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Local behaviors of Fourier expansions for functions of limited regularities

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Abstract

Based on the explicit formula of the pointwise error of Fourier projection approximation and by applying van der Corput-type Lemma, optimal convergence rates for periodic functions with different degrees of smoothness are established. It shows that the convergence rate enjoys a decay rate one order higher in the smooth parts than that at the singularities. In addition, it also depends on the distance from the singularities. Ample numerical experiments illustrate the perfect coincidence with the estimates.

Keywords Fourier expansion · Projection approximation · Pointwise error · Convergence rate · Regularity · Window function

Mathematics Subject Classification (2010) 65N15 · 42A10 · 41A60 · 42A20 · 42A16

1 Introduction

Fourier expansions and projection approximations are powerful tools in various scientific fields [1–3] and in developing numerical methods for ordinary and partial differential equations [4–13].

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Let f be a 2π-periodic function, and $|E_n^f(x)| = |f(x) - S_n^f(x)|$, where $S_n^f(x)$ denotes the Fourier projection of f defined by

$$S_n^f(x) = \frac{a_0^f}{2} + \sum_{k=1}^{n} \left[a_k^f \cos(kx) + b_k^f \sin(kx)\right],\ a_0^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt,$$
$$a_k^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt,\ b_k^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt,\ k = 1, \cdots, n. \tag{1.1}$$

Lebesgue in 1910 showed for f satisfying a Lipschitz condition of order $\alpha \in (0, 1]$ that the uniform error of the Fourier projection decays at rate $\mathcal{O}(n^{-\alpha} \ln n)$ [14]. This error bound was improved by Salem and Zygmund in 1946 to $\mathcal{O}(n^{-\alpha})$ by imposing a monotonic type condition on f [15]. Particularly, Jackson [16] gave a sharp bound $\mathcal{O}(n^{-k-\alpha})$ for the best uniform trigonometric polynomial approximation provided that $f^{(k)}$ satisfies a Lipschitz condition of order $\alpha \in (0, 1]$, from which together with the Fejér's estimate for the Lebesgue constant [17] it leads to the estimate [18]

$$\|E_n^f\|_{L^\infty([-\pi,\pi])} = \mathcal{O}(n^{-k-\alpha} \ln n). \tag{1.2}$$

Moreover, for a function that can be analytically extended to a strip area $\{z \in \mathbb{C} : |\Im(z)| \le a\}$, where a is a positive constant and $\Im(z)$ denotes the imaginary part of z, Paley-Wiener theorem shows that its projection approximation enjoys an exponential convergence rate [5, 19, 20]. Whereas for a limited regular function, it converges at an algebraic rate [20–22].

Theorem 1.1 *Let f be a 2π-periodic function. Then the following holds.*
(1) ([5, 19, 20, Paley-Wiener theorem]) If f is holomorphic with $|f(x)| \le M$ in a strip domain $\{z \in \mathbb{C} : |\Im(z)| \le a\}$, where M and a are some positive constants, then

$$\|E_n^f\|_{L^\infty([-\pi,\pi])} \le \frac{2M\mathrm{e}^{-an}}{\mathrm{e}^a - 1},\ x \in I = [-\pi, \pi].$$

(2) ([20, Theorem 4.2]) If f is $r \ge 1$ times differentiable and $f^{(r)}(x)$ is of bounded variation V on $[-\pi, \pi]$, then

$$\|E_n^f\|_{L^\infty([-\pi,\pi])} \le \frac{V}{\pi r n^r},\ x \in I.$$

The errors mentioned above usually refer to the sense of infinite norm. It is worth noting that the pointwise error is usually an accurate indication for the approximability and approximation effect. It has been attracting much attention, although it is always challenging [18, 23–25, 36].

Wahlbin, one of the deep insights of the local behaviors, considered the local convergence in 1985 for spline L^2 projections, Fourier series and Legendre series in [24].

In particular, for the local behavior of $E_n^f(x_0)$ with $x_0 \in I_{x_0,\delta} \subset I$, where I and $I_{x_0,\delta}$ are bounded intervals and $\delta = \text{dist}\big(x_0, I \setminus I_{x_0,\delta}\big) > 0$, Wahlbin [24] proved the following theorem in the case that f is smooth on interval $I_{x_0,\delta}$ but possibly rough on $I \setminus I_{x_0,\delta}$.

Theorem 1.2 *([24]) Assume that $f \equiv 0$ on $I_{x_0,\delta}$ and let p and q be given nonnegative integers. There exist constants C_1, $C_2 = C_2(q)$, $C_3 = C_3(q)$, $C_4 = C_4(p,q)$ such that for $\delta = \text{dist}(x_0, I \setminus I_{x_0,\delta}) \geq C_4 \frac{\ln n}{n}$,*

$$|E_n^f(x_0)| \leq \begin{cases} C_1\delta^{-1}\omega^p\big[f, C_2\frac{\ln n}{n}; L^1(I)\big] + C_3 n^{-q}\|f\|_{L^1(I)}, & p \geq 1, \\ C_1\delta^{-1}\|f\|_{L^1(I)}, & p = 0. \end{cases}$$

Here $x_0 \in I_{x_0,\delta}$ and $\omega^p(\cdot,\cdot;\cdot)$ denotes the pth modulus of continuity.

The local convergence of Theorem 1.2 was illustrated in [24] by the following numerical example

$$H(x) = \begin{cases} 0, & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 1, & x \in I \setminus [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases} \quad \text{and} \quad f_\alpha(x) = h(x)\sum_{j=1}^{\infty} 2^{-j\alpha} e^{-i2^j x} \quad (0 < \alpha < 1)$$

with

$$h(x) = \begin{cases} 0, & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 1, & x \in I \setminus [-\frac{3\pi}{4}, \frac{3\pi}{4}] \end{cases} \in C^\infty(I),$$

whose convergence rates are

$$|E_n^H(0)| \leq C\frac{\ln n}{n} \quad \text{and} \quad |E_n^{f_\alpha}(0)| \leq Cn^{-\alpha}(\ln n)^\alpha \tag{1.3}$$

for some constant C independent of n, respectively. Both the local convergence rates are sharp, against modulo the logarithmic factors.

Ample numerical examples show that the logarithmic factor "$\ln n$" in (1.2) can be removed. Furthermore, we see from Fig. 1 that the pointwise errors of Fourier

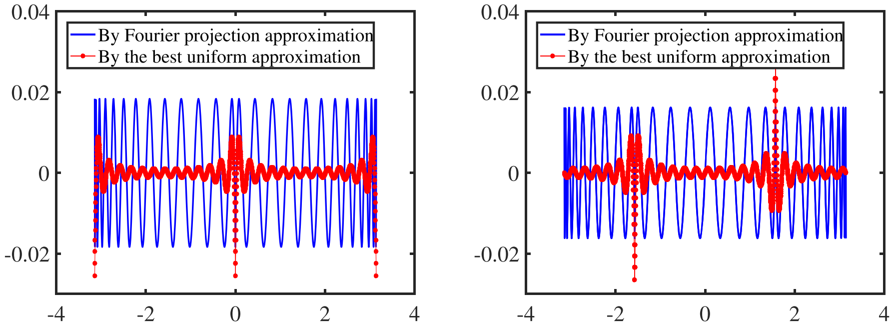

Fig. 1 Comparisons for the pointwise errors of the Fourier projection approximations and the best uniform trigonometric approximations, where $f(x) = |\sin x|$ (left), $f(x) = \arcsin \sin x$ (right), and $n = 24$

projection approximation are much smaller than those of the best uniform approximation in the most area of underlying domain, which excludes a small neighborhood of the singularities. More specifically, the pointwise errors corresponding to the smoother parts of f enjoy a faster convergence rate with an extra order n^{-1} without the logarithmic factor in (1.3).

For the orthogonal polynomial expansions, the logarithmic factor "$\ln n$" in Wahlbin [24] can be removed. Babuška and Hakula [25] in 2019 considered the pointwise errors of projection approximations of the Legendre series for the class of functions of interior singularities like Φ-functions, and obtained an accurate pointwise error estimate without the logarithmic factor in Wahlbin [24] for the step function. More recently, Xiang et al. [26] considered the sharp error bounds on the pointwise errors without the logarithmic factor for Jacobi expansions for generalized Φ-functions with interior or boundary singularities. Wang [27–29] presented the superconvergence orders of pointwise errors corresponding to the smoother parts of Φ-functions for the Chebyshev and Gegenbauer expansion. For more details see [25–29].

However, Fourier projection in many settings remains a favorable choice for the approximation of nonperiodic function [30–32], since it often offers good frequency resolution, and the approximation can be computed numerically via the fast Fourier transform (FFT).

This paper primarily focuses on the decay rates of pointwise errors $E_n^f(x)$ in Fourier projections for functions with varying degrees of smoothness, including some singularities of integer or fractional regularities, as well as logarithmic singularities. We explore the phenomenon of superconvergence mentioned earlier in Fourier approximations for these functions and derive optimal convergence rates theoretically. This implies that the previously mentioned factor "$\ln n$" can be eliminated. Additionally, we provide a detailed analysis of the behaviors of $E_n^f(x)$ around singular points. For convenience, we assume that the function under investigation is 2π-periodic and we specifically focus on $f(x)$ within the cardinal period $I = [-\pi, \pi]$.

The rest of this paper is organized as follows. In Sect. 2, a detailed analysis will be presented for functions with integer regularities. Section 3 will focus on functions with fractional regularities, while also discussing the extension of similar results to cases involving logarithmic singularities. Numerical examples will be provided in both Sect. 2 and Sect. 3 to illustrate the sharpness of the proposed results. Finally, conclusions and discussions are concluded in Sect. 4.

2 Asymptotic behaviors for integer-regular functions

We restrict our attention to the pointwise error $E_n^f(x) = f(x) - S_n^f(x)$. Let $D_n(t)$ denote the n-th Dirichlet kernel

$$D_n(t) = \sum_{k=-n}^{n} \mathrm{e}^{ikt} = \frac{\sin{(n+\frac{1}{2})t}}{\sin{\frac{t}{2}}}. \tag{2.1}$$

It is well known that $S_n^f(x)$ can be represented as a convolution of f and the n-th Dirichlet kernel (2.1), precisely,

$$S_n^f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt. \tag{2.2}$$

The convolution (2.2) will be reduced to an identical transform once that $f(x)$ degenerates into a trigonometric polynomial $T_m(x) = \sum_{k=0}^{m}\left[\alpha_k \cos(kx) + \beta_k \sin(kx)\right]$, $m \leq n$, i. e.,

$$T_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} T_m(t)D_n(x-t), m = 0, 1, \cdots, n.$$

Subsequently, the pointwise error function can be presented by

$$\begin{aligned}
E_n^f(x) =& \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-f(t)]D_n(x-t)dt\\
=& \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-f(x-t)]D_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x,t)\frac{\sin\left[(n+\frac{1}{2})t\right]}{\sin\frac{t}{2}}dt\\
=& \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x,t)\cos(nt)dt + \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x,t)\sin(nt)dt\\
=& a_n^{\varphi(x,t)} + b_n^{\phi(x,t)},
\end{aligned} \tag{2.3}$$

where

$$\varphi(x,t) = \frac{f(x)-f(x-t)}{2},\ \phi(x,t) = \varphi(x,t)\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \frac{f(x)-f(x-t)}{t}\frac{\frac{t}{2}}{\sin\frac{t}{2}}\cos\frac{t}{2}, \tag{2.4}$$

and $\phi(x,0)$ is defined by $\lim_{t\to 0}\phi(x,t)$. Then all we have left is to estimate $a_n^{\varphi(x,t)}$ and $b_n^{\phi(x,t)}$, which depend on the smoothness of $\varphi(x,t)$ and $\phi(x,t)$. Throughout this paper, $\varphi(x,t)$ and $\phi(x,t)$ are regarded as functions of variable t, and x is a fixed parameter. Additionally, the following notations will be used repeatedly:

$$z_{r,k}(x-t) = \partial_t^r\left(\frac{\frac{x-t}{2}}{\sin\frac{x-t}{2}}\right)\partial_t^k\left(\cos\frac{x-t}{2}\right),\ H_{r,k}(x-t) = \sum_{j=0}^{r-k}\frac{r!z_{j,r-k-j}(x-t)}{(r-k-j)!j!}.$$

Firstly, we give a lemma that will be used repeatedly in the subsequent discussion, whose proof can be completed easily by induction.

Lemma 2.1 *Suppose that $\alpha(x)$ and $\beta(t)$ are two functions, and $\beta(t)$ is suitably smooth, then for any nonnegative integer m, it holds that*

$$\partial_t^m\left[\frac{\alpha(x)-\beta(t)}{x-t}\right] = m!\frac{\alpha(x)-\sum_{l=0}^{m}\frac{\beta^{(l)}(t)}{l!}(x-t)^l}{(x-t)^{m+1}}. \tag{2.5}$$

The functions we encounter frequently in practice are sufficiently smooth in the most parts of the underlying domain, except for a few singularities. Inspired by Krylov's method of separating singularities, we restrict our attention on a class of 2π-periodic functions satisfying that

$$f \in C^r(I) \text{ and } f^{(r+l)} \in C\left(I \setminus \{\varsigma\}\right), \ \varsigma \in (-\pi, \pi) \tag{2.6}$$

with $f^{(r+l)}(\varsigma \pm 0) = \lim_{x\to\varsigma^\pm} f^{(r+l)}(\varsigma)$ existing for $l = 1, 2, 3$.

It is enough to consider the case of inner singularity, since the function $f(x)$ is periodic, and a translation can be imposed on f once the singularity lies on the endpoints of $I = [-\pi, \pi]$.

The following lemma concerns the smoothness of $\phi(x,t)$ in (2.4).

Lemma 2.2 *Let f be a 2π-periodic function defined by (2.6). Then for fixed $x \in I$, $\phi(x,t)$ defined by (2.4) is included in $C^r(I)$ when $x \neq \varsigma$, while in $C^{r-1}(I)$ when $x = \varsigma$. Additionally, $\partial_t^r \phi(\varsigma, 0 \pm 0)$ exist.*

Proof From the definition of $\phi(x,t)$, it is obvious that $\phi(x,t)$ has continuous derivatives up to order r at $t \in [-\pi, 0) \cup (0, \pi]$. With the help of Leibniz's Formula, Lemma 2.1 and Taylor's theorem, we have that for $x \neq \varsigma$ and t (t is sufficiently closed to 0 such that x and $x - t$ locate in the same side of ς),

$$\begin{aligned}
\partial_t^r \phi(x,t) &= \sum_{k=0}^{r} \binom{r}{k} \left[\frac{f(x) - f(x-t)}{t}\right]^{(k)} \sum_{j=0}^{r-k} \binom{r-k}{j} z_{j,r-k-j}(t) \\
&= \sum_{k=0}^{r} \frac{f(x) - \sum_{l=0}^{k} \frac{f^{(l)}(x-t)}{l!} t^l}{(-1)^k t^{k+1}} \sum_{j=0}^{r-k} \frac{r! z_{j,r-k-j}(t)}{(r-k-j)! j!} \\
&= \sum_{k=0}^{r} \frac{(-1)^{k+1}}{(k+1)!} f^{(k+1)}[x - (1-\theta)t] H_{r,k}(t) \\
&\to \sum_{k=0}^{r} \frac{(-1)^{k+1}}{(k+1)!} f^{(k+1)}(x) H_{r,k}(0), \ t \to 0,
\end{aligned} \tag{2.7}$$

where

$$H_{r,k}(t) = \sum_{j=0}^{r-k} \frac{r! z_{j,r-k-j}(t)}{(r-k-j)! j!}, \quad z_{j,r-k-j}(t) = \left(\frac{\frac{t}{2}}{\sin \frac{t}{2}}\right)^{(j)} \left(\cos \frac{t}{2}\right)^{(r-k-j)}$$

and $\theta \in (0, 1)$. From (2.7) and the derivative limit theorem, it establishes that $\phi(x, t) \in C^r(I)$ since $z_{k,j}(t)$'s are sufficiently smooth. Analogously, it indicates for $x = \varsigma$ that

$$\partial_t^{r-1}\phi(x, t) = \sum_{k=0}^{r-1} \frac{(-1)^{k+1}}{(k+1)!} f^{(k+1)}[x - (1-\theta)t]H_{r-1,k}(t)$$
$$\to \sum_{k=0}^{r-1} \frac{(-1)^{k+1}}{(k+1)!} f^{(k+1)}(x)H_{r-1,k}(0),\ t \to 0$$

due to that $f^{(k)} \in C(I),\ k = 0, \cdots, r$. Again by the derivative limit theorem, we see that $\phi(\varsigma, t) \in C^{r-1}(I)$.

In addition, for $x = \varsigma$, by (2.7) it yields that $\partial_t^r \phi(\varsigma, 0 \pm 0)$ are well defined by $f^{(k)}(\varsigma \mp 0)$, $k = 0, \cdots, r+1$ and the smoothness of $z_{k,j}(t)$.

Remark 2.1 Lemma 2.2 is also true for $f(x)$ possessing multiple singularities $S = \{\varsigma_i \in I : i = 1, \cdots, s\}$. Assume that the 2π-periodic function $f \in C^r(I)$, and $f^{(r+l)} \in C(I \setminus S)$ with $f^{(r+l)}(\varsigma_i \pm 0),\ l = 1, 2, 3$ existing for $\varsigma_i \in S$. Then $\phi(x, t) \in C^r(I)$ for $x \in I \setminus S$, and $\phi(x, t) \in C^{r-1}(I)$ for $x \in S$. Additionally, $\partial_t^r \phi(\varsigma_i, 0 \pm 0)$ exist for $\varsigma_i \in S$.

From Lemma 2.2 and Remark 2.1, we see that $\phi(x, t)$ in (2.4) inherits almost all of smooth features of f in $\mathbb{R}$ except for $x = \varsigma_i + 2k\pi,\ k \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers). For a special case $f(x) = \arcsin\sin x$, $\phi(x, t) \in C(I)$ when $x \in I \setminus \{\pm\frac{\pi}{2}\}$ and the smoothness is degenerated when $x = \pm\frac{\pi}{2}$, we see from Fig. 2 that $\phi(\pm\frac{\pi}{2}, t)$ have two jump discontinuities in the cardinal interval.

Now we return to our motif: the superconvergence rates of Fourier expansions for limited regular functions.

Theorem 2.1 *Let f be a 2π-periodic function defined by* (2.6). *Then the error defined by* (2.3) *enjoys decay orders*

$$E_n^f(x) = \begin{cases} A(x) \cdot \mathcal{O}(n^{-r-2}),\ x \in I \setminus \{\varsigma\} \\ \mathcal{O}(n^{-r-1}),\ x = \varsigma \end{cases} \tag{2.8}$$

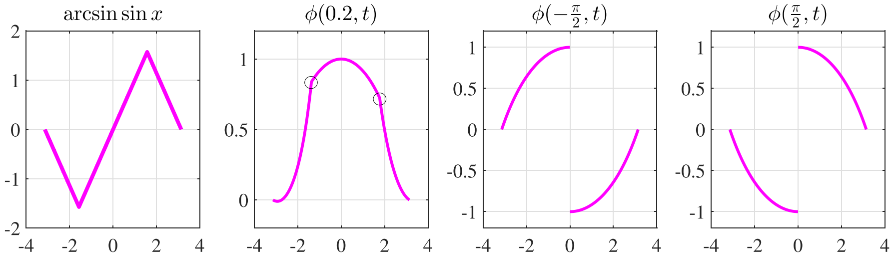

Fig. 2 $\phi(x, t)$ inherits perfectly the smoothness of $f(x) = \arcsin\sin x$ except for $x = 2k\pi \pm \frac{\pi}{2},\ k = 0, \pm 1, \cdots$. The non-smooth points (circled) in the second graph match exactly those in the first graph. The jump discontinuities in the last two graphs owe to the singularities $\pm\frac{\pi}{2}$

with $A(x) = |x-\varsigma|^{-1}$ *for* x *near* ς.

Proof (1) We prove firstly the case $x \neq \varsigma$. From Lemma 2.2 and by integrating by parts repeatedly, $b_n^{\phi(x,t)}(x)$ satisfies that

$$\begin{aligned}\pi n^{r+1}\left|b_n^{\phi(x,t)}\right| =&\left|\int_{-\pi}^{\pi} \partial_t^{r+1}\phi(x,t)\sin(nt-\frac{r+1}{2}\pi)dt\right| \leq \left|\int_{-\pi}^{\pi} \partial_t^{r+1}\phi(x,t)\mathrm{e}^{int}dt\right|\\ =&\left|\left(\int_{-\pi}^{\varsigma}+\int_{\varsigma}^{\pi}\right)\partial_t^{r+1}g(x,t)\mathrm{e}^{-int}dt\right|, \qquad (2.9)\end{aligned}$$

where we used the symmetry of the convolution and set

$$g(x,t) = \frac{f(x)-f(t)}{x-t}\cdot z_{0,0}(x-t).$$

For the case $x \in [-\pi, \varsigma)$: We endow $f^{(r+l)}(x)$ at $x = \varsigma$ with supplementary values $f^{(r+l)}(\varsigma-0)$, $l = 1, 2, 3$. From (2.5) and Taylor's theorem with Lagrange's remainder, it is easy to verify for $t \in [-\pi, \varsigma]$ that

$$\begin{aligned}\partial_t^{r+1}g(x,t) =&\sum_{k=0}^{r+1}\frac{f(x)-\sum_{l=0}^{k}\frac{f^{(l)}(t)}{l!}(x-t)^l}{(x-t)^{k+1}}H_{r+1,k}(x-t)\\ =&\sum_{k=0}^{r+1}\frac{f^{(k+1)}(\xi_k)}{(k+1)!}H_{r+1,k}(x-t) \qquad (2.10)\end{aligned}$$

and

$$\partial_t^{r+2}g(x,t) = \sum_{k=0}^{r+2}\frac{f^{(k+1)}(\eta_k)}{(k+1)!}H_{r+2,k}(x-t)$$

are well defined at $t = x$ too and uniformly bounded since $f^{(k)}(x) \in C[-\pi, \varsigma]$, $k = 0, 1, \cdots, r+3$, where ξ_k and η_k locate between t and x. Therefore

$$\begin{aligned}\left|\int_{-\pi}^{\varsigma}\partial_t^{r+1}g(x,t)\mathrm{e}^{-int}dt\right| \leq&\frac{1}{n}\left\{\left|\partial_t^{r+1}g(x,t)\mathrm{e}^{-int}\Big|_{-\pi}^{\varsigma}\right| + \int_{-\pi}^{\varsigma}\left|\partial_t^{r+2}g(x,t)\right|dt\right\}\\ =&\mathcal{O}(n^{-1}) \qquad (2.11)\end{aligned}$$

holds uniformly for $x \in [-\pi, \varsigma)$.

For the other integral in the parentheses of (2.9), we have

$$\left|\int_{\varsigma}^{\pi}\partial_t^{r+1}g(x,t)\mathrm{e}^{-int}dt\right| \leq \frac{1}{n}\left\{\left|\partial_t^{r+1}g(x,t)\mathrm{e}^{-int}\Big|_{\varsigma}^{\pi}\right| + \int_{\varsigma}^{\pi}\left|\partial_t^{r+2}g(x,t)\right|dt\right\}. \qquad (2.12)$$

In addition, $\partial_t^{r+1} g(x,t)$ can be bounded for some positive number C_1 that

$$\begin{aligned}\left|\partial_t^{r+1} g(x,t)\right| &= \left|\sum_{k=0}^{r+1} \frac{f(x) - \sum_{l=0}^{k} \frac{f^{(l)}(t)}{l!}(x-t)^l}{(x-t)^{k+1}} H_{r+1,k}(x-t)\right| \\ &= \left|\sum_{k=0}^{r+1} \left[\frac{f(x) - \sum_{l=0}^{k-1} \frac{f^{(l)}(t)}{l!}(x-t)^l}{(x-t)^{k+1}} - \frac{f^{(k)}(t)}{k!(x-t)}\right] H_{r+1,k}(x-t)\right| \\ &\leq \frac{C_1}{t-x} \leq \frac{C_1}{|x-\varsigma|}, \quad t \in (\varsigma, \pi], \end{aligned} \tag{2.13}$$

due to that

$$\frac{f(x) - \sum_{l=0}^{k-1} \frac{f^{(l)}(t)}{l!}(x-t)^l}{(x-t)^k} = \frac{\int_t^x f^{(k)}(\tau)(x-\tau)^{k-1} d\tau}{(k-1)!(x-t)^k} = \frac{\sigma_k}{k!}, \tag{2.14}$$

$f^{(k)}(t)$ and $H_{r+1,k}(x-t)$ are bounded on $[\varsigma, \pi]$, where we used Taylor's theorem with Cauchy's integral remainder in the last equality in (2.14), and

$$\min_{\tau \in [t,x]} f^{(k)}(\tau) \leq \sigma_k \leq \max_{\tau \in [t,x]} f^{(k)}(\tau),$$

for all $k = 0, 1, \cdots, r+1$. Thus, it yields that the first term in the brace of (2.12)

$$\left|\partial_t^{r+1} g(x,t) \mathrm{e}^{-int}\Big|_{\varsigma}^{\pi}\right| = \mathcal{O}(|x-\varsigma|^{-1}). \tag{2.15}$$

Similarly, $\partial_t^{r+2} g(x,t)$ can be bounded by

$$\begin{aligned}\left|\partial_t^{r+2} g(x,t)\right| =& \left|\sum_{k=0}^{r+2} \left[\frac{\int_t^x f^{(k-1)}(\tau)(x-\tau)^{k-2} d\tau}{(k-2)!(x-t)^{k+1}}\right.\right. \\ &\left.\left. - \frac{f^{(k-1)}(t)}{(k-1)!(x-t)^2} - \frac{f^{(k)}(t)}{k!(x-t)}\right] H_{r+2,k}(x-t)\right| \\ \leq& \frac{C_2}{t-x} + \frac{C_3}{(t-x)^2}, \quad t \in (\varsigma, \pi] \end{aligned} \tag{2.16}$$

for some positive constants $C_2,\ C_3$, which implies that the integral in the brace of (2.12) satisfies

$$\int_{\varsigma}^{\pi} \left|\partial_t^{r+2} g(x,t)\right| dt = C|x-\varsigma|^{-1} \tag{2.17}$$

for some constant $C > 0$.

Therefore, from (2.12), (2.15) and (2.17) it shows that

$$\left|\int_{\varsigma}^{\pi} \partial_t^{r+1} g(x,t)\mathrm{e}^{-int}dt\right| = |x-\varsigma|^{-1} \cdot \mathcal{O}(n^{-1}). \tag{2.18}$$

Combining (2.9), (2.11) and (2.18) leads to

$$\left|b_n^{\phi(x,t)}\right| = |x-\varsigma|^{-1} \cdot \mathcal{O}(n^{-r-2}),\ x \in (-\pi, \varsigma). \tag{2.19}$$

For the case $x \in (\varsigma, \pi]$: by the same argument, (2.19) also holds for $x \in (\varsigma, \pi]$.

For $a_n^{\varphi(x,t)}$, by a routine exercise we have

$$\begin{aligned} 2\pi\left|a_n^{\varphi(x,t)}\right| \le & 2\left|\int_{-\pi}^{\pi} \varphi(x,t)\mathrm{e}^{int}dt\right| = \left|\int_{-\pi}^{\pi} [f(x)-f(t)]\mathrm{e}^{-int}dt\right| \\ = & \left|\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-int}dt\right| = \mathcal{O}(n^{-r-2}). \end{aligned} \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.3), we deduce (2.8) for $x \neq \varsigma$.

(2) The case $x = \varsigma$ can be checked by integration by parts as follows

$$\begin{aligned} \pi n^r \left|b_n^{\phi(\varsigma,t)}\right| = & \left|\int_{-\pi}^{\pi} \partial_t^r \phi(\varsigma,t)\sin(nt-\frac{r\pi}{2})dt\right| \le \left|\int_{-\pi}^{\pi} \partial_t^r \phi(\varsigma,t)\mathrm{e}^{int}dt\right| \\ = & \left|\int_{-\pi}^{\pi} \partial_t^r g(\varsigma,t)\mathrm{e}^{-int}dt\right| = \left|\left(\int_{-\pi}^{\varsigma} + \int_{\varsigma}^{\pi}\right)\partial_t^r g(\varsigma,t)\mathrm{e}^{-int}dt\right| \\ = & \mathcal{O}(n^{-1}), \end{aligned} \tag{2.21}$$

since

$$\begin{aligned} \left|\partial_t^{\upsilon} g(\varsigma,t)\right| = & \left|\sum_{k=0}^{\upsilon} \frac{f(\varsigma) - \sum_{l=0}^{k} \frac{f^{(l)}(t)}{l!}(\varsigma-t)^l}{(\varsigma-t)^{k+1}} H_{\upsilon,k}(\varsigma-t)\right| \\ \le & \sum_{k=0}^{\upsilon} \frac{\left|f^{(k+1)}(\xi)\right|\left|H_{\upsilon,k}(\varsigma-t)\right|}{(k+1)!} \end{aligned}$$

is bounded for $\upsilon = r,\ r+1$, where ξ locates between ς and t. This implies $E_n^f(\varsigma) = \mathcal{O}(n^{-r-1})$ by noticing that (2.20) also holds for $x = \varsigma$.

Remark 2.2 For f defined by (2.6), it is easy to verify that $f^{(r+1)}(x)$ is of bounded variation and then $\|E_n^f\|_\infty = \mathcal{O}(n^{-r-1})$ by Theorem 1.1.

Remark 2.3 If the non-dominant term $\frac{C_2}{t-x}$ in (2.16) is taken into account, then $A(x)$ becomes a slightly redundant form $O(|x-\varsigma|^{-1}+|\ln|x-\varsigma||)$. For more details on the sharpness, see Fig. 6.

Next, we consider that $f(x)$ has a finite number of singular points in I. Suppose $S=\{\varsigma_1,\ldots,\varsigma_s\}\subseteq I$ is a finite set with

$$-\pi<\varsigma_1<\varsigma_2<\cdots<\varsigma_s<\pi,$$

$f(x)$ is a 2π-periodic function having continuous derivatives up to order r_j at $\varsigma_j\in S$, and sufficiently smooth in $I\setminus S$ with $f^{(r_j+l)}(\varsigma_j\pm 0)$ exiting for all $\varsigma_j\in S, l=1,2,3$. Define $\mu_0=-\pi$, $\mu_s=\pi$, $\mu_\kappa=\frac{\varsigma_\kappa+\varsigma_{\kappa+1}}{2}$, $\kappa=1,\cdots,s-1$ and $U_j=[\mu_{j-1},\mu_j]$, $U_j^\circ=U_j\setminus\{\varsigma_j\}$ for $j=1,2,\cdots,s$.

Corollary 2.1 *Assume $f(x)$ and S are defined by the above, then it holds that*

$$E_n^f(x)=\begin{cases}A_j(x)\cdot\mathcal{O}(n^{-r-2}), & x\in U_j^\circ, & (2.22a)\\ \mathcal{O}(n^{-\min\{r+2,r_j+1\}}), & x=\varsigma_j, & (2.23b)\end{cases}$$

where $r=\min\{r_j: j=1,\cdots,s\}$, $A_j(x)=|x-\varsigma_j|^{-1}$ if $r_j=r$ and $A_j(x)$ can be removed if $r_j\geq r+2$.

Proof It is obvious from Lemma 2.2 that $\partial_t^r\phi(x,t)\in C(I)$ when $x\in I\setminus S$. Suppose $x\in U_j^\circ$, then by the similar method (2.9) for Theorem 2.1 we have that

$$\begin{aligned}\pi n^{r+1}\left|b_n^{\phi(x,t)}\right| &= \left|\int_{-\pi}^{\pi}\partial_t^{r+1}\phi(x,t)\sin\left(nt-\frac{r+1}{2}\pi\right)dt\right|\\ &\leq\left|\left(\sum_{j=1}^{s}\int_{U_j}+\int_{I\setminus\left(\cup_{j=1}^{s}U_j\right)}\right)\partial_t^{r+1}g(x,t)\mathrm{e}^{-int}dt\right|.\end{aligned}$$

Analogous to the proof of Theorem 2.1, we see that

$$\int_{U_j}\partial_t^{r+1}g(x,t)\mathrm{e}^{-int}dt=|x-\varsigma_j|^{-1}\mathcal{O}(n^{-1}),$$
$$\int_{I\setminus\left(\cup_{j=1}^{s}U_j\right)}\partial_t^{r+1}g(x,t)\mathrm{e}^{-int}dt=\mathcal{O}(n^{-1}),$$

where $A_j(x)=|x-\varsigma_j|^{-1}$ if $r_j=r$, and $A_j(x)$ can be removed if $r_j\geq r_j+2$, which leads to (2.22a).

By analogous arguments of Theorem 2.1, (2.22b) can be obtained.

To check the results obtained above numerically, we illustrate the asymptotic orders of $E_n^f(x)$ for $f(x)=|\sin(x-1)|^3\mathrm{e}^{1+\sin x}$ and the zigzag linear function $f(x)=\arcsin\sin x$. See Figs. 3 and 4, respectively.

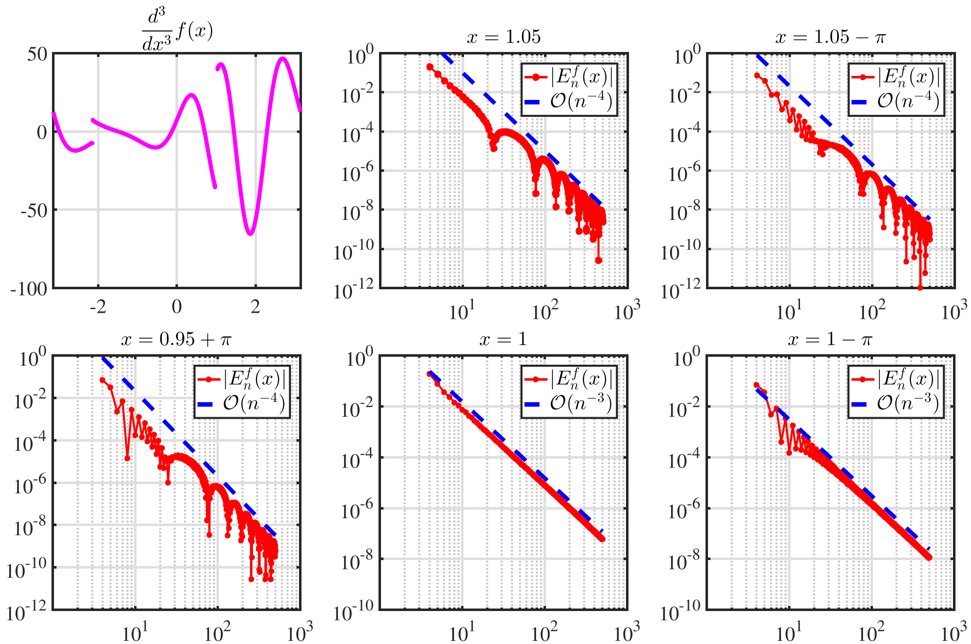

Fig. 3 The graph of three-order derivative of $f(x) = |\sin(x-1)|^3 e^{1+\sin x}$ and the asymptotic orders of pointwise errors $E_n^f(x)$ for its Fourier expansions. $f(x)$ is included in $C^\infty(\mathbb{R} \setminus \{1 + k\pi\}) \cap C^2(\mathbb{R})$, with singular points $x = 1 + k\pi$, $k \in \mathbb{Z}$

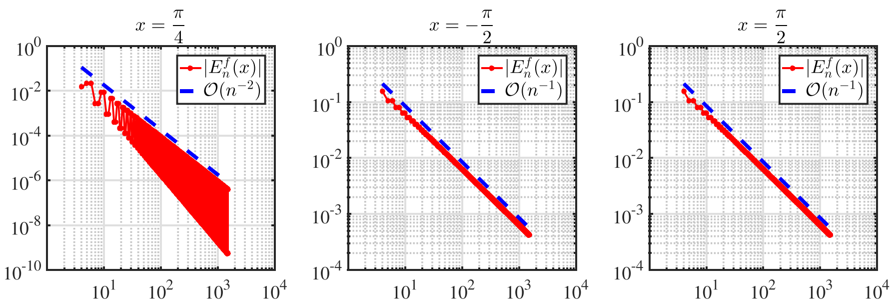

Fig. 4 Pointwise errors of Fourier expansions for function $f(x) = \arcsin \sin x$, which is a zigzag linear function, with singular points $x = 2k\pi \pm \frac{\pi}{2},\ k \in \mathbb{Z}$ (see Fig. 2)

Interestingly, for the functions with some jump discontinuities such as sawtooth function

$$f(x) = -x + \pi \operatorname{sgn}(x),$$

the pointwise error $E_n^f(x)$ corresponding to the smooth parts also enjoys a convergence rate $\mathcal{O}(n^{-1})$, although the Gibbs phenomenon may come as expected at singular points $x = 2k\pi,\ k \in \mathbb{Z}$ (see Fig. 5).

From Figs. 3, 4 and 5 we observe that the convergence orders of $E_n^f(x)$ for different functions with various smooth degrees are completely consistent with the statements of Theorem 2.1 and Corollary 2.1. All of these convergence orders are attainable, which indicate that the estimates above are optimal.

In order to illustrate the behaviors of $A(x)$ in front of $\mathcal{O}(n^{-r-2})$, we further consider the function

$$f(x) = \arcsin \sin x,\ \ x \in [-\pi, \pi]$$

(see Fig. 6). In particular, the demonstrations in the zoomed-in graphs indicate that the estimates for $A(x)$ are much sharp. The consistencies may be more accurate if the logarithm term (Remark 2.3) is taken into account (see the third subplot of Fig. 6).

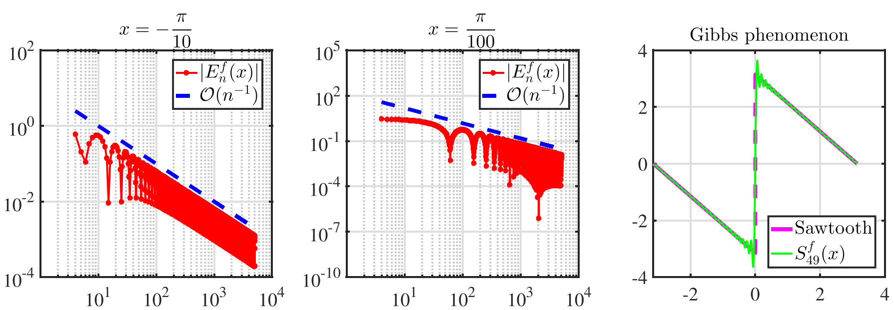

Fig. 5 Pointwise errors of Fourier expansions for sawtooth function $f(x) = -x + \pi \operatorname{sgn}(x)$ with singular points $x = 2k\pi,\ k \in \mathbb{Z}$. The graph of $f(x)$ and the Gibbs phenomenon are sketched in the last subplot

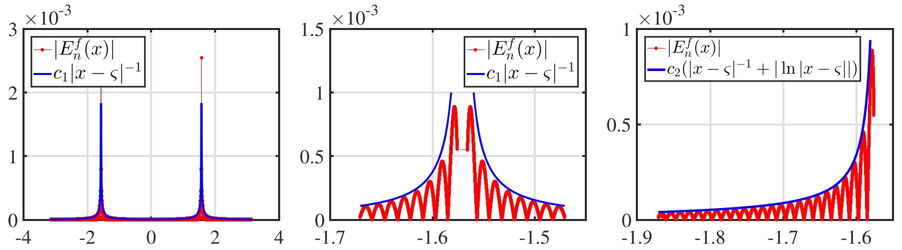

Fig. 6 Pointwise error of $E_n^f(x)$ of Fourier expansion for $f(x) = \arcsin \sin x$ around the singularities, where $c_1 = 1.1e-5$ and $c_2 = 9e-6$. The consistencies of $A(x)$ and $E_n^f(x)$ around the singularities are demonstrated in the zoomed-in graphs (the second and the third). The experiment is performed with $n = 249$

3 Asymptotic behaviors of $E_n^f(x)$ for fractional-regular functions

Generally, the algebraic and logarithmic singularities are two classical types of irregular behaviors, which can be described as $|x-\varsigma_k|^{\lambda_k}$ and $\ln|x-\varsigma_k|$ in a neighborhood I_{ς_k} of ς_k. Subsequently, the present section focuses on the 2π-periodic function $f(x)$ that is sufficiently smooth in I except for $S = \{\varsigma_k : -\pi < \varsigma_1 < \cdots < \varsigma_s < \pi\}$, around which $f(x)$ can be described by

$$f(x) = |x-\varsigma_k|^{\lambda_k} h_k(x), \ x \in I_{\varsigma_k} \tag{3.1}$$

or

$$f(x) = |x-\varsigma_k|^{\lambda_k} h_k(x) \ln|x-\varsigma_k|, \ x \in I_{\varsigma_k}, \tag{3.2}$$

where $I_{\varsigma_k} = (\varsigma_k - \delta, \varsigma_k + \delta) \subset (-\pi, \pi)$, $k = 1, \cdots, s$ with $\delta \in \left(0, \tilde{\delta}\right)$ and

$$\tilde{\delta} = \min\left\{1, \frac{\min_{0\le k\le s} |\varsigma_{k+1} - \varsigma_k|}{2}\right\}, \quad \varsigma_0 = -\pi, \ \varsigma_{s+1} = \pi$$

and all the $h_k(x)$ are sufficiently smooth.

Inspired by the translation invariance of periodic function and the method of separating singularities, we restrict our attention on the case $f(x)$ with only one internal singularity $\varsigma = 0$, without loss of generality. Consequently, the remaining of this section is mainly devoted to the consideration of the type of 2π-periodic functions sufficiently smooth on $I \setminus \{\varsigma\}$, which can be described around the origin by

$$f(x) = |x|^{\lambda} h(x), \ x \in I_0 = (-\delta, \delta) \subset [-\pi, \pi], \tag{3.3}$$

where $\lambda > -1$ and $h(x)$ is sufficiently smooth. In almost exactly the same way, our conclusions can be extended to the case of logarithmic singularity

$$f(x) = |x|^{\lambda} h(x) \ln|x|, \ x \in I_0, \tag{3.4}$$

also $h(x)$ here is sufficiently smooth and $\lambda > -1$.

The n-th remainder of the expansion for $f(x)$ is

$$E_n^f(x) = \sum_{k=n+1}^{\infty} \left[a_k^f \cos(kx) + b_k^f \sin(kx)\right]. \tag{3.5}$$

Obviously, the coefficients a_k^f, b_k^f dominate the error $E_n^f(x)$, so we estimate firstly the convergence orders of a_k^f and b_k^f. Recently, Dominguez, Graham and Kim [33, Lemma 3.1] and Xiang, He and Cho [34, Lemmas 3.1 and 3.2] generalized the van der Corput's Lemma [35].

Lemma 3.1 *([33, 34]) For $\alpha > -1$ and $t \in [0, b]$, $b > 0$, we have for w tending to ∞ that*

$$\int_0^t \tau^\alpha e^{iw\tau} d\tau = \begin{cases} \mathcal{O}(w^{-(1+\alpha)}), & \alpha \in (-1, 0] \\ \mathcal{O}(w^{-1}), & \alpha > 0 \end{cases} \tag{3.6}$$

and

$$\int_0^t \tau^\alpha \ln \tau e^{iw\tau} d\tau = \begin{cases} \mathcal{O}\left(\frac{1+|\ln w|}{w^{1+\alpha}}\right), & \alpha \in (-1, 0], \\ \mathcal{O}(w^{-1}), & \alpha > 0, \end{cases} \tag{3.7}$$

where the constants in $\mathcal{O}$ are independent of w.

Lemma 3.2 *(van der Corput-type Lemma) Let f be defined on $[0, b]$ and $f' \in L^1[0, b]$. Then we have for $\alpha > -1$ and $b > 0$ that*

$$\left|\int_0^b \tau^\alpha f(\tau) e^{iw\tau} d\tau\right| = \mathfrak{D} \cdot \begin{cases} \mathcal{O}(w^{-(1+\alpha)}), & \alpha \in (-1, 0] \\ \mathcal{O}(w^{-1}), & \alpha > 0 \end{cases} \tag{3.8}$$

and

$$\left|\int_0^b \tau^\alpha \ln \tau f(\tau) e^{iw\tau} d\tau\right| = \mathfrak{D} \cdot \begin{cases} \mathcal{O}\left(\frac{1+|\ln w|}{w^{1+\alpha}}\right), & \alpha \in (-1, 0] \\ \mathcal{O}(w^{-1}), & \alpha > 0 \end{cases} \tag{3.9}$$

hold for w tending to ∞, where $\mathfrak{D} = \left(|f(b)| + \int_0^b |f'(\tau)| d\tau\right)$ and the constants in $\mathcal{O}$ are independent of w.

Proof Let $F(\tau) = \int_0^\tau t^\alpha e^{iwt} dt$. Then we have

$$\left|\int_0^b \tau^\alpha f(\tau) e^{iw\tau} d\tau\right| = \left|\int_0^b \tau^\alpha dF(\tau)\right| \le \left[\left|f(b)\right| + \int_0^b \left|f'(\tau)\right| d\tau\right] \|F\|_\infty,$$

which, together with (3.6), follows (3.8). Additionally, we arrive at (3.9) immediately in the same way by setting $F(\tau) = \int_0^\tau t^\alpha \ln t e^{iwt} dt$.

Remark 3.1 It is evident that both (3.6) and (3.8) also hold for w approaching to $-\infty$. Furthermore, when assumptions in Lemma 3.2 are satisfied on $[b, 0]$ $(b < 0)$, (3.6) and (3.8) still hold with the integral limits interchanged, w replaced by $-w$. Meanwhile, (3.7) and (3.9) are true with w in logarithm terms replaced by $|w|$.

With the help of Lemma 3.2 and Remark 3.1, we have that

Lemma 3.3 *Let $f(x)$ be the function defined in (3.3). Then it holds for all $\lambda > -1$ that*

$$a_k^f = \mathcal{O}(k^{-\lambda-1}), \quad b_k^f = \mathcal{O}(k^{-\lambda-1}), \; k \to \infty, \tag{3.10}$$

where the constants in $\mathcal{O}$ are independent of k.

Proof By $v = [\lambda]$ we denote the largest integer not larger than λ. In the case λ not an integer, by integrating by parts repeatedly we have for the projection integral that

$$\begin{aligned}
&\left|\int_{-\pi}^{\pi} f(x) e^{ikx} dx\right| = \frac{(\lambda)_{v+1}}{k^{v+1}}\left|\int_{-\pi}^{\pi} f^{(v+1)}(x) e^{ikx} dx\right| \qquad (3.11)\\
\le&\frac{(\lambda)_{v+1}}{k^{v+1}}\left|\left\{\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi}\right\} f^{(v+1)}(x) e^{ikx} dx\right|\\
=&\frac{(\lambda)_{v+1}}{k^{v+1}}\left|\int_0^\delta \left[x^\lambda h(x)\right]^{(v+1)} e^{ikx} dx + \int_0^\delta \left[x^\lambda h(-x)\right]^{(v+1)} e^{-ikx} dx\right|\\
&+ \mathcal{O}(k^{-v-1}),
\end{aligned}$$

where we used in (3.11) the sufficient smoothness of $f(x)$ on $I \setminus (-\delta, \delta)$ and the Pochhammer symbol $(\lambda)_{v+1} = \lambda(\lambda-1)\cdots(\lambda-v)$ with $(\lambda)_0 = 1$.

From Leibniz's formula it follows that

$$\begin{aligned}
\left[x^\lambda h(x)\right]^{(v+1)} =& x^{\lambda-v-1} \sum_{l=0}^{v+1} \binom{v+1}{l} x^{v-\lambda+1} (x^\lambda)^{(l)} h^{(v-l+1)}(x)\\
=&: (x)^{\lambda-v-1} z(x),
\end{aligned} \tag{3.12}$$

where

$$z(x) = p\left[x, h(x), h'(x), \cdots, h^{(v+1)}(x)\right]$$

is a sufficiently smooth function due to that p is a polynomial of $v+3$ variables.

Then, we obtain immediately from (3.12) and Lemma 3.2 that

$$\left|\int_0^\delta \left[x^\lambda h(x)\right]^{(v+1)} e^{ikx} dx\right| = \left|\int_0^\delta x^{\lambda-v-1} z(x) e^{ikx} dx\right| = \mathcal{O}(k^{-\lambda+v}). \tag{3.13}$$

By analogous arguments, it is easy to obtain from Remark 3.1 that

$$\left|\int_0^{\delta}\left[x^{\lambda}h(-x)\right]^{(v+1)}\mathrm{e}^{-ikx}dx\right| = \mathcal{O}(k^{-\lambda+v}). \tag{3.14}$$

Thus, substituting (3.13) and (3.14) into (3.11) leads to

$$\left|\int_{-\pi}^{\pi} f(x)\mathrm{e}^{ikx}dx\right| = \mathcal{O}(k^{-\lambda-1}). \tag{3.15}$$

For λ being an integer, we obtain (3.15) directly by integration by parts $\lambda+1$ times.

Now we arrive at the conclusion (3.10) by (3.15), since

$$\left|\int_{-\pi}^{\pi} f(x)\mathrm{sc}(kx)dx\right| \le \left|\int_{-\pi}^{\pi} f(x)\mathrm{e}^{ikx}dx\right|,$$

where $\mathrm{sc}(x) = \cos x$ or $\sin x$.

The analogous conclusion for the logarithmic case (3.4) can be checked in the same manner by replacing (3.12) with

$$\begin{aligned}\left[x^{\lambda}h(x)\ln x\right]^{(v+1)} &= x^{\lambda-v-1}\sum_{l=0}^{v+1}\sum_{r=0}^{l}\frac{(v+1)!(\lambda)_r x^{v+1-r}(\ln x)^{(l-r)}}{(v+1-l)!(l-r)!r!}h^{v+1-l}(x)\\ &=: (x)^{\lambda-v-1}\tilde{p}\left[x, \ln x, h(x), h'(x), \cdots, h^{(v+1)}(x)\right], \end{aligned} \tag{3.16}$$

where $\tilde{p}$ is a polynomial of $v+4$ variables, with the degree of $\ln x$ at most 1.

Corollary 3.1 *Let $f(x)$ be the function defined in (3.4). Then it holds for all $\lambda > -1$ that*

$$a_k^f = \mathcal{O}\left(k^{-\lambda-1}\ln k\right), \quad b_k^f = \mathcal{O}\left(k^{-\lambda-1}\ln k\right), \ k \to \infty, \tag{3.17}$$

where the constants in $\mathcal{O}$ are independent of k.

Proof The proof can be completed in the analogous manner by (3.9) and (3.16).

Theorem 3.1 *Suppose $f(x)$ is a function defined in (3.3) or (3.4) with $\lambda > 0$, then*

$$\|E_n^f\|_{L^{\infty}([-\pi,\pi])} = \begin{cases}\mathcal{O}(n^{-\lambda}), & \textit{for } (3.3),\\ \mathcal{O}(n^{-\lambda}\ln n), & \textit{for } (3.4).\end{cases}$$

Proof Substitute (3.10) into (3.5), it holds for some constant G that

$$\|E_n^f\|_{L^{\infty}([-\pi,\pi])} \le G\sum_{k=n+1}^{\infty} k^{-\lambda-1} \le G\int_n^{\infty} x^{-\lambda-1}dx = \mathcal{O}(n^{-\lambda}).$$

Similarly, by substituting (3.17) into (3.5) and when $\ln n \geq \frac{1}{\lambda+1}$, we have

$$\|E_n^f\|_{L^\infty([-\pi,\pi])} \leq G \sum_{k=n+1}^{\infty} k^{-\lambda-1} \ln k \leq G \int_n^{\infty} x^{-\lambda-1} \ln x dx = \mathcal{O}(n^{-\lambda} \ln n).$$

In the remainder of this section, we will focus on the pointwise error $E_n^f(x)$ corresponding to $f(x)$ defined by (3.3) based on the smoothness of $\phi(x,t)$ and the van der Corput-type Lemma. It is obvious from Remark 2.1 that $\phi(x,t)$ w.r.t. $f(x)$ is included in $C^{v}(I)$ when $x \in [-\pi, 0) \cup (0, \pi]$, and in $C^{v-1}(I)$ when $x = 0$, where $v = [\lambda]$.

Theorem 3.2 *Let $f(x)$ be a function defined in (3.3). Then the corresponding pointwise error $E_n^f(x)$ enjoys the following decay orders*

$$E_n^f(x) = \begin{cases} A(x) \cdot \mathcal{O}(n^{-\lambda-1}), & x \in I_0 \setminus \{0\}, \quad \lambda > -1, & (3.19a) \\ \mathcal{O}(n^{-\lambda-1}), & x \in I \setminus I_0, \quad \lambda > -1, & (3.19b) \\ \mathcal{O}(n^{-\lambda}), & x = 0, \quad \lambda > 0, & (3.19c) \end{cases}$$

where $A(x) = |x|^{-1}$ for x near 0.

Proof By $C,\ C'$ we denote some positive constants that may be unequal in different places in the current proof. We focus our proof solely on the case where λ is not an integer, as the cases of $\lambda = 0, 1, \cdots$ can be reduced to the specific examples covered in Corollary 2.1.

(1) We first consider the case $x \in [-\pi, 0) \cup (0, \pi]$.

- **For the decay order of $b_n^{\phi(x,t)}$:** According to Lemma 2.2, we have for $x \in [-\pi, 0) \cup (0, \pi]$ that

$$|b_n^{\phi(x,t)}| = \frac{1}{\pi n^{v+1}} \left| \int_{-\pi}^{\pi} \partial_t^{v+1} \phi(x,t) \sin\left(nt - \frac{v+1}{2}\pi\right) dt \right|$$
$$\leq \frac{1}{\pi n^{v+1}} \left| \left(\int_0^{\delta} + \int_{\delta}^{\pi} + \int_{-\delta}^{0} + \int_{-\pi}^{-\delta} \right) \partial_t^{v+1} g(x,t) \mathrm{e}^{int} dt \right|. \quad (3.20)$$

In the case $x \in (0, \delta)$: For the first integral in the parentheses of (3.20), it is obvious that

$$\int_0^{\delta} \partial_t^{v+1} g(x,t) \mathrm{e}^{int} dt = \int_0^{\delta} \partial_t^{v+1} \left[\frac{x^{\lambda} h(x) - t^{\lambda} h(t)}{x-t} z_{0,0}(x-t) \right] \mathrm{e}^{int} dt$$
$$= \int_0^{\delta} \partial_t^{v+1} \left[t^{\lambda} \frac{h(x) - h(t)}{x-t} z_{0,0}(x-t) \right] \mathrm{e}^{int} dt$$
$$+ h(x) \int_0^{\delta} \partial_t^{v+1} \left[\frac{x^{\lambda} - t^{\lambda}}{x-t} z_{0,0}(x-t) \right] \mathrm{e}^{int} dt. \quad (3.21)$$

From Lemma 3.2 we have that for the first integral in the last formula of (3.21)

$$\int_0^\delta \partial_t^{v+1}\Big[t^\lambda \cdot \frac{h(x)-h(t)}{x-t}\cdot z_{0,0}(x-t)\Big]\mathrm{e}^{int}dt = \int_0^\delta t^{\lambda-v-1}u(x,t)\mathrm{e}^{int}dt \\ =\mathcal{O}\big(n^{v-\lambda}\big), \tag{3.22}$$

since

$$u(x,t) = \sum_{k=0}^{v+1}\sum_{j=0}^{k}\sum_{l=0}^{j}\frac{(v+1)!(\lambda)_{v+1-k}t^k z_{j-l,l}(x-t)}{(v+1-k)!(k-j)!(j-l)!l!}\left(\frac{h(x)-h(t)}{x-t}\right)^{(k-j)}$$

is smooth.

For the second integral in the last formula of (3.21), we get from (2.5) and Taylor's theorem that

$$\int_0^\delta \partial_t^{v+1}\Big[\frac{x^\lambda - t^\lambda}{x-t}z_{0,0}(x-t)\Big]\mathrm{e}^{int}dt \\ =\Big(\sum_{k=0}^{v-2}+\sum_{k=v-1}^{v+1}\Big)\int_0^\delta \sigma_k(x,t)H_{v+1,k}(x-t)\mathrm{e}^{int}dt, \tag{3.23}$$

where

$$\sigma_k(x,t) = \frac{x^\lambda - \sum_{l=0}^{k}\frac{(\lambda)_l t^{\lambda-l}}{l!}(x-t)^l}{(x-t)^{k+1}} \tag{3.24}$$

satisfies for $t\in[0,\delta]$ that

$$\big|\sigma_k(x,t)\big| = \left|\frac{(\lambda)_{k+1}\xi_k^{\lambda-k-1}}{(k+1)!}\right| \le C,\quad k=0,\cdots,v-1, \tag{3.25a}$$

$$\frac{\big|\partial_t\sigma_k(x,t)\big|}{k+1} = \left|\frac{x^\lambda - \sum_{l=0}^{k+1}\frac{(\lambda)_l t^{\lambda-l}}{l!}(x-t)^l}{(x-t)^{k+2}}\right| \\ = \left|\frac{(\lambda)_{k+2}\xi_{k+1}^{\lambda-k-2}}{(k+2)!}\right| \le C,\quad k=0,\cdots,v-2 \tag{3.25b}$$

with ξ_k locating between x and t. Then from Lemma 3.2 it yields that

$$\sum_{k=0}^{v-2}\int_0^\delta \sigma_k(x,t)H_{v+1,k}(x-t)\mathrm{e}^{int}dt = \mathcal{O}(n^{-1}). \tag{3.26}$$

Whereas, it holds for $k=v, v+1$ and $t>0$ that

$$\mu_k(x,t) := t^{v+1-\lambda}\sigma_k(x,t) = \frac{(\lambda)_{k+1}\xi_k^{\lambda-k-1}}{(k+1)!t^{\lambda-v-1}}\begin{cases} >0, & k=v\\ <0, & k=v+1\end{cases} \tag{3.27}$$

with some ξ_k between x and t. Analogous to [26, Appendix A] by an elementary proof, we can show that $\partial_t \mu_k(x,t) \geq 0$ and then $\mu_k(x,t)$ is monotonically increasing w.r.t. $t \geq 0$ for $k = v, v+1$. Therefore, we get from (3.27) and the smoothness of $H_{r,k}(x-t)$ that

$$\max_{t\in[0,\delta]} |\mu_v(x,t)| \leq C'\left(\frac{t}{\xi_v}\right)^{v+1-\lambda} \begin{cases} \leq C' \frac{t^{v+1-\lambda}}{x} \leq \frac{C}{x}, & t > x, \\ \leq C', & t \leq x, \end{cases} \tag{3.28a}$$

$$\max_{t\in[0,\delta]} |\mu_{v+1}(x,t)| \leq C' \lim_{t\to 0} |\mu_{v+1}(x,t)| = C' \frac{(\lambda)_{v+1}}{(v+1)!x}, \tag{3.28b}$$

$$\begin{aligned} &\int_0^\delta \left|\partial_t\left[\mu_k(x,t)H_{v+1,k}(x-t)\right]\right| dt \leq C' \int_0^\delta \left[\left|\partial_t \mu_k(x,t)\right| + \left|\mu_k(x,t)\right|\right] dt \\ &\leq C'\left|\mu_k(x,\delta) - \mu_k(x,0)\right| + \int_0^\delta \left|\mu_k(x,t)\right| dt \leq \frac{C}{x}, \;\; k = v, \;\; v+1. \end{aligned} \tag{3.28c}$$

With (3.28) in the hand, we have immediately by Lemma 3.2 that

$$\begin{aligned} \int_0^\delta \sigma_k(x,t)H_{v+1,k}(x-t)\mathrm{e}^{int}dt &= \int_0^\delta t^{\lambda-v-1}\mu_k(x,t)H_{v+1,k}(x-t)\mathrm{e}^{int}dt \\ &= |x|^{-1}\mathcal{O}\left(n^{v-\lambda}\right), \quad k = v, \;\; v+1. \end{aligned} \tag{3.29}$$

Additionally, from (3.25a) and (3.29), it holds that

$$\begin{aligned} &\left|\int_0^\delta \sigma_{v-1}(x,t)H_{v+1,v-1}(x-t)\mathrm{e}^{int}dt\right| \\ &\leq \frac{1}{n}\left[C + v\left|\int_0^\delta \sigma_v(x,t)H_{v+1,v-1}(x-t)\mathrm{e}^{int}dt\right|\right] \leq |x|^{-1}\mathcal{O}(n^{-1}). \end{aligned} \tag{3.30}$$

Combining (3.23), (3.26),(3.29) and (3.30) yields that

$$\int_0^\delta \partial_t^{v+1}\left[\frac{x^\lambda - t^\lambda}{x-t} z_{0,0}(x-t)\right]\mathrm{e}^{int}dt = |x|^{-1}\mathcal{O}\left(n^{v-\lambda}\right). \tag{3.31}$$

Then, the following decay order is an immediate result by substituting (3.22) and (3.31) into (3.21)

$$\int_0^\delta \partial_t^{v+1} g(x,t)\mathrm{e}^{int}dt = |x|^{-1}\mathcal{O}\left(n^{v-\lambda}\right). \tag{3.32}$$

For the second integral in the last formula of (3.20), we have for $x \in (0, \frac{\delta}{2})$ that

$$\int_{\delta}^{\pi} \partial_t^{v+1} g(x,t)\mathrm{e}^{int} dt = \int_{\delta}^{\pi} \partial_t^{v+1}\left[\frac{f(x)-f(t)}{x-t} z_{0,0}(x-t)\right]\mathrm{e}^{int} dt$$
$$= x^{\lambda} h(x) \int_{\delta}^{\pi} \partial_t^{v+1}\left[\frac{z_{0,0}(x-t)}{x-t}\right]\mathrm{e}^{int} dt - \int_{\delta}^{\pi} \partial_t^{v+1}\left[\frac{f(t)}{x-t} z_{0,0}(x-t)\right]\mathrm{e}^{int} dt$$
$$= x^{\lambda}\mathcal{O}(n^{-1}), \tag{3.33}$$

and for $x \in [\frac{\delta}{2}, \delta)$ that

$$\int_{\delta}^{\pi} \partial_t^{v+1} g(x,t)\mathrm{e}^{int} dt = \int_{\delta}^{\pi} \partial_t^{v+1}\left[\frac{f(x)-f(t)}{x-t} z_{0,0}(x-t)\right]\mathrm{e}^{int} dt$$
$$= \mathcal{O}(n^{-1}), \tag{3.34}$$

by the sufficient smoothness of $f(t)$ and $z_{0,0}(x-t)$.
For the third integral in the parentheses of (3.20), we have by Remark 3.1 that

$$\int_{-\delta}^{0} \partial_t^{v+1} g(x,t)\mathrm{e}^{int} dt = \int_{-\delta}^{0} \partial_t^{v+1}\left[\frac{x^{\lambda}h(x)-(-t)^{\lambda}h(t)}{x-t} z_{0,0}(x-t)\right]\mathrm{e}^{int} dt$$
$$= h(x)\int_{-\delta}^{0} \partial_t^{v+1}\left[\frac{x^{\lambda}-(-t)^{\lambda}}{x-t} z_{0,0}(x-t)\right]\mathrm{e}^{int} dt$$
$$+ \int_{-\delta}^{0} \partial_t^{v+1}\left[(-t)^{\lambda}\frac{h(x)-h(t)}{x-t} z_{0,0}(x-t)\right]\mathrm{e}^{int} dt$$
$$= \sum_{k=0}^{v+1}\int_{-\delta}^{0} (-t)^{\lambda-v-1} Q_k(x,t) H_{v+1,k}(x-t)\mathrm{e}^{int} dt + \mathcal{O}(n^{v-\lambda}), \tag{3.35}$$

where

$$Q_k(x,t) = (-t)^{v+1-\lambda}\cdot\frac{x^{\lambda}-\sum_{l=0}^{k}\frac{(-1)^l(\lambda)_l(-t)^{\lambda-l}}{l!}(x-t)^l}{(x-t)^{k+1}}$$
$$= \frac{(x-t)^{v+1-k}}{x-t}\left[\frac{x^{\lambda}-(-t)^{\lambda}}{(x-t)^{\lambda}}\left(\frac{-t}{x-t}\right)^{v+1-\lambda}\right.$$
$$\left. + \sum_{l=1}^{k}\frac{(-1)^{l+1}(\lambda)_l}{l!}\left(\frac{-t}{x-t}\right)^{v+1-l}\right] \tag{3.36}$$

satisfies

$$\max_{t\in[-\delta,0]}|Q_k(x,t)| \le \frac{C}{x-t}, \quad \max_{t\in[-\delta,0]}\left|\partial_t Q_k(x,t)\right| \le \frac{C}{(x-t)^2} \tag{3.37}$$

for $k = 0, 1, \cdots, v+1$. We obtain from (3.35), (3.37) and Remark 3.1 that

$$\int_{-\delta}^{0} \partial_t^{v+1} g(x,t) \mathrm{e}^{int} dt = |x|^{-1} \mathcal{O}(n^{v-\lambda}). \tag{3.38}$$

For the forth integral in the parentheses of (3.20), it is obvious by the sufficient smoothness of $f(t)$ on $[\pi, -\delta]$ that

$$\begin{aligned}\int_{-\pi}^{-\delta} \partial_t^{v+1} g(x,t) \mathrm{e}^{int} dt =& \int_{-\pi}^{-\delta} \partial_t^{v+1} \left[\frac{x^\lambda h(x) - f(t)}{x-t} z_{0,0}(x-t) \right] \mathrm{e}^{int} dt \\ =& \mathcal{O}(n^{-1}). \end{aligned} \tag{3.39}$$

Substituting (3.32), (3.33) or (3.34), (3.38) and (3.39) into (3.20) leads to

$$b_n^{\phi(x,t)} = |x|^{-1} \mathcal{O}(n^{-\lambda-1}), \ x \in (0, \delta). \tag{3.40}$$

In the cases of x locating in $(-\delta, 0), \ [\delta, \pi], \ [-\pi, -\delta]$**:** For the case $x \in (-\delta, 0)$ it can be proved in the exactly same way above that

$$b_n^{\phi(x,t)} = |x|^{-1} \mathcal{O}(n^{-\lambda-1}), \ x \in (-\delta, 0), \tag{3.41}$$

and for cases $x \in [\delta, \pi], \ [-\pi, -\delta]$, we have from Lemma 3.2 that

$$b_n^{\phi(x,t)} = \mathcal{O}(n^{-\lambda-1}), \ x \in I \setminus I_0, \tag{3.42}$$

since $f(t)$ is sufficiently smooth on $I \setminus I_0$ and can be described by $|t|^\alpha h(t)$ on $I_0 = (-\delta, \delta)$.

- **For the decay order of $a_n^{\varphi(x,t)}$:** By Lemma 3.3, it is obvious that

$$\begin{aligned} a_n^{\varphi(x,t)} =& \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x,t) \cos(nt) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(x) - f(t) \right] \cos\left[n(x-t) \right] dt \\ =& -\frac{\cos(nx)}{2\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt - \frac{\sin(nx)}{2\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \\ =& -\frac{\cos(nx)}{2} a_n^f - \frac{\sin(nx)}{2} b_n^f = \mathcal{O}(n^{-\lambda-1}). \end{aligned} \tag{3.43}$$

Now, we arrive at the conclusion (3.19a) by substituting (3.40), (3.41) and (3.43) into (2.3), arrive at (3.19b) by (3.42) and (3.43) into (2.3).

(2) We consider the cases of x coinciding with the singularity 0.

For $\lambda > 0$, we have by integrating by parts repeatedly and Lemma 3.2 that

$$\begin{aligned}
\pi\left|b_n^{\phi(0,t)}\right| \le& \left|\int_{-\pi}^{\pi}\phi(0,t)\mathrm{e}^{int}dt\right| = \frac{1}{2}\left|\int_{-\pi}^{\pi}f(-t)\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\mathrm{e}^{int}dt\right|\\
=&\frac{1}{2n^{v}}\left|\int_{I\setminus(-\delta,\delta)}\left[f(-t)\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\right]^{(v)}\mathrm{e}^{int}dt+\int_{-\delta}^{\delta}\left[|t|^{\lambda}h(-t)\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\right]^{(v)}\mathrm{e}^{int}dt\right|\\
\le&\mathcal{O}(n^{-v-1})+\frac{1}{n^{v}}\left|\int_{0}^{\delta}\left[t^{\lambda}h(-t)z_{0,0}(t)\right]^{(v)}\mathrm{e}^{int}dt\right|\\
&+\frac{1}{n^{v}}\left|\int_{0}^{\delta}\left[t^{\lambda}h(t)z_{0,0}(t)\right]^{(v)}\mathrm{e}^{-int}dt\right|\\
=&\frac{1}{n^{v}}\left|\int_{0}^{\delta}t^{\lambda-v-1}\tilde{w}_1(t)\mathrm{e}^{int}dt\right|+\frac{1}{n^{v}}\left|\int_{0}^{\delta}t^{\lambda-v-1}\tilde{w}_2(t)\mathrm{e}^{-int}dt\right|+\mathcal{O}(n^{-v-1})\\
=&\frac{1}{n^{v}}\mathcal{O}(n^{v-\lambda})=\mathcal{O}(n^{-\lambda}),
\end{aligned}\tag{3.44}$$

where $\tilde{w}_1(t)$ and $\tilde{w}_2(t)$ are smooth functions on $[0,\delta]$.

Then the conclusion (3.19c) is obtained by combining (2.3), (3.43) and (3.44).

By the same approach of separating irregularities for Corollary 2.1 and the same method of the proof for Theorem 3.2, we have that for the case (3.1) with multiple irregularities

Corollary 3.2 *Let $f(x)$ be a function defined in (3.1). Then the corresponding pointwise error $E_n^f(x)$ enjoys the following decay orders*

$$E_n^f(x)=\begin{cases}A_k(x)\cdot\mathcal{O}(n^{-\lambda-1}),\ x\in I_{\varsigma_k}\setminus\{\varsigma_k\},\lambda>-1, & (3.45a)\\ \mathcal{O}(n^{-\lambda-1}),\ x\in I\setminus\left(\cup_{k=1}^{s}I_{\varsigma_k}\right),\lambda>-1, & (3.45b)\\ \mathcal{O}(n^{-\min\{\lambda_k,\lambda+1\}}),\ x=\varsigma_k,\ \lambda_k>0, & (3.45c)\end{cases}$$

where $\lambda=\min_{1\le k\le s}\{\lambda_k\}$, $A_k(x)=|x-\varsigma_k|^{-1}$ for x near ς_k and $A_k(x)$ can be removed if $\lambda_k\ge\lambda+1$.

Proof Similarly to the proof of Corollary 2.1 and by the same definition of U_k in Corollary 2.1 we have

$$\pi n^{v+1}\left|b_n^{\phi(x,t)}\right|\le\left|\sum_{k=1}^{s}\int_{U_k}+\int_{I\setminus(\cup_{k=1}^{s}U_k)}\partial_t^{v+1}g(x,t)\mathrm{e}^{-int}dt\right|,$$

$$\int_{I\setminus(\cup_{k=1}^{s}U_k)}\partial_t^{v+1}g(x,t)\mathrm{e}^{-int}dt=\mathcal{O}(n^{-1})$$

and by the same method of the proof for Theorem 3.2 it holds that

$$\int_{U_k}\partial_t^{v+1}g(x,t)\mathrm{e}^{-int}dt=\int_{\mu_{k-1}}^{\mu_k}\partial_t^{v+1}g(x,t)\mathrm{e}^{-int}dt=|x-\varsigma_k|^{-1}\mathcal{O}(n^{v-\lambda}),$$

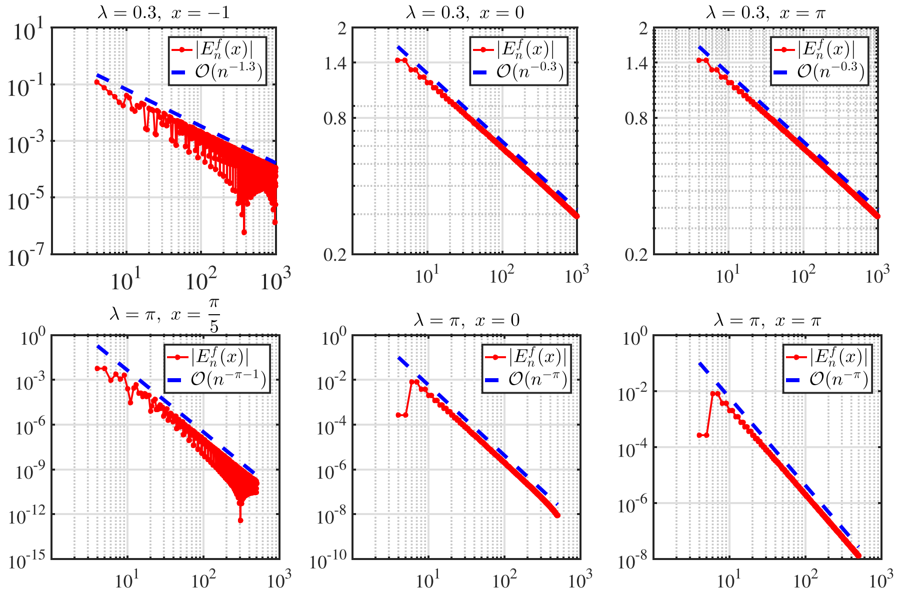

Fig. 7 Pointwise errors of Fourier expansions for function $f(x) = |\sin x|^\lambda e^{1+\sin x}$, with $\lambda = 0.3$ (the first row) and π (the second row)

where we translated the integral $\int_{U_k}$ onto a neighborhood of 0 by the change of variables $\tau = t - \varsigma_k$, the upper and lower integral bounds $\pm\pi$ should be changed into $\varsigma_k - \mu_{k-1}$ and $\mu_k - \varsigma_k$, respectively.

Then by the same approach of Theorem 3.2 the proof is completed.

To observe the error bounds (3.19) and (3.45) numerically, we consider the concrete function

$$f(x) = |\sin x|^\lambda e^{1+\sin x},\ \ x \in [-\pi, \pi]$$

with various values for λ (see Fig. 7). All of these convergence orders are attainable, which indicate that the estimates (3.19) and (3.45) are optimal.

In order to illustrate the behavior of $A(x)$ in front of $\mathcal{O}(n^{-\lambda-1})$, we concern the special case

$$f(x) = \left|\sin x\right|^{1.5} e^{1+\sin x}$$

(see Fig. 8). Again, the demonstration in the zoomed-in graph indicates that the estimates for $A(x)$ are sharp.

We also consider the following function as an example with different irregularities at various singularities

$$f(x) = |x|^\lambda h(x),\ \ x \in [-\pi, \pi], \tag{3.46}$$

where $h(x)$ is a 2π-periodic sufficiently smooth function.

Different from the Jacobi projection approximation, the two endpoints may be two corner points after a periodic extension being imposed on $f(x)$, which may result in

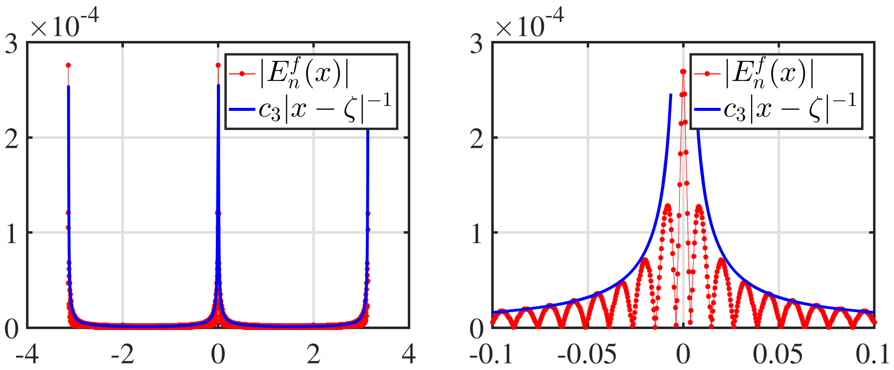

Fig. 8 Pointwise error of $E_n^f(x)$ of Fourier expansion for $f(x) = |\sin x|^{1.5} e^{1+\sin x}$, $x \in [-\pi, \pi]$ around the singularities, where $c_3 = 1.6\text{e}-6$. The consistencies of $A(x)$ and $E_n^f(x)$ around the singularities are demonstrated in the zoomed-in graph (the second). The experiment is performed with $n = 249$

a catastrophic deceleration on the decay rate of $E_n^f(x)$ (see Fig. 9). The convergence rate depends largely on the scale of λ when $0 < \lambda < 1$, and suffered from the endpoint singularities $x = \pm\pi$ when $\lambda > 1$ (see Corollary 3.3 and Fig. 11).

Based on the smooth degree of $\phi(x, t)$ and Corollary 3.2, it follows immediately that

Corollary 3.3 *Let $f(x)$ be defined in (3.46) with $\lambda > 0$. Then the pointwise error $E_n^f(x)$ enjoys the decay orders*

$$E_n^f(x) = \begin{cases} \tilde{A}(x) \cdot \mathcal{O}(n^{-2}), \ x \in (-\pi, \pi), \\ \mathcal{O}(n^{-1}), \ x = \pm\pi, \end{cases} \quad \lambda \geq 2$$

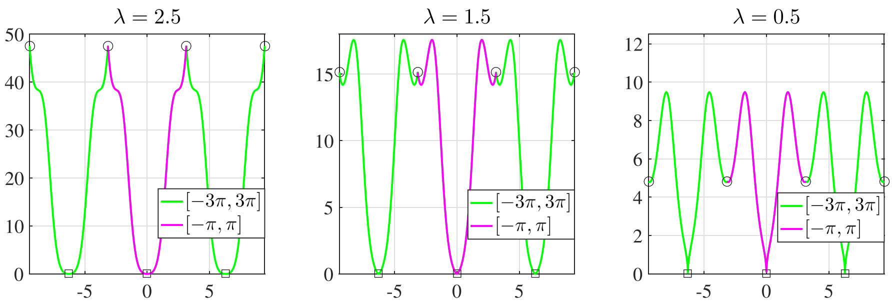

Fig. 9 The periodically extended functions of $f(x) = |x|^{\lambda} e^{1+\sin^2 x}$ with $\lambda = 2.5$ (left), 1.5 (middle) and 0.9 (right). The circled points are corner points owed to periodic extensions, and the squared points are singularities dependent on the size of λ

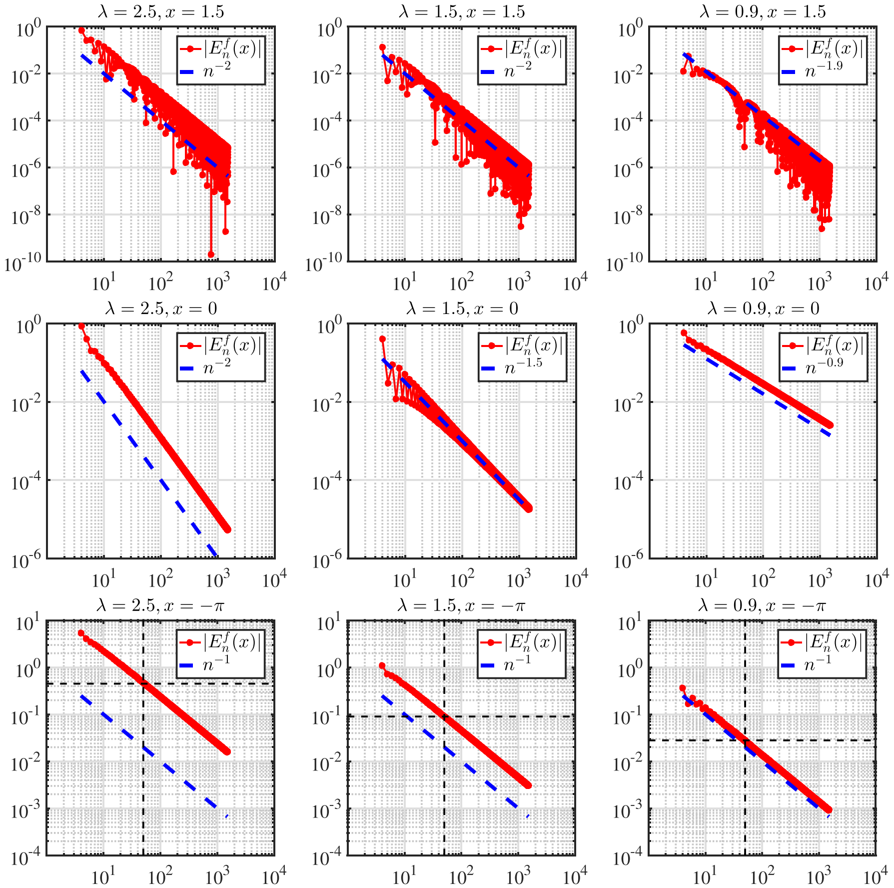

Fig. 10 Pointwise errors of Fourier expansions for function $f(x) = |x|^{\lambda} e^{1+\sin^2 x}$, with $\lambda = 2.5$ (the first column), 1.5 (the second column) and 0.9 (the third column)

with $\tilde{A}(x) = |x - \varsigma|^{-1}$ *for* x *near* $\varsigma = \pm\pi$ *and*

$$E_n^f(x) = \begin{cases} A(x) \cdot \mathcal{O}(n^{-2}), & x \in (-\pi, 0) \cup (0, \pi), \\ \mathcal{O}(n^{-\lambda}), & x = 0, \\ \mathcal{O}(n^{-1}), & x = \pm\pi, \end{cases} \qquad 1 < \lambda < 2,$$

$$E_n^f(x) = \begin{cases} A(x) \cdot \mathcal{O}(n^{-\lambda-1}), & x \in (-\pi, 0) \cup (0, \pi), \\ \mathcal{O}(n^{-\lambda}), & x = 0, \\ \mathcal{O}(n^{-1}), & x = \pm\pi, \end{cases} \qquad 0 < \lambda \le 1$$

with $A(x) = |x - \varsigma|^{-1}$ *for* x *near* $\varsigma = 0, \pm\pi$*, respectively.*

To illustrate Corollary 3.3 visually, we consider function $f(x) = |x|^{\lambda} \mathrm{e}^{1+\sin^2 x}$ with various values for λ. From Fig. 10, we see that the convergence orders are also in accordance with the statements of Corollary 3.3, and all of the estimated orders are attainable. From Fig. 11 and the third row of Fig. 10, we observe that the pointwise errors $E_n^f(\pm\pi)$ increase instead of decreasing as λ growing up, though still $E_n^f(\pm\pi)$ enjoy the convergence order $\mathcal{O}(n^{-1})$. A reasonable explanation would be that the corners (circled points in Fig. 9) of periodically extended functions become more acute as λ growing up, which erodes the smoothness of functions at the corners.

An appropriate therapy for this abnormal phenomenon is Fourier extensions, also called Fourier continuation (see [31, 32, 36] for details). Also, the Jacobi projection approximation is a really great choice, for which it holds [26]

$$\begin{aligned} &|E_n^f(x;\alpha,\beta)| \le C(x) n^{-\lambda-1},\ x \in (-1,0)\cup(0,1); \quad |E_n^f(0;\alpha,\beta)| \le C n^{-\lambda}; \\ &|E_n^f(1;\alpha,\beta)| \le C n^{-\lambda+\alpha-\frac{1}{2}}; \quad |E_n^f(-1;\alpha,\beta)| \le C n^{-\lambda+\beta-\frac{1}{2}}. \end{aligned}$$

On the account of the nice frequency resolution and the facility of FFT, Fourier expansion generally remains a favorable choice. An alternative efficient approach is to mollify the corners in Fig. 9 by multiplying $f(x)$ with a window function [30, 37], which eliminates the corners and may get higher convergence rates of the Fourier projection. For example, we often employ the C^{∞}-bump window

$$\omega_\rho(x) = \begin{cases} 1, & x \in [-\rho,\rho] \subset [-\pi,\pi], \\ \left[\exp\left(\frac{1}{\pi-|x|} + \frac{1}{\rho-|x|}\right) + 1\right]^{-1}, & x \in (-\pi,\pi)\setminus[-\rho,\rho], \\ 0, & \text{otherwise}, \end{cases} \tag{3.47}$$

and Tuckey window

$$T_\alpha(x) = \mathbf{1}_{[0,(1-\alpha)\pi]}(|x|) + \frac{1}{2}\Big[1 - \cos\Big(\frac{|x|-\pi}{\alpha}\Big)\Big]\mathbf{1}_{[(1-\alpha)\pi,\pi]}(|x|),\ 0 < \alpha < 1,$$

which force both ends of f to decay rapidly to 0. Then the windowed function $\omega_\rho f$ or $T_\alpha f$ is extended periodically, thus the corners (circled points in Fig. 9) of extensions of

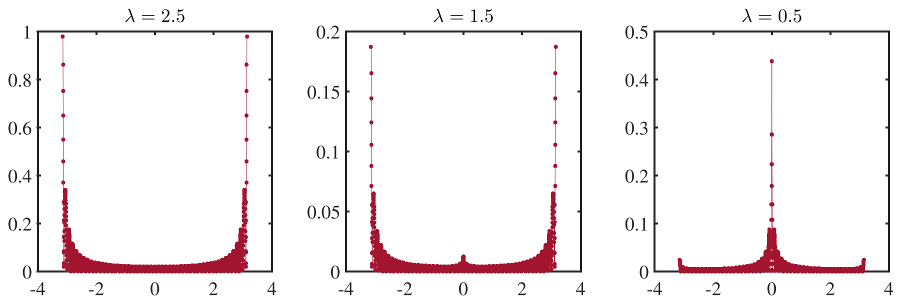

Fig. 11 The pointwise errors $|E_n^f(x)|$ of Fourier expansions for $f(x) = |x|^{\lambda} \mathrm{e}^{1+\sin^2 x}$ with $\lambda = 2.5$ (left), 1.5 (middle) and 0.9 (right). The experiments are performed with $n = 24$

$f(x)$ are eliminated, and now the smoothness is spoiled only by the inner singularities of $f(x)$. Figure 12 exhibits the numerical experiments for $f(x) = |x|^{1.5}e^{1+\sin^2 x}$ windowed by $\omega_\rho(x)$ with $\rho = 0.5$, which match perfectly the statements in Theorem 3.2, Corollaries 3.2 and 3.3. Furthermore, the window functions can be employed to eliminate the jump discontinuities at boundaries, which may lead to Gibbs phenomenon after a periodic extension (see Fig. 12).

Additionally, numerous windows have been developed for various functions (or signals), see [30, 37–39]. The essential features of window functions are that they are sufficiently smooth on $\mathbb{R}$ and equal to 1 on a closed subset $[-\rho, \rho]$ of their supports $[-\pi, \pi]$, that is, the general window $\varpi_\rho(x)$ should satisfy

$$\begin{aligned} &(1)\ 0 \le \varpi_\rho(x) \le 1, \text{ for } x \in (-\pi, \pi), \\ &(2)\ \varpi_\rho(x), \text{ for } x \in \mathbb{R} \setminus (-\pi, \pi), \\ &(3)\ \varpi_\rho(x) = 1, \text{ for } x \in [-\rho, \rho]. \end{aligned} \tag{3.48}$$

Window's constant value on $[-\rho, \rho]$ leaves the original function intact on the closed subset, and the regularity of windowed function is improved by the smooth and fast fall off near the endpoints. For more details on the windowed function $\varpi_\rho(x) f(x)$, refer to [30, Corollary 4.5, Theorem 4.6].

Specially, the C^∞-bump window $\omega_\rho(x)$ has infinitely smooth derivatives, which decay exponentially near $x = \pm\pi$, $\pm\rho$ and vanish on $\mathbb{R} \setminus \{x : \rho \le |x|\}$. Consequently, multiplying $f(x)$ by $\omega_\rho(x)$ erases the corners or jump discontinuities of extensions of $f(x)$, without damage to the smoothness of other areas, which results in that the statements of Theorem 3.2 and Corollary 3.2 are also applicable to $\omega_\rho(x) f(x)$.

The preceding discussion presents a methodology for obtaining the optimal pointwise convergence rates of Fourier projection approximations for functions that are

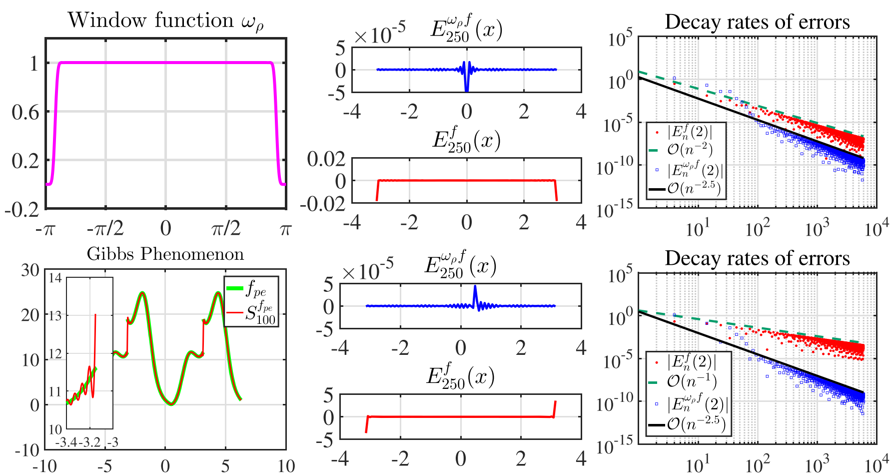

Fig. 12 Experiments for functions $f(x) = |x|^{1.5}e^{1+\sin^2 x}$ (the first row) and $f(x) = |x - 0.5|^{1.5}e^{1+\sin^2 x}$ (the second row) mollified by $\omega_\rho(x)$, $\rho = 0.5$ (the first subplot). Notation f_{pe} refers to the periodic extension of f

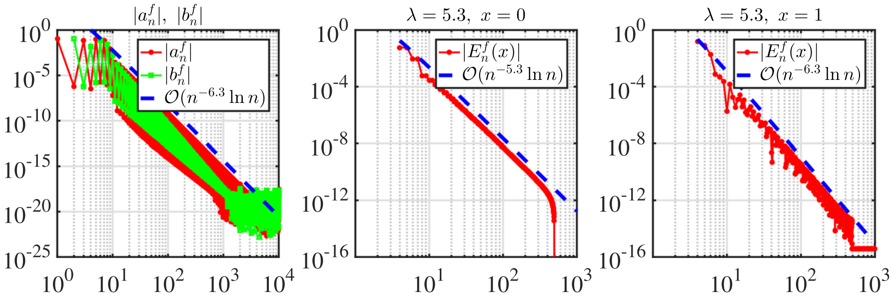

Fig. 13 The decay behaviors of coefficients a_n^f and b_n^f (left) and the pointwise errors $|E_n^f(x)|$ of Fourier expansions for $f(x) = \mathrm{e}^{1+\sin x}|\sin x|^\lambda \ln|\sin x|$ with $\lambda = 5.3$ at the finitely regular point (middle) and the sufficiently smooth point (right). In order to facilitate the use of Chebfun, we added an epsilon term to the logarithm term in the numerical experiment, that is, $f(x) = \mathrm{e}^{1+\sin x}|\sin x|^\lambda \ln(\mathrm{esp} + |\sin x|)$

sufficiently smooth in the majority of the interval I, except for a few isolate finitely regular points characterized by $|x - \varsigma_k|^{\lambda_k}$. Similarly, by employing the logarithmic versions of generalized van der Corput's Lemma (see (3.7) in Lemma 3.1 and (3.9) in Lemma 3.2), we can arrive at the similar conclusions (only one more logarithmic term $\ln n$ attached) for functions exhibiting a different type of singularity, namely $|x - \varsigma_k|^{\lambda_k} \ln|x - \varsigma_k|$ as in (3.2).

Theorem 3.3 *Let $f(x)$ be a function defined in (3.2). Then the corresponding pointwise error $E_n^f(x)$ enjoys the following decay orders*

$$E_n^f(x) = \begin{cases} A_k(x) \cdot \mathcal{O}(n^{-\lambda-1}\ln n), & x \in I_{\varsigma_k} \setminus \{\varsigma_k\}, \quad \lambda > -1, \\ \mathcal{O}(n^{-\lambda-1}\ln n), & x \in I \setminus \left(\cup_{k=1}^s I_{\varsigma_k}\right), \quad \lambda > -1, \\ \mathcal{O}(n^{-\min\{\lambda_k, \lambda+1\}}\ln n), & x = \varsigma_k, \ \lambda_k > 0, \end{cases}$$

where $\lambda = \min_{1\le k\le s}\{\lambda_k\}$, $A_k(x) = |x - \varsigma_k|^{-1}$ for x near ς_k and $A_k(x)$ can be removed if $\lambda_k \ge \lambda + 1$.

We also illustrate this extension by the function

$$f(x) = e^{1+\sin x}|\sin x|^\lambda \ln|\sin x|, \ x \in [-\pi, \pi]$$

in Fig. 13, which displays both of decay orders of coefficients a_n^f and b_n^f and the pointwise errors $E_n^f(x)$, in good agreement with the theoretical results in Corollary 3.1 and Theorem 3.3.

4 Conclusions and discussions

In most harmonic analysis textbooks, one of conclusions on Fourier expansions usually goes as that, the truncated expansions for a periodic continuous function does not always converge, and the expansion for a v-th differentiable function enjoys convergence order $\mathcal{O}(n^{-v})$. Afterwards, some extreme examples will be given to illustrate the conclusion. Nevertheless, the functions we encounter in many settings are sufficiently smooth in the most parts of underlying domain, except for a few singularities. One may conclude from the previous discussions that the approximate quality corresponding to the smooth parts of $f(x)$ is much better than that around the contained singularities. So the computing costs can be reduced substantially when one focus on local approximations. Additionally, the results obtained describe again the localization theorem of Fourier series from the view of convergence rate. It is surprised to find that the superconvergence proved above is exactly similar to the phenomenon of Chebyshev interpolation, which is referred to as the third of the six myths by Trefethen in [40, Myth 3].

For aperiodic functions, Fourier transform plays an important role in spectrum analysis and signal reconstruction. The truncated Fourier integral can be represented by

$$\tilde{f}(x) = \frac{1}{2\pi}\int_{-M}^{M} \mathrm{e}^{iwx}dw \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-iwt}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-t)D_M(t)dt,$$

where $D_M(t) = \frac{2\sin(Mt)}{t}$ is the M-kernel of Fourier Transform. Since

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)D_M(t)dt,$$

one has a similar pointwise error formula as (2.3)

$$E_M^f(x) = f(x) - \tilde{f}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)-f(x-t)}{t}\sin(Mt)dt.$$

Hence, the similar explorations completed in previous sections can be developed for Fourier Transforms.

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Declarations

Conflict of interest The authors declare no competing interests.

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