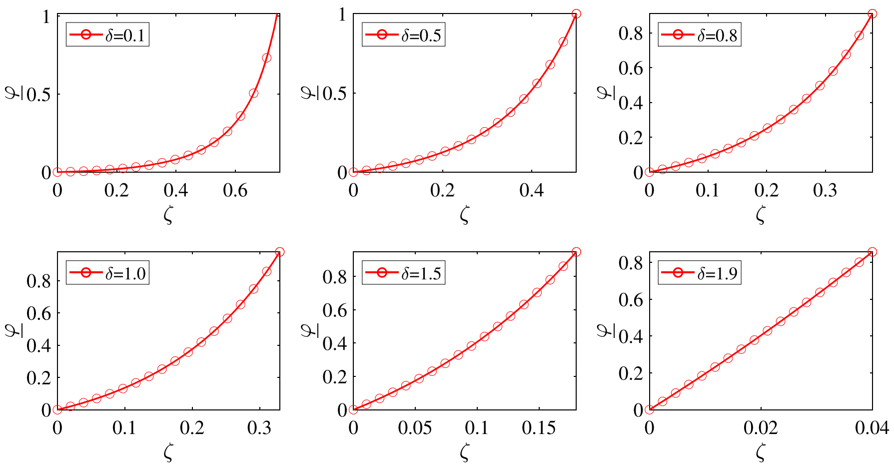


Fig. 2 The relationship between parameters ζ, δ, and $\underline{\varphi}$

By $s_{n+1}=(1-\varphi)u_n+\varphi q_n$ (noting that $\varphi\in(0,1]$), one sees that

$$\|q_n-u_n\|=\frac{1}{\varphi}\|s_{n+1}-u_n\|\geq\|s_{n+1}-u_n\|. \tag{3.29}$$

Substituting (3.29) into (3.28), we obtain

$$\|s_{n+1}-p\|^2\leq\|u_n-p\|^2-\varphi\frac{2-\delta}{\delta}\|s_{n+1}-u_n\|^2. \tag{3.30}$$

It follows from the definition of u_n that

$$\begin{aligned}\|s_{n+1}-u_n\|^2&=\|s_{n+1}-s_n-\zeta(s_n-s_{n-1})\|^2\\&=\|s_{n+1}-s_n\|^2+\zeta^2\|s_n-s_{n-1}\|^2-2\zeta\langle s_{n+1}-s_n,s_n-s_{n-1}\rangle\\&\geq\|s_{n+1}-s_n\|^2+\zeta^2\|s_n-s_{n-1}\|^2-2\zeta\|s_{n+1}-s_n\|\|s_n-s_{n-1}\|\\&\geq(1-\zeta)\|s_{n+1}-s_n\|^2+\left(\zeta^2-\zeta\right)\|s_n-s_{n-1}\|^2.\end{aligned} \tag{3.31}$$

Substituting (3.17) and (3.31) into (3.30), we obtain

$$\begin{aligned}\|s_{n+1}-p\|^2\leq{}&(1+\zeta)\|s_n-p\|^2-\zeta\|s_{n-1}-p\|^2+\zeta(1+\zeta)\|s_n-s_{n-1}\|^2\\&-\varphi\frac{2-\delta}{\delta}\left((1-\zeta)\|s_{n+1}-s_n\|^2+\left(\zeta^2-\zeta\right)\|s_n-s_{n-1}\|^2\right).\end{aligned} \tag{3.32}$$

This implies that

$$\begin{aligned}&\|s_{n+1}-p\|^2-\zeta\|s_n-p\|^2+\left[(1+\zeta)\zeta-\varphi\frac{2-\delta}{\delta}\left(\zeta^2-\zeta\right)\right]\|s_{n+1}-s_n\|^2\\&\leq\|s_n-p\|^2-\zeta\|s_{n-1}-p\|^2+\left[(1+\zeta)\zeta-\varphi\frac{2-\delta}{\delta}\left(\zeta^2-\zeta\right)\right]\|s_n-s_{n-1}\|^2\\&-\left[\varphi\frac{2-\delta}{\delta}(1-\zeta)-(1+\zeta)\zeta+\varphi\frac{2-\delta}{\delta}\left(\zeta^2-\zeta\right)\right]\|s_{n+1}-s_n\|^2.\end{aligned} \tag{3.33}$$

Let

$$\Delta_n:=\|s_n-p\|^2-\zeta\|s_{n-1}-p\|^2+\left[(1+\zeta)\zeta-\varphi\frac{2-\delta}{\delta}\left(\zeta^2-\zeta\right)\right]\|s_n-s_{n-1}\|^2$$

and

$$\epsilon:=\varphi\frac{2-\delta}{\delta}(1-\zeta)-(1+\zeta)\zeta+\varphi\frac{2-\delta}{\delta}\left(\zeta^2-\zeta\right).$$

From (3.33), we reduce to

$$\Delta_{n+1}-\Delta_n\leq-\epsilon\|s_{n+1}-s_n\|^2. \tag{3.34}$$

Since $\zeta \in (0,1)$, $\varphi \in (0,1]$, and $\delta \in (0,1)$, we find that

$$(1+\zeta)\,\zeta - \varphi\frac{2-\delta}{\delta}\left(\zeta^2-\zeta\right) \geq 0.$$

Note that $\epsilon > 0$ by means of equation (3.26). Combining (3.34) and Lemma 2.5, we arrive at $\sum_{n=1}^{\infty}\|s_{n+1}-s_n\|^2 < +\infty$. Thus, we have $\lim_{n\to\infty}\|s_{n+1}-s_n\| = 0$. From the definition of u_n, one has

$$\|s_{n+1}-u_n\|^2 = \|s_{n+1}-s_n\|^2 + \zeta^2\|s_n-s_{n-1}\|^2 - 2\zeta\langle s_{n+1}-s_n, s_n-s_{n-1}\rangle .$$

Therefore, we obtain

$$\lim_{n\to\infty}\|s_{n+1}-u_n\| = 0. \tag{3.35}$$

From (3.32), one can check that

$$\begin{aligned}\|s_{n+1}-p\|^2 \leq \|s_n-p\|^2 + \zeta\left(\|s_n-p\|^2 - \|s_{n-1}-p\|^2\right)\\ + \left(\zeta(1+\zeta) - \varphi\frac{2-\delta}{\delta}\left(\zeta^2-\zeta\right)\right)\|s_n-s_{n-1}\|^2 .\end{aligned} \tag{3.36}$$

Using $\sum_{n=1}^{\infty}\|s_n-s_{n-1}\|^2 < +\infty$, (3.36), and Lemma 2.3, we have $\lim_{n\to\infty}\|s_n-p\|^2 = l$. Combining (3.7), (3.29), and (3.35), we infer that $\lim_{n\to\infty}\|t_n-u_n\| = 0$. By the definition of u_n, one sees that $\lim_{n\to\infty}\|u_n-s_n\| = \lim_{n\to\infty}\zeta\|s_n-s_{n-1}\| = 0$. The sequence $\{s_n\}$ converges weakly to an element of Ω by using a similar statement of Theorem 3.1. □

To conclude this section, we introduce a strongly convergent relaxed forward-backward algorithm for solving monotone inclusion problems in real Hilbert spaces. More precisely, the iterative scheme is shown in Algorithm 6.

Theorem 3.3 *Assume that Conditions (C1), (C2), and (C5) hold and $\{s_n\}$ is generated by Algorithm 6. Then, $\{s_n\}$ converges strongly to an element $p \in \Omega$, where $p = P_{\Omega}(0)$.*

(C5) *Let $\{\epsilon_n\}$ and $\{\beta_n\} \subset (0,1)$ be two positive sequences satisfy the following condition:*

$$\lim_{n\to\infty}\beta_n = 0, \quad \sum_{n=1}^{\infty}\beta_n = \infty, \quad \textit{and} \quad \lim_{n\to\infty}\frac{\epsilon_n}{\beta_n} = 0. \tag{3.38}$$

Proof Combining (3.6) and (3.27), we have

$$\|s_{n+1}-p\|^2 \leq \|u_n-p\|^2 - \varphi\frac{2-\delta}{\delta}\|q_n-u_n\|^2, \quad \forall p \in \Omega. \tag{3.39}$$

Algorithm 3.3

Initialization: Given $\chi_1 > 0$, $\zeta > 0$, $\kappa \in (0,1)$, $\varphi \in (0,1)$, and $\delta \in (0,2)$. Let $\{\epsilon_n\}$ and $\{\beta_n\} \subset (0,1)$ satisfy (3.38). Let $\{\xi_n\}$ and $\{\tau_n\}$ meet Condition (C3). Select initial points $s_0, s_1 \in \mathcal{H}$ and set $n := 1$.
Iterative Steps: Given the iterates s_n, s_{n-1}, perform the following steps.
Step 1. Compute

$$u_n = (1-\beta_n)(s_n + \zeta_n (s_n - s_{n-1})),$$

where

$$\zeta_n = \begin{cases} \min\left\{\zeta, \dfrac{\epsilon_n}{\|s_n - s_{n-1}\|}\right\} & \text{if } s_n \neq s_{n-1}; \\ \zeta & \text{otherwise.} \end{cases} \tag{3.37}$$

Step 2. Compute

$$t_n = (I + \chi_n B)^{-1} (I - \chi_n A) u_n.$$

If $t_n = u_n$ then stop and $t_n \in \Omega$. Otherwise, go to *Step 3*.
Step 3. Compute

$$q_n = u_n - \delta\theta_n g_n,$$

where g_n and θ_n are defined in (3.2).
Step 4. Compute

$$s_{n+1} = (1-\varphi)u_n + \varphi q_n.$$

Update χ_{n+1} by (3.3).
Set $n := n+1$ and go to *Step 1*.

By the definition of u_n, one has

$$\begin{aligned} \|u_n - p\| &= \|(1-\beta_n)(s_n + \zeta_n (s_n - s_{n-1})) - p\| \\ &= \|(1-\beta_n)(s_n - p) + (1-\beta_n)\zeta_n (s_n - s_{n-1}) - \beta_n p\| \\ &\le (1-\beta_n)\|s_n - p\| + (1-\beta_n)\zeta_n \|s_n - s_{n-1}\| + \beta_n \|p\| \\ &= (1-\beta_n)\|s_n - p\| + \beta_n \left[(1-\beta_n)\frac{\zeta_n}{\beta_n}\|s_n - s_{n-1}\| + \|p\|\right]. \end{aligned} \tag{3.40}$$

Using (3.37) and (3.38), one sees that

$$\lim_{n\to\infty} \frac{\zeta_n}{\beta_n}\|s_n - s_{n-1}\| \le \lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0.$$

This follows that $\lim_{n\to\infty}\left((1-\beta_n)\frac{\zeta_n}{\beta_n}\|s_n - s_{n-1}\| + \|p\|\right) = \|p\|$. Hence, there exists a positive constant M such that

$$(1-\beta_n)\frac{\zeta_n}{\beta_n}\|s_n - s_{n-1}\| + \|p\| \le M, \quad \forall n \ge 1. \tag{3.41}$$

It implies form (3.40) and (3.41) that

$$\|u_n - p\| \le (1-\beta_n)\|s_n - p\| + \beta_n M. \tag{3.42}$$

Thanks to $\varphi \in (0,1]$ and $\delta \in (0,2)$, by using (3.39), one obtains

$$\|s_{n+1}-p\| \leq \|u_n-p\|, \quad \forall n \geq 1. \tag{3.43}$$

Combining (3.42) and (3.43), we deduce that

$$\begin{aligned}\|s_{n+1}-p\| &\leq (1-\beta_n)\|s_n-p\|+\beta_n M \\ &\leq \max\{\|s_n-p\|, M\} \leq \cdots \leq \max\{\|s_1-p\|, M\}.\end{aligned}$$

This yields that the sequence $\{s_n\}$ is bounded. So $\{u_n\}$, $\{q_n\}$, and $\{t_n\}$ are also bounded.

From (3.42) (noting that $\{\beta_n\} \subset (0,1)$), one infers that

$$\begin{aligned}\|u_n-p\|^2 &\leq (1-\beta_n)^2\|s_n-p\|^2+2\beta_n(1-\beta_n)M\|s_n-p\|+\beta_n^2M^2 \\ &\leq \|s_n-p\|^2+\beta_n M_1,\end{aligned} \tag{3.44}$$

where $M_1 := \max\left\{2(1-\beta_n)M\|s_n-p\|+\beta_n M^2 : n \in \mathbb{N}\right\}$. Substituting (3.44) into (3.39), we conclude that

$$\varphi\frac{2-\delta}{\delta}\|q_n-u_n\|^2 \leq \|s_n-p\|^2-\|s_{n+1}-p\|^2+\beta_n M_1. \tag{3.45}$$

Using (3.43) and the definition of u_n, we have

$$\begin{aligned}\|s_{n+1}-p\|^2 &\leq \|u_n-p\|^2 \\ &= \|(1-\beta_n)(s_n-p)+(1-\beta_n)\zeta_n(s_n-s_{n-1})-\beta_n p\|^2 \\ &\leq \|(1-\beta_n)(s_n-p)+(1-\beta_n)\zeta_n(s_n-s_{n-1})\|^2+2\beta_n\langle -p, u_n-p\rangle \\ &\leq (1-\beta_n)^2\|s_n-p\|^2+2(1-\beta_n)\zeta_n\|s_n-p\|\|s_n-s_{n-1}\|+\zeta_n^2\|s_n-s_{n-1}\|^2 \\ &\quad +2\beta_n\langle -p, u_n-s_{n+1}\rangle+2\beta_n\langle -p, s_{n+1}-p\rangle.\end{aligned}$$

This yields that

$$\begin{aligned}&\|s_{n+1}-p\|^2 \\ &\leq (1-\beta_n)\|s_n-p\|^2+\beta_n\Big[2(1-\beta_n)\|s_n-p\|\frac{\zeta_n}{\beta_n}\|s_n-s_{n-1}\| \\ &\quad +\zeta_n\|s_n-s_{n-1}\|\frac{\zeta_n}{\beta_n}\|s_n-s_{n-1}\|+2\|p\|\|u_n-s_{n+1}\|+2\langle p, p-s_{n+1}\rangle\Big].\end{aligned} \tag{3.46}$$

Next, we show that $\left\{\|s_n-p\|^2\right\}$ converges to zero. To prove this, we need to rely on Lemma 2.6. We now assume that $\left\{\|s_{n_k}-p\|\right\}$ is a subsequence of $\{\|s_n-p\|\}$ such that

$$\liminf_{k\to\infty}\left(\|s_{n_k+1}-p\|-\|s_{n_k}-p\|\right) \geq 0.$$

Then,

$$\liminf_{k\to\infty}\left(\left\|s_{n_k+1}-p\right\|^2-\left\|s_{n_k}-p\right\|^2\right)$$
$$=\liminf_{k\to\infty}\left[\left(\left\|s_{n_k+1}-p\right\|-\left\|s_{n_k}-p\right\|\right)\left(\left\|s_{n_k+1}-p\right\|+\left\|s_{n_k}-p\right\|\right)\right]\geq 0.$$

This together with (3.38) and (3.45) yields

$$\begin{aligned}\limsup_{k\to\infty}\varphi\frac{2-\delta}{\delta}\left\|q_{n_k}-u_{n_k}\right\|^2 &\leq \limsup_{k\to\infty}\left[\left\|s_{n_k}-p\right\|^2-\left\|s_{n_k+1}-p\right\|^2+\beta_{n_k}M_1\right]\\ &\leq \limsup_{k\to\infty}\left[\left\|s_{n_k}-p\right\|^2-\left\|s_{n_k+1}-p\right\|^2\right]+\limsup_{k\to\infty}\beta_{n_k}M_1\\ &= -\liminf_{k\to\infty}\left[\left\|s_{n_k+1}-p\right\|^2-\left\|s_{n_k}-p\right\|^2\right]\\ &\leq 0,\end{aligned}$$

which means that

$$\lim_{k\to\infty}\left\|q_{n_k}-u_{n_k}\right\|=0.$$

This combining with (3.7) implies that

$$\lim_{k\to\infty}\left\|t_{n_k}-u_{n_k}\right\|=0. \tag{3.47}$$

By the definition of s_{n+1}, one sees that

$$\lim_{k\to\infty}\left\|s_{n_k+1}-u_{n_k}\right\|=\lim_{k\to\infty}\varphi\left\|q_{n_k}-u_{n_k}\right\|=0. \tag{3.48}$$

Using the definition of u_n, one has

$$\begin{aligned}\left\|s_{n_k}-u_{n_k}\right\| &= \left\|\left(1-\beta_{n_k}\right)\zeta_{n_k}\left(s_{n_k}-s_{n_k-1}\right)-\beta_{n_k}s_{n_k}\right\|\\ &\leq \beta_{n_k}\left[\left(1-\beta_{n_k}\right)\frac{\zeta_{n_k}}{\beta_{n_k}}\left\|s_{n_k}-s_{n_k-1}\right\|+\left\|s_{n_k}\right\|\right].\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty}\left\|s_{n_k}-u_{n_k}\right\|=0. \tag{3.49}$$

Combining (3.48) and (3.49), we deduce that

$$\lim_{k\to\infty}\left\|s_{n_k+1}-s_{n_k}\right\|=0. \tag{3.50}$$

Since $\left\{s_{n_k}\right\}$ is bounded, there exists a subsequence $\{s_{n_{k_j}}\}$ of $\left\{s_{n_k}\right\}$ such that $\{s_{n_{k_j}}\}$ converges weakly to z^* as $j\to\infty$. By means of (3.49), one obtains $u_{n_k}\rightharpoonup z^*$. This combining with (3.47) and Lemma 3.3 implies that $z^*\in\Omega$. By using (2.1) and the definition of $p=P_\Omega(0)$, we find that

$$\limsup_{k\to\infty}\left\langle p, p-s_{n_k}\right\rangle=\lim_{j\to\infty}\langle p, p-s_{n_{k_j}}\rangle=\left\langle p, p-z^*\right\rangle\leq 0. \tag{3.51}$$

It follows from (3.50) and (3.51) that

$$\limsup_{k\to\infty}\langle p, p - s_{n_k+1}\rangle \leq 0.$$

This together with (3.46), (3.48), $\lim_{n\to\infty}\frac{\zeta_n}{\beta_n}\|s_n - s_{n-1}\| = 0$, and Lemma 2.6 yields that $\lim_{n\to\infty}\|s_n - p\| = 0$. That is $s_n \to p$ as $n \to \infty$. □

4 *R*-linear convergence

In this section, our goal is to establish the R-linear convergence of Algorithm 4 and Algorithm 5 under the condition that the multi-valued operator $B : \mathcal{H} \to 2^{\mathcal{H}}$ satisfies strong monotonicity. First, we recall the definition of R-linear convergence.

Definition 4.1 ([21]) A sequence $\{s_n\}$ in $\mathcal{H}$ is said to `converge R-linearly` to p with rate $\rho \in [0, 1)$ if there exists a constant $c > 0$ such that $\|s_n - p\| \leq c\rho^n$, $\forall n \in \mathbb{N}$.

We now replace Condition (C2) in Sect. 3 with the following Condition (C2)′.

(C2)′ The operator $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous and monotone and operator $B : \mathcal{H} \to 2^{\mathcal{H}}$ is ν-strongly monotone.

The following Theorem 4.1 demonstrates that Algorithm 4 can obtain R-linear convergence provided that the parameters satisfy certain conditions.

Theorem 4.1 *Let $\{s_n\}$ be any sequence generated by Algorithm 4 and Conditions (C1), (C2)′, and (C3) hold. Let $\varphi \in (0, \frac{1}{2}]$. Choose ζ, δ, and κ such that*

$$\epsilon := \min\left\{\frac{2\delta - \delta^2}{2}\frac{(1-\kappa)^2}{(1+\kappa)^2}, \nu\delta\frac{1-\kappa}{(1+\kappa)^2}\frac{\kappa}{L}\right\} > \frac{\zeta}{1+\zeta}. \tag{4.1}$$

Then, $\{s_n\}$ converges to an element $p \in \Omega$ with an R-linear rate.

Proof It follows from $t_n = (I + \chi_n B)^{-1}(I - \chi_n A)u_n$ that $(I - \chi_n A)u_n \in (I + \chi_n B)t_n$. This implies that

$$\chi_n^{-1}(u_n - t_n - \chi_n A u_n) \in Bt_n. \tag{4.2}$$

Since $0 \in (A + B)p$, one has $-Ap \in Bp$. This combining with (4.2) and the fact that B is strongly monotone with constant ν produces

$$\langle u_n - t_n - \chi_n A u_n + \chi_n A p, t_n - p\rangle \geq \nu\chi_n\|t_n - p\|^2.$$

From the monotonicity of A, we have

$$\begin{aligned}\langle u_n - t_n - \chi_n(Au_n - At_n), t_n - p\rangle &\geq \nu\chi_n\|t_n - p\|^2 + \chi_n\langle At_n - Ap, t_n - p\rangle\\ &\geq \nu\chi_n\|t_n - p\|^2.\end{aligned} \tag{4.3}$$

Using the definition of θ_n, (3.4), and (3.13), one obtains

$$\frac{\langle u_n - t_n, g_n\rangle}{\|g_n\|^2} \le \frac{\|u_n - t_n\|}{\|g_n\|} \le \frac{1}{1 - \frac{\kappa\chi_n}{\chi_{n+1}}}$$

and

$$\frac{\langle u_n - t_n, g_n\rangle}{\|g_n\|^2} \ge \frac{\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)\|u_n - t_n\|^2}{\|g_n\|^2} \ge \frac{\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}. \tag{4.4}$$

Combining (3.7), (3.8), (3.9), (4.3), and (4.4), we find that

$$\begin{aligned}
\|q_n - p\|^2 &\le \|u_n - p\|^2 - \frac{2-\delta}{\delta}\|q_n - u_n\|^2 - 2\nu\delta\theta_n\chi_n\|t_n - p\|^2 \\
&\le \|u_n - p\|^2 - \frac{2-\delta}{\delta}\frac{\left[\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)\delta\right]^2}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}\|u_n - t_n\|^2 \\
&\quad - 2\nu\delta\chi_n\frac{\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}\|t_n - p\|^2 .
\end{aligned}$$

Let $\epsilon := \min\left\{\frac{2\delta - \delta^2}{2}\frac{(1-\kappa)^2}{(1+\kappa)^2}, \nu\delta\frac{1-\kappa}{(1+\kappa)^2}\frac{\kappa}{L}\right\}$. Note that $\epsilon < \frac{1}{2}$. From Lemma 3.1, we have

$$\lim_{n\to\infty}\frac{2-\delta}{\delta}\frac{\left[\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)\delta\right]^2}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2} = (2\delta - \delta^2)\frac{(1-\kappa)^2}{(1+\kappa)^2} \ge 2\epsilon$$

and

$$\lim_{n\to\infty}\nu\delta\chi_n\frac{\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2} \ge \nu\delta\frac{1-\kappa}{(1+\kappa)^2}\min\left\{\chi_1, \frac{\kappa}{L}\right\} \ge \nu\delta\frac{1-\kappa}{(1+\kappa)^2}\frac{\kappa}{L} \ge \epsilon.$$

Therefore, there exists $N \in \mathbb{N}$ such that

$$\frac{2-\delta}{\delta}\frac{\left[\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)\delta\right]^2}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2} \ge 2\epsilon, \quad \nu\delta\chi_n\frac{\left(1 - \frac{\kappa\chi_n}{\chi_{n+1}}\right)}{\left(1 + \frac{\kappa\chi_n}{\chi_{n+1}}\right)^2} \ge 2\epsilon, \quad \forall n \ge N.$$

Thus,

$$\begin{aligned}\|q_n - p\|^2 &\le \|u_n - p\|^2 - 2\epsilon \|u_n - t_n\|^2 - 2\epsilon \|t_n - p\|^2 \\ &\le \Big\| u_n - p|^2 - \epsilon \Big\| u_n - p\|^2 \\ &= (1-\epsilon) \|u_n - p\|^2, \quad \forall n \ge N.\end{aligned}$$

This together with the definition of s_{n+1} yields that

$$\begin{aligned}\|s_{n+1} - p\|^2 &= \|(1-\varphi)(s_n - p) + \varphi (q_n - p)\|^2 \\ &= (1-\varphi) \|s_n - p\|^2 + \varphi \|q_n - p\|^2 - (1-\varphi)\varphi \|s_n - q_n\|^2 \\ &= (1-\varphi) \|s_n - p\|^2 + \varphi \|q_n - p\|^2 - \frac{1-\varphi}{\varphi} \|s_{n+1} - s_n\|^2 \\ &\le (1-\varphi) \|s_n - p\|^2 + \varphi(1-\epsilon) \|u_n - p\|^2 - \frac{1-\varphi}{\varphi} \|s_{n+1} - s_n\|^2, \quad \forall n \ge N.\end{aligned}$$

By the definition of u_n, we also have

$$\begin{aligned}\|u_n - p\|^2 &= \|(1+\zeta)(s_n - p) - \zeta (s_{n-1} - p)\|^2 \\ &= (1+\zeta) \|s_n - p\|^2 - \zeta \|s_{n-1} - p\|^2 + \zeta(1+\zeta) \|s_n - s_{n-1}\|^2.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|s_{n+1} - p\|^2 &\le (1-\varphi) \|s_n - p\|^2 - \frac{1-\varphi}{\varphi} \|s_{n+1} - s_n\|^2 \\ &\quad + \varphi(1-\epsilon) \left[(1+\zeta) \|s_n - p\|^2 - \zeta \|s_{n-1} - p\|^2 + \zeta(1+\zeta) \|s_n - s_{n-1}\|^2\right] \\ &= \underbrace{(1 - \varphi(1 - (1-\epsilon)(1+\zeta)))}_{=\sigma_1} \|s_n - p\|^2 - \varphi\zeta(1-\epsilon) \|s_{n-1} - p\|^2 \\ &\quad + \underbrace{\varphi\zeta(1-\epsilon)(1+\zeta)}_{=\sigma_2} \|s_n - s_{n-1}\|^2 - \frac{1-\varphi}{\varphi} \|s_{n+1} - s_n\|^2 \\ &\le \sigma_1 \|s_n - p\|^2 + \sigma_2 \|s_n - s_{n-1}\|^2 - \frac{1-\varphi}{\varphi} \|s_{n+1} - s_n\|^2, \quad \forall n \ge N.\end{aligned}$$

Thanks to $\varphi \in (0, \frac{1}{2}]$, it implies $\frac{1-\varphi}{\varphi} \ge 1$. Thus, we have

$$\begin{aligned}\|s_{n+1} - p\|^2 + \|s_{n+1} - s_n\|^2 &\le \|s_{n+1} - p\|^2 + \frac{1-\varphi}{\varphi} \|s_{n+1} - s_n\|^2 \\ &\le \sigma_1 \|s_n - p\|^2 + \sigma_2 \|s_n - s_{n-1}\|^2, \quad \forall n \ge N.\end{aligned}$$

This follows that

$$\|s_{n+1} - p\|^2 + \|s_{n+1} - s_n\|^2 \le \sigma_1 \left[\|s_n - p\|^2 + \frac{\sigma_2}{\sigma_1} \|s_n - s_{n-1}\|^2\right], \quad \forall n \ge N.$$

By (4.1), one sees that $(1-\epsilon)(1+\zeta)\in(0,1)$, which further yields that $\sigma_1\in(0,1)$ and $\frac{\sigma_2}{\sigma_1}\in(0,1)$. Hence, we have

$$\|s_{n+1}-p\|^2+\|s_{n+1}-s_n\|^2\le\sigma_1\left(\|s_n-p\|^2+\|s_n-s_{n-1}\|^2\right),\quad\forall n\ge N.$$

Let $a_n:=\|s_n-p\|^2+\|s_n-s_{n-1}\|^2$. It follows that

$$\|s_{n+1}-p\|^2\le a_{n+1}\le\sigma_1 a_n\le\sigma_1^{n-N+1}a_N=\sigma_1^{1-N}a_N\sigma_1^n.$$

This implies that the sequence $\{s_n\}$ converges R-linearly to p. □

Before starting to prove the R-linear convergence of Algorithm 5, we need the following lemma.

Lemma 4.1 *Let $\{s_n\}$ be formed by Algorithm 5 and let $p\in\Omega$. Assume that $\delta\in[1,2)$ and $\varphi\in[\frac{2}{\delta}-1,1]$. Then, for all $\xi\in(0,1)$, there exists $N\in\mathbb{N}$ and $\omega=\omega(\xi)$, $\epsilon=\epsilon(\xi)$ such that $\omega,\epsilon\in(0,1)$ and*

$$\|s_{n+1}-p\|^2\le\omega\,\|u_n-p\|^2-\epsilon\,\|s_{n+1}-u_n\|^2,\quad\forall n\ge N. \tag{4.5}$$

Proof Using (3.27) and the similar proof to that in Theorem 4.1, one has

$$\|s_{n+1}-p\|^2\le\|u_n-p\|^2-\varphi\frac{2-\delta}{\delta}\,\|q_n-u_n\|^2-2\varphi\nu\delta\theta_n\chi_n\,\|t_n-p\|^2,\quad\forall p\in\Omega. \tag{4.6}$$

From the definition of θ_n, s_{n+1}, and q_n, one sees that

$$\|s_{n+1}-u_n\|=\varphi\|q_n-u_n\|=\varphi\delta\theta_n\|g_n\|\le\varphi\delta\|u_n-t_n\|. \tag{4.7}$$

Combining (3.7), (4.6), and (4.7), one finds that

$$\begin{aligned}
&\|s_{n+1}-p\|^2\\
&\le\|u_n-p\|^2-\underbrace{\varphi\frac{2-\delta}{\delta}\frac{\left[\left(1-\frac{\kappa\chi_n}{\chi_{n+1}}\right)\delta\right]^2}{\left(1+\frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}}_{=\sigma_n}\|u_n-t_n\|^2-2\varphi\nu\delta\theta_n\chi_n\,\|t_n-p\|^2\\
&=\|u_n-p\|^2-\sigma_n\xi\,\|t_n-u_n\|^2-\sigma_n(1-\xi)\,\|t_n-u_n\|^2\\
&\quad-2\varphi\nu\delta\theta_n\chi_n\,\|t_n-p\|^2\\
&\le\|u_n-p\|^2-\sigma_n\xi\,\|t_n-u_n\|^2-\frac{\sigma_n(1-\xi)}{\varphi^2\delta^2}\|s_{n+1}-u_n\|^2\\
&\quad-2\varphi\nu\delta\theta_n\chi_n\,\|t_n-p\|^2,\quad\text{where }\xi\in(0,1).
\end{aligned} \tag{4.8}$$

By using (4.4) and (4.8), one obtains

$$
\begin{aligned}
\|s_{n+1}-p\|^2 \leq \|u_n-p\|^2 - \sigma_n\xi\,\|t_n-u_n\|^2 - \frac{\sigma_n(1-\xi)}{\varphi^2\delta^2}\|s_{n+1}-u_n\|^2 \\
- 2\varphi\nu\delta\chi_n \frac{\left(1-\frac{\kappa\chi_n}{\chi_{n+1}}\right)}{\left(1+\frac{\kappa\chi_n}{\chi_{n+1}}\right)^2}\|t_n-p\|^2 .
\end{aligned}
\tag{4.9}
$$

Let $\gamma := \min\left\{\frac{\sigma\xi}{2}, \varphi\nu\delta\chi\frac{1-\kappa}{(1+\kappa)^2}\right\}$, where $\chi := \lim_{n\to\infty}\chi_n$ and $\sigma := \lim_{n\to\infty}\sigma_n$. Note that $\sigma\in(0,1)$ and $\gamma\in(0,1)$. Then, we observe that

$$
\begin{aligned}
&\lim_{n\to\infty}\sigma_n\xi = \sigma\xi \geq 2\gamma,\\
&\lim_{n\to\infty}\frac{\sigma_n(1-\xi)}{\varphi^2\delta^2} = \frac{\sigma(1-\xi)}{\varphi^2\delta^2},\\
&\lim_{n\to\infty}\varphi\nu\delta\chi_n\frac{\left(1-\frac{\kappa\chi_n}{\chi_{n+1}}\right)}{\left(1+\frac{\kappa\chi_n}{\chi_{n+1}}\right)^2} = \varphi\nu\delta\chi\frac{1-\kappa}{(1+\kappa)^2} \geq \gamma.
\end{aligned}
$$

Thus, there exists $N\in\mathbb{N}$ such that

$$
\sigma_n\xi \geq 2\gamma,\quad \varphi\nu\delta\chi_n\frac{\left(1-\frac{\kappa\chi_n}{\chi_{n+1}}\right)}{\left(1+\frac{\kappa\chi_n}{\chi_{n+1}}\right)^2} \geq \gamma,\quad \forall n\geq N.
$$

Let $\omega := 1-\gamma$ and $\epsilon := \frac{\sigma(1-\xi)}{\varphi^2\delta^2}$. Since $\xi\in(0,1)$, $\delta\in[1,2)$, and $\varphi\in[\frac{2}{\delta}-1,1]$, one has $\omega\in(0,1)$ and $\epsilon\in(0,1)$. In view of (4.9), one concludes that

$$
\begin{aligned}
\|s_{n+1}-p\|^2 &\leq \|u_n-p\|^2 - 2\gamma\,\|t_n-u_n\|^2 - \epsilon\,\|s_{n+1}-u_n\|^2 - 2\gamma\,\|t_n-p\|^2\\
&= \|u_n-p\|^2 - \epsilon\,\|s_{n+1}-u_n\|^2 - 2\gamma\left(\|t_n-u_n\|^2+\|t_n-p\|^2\right)\\
&\leq \|u_n-p\|^2 - \epsilon\,\|s_{n+1}-u_n\|^2 - \gamma\,\|u_n-p\|^2\\
&= \omega\,\|u_n-p\|^2 - \epsilon\,\|s_{n+1}-u_n\|^2,\quad \forall n\geq N.
\end{aligned}
$$

This completes the proof. □

Using the technique in [23], we obtain the following result.

Theorem 4.2 *Let $\{s_n\}$ be any sequence generated by Algorithm 5 and Conditions (C1), (C2)′, and (C4) hold. Let $\kappa\in(0,1)$, $\xi\in(0,1)$, $\delta\in[1,2)$, and $\varphi\in[\frac{2}{\delta}-1,1]$ such that*

$$
0\leq\zeta\leq\frac{\omega\epsilon}{\omega\epsilon+2\omega+\epsilon},
\tag{4.10}
$$

where ω and ϵ are defined in Lemma 4.1. *Then, $\{s_n\}$ converges to an element $p \in \Omega$ with an R-linear rate.*

Proof By using the definition of u_n, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|(1+\zeta)(s_n - p) - \zeta(s_{n-1} - p)\|^2 \\ &= (1+\zeta)\|s_n - p\|^2 - \zeta\|s_{n-1} - p\|^2 + \zeta(1+\zeta)\|s_n - s_{n-1}\|^2\end{aligned}$$

and

$$\begin{aligned}\|s_{n+1} - u_n\|^2 &= \|s_{n+1} - s_n - \zeta(s_n - s_{n-1})\|^2 \\ &\geq \|s_{n+1} - s_n\|^2 + \zeta^2\|s_n - s_{n-1}\|^2 - 2\zeta\|s_{n+1} - s_n\|\|s_n - s_{n-1}\| \\ &\geq \|s_{n+1} - s_n\|^2 + \zeta^2\|s_n - s_{n-1}\|^2 - \zeta\|s_{n+1} - s_n\|^2 - \zeta\|s_n - s_{n-1}\|^2 \\ &= (1-\zeta)\|s_{n+1} - s_n\|^2 - \zeta(1-\zeta)\|s_n - s_{n-1}\|^2.\end{aligned}$$

Combining these inequalities with (4.5), we obtain

$$\begin{aligned}\|s_{n+1} - p\|^2 \leq\ & \omega(1+\zeta)\|s_n - p\|^2 - \omega\zeta\|s_{n-1} - p\|^2 + \omega\zeta(1+\zeta)\|s_n - s_{n-1}\|^2 \\ & - \epsilon(1-\zeta)\|s_{n+1} - s_n\|^2 + \epsilon\zeta(1-\zeta)\|s_n - s_{n-1}\|^2, \quad \forall n \geq N.\end{aligned}$$

This is equivalent to

$$\begin{aligned}& \|s_{n+1} - p\|^2 - \omega\zeta\|s_n - p\|^2 + \epsilon(1-\zeta)\|s_{n+1} - s_n\|^2 \\ & \leq \omega\left[\|s_n - p\|^2 - \zeta\|s_{n-1} - p\|^2 + \epsilon(1-\zeta)\|s_n - s_{n-1}\|^2\right] \\ & \quad - (\omega\epsilon(1-\zeta) - \omega\zeta(1+\zeta) - \epsilon\zeta(1-\zeta))\|s_n - s_{n-1}\|^2, \quad \forall n \geq N.\end{aligned}$$

Set

$$\Gamma_n := \|s_n - p\|^2 - \zeta\|s_{n-1} - p\|^2 + \epsilon(1-\zeta)\|s_n - s_{n-1}\|^2.$$

Since $\omega \in (0,1)$, one finds that

$$\begin{aligned}\Gamma_{n+1} &\leq \|s_{n+1} - p\|^2 - \omega\zeta\|s_n - p\|^2 + \epsilon(1-\zeta)\|s_{n+1} - s_n\|^2 \\ &\leq \omega\Gamma_n - (\omega\epsilon(1-\zeta) - \omega\zeta(1+\zeta) - \epsilon\zeta(1-\zeta))\|s_n - s_{n-1}\|^2, \quad \forall n \geq N.\end{aligned}$$

From $\zeta \in (0,1)$ and (4.10), we have

$$\omega\epsilon(1-\zeta) - \omega\zeta(1+\zeta) - \epsilon\zeta(1-\zeta) \geq \omega\epsilon(1-\zeta) - 2\omega\zeta - \epsilon\zeta \geq 0.$$

This implies that

$$\Gamma_{n+1} \leq \omega\Gamma_n, \quad \forall n \geq N.$$

Now, we show that $\Gamma_n > 0$ for all $n \geq N$. From (4.10), we have

$$\zeta \leq \frac{\omega\epsilon}{\omega\epsilon + 2\omega + \epsilon} < \frac{\epsilon}{2+\epsilon},$$

which further yields that

$$\frac{\epsilon\,(1-\zeta)}{2}-\zeta>0.$$

Using the definition of $\varGamma_n$, we have

$$\begin{aligned}\varGamma_n &= (1-\epsilon\,(1-\zeta))\,\|s_n-p\|^2+\epsilon\,(1-\zeta)\left(\|s_n-p\|^2+\|s_n-s_{n-1}\|^2\right)-\zeta\,\|s_{n-1}-p\|^2\\ &\ge (1-\epsilon\,(1-\zeta))\,\|s_n-p\|^2+\left(\frac{\epsilon\,(1-\zeta)}{2}-\zeta\right)\|s_{n-1}-p\|^2\\ &\ge (1-\epsilon\,(1-\zeta))\,\|s_n-p\|^2>0.\end{aligned}$$

Hence,

$$\varGamma_{n+1}\le\omega\varGamma_n\le\cdots\le\omega^{n-N+1}\varGamma_N.$$

That is

$$\|s_n-p\|^2\le\frac{\varGamma_N}{(1-\epsilon\,(1-\zeta))\,\omega^N}\omega^n.$$

This implies that $\{s_n\}$ converges R-linearly to p, as desired. □

Remark 4.1 We have the following comments regarding the R-linear convergence analysis of the proposed algorithms.

(i) We do not include the R-linear convergence results of Algorithm 6 in this section. The key reason behind this limitation lies in the specific characteristics and adjustments made in Algorithm 6 to ensure strong convergence. While these adjustments may enhance the algorithm's ability to converge to the solution, they may also introduce complexities or trade-offs that hinder the attainment of R-linear convergence.

(ii) Our R-linear convergence results were obtained under specific step size choices, which may appear more restrictive. We acknowledge the importance of step size selection in iterative optimization algorithms and assure that our choices were made after thorough analysis to ensure convergence and effectiveness.

(iii) We made stricter assumptions on the parameters to obtain R-linear convergence results for Algorithms 4 and 5. The stronger correlation between parameters emerged from rigorous convergence analysis and imposed tighter constraints to provide robust theoretical guarantees. This was done to strike a balance between computational efficiency and convergence performance.

5 Numerical experiments

In this section, we provide the application of the proposed algorithms to signal and image recovery problems to demonstrate their computational efficiency and advantages. All programs were performed in MATLAB 2018a on a personal computer with RAM 8.00 GB.

Example 5.1 (**Signal Recovery Problem**) The signal recovery problem involves reconstructing an original signal from its degraded or partial form that occurs frequently in image processing, audio processing, and communication systems. Let

$\mathbf{x} \in \mathbb{R}^N$ with k ($k \ll N$, i.e., $\mathbf{x}$ is sparse) non-zero elements be the original signal, $\mathbf{C} : \mathbb{R}^{M\times N}$ be a bounded linear operator, and ϵ be the noise data during transmission. The noise signal $\mathbf{y} \in \mathbb{R}^M$ is assumed to be obtained by the following variation:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \epsilon. \tag{5.1}$$

Figure 3 visually shows the matrix structure expression of the model (5.1). The signal recovery process can be represented as a mapping from the degraded signal to the recovered signal, aiming to reduce the discrepancy between the two signals. This can be formulated mathematically as an optimization problem, where the objective is to minimize the difference between the degraded and recovered signals and the optimization variables are the parameters of the recovery algorithm. That is, we can solve the model (5.1) by formulating the following unconstrained optimization problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - \mathbf{C}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|_1 \le t, \tag{5.2}$$

where t is a positive constant. Note that the problem (5.2) described above can be regarded as a special case of the monotone inclusion problem (1.1) (see [13, Section 4.3] for more details).

We use the proposed Algorithms 4, 5 and 6 as well as the ones in the literature [3, 10, 11] to solve the problem (5.2). The parameters of the proposed algorithms and the compared ones are set as follows.

- Set $\chi_1 = 1$, $\kappa = 0.5$, $\delta = 1.5$, $\xi_n = 1 + \frac{1}{(n+1)^2}$, and $\tau_n = \frac{1}{n+1}$ for the proposed Algorithms 4, 5 and 6. Choose $\zeta = 0.2$ and $\varphi = 0.7$ for our Algorithm 6. Select $\zeta = 0.15$ and $\varphi = 0.9$ for our Algorithm 5. Take $\zeta = 0.5$, $\varphi = 0.9$, $\beta_n = \frac{1}{100(n+1)}$, and $\epsilon_n = \frac{1}{(n+1)^2}$ for our Algorithm 6.
- Pick $\chi_1 = 1$, $\kappa = 0.5$, $\alpha_n = \frac{1}{n+1}$, and $\beta_n = 0.5(1-\alpha_n)$ for GT Algorithm 1 [3]. Set $\zeta_n = 0.2$, $\delta = 1.5$, and $\chi_n = \frac{0.3}{L}$ (where $L = \|C\|^2$) for GTV Algorithm 1 [10]. Take $\chi_1 = 1$, $\kappa = 0.5$, $\zeta = 0.2$, $\varphi = 0.4$, and $\tau_n = \frac{1}{n+1}$ for TCPDL Algorithm 1 [11].

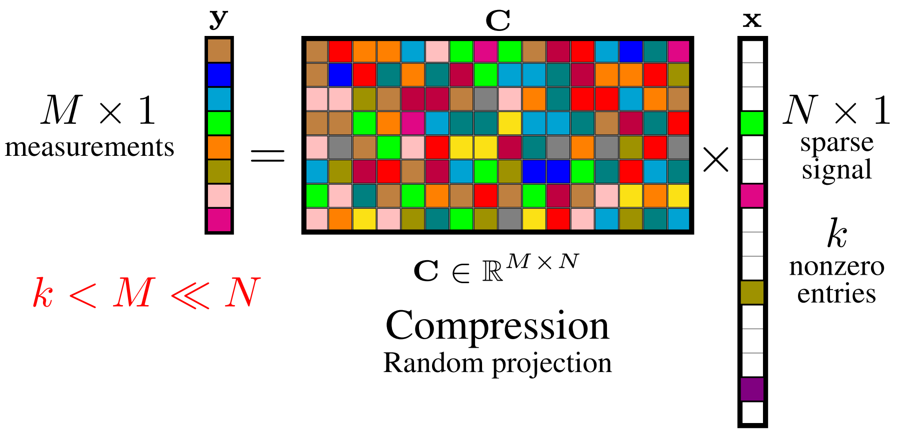


Fig. 3 Geometric interpretation of model (5.1)

In our numerical experiments, the clean signal $\mathbf{x} \in \mathbb{R}^N$ contains k ($k \ll N$) randomly created ± 1 spikes. The matrix $\mathbf{C} : \mathbb{R}^{M \times N}$ is produced using a standard normal distribution with zero mean and unit variance, and the rows are then orthonormalized. Let ϵ be the white Gaussian noise with variance 10^{-4}. Then, the observation $\mathbf{y}$ is generated by (5.1). With the starting signals $\mathbf{s_0} = \mathbf{s_1} = \mathbf{0}$, the recovery procedure begins and terminates after 100 iterations. We use the mean squared error $\text{MSE} = (1/N)\,\|\mathbf{x}^* - \mathbf{x}\|^2$ ($\mathbf{x}^*$ is an estimated signal of $\mathbf{x}$) to measure the restoration accuracy of all algorithms. In our first test, we set $M = 512$, $N = 1024$, and $k = 100$. The original signal and the noisy signal are shown in Fig. 4. The recovery results using the proposed algorithms are presented in Fig. 5. The variation of MSE with the number of iterations for all algorithms is illustrated in Fig. 6.

Finally, we did all algorithms for solving the signal recovery problem (5.1) in different dimensions and with different sparsity. Their results are shown in Table 1 and Fig. 7. Notice that we do not present the test results for GT Algorithm 1 [3] in Table 1 because it performs poorly (see the variation of MSE for GT Algorithm 1 [3] in Fig. 7).

Remark 5.1 From Figs. 4, 5, 6, and 7 and Table 1, it can be seen that the proposed Algorithms 4, 5 and 6 can solve the signal processing problem as shown in model (5.1) very well. In addition, the presented algorithms converge faster than the methods in the literature [3, 10, 11] and their performance is robust (i.e., it is independent of the size of the dimensionality and sparsity). In conclusion, the algorithms presented in

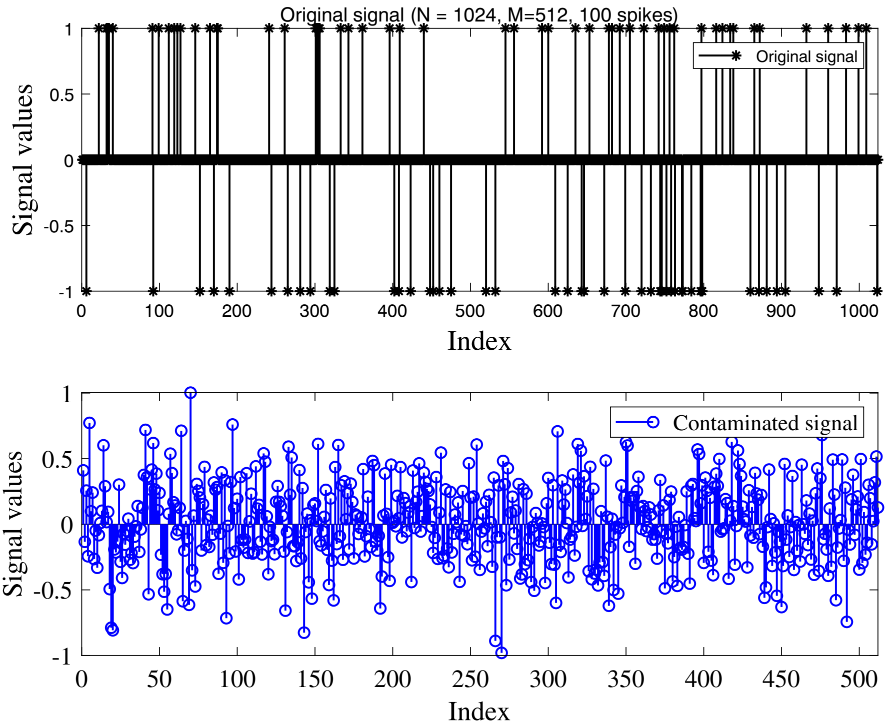
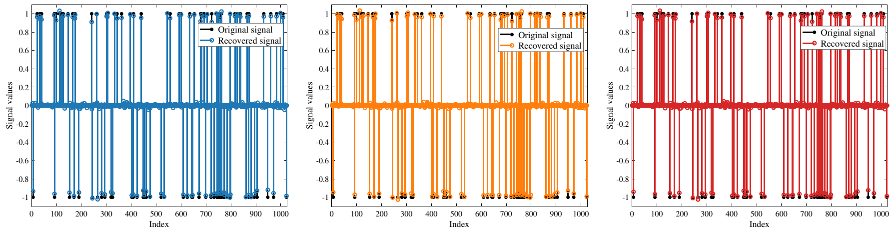


Fig. 4 Original signal and noise signal generated by model (5.1)



(a) Recovered by our Alg. 3.1 (b) Recovered by our Alg. 3.2 (c) Recovered by our Alg. 3.3

Fig. 5 The original signal and the signal recovered by our algorithms

this paper are efficient and useful. On the other hand, although the computational time of the proposed algorithms is not always the minimum (as indicated in Table 1), the difference in time between our proposed algorithms and the comparison algorithms is negligible in most cases. This illustrates that our proposed algorithms can achieve higher precision within the same number of iterations, despite the minimal difference in computational time compared to the comparison algorithms. Essentially, our algorithms demonstrate comparable complexity to the comparison algorithms while offering improved accuracy.

Example 5.2 (**Image Restoration Problem**) The image restoration problem can be represented as the following model:

$$\mathbf{Cx} = \mathbf{b} + \mathbf{v}, \tag{5.3}$$

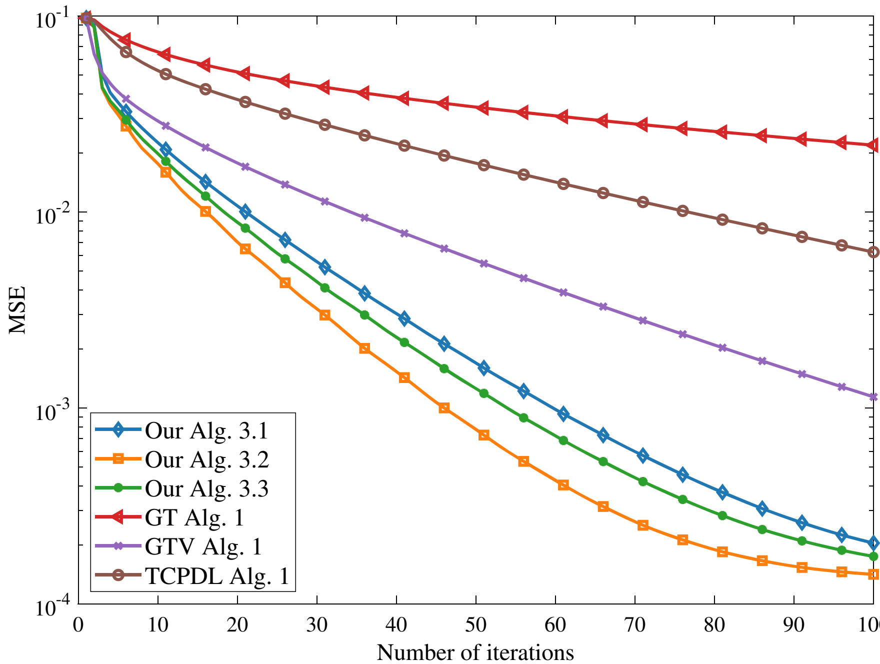
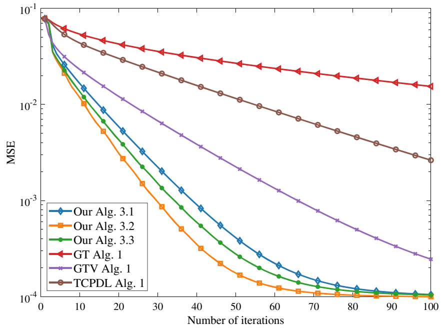


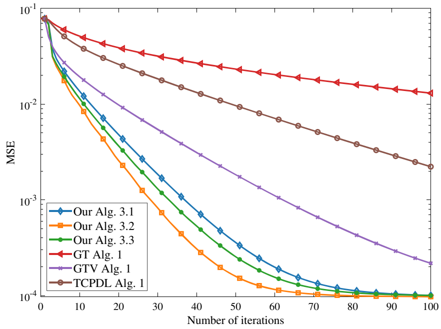
Fig. 6 The variation of MSE with the number of iterations for all algorithms at $M = 512$, $N = 1024$, $k = 100$

Table 1 The numerical results of all algorithms for solving (5.1) under different situations

Algorithms	$M = 256, N = 512$ $k = 20$		$M = 256, N = 512$ $k = 40$		$M = 512, N = 1024$ $k = 40$		$M = 512, N = 1024$ $k = 80$	
	MSE $(\times 10^{-4})$	Time (s)	MSE $(\times 10^{-3})$	Time (s)	MSE $(\times 10^{-4})$	Time (s)	MSE $(\times 10^{-3})$	Time (s)
Our Algorithm 4	0.4258	0.0136	0.1045	0.0156	0.4645	0.0440	0.1003	0.0399
Our Algorithm 5	0.4258	0.0132	0.0997	0.0158	0.4645	0.0407	0.0974	0.0395
Our Algorithm 6	0.4312	0.0139	0.1038	0.0181	0.4705	0.0411	0.1002	0.0399
GTV Algorithm 1 [10]	0.4264	0.0156	0.2378	0.0142	0.4653	0.0353	0.2115	0.0393
TCPDL Algorithm 1 [11]	0.8265	0.0141	2.5557	0.0163	0.9976	0.0423	2.1614	0.0378



(a) $M = 256, N = 512, k = 40$



(b) $M = 512, N = 1024, k = 80$

Fig. 7 The variation of MSE with the number of iterations for all algorithms in two cases

where $\mathbf{C}$ is a convolution matrix of size $m \times k$, $\mathbf{x}$ is the original image data in $\mathbb{R}^k$, $\mathbf{b}$ is the degraded image data in $\mathbb{R}^m$, and $\mathbf{v}$ is the noise vector in $\mathbb{R}^m$. The problem (5.3) can be represented as a constrained optimization model, and the objective is to minimize $f(\mathbf{x}) = \|\mathbf{C}\mathbf{x} - \mathbf{b}\|^2$ subject to $\mathbf{x} \in \mathbb{C}$. This model can be transformed into the split feasibility problem by defining $\mathbb{C}$ as a box in $\mathbb{R}^k$ and $\mathbb{Q}$ as either $\{\mathbf{b}\}$ if $\mathbf{v} = \mathbf{0}$ (no noise added), or as a set $\mathbb{Q} = \{\mathbf{y} \in \mathbb{R}^m \mid \|\mathbf{y} - (\mathbf{b} + \mathbf{v})\| \leq \varepsilon\}$ for small enough $\varepsilon > 0$.

In this experiment, we selected three grayscale images with a size of 515×512 as the test subjects. In this case, the range of $\mathbb{C}$ is from 0 to 1. The test images undergo two stages of degradation: first, a 9×9 Gaussian blur with a standard deviation of 2 is applied, followed by addition of a zero-mean Gaussian white noise with a standard deviation of 10^{-4}. Next, we use the same algorithms and keep the same parameters as in Example 5.1 to solve problem (5.3). The signal-to-noise ratio (SNR) in decibels and the structural similarity index (SSIM) are used to measure the quality of the reconstructed image compared to the original image. The SNR is defined as

$$\mathrm{SNR} := 20 \log_{10} \frac{\|\mathbf{x}\|}{\|\tilde{\mathbf{x}} - \mathbf{x}\|},$$

where $\mathbf{x}$ is an original image and $\tilde{\mathbf{x}}$ is a restored image. The calculation of SSIM directly calls the function "`ssimval=ssim(`$\tilde{\mathbf{x}}$`,`$\mathbf{x}$`)`" in MATLAB. A higher SNR

Fig. 8 The original Cameraman image, the degraded image, and the image recovered by our Algorithm 4

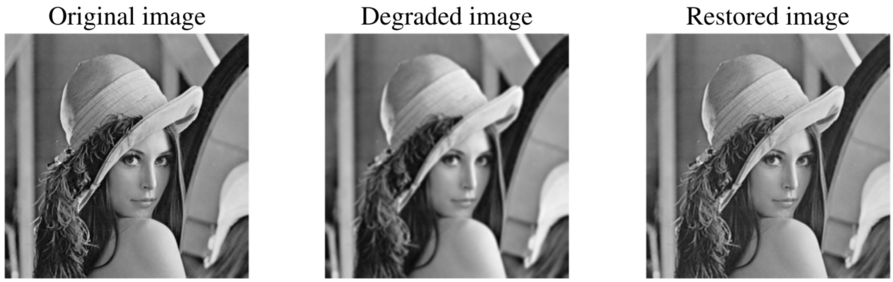


Fig. 9 The original Lena image, the degraded image, and the image recovered by our Algorithm 5

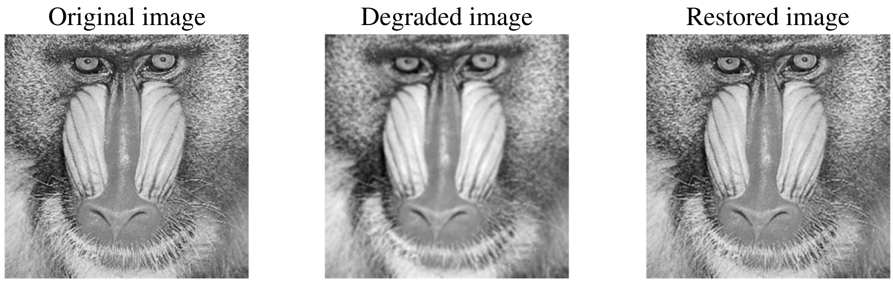
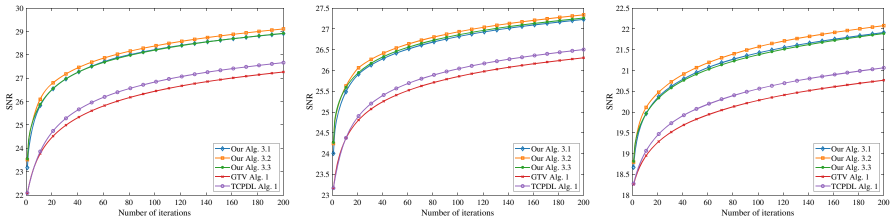
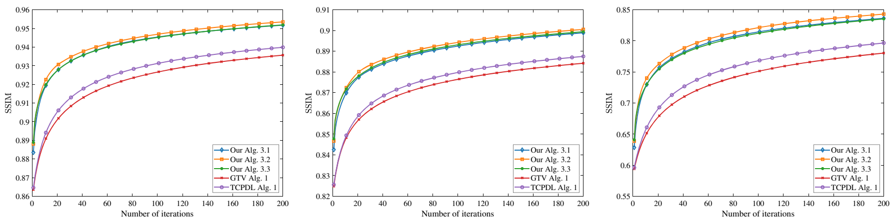


Fig. 10 The original Mandril image, the degraded image, and the image recovered by our Algorithm 6



(a) Cameraman image (b) Lena image (c) Mandril image

Fig. 11 The variation of SNR for all algorithms with three images



(a) Cameraman image (b) Lena image (c) Mandril image

Fig. 12 The variation of SSIM for all algorithms with three images

Table 2 Numerical results for all algorithms under different images

Algorithms	Cameraman		Lena		Mandril	
	SNR	SSIM	SNR	SSIM	SNR	SSIM
Our Alg. 3.1	28.9115	0.9518	27.2279	0.8989	21.9129	0.8364
Our Alg. 3.2	29.1107	0.9535	27.3386	0.9005	22.0787	0.8431
Our Alg. 3.3	28.9218	0.9519	27.2621	0.8994	21.8878	0.8353
GT Alg. 1 [3]	25.4751	0.9263	25.0525	0.8753	19.9984	0.7446
GTV Alg. 1 [10]	27.2681	0.9357	26.3045	0.8842	20.7651	0.7804
TCPDL Alg. [11]	27.6678	0.9399	26.5029	0.8875	21.0635	0.7966

value indicates better reconstruction quality, while a SSIM value closer to 1 means that the restored image is more similar to the original image. The initial points for all algorithms are $\mathbf{s_0} = \mathbf{s_1} = \mathbf{b}$, and the iteration stops after 200 iterations. Figures 8, 9, and 10 show the original three test images, the degraded images, and the images restored by our algorithms, respectively. The variation of SNR and SSIM with the number of iterations for all algorithms under the three test images is plotted in Figs. 11 and 12, respectively. Finally, the SNR and SSIM values of all algorithms after executing 200 iterations for the three test images are presented in Table 2.

Remark 5.2 It can be intuitively seen from Figs. 8, 9, and 10 that the three algorithms proposed in this paper can effectively solve the image denoising problem. On the other hand, our three algorithms have higher SNR and SSIM values than the comparison algorithms in references [3, 10, 11] (refer to Figs. 11 and 12 and Table 2), which means that our algorithms have higher computational efficiency in processing such problems when appropriate parameters are selected.

6 Conclusions

In this paper, we propose three new algorithms for solving the monotone inclusion problems. These approaches are obtained based on the inertial method, the forward-backward algorithm, the projection and contraction algorithm, and the relaxation method. In the framework of real Hilbert spaces, the weak convergence of the proposed Algorithms 4 and 5 and the strong convergence of the suggested Algorithm 6 are established under the condition that the parameters and operators satisfy some suitable conditions. Furthermore, the R-linear convergence rates of the proposed Algorithms 4 and 5 are also proved in the case where the multi-valued operator meets strong monotonicity. Finally, we apply the proposed algorithms to signal processing problems and image recovery problems. The results of this paper improved and extended many relevant algorithms in this field.

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Data Availability Data sharing is not applicable for this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of Interest The authors declare no competing interests.

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