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Fast numerical integration of highly oscillatory Bessel transforms with a Cauchy type singular point and exotic oscillators

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Abstract

In this article, we propose an efficient hybrid method to calculate the highly oscillatory Bessel integral $\int_0^1 \frac{f(x)}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x$ with the Cauchy type singular point, where $0 < \tau < 1, m \geq 0, 2\gamma \in N^+$. The hybrid method is established by combining the complex integration method with the Clenshaw– Curtis– Filon– type method. Based on the special transformation of the integrand and the additivity of the integration interval, we convert the integral into three integrals. The explicit formula of the first one is expressed in terms of the Meijer G function. The second is computed by using the complex integration method and the Gauss– Laguerre quadrature rule. For the third, we adopt the Clenshaw– Curtis– Filon– type method to obtain the quadrature formula. In particular, the important recursive relationship of the required modified moments is derived by utilizing the Bessel equation and the properties of Chebyshev polynomials. Importantly, the strict error analysis is performed by a large amount of theoretical analysis. Our proposed methods only require a few nodes and interpolation multiplicities to achieve very high accuracy. Finally, numerical examples are provided to verify the validity of our theoretical analysis and the accuracy of the proposed methods.

Keywords Bessel function · Cauchy type singular point · Clenshaw– Curtis– Filon– type method · Complex integration method · Chebyshev polynomials · Error analysis

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1 Introduction

In many application fields, such as quantum mechanics, celestial mechanics, electrodynamics, signal processing, medical imaging and fluid mechanics, highly oscillatory problems are involved [3, 4, 6, 7, 18]. The highly oscillatory Bessel integrals with a Cauchy type singular point and exotic oscillators are defined by

$$I[f] = \int_0^1 \frac{f(x)}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x, \tag{1.1}$$

where $0 < \tau < 1, \omega \gg 1, 2\gamma$ is a positive integer, $f(x)$ denotes a smooth function on $[0, 1]$ and $J_m(x)$ denotes the first kind of Bessel function of order m. We can deduce from [11] that the integral (1.1) exists if the function $f(x)$ satisfies Hölder's condition on the interval $[0, 1]$. Furthermore, the integral (1.1) is understood in the sense of principal value and is defined as follows

$$I[f] = \lim_{\varepsilon\to 0^+}\left(\int_0^{\tau-\varepsilon} + \int_{\tau+\varepsilon}^1\right)\frac{f(x)}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x.$$

The integral (1.1) has three significant characteristics. It is clear that the integral has a Cauchy type singularity at $x = \tau$. Moreover, if frequency ω is high enough, the integrand becomes very oscillatory. In addition, The oscillator x^γ has zero and stationary points at $x = 0$. It is highly time consuming for very large values of ω to use the traditional quadrature method such as Gaussian quadrature rules. Therefore, the calculation of the integral (1.1) becomes very difficult. We must investigate an effective quadrature rule to approximate the integral.

In recent years, a few researchers devoted to studying the calculation of highly oscillatory Bessel integrals. For the general form of $g(x)$, there were some methods [31, 36] to compute the integral $\int_a^b f(x)J_v(\omega g(x))\mathrm{d}x$. For the simple form of $g(x) = x$, many methods were also used to calculate the integral $\int_a^b f(x)J_v(\omega x)\mathrm{d}x$, for example, the asymptotic method [35], the numerical steepest descent method [8, 10], the Clenshaw–Curtis–Filon–type method [37], the modified Clenshaw–Curtis method [27] and the Levin-type method [34, 35]. Wang et al. [32] studied the asymptotics and fast computation of the following onesided oscillatory Hilbert transforms of the form

$$\int_0^{+\infty} \frac{f(t)}{t-x} e^{i\omega t}\mathrm{d}t,\ \omega > 0,\ x \geq 0.$$

Many methods [9, 13, 14, 20, 24, 29, 30, 32] were utilized for computing the Cauchy principal value integral of oscillating function

$$\int_{-1}^1 \frac{f(x)}{x-\tau} e^{i\omega x}\mathrm{d}x,\ -1 < \tau < 1. \tag{1.2}$$

Further, Xu et al. [39] constructed two efficient methods to calculate the Hilbert transform with the oscillatory Bessel function, as follows

$$\int_0^{+\infty} \frac{f(x)}{x-\tau} J_v(\omega x)\mathrm{d}x,\ 0<\tau<+\infty. \tag{1.3}$$

However, these methods cannot directly calculate the integral (1.1). In order to accurately and efficiently compute the integral (1.1), we must improve, modify or combine these methods.

In this paper, we first rewrite the integral (1.1) as

$$\int_0^1 \frac{f(x)}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x = f(\tau)\int_0^1 \frac{1}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x + \int_0^1 F(x) J_m(\omega x^\gamma)\mathrm{d}x, \tag{1.4}$$

where

$$F(x) = \begin{cases} \frac{f(x)-f(\tau)}{x-\tau}, & x \neq \tau, \\ f'(\tau), & x = \tau, \end{cases}$$

is a continuously differentiable function if $f(x) \in C^\infty[0,1]$. Then the first integral on the right of (1.4) is transformed as follows

$$\int_0^1 \frac{1}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x = \int_0^{+\infty} \frac{1}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x - \int_1^{+\infty} \frac{1}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x. \tag{1.5}$$

We let $I_{11} = \int_0^{+\infty} \frac{1}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x$ and $I_{12} = \int_1^{+\infty} \frac{1}{x-\tau} J_m(\omega x^\gamma)\mathrm{d}x$. According to the relationship between Meijer G function and Bessel function, the integral I_{11} can be explicitly computed via the Meijer G function. For the integral I_{12}, we establish new steepest descent integration paths to transform the integral into two infinite integrals on [0,+∞) on the basis of analytic continuation, and then use the Gauss–Laguerre quadrature rule to compute the resulting two infinite integrals. We use the Clenshaw–Curtis–Filon–type method to calculate the integral $\int_0^1 F(x) J_m(\omega x^\gamma)\mathrm{d}x$ in (1.4). First, the quadrature formula is obtained by interpolating $F(x)$ at the Clenshaw-Curtis point. Then, the recursive relationship of the modified moments is derived by utilizing integration by parts, the Bessel equation and the properties of Chebyshev polynomials. Finally, employing explicit formula can effectively calculate the values of the initial modified moments.

The structure of this paper is as follows. In Section 2, we first derive the explicit formula of the integral I_{11}. Then, we use the complex integration method to compute the integral I_{12}. Finally, we adopt the Clenshaw–Curtis–Filon–type method for calculating the integral $\int_0^1 F(x) J_m(\omega x^\gamma)\mathrm{d}x$. Moreover, the very useful recursive relationship of the required modified moments is deduced through detailed theoretical analysis. In Section 3, we carry out strict error analysis by a large amount of theoretical derivation. In Section 4, we verify the correctness of the theoretical analysis through numerical experiments. Meanwhile, these numerical experiments show the accuracy and efficiency of the proposed methods. The conclusion of this paper is given in the final.

2 Hybrid method

In this section, we devise a hybrid method. First, the explicit formula of I_{11} is given by the following lemma.

Lemma 2.1 *If* 2γ *is a positive integer, then it is true that*

$$I_{11}=\int_0^{+\infty}\frac{1}{x-\tau}J_m(\omega x^\gamma)\mathrm{d}x=-\pi G^{1+2\gamma,2\gamma}_{4\gamma,2+4\gamma}\left(\frac{1}{4}\omega^2\tau^{2\gamma}\left|\begin{matrix}0,\frac{1}{2\gamma},\frac{2}{2\gamma},\cdots,\frac{2\gamma-1}{2\gamma},-\frac{1}{4\gamma},\frac{1}{4\gamma},\cdots,\frac{4\gamma-3}{4\gamma}\\0,\frac{1}{2\gamma},\frac{2}{2\gamma},\cdots,\frac{2\gamma-1}{2\gamma},\frac{m}{2},-\frac{m}{2},-\frac{1}{4\gamma},\frac{1}{4\gamma},\cdots,\frac{4\gamma-3}{4\gamma}\end{matrix}\right.\right), \tag{2.1}$$

where $G^{p,q}_{u,v}(\cdot)$ is the Meijer G function [5, p. 206].

Proof We can know from [5, p. 219] and [16],

$$J_m(\omega x^\gamma)=G^{1,0}_{0,2}\left(\frac{1}{4}\omega^2x^{2\gamma}\left|_{\frac{1}{2}m,-\frac{1}{2}m}\right.\right), \tag{2.2}$$

$$\int_0^{+\infty}\frac{t^{\alpha-1}}{t-\tau}G^{u,v}_{p,q}\left(at^l\left|\begin{matrix}a_1,a_2,\cdots,a_v,a_{v+1},\cdots,a_p\\b_1,b_2,\cdots,b_u,b_{u+1},\cdots,b_q\end{matrix}\right.\right)\mathrm{d}t$$
$$=-\pi\tau^{\alpha-1}G^{u+l,v+l}_{p+2l,q+2l}\left(a\tau^l\left|\begin{matrix}\frac{1-\alpha}{l},\cdots,\frac{l-\alpha}{l},a_1,\cdots,a_v,a_{v+1},\cdots,a_p,\frac{-\alpha+\frac{1}{2}}{l},\cdots,\frac{l-\alpha-\frac{1}{2}}{l}\\\frac{1-\alpha}{l},\cdots,\frac{l-\alpha}{l},b_1,\cdots,b_u,b_{u+1},\cdots,b_q,\frac{-\alpha+\frac{1}{2}}{l},\cdots,\frac{l-\alpha-\frac{1}{2}}{l}\end{matrix}\right.\right). \tag{2.3}$$

We let $\alpha=1,\ l=2\gamma, a=\frac{1}{4}\omega^2$ in (2.3) and substitute (2.2) into the integral I_{11}. Then, employing (2.3), we can obtain the result.

In Matlab, the Meijer G function $G^{m,n}_{p,q}\left(z\left|\begin{matrix}a_1,\cdots,a_n,a_{n+1},\cdots,a_p\\b_1,\cdots,b_m,b_{m+1},\cdots,b_q\end{matrix}\right.\right)$ can be computed by invoking the built-in function

MeijerG($[a_1,a_2,\cdots,a_n],[a_{n+1},a_{n+2},\cdots,a_p],[b_1,b_2,\cdots,b_m],[b_{m+1},b_{m+2},\cdots,b_q],z$).

In Maple, the Meijer G function is implemented as

MeijerG($[[a_1,a_2,\cdots,a_n],[a_{n+1},a_{n+2},\cdots,a_p]],[[b_1,b_2,\cdots,b_m],[b_{m+1},b_{m+2},\cdots,b_q]],z$),

and it is suitable for both symbolic and numerical manipulation and its value can be evaluated with an arbitrary precision. The Meijer G function can be computed with the

Matlab code **MeijerG**.**m** [26]. Moreover, according to [17], as $|z| \to \infty$, the Meijer G function has the following asymptotic formula

$$
\begin{aligned}
&G^{m,n}_{p,p+2}\left(z\Big|^{a_1,\cdots,a_n,a_{n+1},\cdots,a_p}_{b_1,\cdots,b_m,b_{m+1},\cdots,b_{p+2}}\right) \propto \\
&\frac{1}{2}\pi^{m+n-p-\frac{3}{2}}\sum_{k=1}^{m}\frac{\prod_{j=n+1}^{p}\sin((a_j-b_k)\pi)}{\prod_{j=1,j\neq k}^{m}\sin((b_j-b_k)\pi)}z^{b_k}((-1)^{p-m-n-1}z)^{\chi-b_k}\left(e^{-i(\pi(\chi-b_k)+2\sqrt{(-1)^{p-m-n-1}z})}\right.\\
&\left.(1+O(\frac{1}{\sqrt{(-1)^{p-m-n-1}|z|}}))+e^{-i(\pi(\chi-b_k)+2\sqrt{(-1)^{p-m-n-1}z})}(1+O(\frac{1}{\sqrt{(-1)^{p-m-n-1}|z|}}))\right)+\\
&\pi^{m+n+p}\sum_{k=1}^{m}\frac{\prod_{j=n+1}^{p}\sin((a_j-b_k)\pi)}{\prod_{j=1,j\neq k}^{m}\sin((b_j-b_k)\pi)}z^{b_k}\sum_{i=1}^{p}\frac{\prod_{s=1,s\neq i}^{p}\Gamma(a_j-a_s)}{\sin((a_i-b_k)\pi)\prod_{s=1}^{p+2}\Gamma(a_i-b_s)}\\
&((-1)^{p-m-n-1}z)^{a_i-b_k-1}(1+O(\frac{1}{|z|})), \qquad (2.4)
\end{aligned}
$$

where $\chi = \frac{1}{2}(\sum_{j=1}^{p+2} b_j - \sum_{j=1}^{p} a_j - \frac{1}{2})$. The Meijer G function can be also computed by truncating the above asymptotic formula (2.4).

Next, we use the complex integration method to calculate the second integral I_{12} on the right of (1.5). We set $t = x^\gamma$, and the integral is transformed as

$$I_{12} = \frac{1}{\gamma}\int_1^{+\infty}\frac{x^{\frac{1}{\gamma}-1}}{x^{\frac{1}{\gamma}}-\tau}J_m(\omega x)\mathrm{d}x. \qquad (2.5)$$

From [33, p. 386], when $m \geq 0$, the relationship between the Bessel function and the Whittaker's functions is as follows

$$J_m(x) = \frac{1}{(2\pi x)^{\frac{1}{2}}}\left\{e^{\frac{1}{2}(m+\frac{1}{2})\pi i}W_{0,m}(2ix) + e^{-\frac{1}{2}(m+\frac{1}{2})\pi i}W_{0,m}(-2ix)\right\}. \qquad (2.6)$$

With the aid of (2.6), the integral I_{12} can be rewritten as

$$
\begin{aligned}
I_{12} =&\frac{1}{\gamma(2\pi\omega)^{\frac{1}{2}}}e^{\frac{1}{2}(m+\frac{1}{2})\pi i}\int_1^{+\infty}\frac{x^{\frac{1}{\gamma}-\frac{3}{2}}}{x^{\frac{1}{\gamma}}-\tau}e^{i\omega x}W_{0,m}(2i\omega x)e^{-i\omega x}\mathrm{d}x\\
&+\frac{1}{\gamma(2\pi\omega)^{\frac{1}{2}}}e^{-\frac{1}{2}(m+\frac{1}{2})\pi i}\int_1^{+\infty}\frac{x^{\frac{1}{\gamma}-\frac{3}{2}}}{x^{\frac{1}{\gamma}}-\tau}e^{-i\omega x}W_{0,m}(-2i\omega x)e^{i\omega x}\mathrm{d}x. \qquad (2.7)
\end{aligned}
$$

From [2, pp. 505 and 508], we get the asymptotic formula of the Whittaker's function as follows

$$
\begin{aligned}
W_{\alpha,\beta}(t) \sim & t^{\alpha}e^{-\frac{t}{2}}\left\{1+\sum_{n=1}^{N}(-1)^n\frac{(\frac{1}{2}-\alpha+\beta)_n(\frac{1}{2}-\alpha-\beta)_n}{n!t^n}\right\}\\
&+O(|t|^{\alpha-N-1}e^{-\frac{t}{2}}), \quad |t| \to \infty. \qquad (2.8)
\end{aligned}
$$

Theorem 2.1 *Suppose that*

$$G_1(x) = \frac{x^{\frac{1}{\gamma}-\frac{3}{2}}}{x^{\frac{1}{\gamma}} - \tau} e^{-i\omega x} W_{0,m}(-2i\omega x), \tag{2.9}$$

$$G_2(x) = \frac{x^{\frac{1}{\gamma}-\frac{3}{2}}}{x^{\frac{1}{\gamma}} - \tau} e^{i\omega x} W_{0,m}(2i\omega x). \tag{2.10}$$

Then the integral I_{12} can be rewritten as

$$\begin{aligned} I_{12} =& \frac{1}{\gamma(2\pi\omega)^{\frac{1}{2}}} \Bigg\{ e^{-\frac{1}{2}(m+\frac{1}{2})\pi i} \frac{ie^{i\omega}}{\omega} \int_0^{+\infty} G_1(1+\frac{it}{\omega}) e^{-t} \mathrm{d}t \\ & - e^{\frac{1}{2}(m+\frac{1}{2})\pi i} \frac{ie^{-i\omega}}{\omega} \int_0^{+\infty} G_2(1-\frac{it}{\omega}) e^{-t} \mathrm{d}t \Bigg\}. \end{aligned} \tag{2.11}$$

Proof Based on (2.7), we rewrite the integral I_{12} as

$$I_{12} = \frac{1}{\gamma(2\pi\omega)^{\frac{1}{2}}} \left\{ e^{-\frac{1}{2}(m+\frac{1}{2})\pi i} I_1' + e^{\frac{1}{2}(m+\frac{1}{2})\pi i} I_2' \right\}, \tag{2.12}$$

where

$$I_1' = \int_1^{+\infty} G_1(x) e^{i\omega x} \mathrm{d}x, \tag{2.13}$$

$$I_2' = \int_1^{+\infty} G_2(x) e^{-i\omega x} \mathrm{d}x. \tag{2.14}$$

In the following, we only calculate I_1'. Similarly, I_2' can be easily calculated. The following is the definition of the region P:

$$P = \left\{ z \in C \middle| |z-1| \le R,\ 0 \le \arg(z) \le \frac{\pi}{2} \right\}.$$

Both $G_1(x)$ and $G_2(x)$ are analytic functions in P. The boundary of the region P can be thought of as a closed contour Γ. Through the Cauchy's residue theorem [1, p. 206], we have

$$\oint_\Gamma G_1(x) e^{i\omega x} \mathrm{d}x = 0. \tag{2.15}$$

We let

$$\Gamma = \bigcup_{k=1}^{3} \Gamma_k,$$

where

$$\Gamma_1 : \eta_1(t) = t, \quad t : 1 \to 1 + R,$$
$$\Gamma_2 : \eta_2(\theta) = 1 + Re^{i\theta}, \quad \theta : 0 \to \frac{\pi}{2},$$
$$\Gamma_3 : \eta_3(t) = 1 + it, \quad t : R \to 0,$$

are shown on Fig. 1 (left). Then, we respectively consider every integral on contour $\Gamma_1, \Gamma_2, \Gamma_3$.

On the basis of Eq. (2.8), as $|z| \to +\infty$, it follows that

$$zG_1(z) \sim \frac{z^{\frac{1}{\gamma}-\frac{1}{2}}}{z^{\frac{1}{\gamma}} - \tau} \left\{ 1 + \sum_{n=1}^{N} (-1)^n \frac{(\frac{1}{2}+m)_n(\frac{1}{2}-m)_n}{n!(-2i\omega z)^n} \right\} + O(|z|^{-N-\frac{3}{2}}),$$
$$zG_2(z) \sim \frac{z^{\frac{1}{\gamma}-\frac{1}{2}}}{z^{\frac{1}{\gamma}} - \tau} \left\{ 1 + \sum_{n=1}^{N} (-1)^n \frac{(\frac{1}{2}+m)_n(\frac{1}{2}-m)_n}{n!(2i\omega z)^n} \right\} + O(|z|^{-N-\frac{3}{2}}).$$

Therefore, there exist two constants C_1 and C_2, such that

$$|G_1(z)| \le \frac{C_1}{|z|}, \ |G_2(z)| \le \frac{C_2}{|z|}, \ as\ |z| \to +\infty. \tag{2.16}$$

Firstly, using Jordan inequality [1, p. 223]

$$\frac{2}{\pi}\theta \le \sin\theta, \ \theta \in \left[0, \frac{\pi}{2}\right],$$

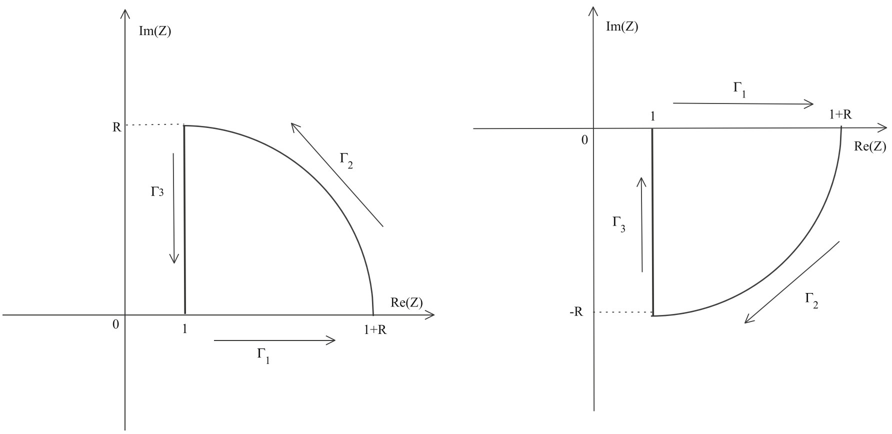

Fig. 1 The integration paths of I_1', which involve the kernel $e^{i\omega x}$, are shown in the left figure. The integration paths of I_2', which involve the kernel $e^{-i\omega x}$, are shown in the right figure

we obtain

$$\begin{aligned}\left|\oint_{\Gamma_2} G_1(x)e^{i\omega x}\mathrm{d}x\right| &= \left|\int_0^{\frac{\pi}{2}} G_1(1+Re^{i\theta})e^{i\omega(1+Re^{i\theta})}\mathrm{d}(1+Re^{i\theta})\right| \\ &\leq R\left|\int_0^{\frac{\pi}{2}} G_1(1+Re^{i\theta})e^{-\omega R\sin\theta}\mathrm{d}\theta\right| \\ &\leq R\frac{C_1}{|1+Re^{i\theta}|}\int_0^{\frac{\pi}{2}} e^{-\frac{2\omega R\theta}{\pi}}\mathrm{d}\theta \\ &\leq C_1\frac{\pi}{2\omega R}(1-e^{-\omega R}) \to 0,\ R\to+\infty. \end{aligned} \tag{2.17}$$

Then, we can express the integral on Γ_3 as

$$\begin{aligned}\oint_{\Gamma_3} G_1(x)e^{i\omega x}\mathrm{d}x &= -ie^{i\omega}\int_0^R G_1(1+it)e^{-\omega t}\mathrm{d}t \\ &= -\frac{i}{\omega}e^{i\omega}\int_0^{\omega R} G_1(1+\frac{it}{\omega})e^{-t}\mathrm{d}t. \end{aligned} \tag{2.18}$$

According to (2.15)-(2.18), we can get

$$\begin{aligned} I_1' &= \int_1^{+\infty} G_1(x)e^{i\omega x}\mathrm{d}x \\ &= \lim_{R\to+\infty}\oint_{\Gamma_1} G_1(x)e^{i\omega x}\mathrm{d}x \\ &= -\lim_{R\to+\infty}\oint_{\Gamma_2+\Gamma_3} G_1(x)e^{i\omega x}\mathrm{d}x \\ &= \frac{ie^{i\omega}}{\omega}\int_0^{+\infty} G_1(1+\frac{it}{\omega})e^{-t}\mathrm{d}t. \end{aligned} \tag{2.19}$$

Similarly, we can calculate the integral I_2' according to the contour path in the right of Fig. 1. The integral I_2' can be expressed as follows

$$I_2' = -\frac{ie^{-i\omega}}{\omega}\int_0^{+\infty} G_2(1-\frac{it}{\omega})e^{-t}\mathrm{d}t. \tag{2.20}$$

We can obtain (2.11) by substituting (2.19) and (2.20) into (2.12).

We approximate the integral (2.11) by the Gauss–Laguerre quadrature rule. The n–points Gauss–Laguerre quadrature rule of the integral $\int_0^{+\infty}\phi(x)e^{-x}\mathrm{d}x$ is denoted as [11, p. 222]

$$\int_0^{+\infty}\phi(x)e^{-x}\mathrm{d}x \approx \sum_{k=1}^{n} w_k\phi(x_k),$$

where x_k and w_k are respectively node and weight. Then, we use the above formula to obtain the complex integration method of the integral I_{12} as follows

$$
\begin{aligned}
Q_{12} =& \frac{1}{\gamma(2\pi\omega)^{\frac{1}{2}}}\left\{e^{-\frac{1}{2}(m+\frac{1}{2})\pi i}\frac{ie^{i\omega}}{\omega}\sum_{k=1}^{n} w_k G_1(1+\frac{it_k}{\omega}) - e^{\frac{1}{2}(m+\frac{1}{2})\pi i}\frac{ie^{-i\omega}}{\omega}\sum_{k=1}^{n} w_k G_2(1-\frac{it_k}{\omega})\right\}\\
=& \frac{1}{\gamma(2\pi\omega)^{\frac{1}{2}}}\left\{e^{-\frac{1}{2}(m+\frac{1}{2})\pi i}\frac{i}{\omega}\sum_{k=1}^{n} w_k \frac{(1+\frac{it_k}{\omega})^{\frac{1}{\gamma}-\frac{3}{2}}}{(1+\frac{it_k}{\omega})^{\frac{1}{\gamma}}-\tau} e^{t_k} W_{0,m}(-2i\omega+2t_k)\right.\\
&\left. - e^{\frac{1}{2}(m+\frac{1}{2})\pi i}\frac{i}{\omega}\sum_{k=1}^{n} w_k \frac{(1-\frac{it_k}{\omega})^{\frac{1}{\gamma}-\frac{3}{2}}}{(1-\frac{it_k}{\omega})^{\frac{1}{\gamma}}-\tau} e^{t_k} W_{0,m}(2i\omega+2t_k)\right\},
\end{aligned}
\tag{2.21}
$$

where the Whittaker's function $W_{\alpha,\beta}(t)$ can be computed by either truncating the above asymptotic formula (2.8) or invoking '$whittakerW(\alpha, \beta, t)$' in Matlab.

Next, we apply the Clenshaw–Curtis–Filon–type method to calculate the integral $I_{13} = \int_0^1 F(x)J_m(\omega x^{\gamma})\mathrm{d}x$. Let $P_{N_1+2s}(x) = \sum_{n=0}^{N_1+2s} d_n^{(s)} T_n^*(x)$ be the Hermite interpolation polynomial of $F(x)$ at the Clenshaw-Curtis points

$$
x_i = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{i\pi}{N_1}\right),\ i = 0, 1, \cdots, N_1,
$$

where s is a non-negative integer, $T_n^*(x)$ represents the Chebyshev polynomial of the first kind shifted to the range [0,1]. The interpolation polynomial $P_{N_1+2s}(x)$ satisfies the following relationships

$$
P_{N_1+2s}^{(k)}(0) = F^{(k)}(0),\ P_{N_1+2s}(x_j) = F(x_j),\ P_{N_1+2s}^{(k)}(1) = F^{(k)}(1),\ k = 0, 1, \cdots, s,\ j = 1, 2, \cdots, N_1 - 1.
$$

The coefficient $d_n^{(s)}$ can be effectively computed by a modified algorithm [37, 38]. The algorithm is based on the Fast Fourier Transform. Then the Clenshaw–Curtis—Filon–type method for the integral I_{13} can be expressed as

$$
\begin{aligned}
Q_{N_1}^{CCF}[F] &= \int_0^1 P_{N_1+2s}(x)J_m(\omega x^{\gamma})\mathrm{d}x\\
&= \sum_{n=0}^{N_1+2s} d_n^{(s)} M_n(m, \omega, \gamma),
\end{aligned}
\tag{2.22}
$$

where

$$
M_n(m, \omega, \gamma) = \int_0^1 T_n^*(x)J_m(\omega x^{\gamma})\mathrm{d}x,
\tag{2.23}
$$

are known as the modified moments.

In the following, we utilize the Bessel equation and Chebyshev polynomial to derive a homogeneous recurrence relationship for the modified moments (2.23). It is applicable for all positive integers 2γ.

Theorem 2.2 *The modified moments (2.23) satisfy the following recursive relationship*

$$(n^2+6n+9-\gamma^2m^2)M_{n+2}(m,\omega,\gamma)+(2n+4+4\gamma^2m^2)M_{n+1}(m,\omega,\gamma)-(2n^2-6+6\gamma^2m^2)$$
$$M_n(m,\omega,\gamma)-(2n-4-4\gamma^2m^2)M_{n-1}(m,\omega,\gamma)+(n^2-6n+9-\gamma^2m^2)M_{n-2}(m,\omega,\gamma)$$
$$+\gamma^2\frac{\omega^2}{2^{2\gamma}}\sum_{j=0}^{2\gamma}C_{2\gamma}^j2^{-j}\Big[\sum_{k=0}^{j+2}C_{j+2}^kM_{n+j+2-2k}(m,\omega,\gamma)-4\sum_{k=0}^{j+1}C_{j+1}^kM_{n+j+1-2k}(m,\omega,\gamma)$$
$$+4\sum_{k=0}^{j}C_j^kM_{n+j-2k}(m,\omega,\gamma)\Big]=0, \tag{2.24}$$

where $C_j^k=\frac{j!}{k!(j-k)!}$.

Proof Firstly, we rewrite the modified moments as

$$M_n(m,\omega,\gamma)=\frac{1}{2}\int_{-1}^{1}T_n(x)J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^\gamma\Big)\mathrm{d}x,$$

where $T_n(x)$ is the Chebyshev polynomial of degree n. According to the Bessel equation [2, p. 358]

$$z^2\frac{\mathrm{d}^2J_m(z)}{\mathrm{d}z^2}+z\frac{\mathrm{d}J_m(z)}{\mathrm{d}z}+(z^2-m^2)J_m(z)=0,$$

we can obtain

$$\frac{(1+x)^2}{\gamma^2}\Big[J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^\gamma\Big)\Big]''+\frac{(1+x)}{\gamma^2}\Big[J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^\gamma\Big)\Big]'$$
$$-\Big(m^2-\omega^2\Big(\frac{1+x}{2}\Big)^{2\gamma}\Big)\Big[J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^\gamma\Big)\Big]=0. \tag{2.25}$$

We let

$$K_1=4\int_{-1}^{1}(1+x)^2(1-x)^2T_n(x)\Big[J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^\gamma\Big)\Big]''\mathrm{d}x, \tag{2.26}$$

$$K_2=4\int_{-1}^{1}(1+x)(1-x)^2T_n(x)\Big[J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^\gamma\Big)\Big]'\mathrm{d}x, \tag{2.27}$$

$$K_3=4\gamma^2\int_{-1}^{1}(1-x)^2\Big[m^2-\omega^2\Big(\frac{1+x}{2}\Big)^{2\gamma}\Big]T_n(x)\Big[J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^\gamma\Big)\Big]\mathrm{d}x. \tag{2.28}$$

From (2.25), we can easily deduce that K_1, K_2 and K_3 satisfy the following relationship

$$K_1 + K_2 - K_3 = 0. \tag{2.29}$$

Applying integration by parts to (2.26) and (2.27), we have

$$\begin{aligned}
K_1 &= 4\int_{-1}^{1}\Big[(1+x)^2(1-x)^2T_n(x)\Big]'' J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^{\gamma}\Big)\mathrm{d}x,\\
K_2 &= -4\int_{-1}^{1}\Big[(1+x)(1-x)^2T_n(x)\Big]' J_m\Big(\omega\Big(\frac{1+x}{2}\Big)^{\gamma}\Big)\mathrm{d}x.
\end{aligned}$$

For K_1, K_2 and K_3, utilizing the properties of the first kind of Chebyshev polynomials [23]

$$\begin{aligned}
(1-x^2)T_n{}'(x) &= \frac{n}{2}(T_{n-1}(x) - T_{n+1}(x)),\\
x^m T_n(x) &= 2^{-m}\sum_{j=0}^{m} C_m^j T_{n+m-2j}(x),
\end{aligned}$$

we can obtain

$$K_1 = (n^2+7n+12)M_{n+2}(m,\omega,\gamma) - (2n^2-8)M_n(m,\omega,\gamma) + (n^2-7n+12)M_{n-2}(m,\omega,\gamma), \tag{2.30}$$

$$\begin{aligned}
K_2 = &-(n+3)M_{n+2}(m,\omega,\gamma) + 2(n+2)M_{n+1}(m,\omega,\gamma) - 2M_n(m,\omega,\gamma)\\
&- 2(n-2)M_{n-1}(m,\omega,\gamma) + (n-3)M_{n-2}(m,\omega,\gamma),
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
K_3 = &\gamma^2 m^2\Big[M_{n+2}(m,\omega,\gamma) - 4M_{n+1}(m,\omega,\gamma) + 6M_n(m,\omega,\gamma) - 4M_{n-1}(m,\omega,\gamma)\\
&+ M_{n-2}(m,\omega,\gamma)\Big] - \gamma^2\frac{\omega^2}{2^{2\gamma}}\sum_{j=0}^{2\gamma} C_{2\gamma}^j 2^{-j}\Big[\sum_{k=0}^{j+2} C_{j+2}^k M_{n+j+2-2k}(m,\omega,\gamma)\\
&- 4\sum_{k=0}^{j+1} C_{j+1}^k M_{n+j+1-2k}(m,\omega,\gamma) + 4\sum_{k=0}^{j} C_j^k M_{n+j-2k}(m,\omega,\gamma)\Big].
\end{aligned} \tag{2.32}$$

Finally, we can get (2.24) by substituting (2.30), (2.31) and (2.32) into (2.29).

Since $T_{-j}(t) = T_j(t)$, $j = 1, 2, \cdots$, we have that $T^*_{-j}(t) = T^*_j(t)$ and $M_{-j}(m,\omega,\gamma) = M_j(m,\omega,\gamma)$. In addition, if the first few terms are known, the next few terms can be obtained by recursion of the previous terms. For example, when $\gamma = 1$ and $M_n(m,\omega,\gamma)$, $n = 0, 1, 2, 3, 4, 5$ are known, we can obtain $M_6(m,\omega,\gamma)$, $M_7(m,\omega,\gamma)$, $M_8(m,\omega,\gamma)$ and $M_9(m,\omega,\gamma)$. Further, $M_n(m,\omega,\gamma)$, $n = 10, 11, \cdots$ can be computed by forward recursion. By substituting $T_0^*(t) = 1$, $T_1^*(t) = 2t - 1$, $T_2^*(t) = 8t^2 - 8t + 1$, $T_3^*(t) = 32t^3 - 48t^2 + 18t - 1$, $T_4^*(t) = 128t^4 - 256t^3 + 160t^2 - 32t + 1$ and $T_5^*(t) = 512t^5 - 1280t^4 +$

$1120t^3 - 400t^2 + 50t - 1$ into $M_n(m,\omega,\gamma) = \int_0^1 T_n^*(t)J_m(\omega t^\gamma)\mathrm{d}t$, we obtain the initial modified moments

$$\begin{aligned}
M_0(m,\omega,\gamma) &= I_0(m,\omega,\gamma),\\
M_1(m,\omega,\gamma) &= 2I_1(m,\omega,\gamma) - I_0(m,\omega,\gamma),\\
M_2(m,\omega,\gamma) &= 8I_2(m,\omega,\gamma) - 8I_1(m,\omega,\gamma) + I_0(m,\omega,\gamma),\\
M_3(m,\omega,\gamma) &= 32I_3(m,\omega,\gamma) - 48I_2(m,\omega,\gamma) + 18I_1(m,\omega,\gamma) - I_0(m,\omega,\gamma),\\
M_4(m,\omega,\gamma) &= 128I_4(m,\omega,\gamma) - 256I_3(m,\omega,\gamma) + 160I_2(m,\omega,\gamma)\\
&\quad -32I_1(m,\omega,\gamma) + I_0(m,\omega,\gamma),\\
M_5(m,\omega,\gamma) &= 512I_5(m,\omega,\gamma) - 1280I_4(m,\omega,\gamma) + 1120I_3(m,\omega,\gamma)\\
&\quad -400I_2(m,\omega,\gamma) + 50I_1(m,\omega,\gamma) - I_0(m,\omega,\gamma).
\end{aligned}$$

Here, the computation of the moments $I_j(m,\omega,\gamma) = \int_0^1 x^j J_m(\omega x^\gamma)\mathrm{d}x$ are as follows.

Firstly, we can know from [22, p. 44]

$$\int_0^1 t^j J_m(\omega t)\mathrm{d}t = \frac{\omega^m}{2^m(m+j+1)\Gamma(m+1)}\,{}_1F_2\Big(\frac{1}{2}(m+j+1);\,\frac{1}{2}(m+j+3);\,m+1;\,-\frac{1}{4}\omega^2\Big).$$

Letting $x^\gamma = t$, we can obtain

$$\begin{aligned}
I_j(m,\omega,\gamma) &= \frac{1}{\gamma}\int_0^1 t^{\frac{j+1}{\gamma}-1} J_m(\omega t)\mathrm{d}t\\
&= \frac{1}{\gamma}\frac{\omega^m}{2^m(m+\frac{j+1}{\gamma})\Gamma(m+1)}\,{}_1F_2\Big(\frac{1}{2}(m+\frac{j+1}{\gamma});\,\frac{1}{2}(m+\frac{j+1}{\gamma}+2);\,m+1;\,-\frac{1}{4}\omega^2\Big)\\
&= \frac{\omega^m}{2^m(\gamma m+j+1)\Gamma(m+1)}\,{}_1F_2\Big(\frac{1}{2}(m+\frac{j+1}{\gamma});\,\frac{1}{2}(m+\frac{j+1}{\gamma}+2);\,m+1;\,-\frac{1}{4}\omega^2\Big).
\end{aligned} \tag{2.33}$$

Here, ${}_pF_{p+1}$ is the hypergeometric function, which has the following asymptotic formula [15]

$$\begin{aligned}
&{}_pF_{p+1}(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_{p+1};z) \propto\\
&\frac{\prod_{j=1}^{p+1}\Gamma(b_j)}{2\sqrt{\pi}\prod_{k=1}^{p}\Gamma(a_k)}(-z)^\chi\left(e^{i(\pi\chi+2\sqrt{-z})}\Big(1+O\Big(\frac{1}{\sqrt{-z}}\Big)\Big)+e^{-i(\pi\chi+2\sqrt{-z})}\Big(1+O\Big(\frac{1}{\sqrt{-z}}\Big)\Big)\right)\\
&+\frac{\prod_{j=1}^{p+1}\Gamma(b_j)}{\prod_{k=1}^{p}\Gamma(a_k)}\sum_{k=1}^{p}\frac{\Gamma(a_k)\prod_{j=1,j\neq k}^{p}\Gamma(a_j-a_k)}{\prod_{j=1}^{p+1}\Gamma(b_j-a_k)}(-z)^{-a_k}\Big(1+O\Big(\frac{1}{z}\Big)\Big),
\end{aligned}$$

where $|z| \to \infty$; $\chi = \frac{1}{2}(\sum_{k=1}^{p} a_k - \sum_{j=1}^{p+1} b_j + \frac{1}{2})$; $j, k \in Z$; $1 \le j, k \le p$; $a_j - a_k \notin Z$. The hypergeometric function can be computed by either truncating the above asymptotic formula or invoking '$hypergeom(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_{p+1};z)$' in Matlab.

Generally, both forward recursion and backward recursion are unstable. If ω is large, the instability is less pronounced. For larger ω, if $n \le \frac{\omega}{2}$, it is accurate to compute $M_n(m,\omega,\gamma)$ by forward recursion. However, when $n > \frac{\omega}{2}$, the loss of important figures increases and the forward recursion is no longer applicable. In this case, we compute the modified moments by either using Oliver's method [25] or Lozier's method [21]. This means that (2.24) has to be solved as a boundary value problem. In this paper, we primarily consider the moderate and large ω and employ the forward recursion to accurately calculate the values of the modified moments.

And then, as $n > \frac{\omega}{2}$, we provide a table to illustrate the numerical instability of forward recursion.

From Table 1, we can see that the forward recursion becomes unstable when $n > \frac{\omega}{2}$. In this case, as n becomes large, the values of modified moments $M_n(m, \omega, \gamma)$ obtained by the forward recursion become more inaccurate. The recursive relationship (2.24) may be considered as a homogeneous difference equation. The numerical stability of forward recursion depends on the behaviour of all solutions of these difference equations, especially on the asymptotic behaviour for $n \to \infty$ (see Gautschi [12], Oliver [25]). In the future research, we will consider how to study the asymptotic behavior of all solutions of these difference equations.

In the following, as $n \le \frac{\omega}{2}$, we use forward recursion for the recursion relationship (2.24) to compute the values of the modified moments $M_n(m, \omega, \gamma)$. Tables 2 and 3 verify the accuracy of these values.

As $n \le \frac{\omega}{2}$, practical experiments illustrate that $M_n(m, \omega, \gamma)$ can be computed accurately by using forward recursion for the recursion relationship (2.24). In the numerical experiments in Section 4, we only need to take a small number of nodes to achieve high accuracy for the moderate and large ω. Therefore, a small number of the modified moments $M_n(m, \omega, \gamma)$ are sufficient.

So far, we can get the quadrature formula of the integral (1.1) as follows

$$Q[f] = f(\tau)(I_{11} - Q_{12}) + Q_{N_1}^{CCF}[F], \tag{2.34}$$

where I_{11} is calculated by using (2.1), Q_{12} and $Q_{N_1}^{CCF}[F]$ are given in (2.21) and (2.22), respectively.

3 Error analysis

Firstly, we analyze the error of the quadrature formula Q_{12}.

Table 1 The absolute errors for the calculation of $M_n(m, \omega, \gamma)$ by the forward recursion for the recursion relationship (2.24) with $m = 2,\ \gamma = \frac{1}{2}$

n	$\omega = 6$	$\omega = 8$	$\omega = 10$	$\omega = 11$	$\omega = 13$
$n = 11$	1.56×10^{-6}	4.34×10^{-8}	2.08×10^{-9}	6.88×10^{-10}	1.71×10^{-11}
$n = 12$	9.56×10^{-5}	1.45×10^{-6}	4.29×10^{-8}	1.14×10^{-8}	1.92×10^{-10}
$n = 13$	6.93×10^{-3}	5.78×10^{-5}	1.06×10^{-6}	2.28×10^{-7}	2.62×10^{-9}
$n = 14$	5.85×10^{-1}	2.69×10^{-3}	3.06×10^{-5}	5.37×10^{-6}	4.24×10^{-8}

The values of the initial modified moments $M_n(m, \omega, \gamma),\ n = 0, 1, 2, 3, 4, 5, 6$, can be calculated by using $T_n^*(t)$ in terms of powers of t [23, subsections 2.3.2, 1.3.1, Eqs. 2.16, 1.21] and the formula (2.33)

Table 2 The absolute errors for the calculation of $M_n(m, \omega, \gamma)$ by the forward recursion for the recursion relationship (2.24) with $m = 4,\ \gamma = \frac{1}{2}$

n	$\omega = 50$	$\omega = 100$	$\omega = 200$	$\omega = 300$	$\omega = 500$
$n = 6$	3.55×10^{-15}	7.24×10^{-17}	4.90×10^{-17}	1.62×10^{-17}	3.17×10^{-17}
$n = 7$	8.88×10^{-15}	1.67×10^{-16}	8.20×10^{-17}	4.67×10^{-17}	1.00×10^{-16}
$n = 8$	9.18×10^{-15}	3.15×10^{-16}	4.70×10^{-17}	6.70×10^{-17}	1.52×10^{-16}
$n = 9$	1.04×10^{-14}	4.83×10^{-16}	2.89×10^{-16}	1.27×10^{-16}	2.86×10^{-16}

The values of the initial modified moments $M_n(m, \omega, \gamma),\ n = 0, 1, 2, 3, 4, 5$, can be calculated by using $T_n^*(t)$ in terms of powers of t [23, subsections 2.3.2, 1.3.1, Eqs. 2.16, 1.21] and the formula (2.33)

Theorem 3.1 *The absolute error of the quadrature formula* (2.21) *for the integral* I_{12} *satisfies*

$$\left|I_{12} - Q_{12}\right| = O\left(\omega^{-2n-\frac{3}{2}}\right),\ \omega \to +\infty. \tag{3.1}$$

Proof Firstly, we know that Bessel function and Hankel function have the following relationship [2, p. 358]

$$J_m(x) = \frac{1}{2}[H_m^{(1)}(x) + H_m^{(2)}(x)],$$

with

$$H_m^{(1)}(x) = \sqrt{\frac{2}{\pi x}} e^{-i\frac{\pi}{2}(m+\frac{1}{2})} W_{0,m}(-2ix),$$
$$H_m^{(2)}(x) = \sqrt{\frac{2}{\pi x}} e^{i\frac{\pi}{2}(m+\frac{1}{2})} W_{0,m}(2ix).$$

According to the above relationships, we can rewrite (2.11) as

$$I_{12} = \frac{ie^{i\omega}}{2\gamma\omega} \int_0^{+\infty} Q_1(1 + \frac{it}{\omega})e^{-t}\mathrm{d}t - \frac{ie^{-i\omega}}{2\gamma\omega} \int_0^{+\infty} Q_2(1 - \frac{it}{\omega})e^{-t}\mathrm{d}t, \tag{3.2}$$

Table 3 The absolute errors for the calculation of $M_n(m, \omega, \gamma)$ by the forward recursion for the recursion relationship (2.24) with $m = 1,\ \gamma = 2$

n	$\omega = 100$	$\omega = 150$	$\omega = 200$	$\omega = 600$	$\omega = 800$
$n = 12$	4.56×10^{-13}	7.93×10^{-14}	9.50×10^{-13}	2.58×10^{-13}	1.95×10^{-13}
$n = 13$	1.27×10^{-12}	2.68×10^{-13}	3.64×10^{-12}	3.28×10^{-13}	6.59×10^{-13}
$n = 14$	3.10×10^{-12}	6.78×10^{-13}	9.58×10^{-12}	4.68×10^{-13}	1.67×10^{-12}
$n = 15$	6.71×10^{-12}	1.57×10^{-12}	2.32×10^{-11}	4.52×10^{-14}	3.89×10^{-12}

The values of the initial modified moments $M_n(m, \omega, \gamma),\ n = 0, 1, 2, \cdots, 11$, can be calculated by using $T_n^*(t)$ in terms of powers of t [23, subsections 2.3.2, 1.3.1, Eqs. 2.16, 1.21] and the formula (2.33) where

$$Q_1(x) = \frac{H_m^{(1)}(\omega x)T(x)}{e^{i\omega x}},$$
$$Q_2(x) = H_m^{(2)}(\omega x)T(x)e^{i\omega x},$$
$$T(x) = \frac{x^{\frac{1}{\gamma}-1}}{x^{\frac{1}{\gamma}} - \tau}.$$

At the same time, we can rewrite (2.21) as

$$Q_{12} = \frac{ie^{i\omega}}{2\gamma\omega}\sum_{k=1}^{n} w_k Q_1(1+\frac{ix_k}{\omega}) - \frac{ie^{-i\omega}}{2\gamma\omega}\sum_{k=1}^{n} w_k Q_2(1-\frac{ix_k}{\omega}). \tag{3.3}$$

We let

$$E_1 = \frac{ie^{i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} Q_1(1+\frac{it}{\omega})e^{-t}\mathrm{d}t - \sum_{k=1}^{n} w_k Q_1(1+\frac{ix_k}{\omega})\Big], \tag{3.4}$$

$$E_2 = \frac{ie^{-i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} Q_2(1-\frac{it}{\omega})e^{-t}\mathrm{d}t - \sum_{k=1}^{n} w_k Q_2(1-\frac{ix_k}{\omega})\Big]. \tag{3.5}$$

Then, we can obtain

$$\begin{aligned} |I_{12} - Q_{12}| =& \Bigg|\frac{ie^{i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} Q_1(1+\frac{it}{\omega})e^{-t}\mathrm{d}t - \sum_{k=1}^{n} w_k Q_1(1+\frac{ix_k}{\omega})\Big] \\ &- \frac{ie^{-i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} Q_2(1-\frac{it}{\omega})e^{-t}\mathrm{d}t - \sum_{k=1}^{n} w_k Q_2(1-\frac{ix_k}{\omega})\Big]\Bigg| \\ =& \big|E_1 - E_2\big|. \end{aligned} \tag{3.6}$$

From [28], for large $|x|$, it follows that

$$H_m^{(1)}(x)e^{-ix} = \sqrt{\frac{2}{\pi x}}e^{-i(\frac{m\pi}{2}+\frac{\pi}{4})}\sum_{j=0}^{2n+1}\frac{(\frac{1}{2}+m)_j(\frac{1}{2}-m)_j}{j!(2ix)^j} + O(|x|^{-2n-\frac{5}{2}}), \tag{3.7}$$

where $(a)_j$ is defined by

$$(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)} = \begin{cases} 1, & j=0, \\ a(a+1)(a+2)\cdots(a+j), & j \geq 1. \end{cases} \tag{3.8}$$

We define $D_\omega(u) = T(1+iu)$, $u = \frac{t}{\omega}$, $W_\omega(t) = \sqrt{\frac{2}{\pi}}e^{-i(\frac{m\pi}{2}+\frac{\pi}{4})}\sum_{j=0}^{2n+1}\frac{(\frac{1}{2}+m)_j(\frac{1}{2}-m)_j}{j!(2i)^j(\omega+it)^{j+\frac{1}{2}}}$. We apply the Hermite interpolation to the function $D_\omega(\frac{t}{\omega})W_\omega(t)$, and construct the interpolation polynomial $P_{2n-1}(\frac{t}{\omega})$, which satisfies that

$$P_{2n-1}(\frac{x_k}{\omega}) = D_\omega(\frac{x_k}{\omega})W_\omega(x_k),$$
$$P'_{2n-1}(\frac{x_k}{\omega}) = \Big[D_\omega(\frac{t}{\omega})W_\omega(t)\Big]'\Big|_{t=x_k}.$$

Thus, using the remainder formula of Hermite interpolation, we can get that

$$\begin{aligned} E &= \int_0^{+\infty}\Big[D_\omega(\frac{t}{\omega})W_\omega(t) - P_{2n-1}(\frac{t}{\omega})\Big]e^{-t}\mathrm{d}t \\ &= \int_0^{+\infty}\frac{\Big[D_\omega(\frac{t}{\omega})W_\omega(t)\Big]^{(2n)}\Big|_{t=\zeta}}{(2n)!}Ln^2(t)e^{-t}\mathrm{d}t, \end{aligned} \tag{3.9}$$

where $Ln(x) = \prod_{j=1}^{n}(x - x_j)$, $\zeta \in (0, \max(t, x_n))$. Thus, according to (3.7) and (3.9), we have

$$\begin{aligned} E_1 =& \frac{ie^{i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} Q_1(1+\frac{it}{\omega})e^{-t}\mathrm{d}t - \sum_{k=1}^{n} w_k Q_1(1+\frac{ix_k}{\omega})\Big] \\ =& \frac{ie^{i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} D_\omega(\frac{t}{\omega})H_m^{(1)}(\omega+it)e^{-i\omega+t}e^{-t}\mathrm{d}t - \sum_{k=1}^{n} w_k D_\omega(\frac{x_k}{\omega})H_m^{(1)}(\omega+ix_k)e^{-i\omega+x_k}\Big] \\ =& \frac{ie^{i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} e^{-t}D_\omega(\frac{t}{\omega})W_\omega(t)\mathrm{d}t - \sum_{k=1}^{n} w_k D_\omega(\frac{x_k}{\omega})W_\omega(x_k) \\ &+ \int_0^{+\infty} e^{-t}D_\omega(\frac{t}{\omega})O(|\omega+it|^{-2n-\frac{5}{2}})\mathrm{d}t + O(|\omega+ix_k|^{-2n-\frac{5}{2}})\Big] \\ =& \frac{ie^{i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty}\frac{\Big[D_\omega(\frac{t}{\omega})W_\omega(t)\Big]^{(2n)}\Big|_{t=\zeta}}{(2n)!}Ln^2(t)e^{-t}\mathrm{d}t + \int_0^{+\infty} e^{-t}D_\omega(\frac{t}{\omega})O(|\omega+it|^{-2n-\frac{5}{2}})\mathrm{d}t \\ &+ O(|\omega+ix_k|^{-2n-\frac{5}{2}})\Big]. \end{aligned} \tag{3.10}$$

For the second integral in the penultimate line of (3.10), we have

$$\int_0^{+\infty} e^{-t}D_\omega(\frac{t}{\omega})O(|\omega+it|^{-2n-\frac{5}{2}})\mathrm{d}t = O(\omega^{-2n-\frac{5}{2}}),\ \omega \to +\infty. \tag{3.11}$$

In addition, it can be derived that

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}W_\omega(t) = O(\omega^{-k-\frac{1}{2}}),\ \omega \to +\infty. \tag{3.12}$$

According to (3.12), applying the Leibniz's Theorem [2, p. 12], the first integral in the penultimate line of (3.10) can be written as

$$\int_0^{+\infty} \frac{\left[D_\omega(\frac{t}{\omega})W_\omega(t)\right]^{(2n)}\Big|_{t=\zeta}}{(2n)!} Ln^2(t)e^{-t}\mathrm{d}t = \frac{1}{\omega^{2n+\frac{1}{2}}} \int_0^{+\infty} \frac{R(\zeta)}{(2n)!} Ln^2(t)e^{-t}\mathrm{d}t,$$

where $\left[D_\omega(\frac{t}{\omega})W_\omega(t)\right]^{(2n)}\Big|_{t=\zeta} = \frac{1}{\omega^{2n+\frac{1}{2}}} R(\zeta)$. We use the Cauchy's test to obtain that the integral $\int_0^{+\infty} \frac{R(\zeta)}{(2n)!} Ln^2(t)e^{-t}\mathrm{d}t$ is convergent. Therefore, we can get

$$\int_0^{+\infty} \frac{\left[D_\omega(\frac{t}{\omega})W_\omega(t)\right]^{(2n)}\Big|_{t=\zeta}}{(2n)!} Ln^2(t)e^{-t}\mathrm{d}t = O\Big(\frac{1}{\omega^{2n+\frac{1}{2}}}\Big),\ \omega \to +\infty. \quad (3.13)$$

Thus, based on (3.10), (3.11) and (3.13), we have

$$\begin{aligned} E_1 &= \frac{ie^{i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} Q_1(1+\frac{it}{\omega})e^{-t}\mathrm{d}t - \sum_{k=1}^{n} w_k Q_1(1+\frac{ix_k}{\omega})\Big] \\ &= O\Big(\omega^{-2n-\frac{3}{2}}\Big),\ \omega \to +\infty. \end{aligned} \quad (3.14)$$

Similarly, we can also get

$$\begin{aligned} E_2 &= \frac{ie^{-i\omega}}{2\gamma\omega}\Big[\int_0^{+\infty} Q_2(1-\frac{it}{\omega})e^{-t}\mathrm{d}t - \sum_{k=1}^{n} w_k Q_2(1-\frac{ix_k}{\omega})\Big] \\ &= O\Big(\omega^{-2n-\frac{3}{2}}\Big),\ \omega \to +\infty. \end{aligned} \quad (3.15)$$

Finally, we obtain

$$\begin{aligned} \Big|I_{12} - Q_{12}\Big| &= \Big|E_1 - E_2\Big| \\ &= O\Big(\omega^{-2n-\frac{3}{2}}\Big),\ \omega \to +\infty. \end{aligned}$$

Next, we analyze the error of the quadrature formula $Q_{N_1}^{CCF}[F]$.

Lemma 3.1 *When* $\omega \to +\infty, 2\gamma \in N^+, \Re(\mu) > -1, \Re(v) > -1$, *it holds that [19]*

$$\int_0^u x^v(u-x)^\mu J_m(\omega x^\gamma)\mathrm{d}x = O\left(\omega^{-\min\left\{\mu+\frac{3}{2},\frac{v+1}{\gamma}\right\}}\right). \quad (3.16)$$

The following result can be obtained from Lemma 3.1.

Theorem 3.2 *Assume that f is adequately smooth on the range* [0, 1]. *Then the absolute error of the quadrature formula* (2.22) *for the integral I_{13} satisfies*

$$\left|I_{13} - Q_{N_1}^{CCF}[F]\right| = O\left(\omega^{-\min\left\{s+\frac{5}{2}, \frac{s+2}{\gamma}\right\}}\right), \ \omega \to +\infty. \tag{3.17}$$

Proof First, we define the function $\varphi(x) = \frac{F(x)-P_{N_1+2s}(x)}{x^{s+1}(1-x)^{s+1}}$. For $k = 0, 1, \cdots, s+2$, it is relatively simple to obtain that

$$\varphi^{(k)}(x) = \left[\frac{F(x) - P_{N_1+2s}(x)}{x^{s+1}(1-x)^{s+1}}\right]^{(k)}, \ x \in (0, 1),$$

and

$$\varphi^{(k)}(0) = \lim_{t\to 0^+}\left[\frac{F(t) - P_{N_1+2s}(t)}{t^{s+1}(1-t)^{s+1}}\right]^{(k)},$$
$$\varphi^{(k)}(1) = \lim_{t\to 1^-}\left[\frac{F(t) - P_{N_1+2s}(t)}{t^{s+1}(1-t)^{s+1}}\right]^{(k)}.$$

We let

$$\varphi_1(x) = (s+1)\varphi(x) - \varphi'(x)(1-x),$$
$$\varphi_k(x) = (s-k+2)\varphi_{k-1}(x) - \varphi'_{k-1}(x)(1-x), \quad 2 \le k \le s+1,$$

and

$$G_1(x) = \int_0^x t^{s+1} J_m(\omega t^\gamma)\mathrm{d}t,$$
$$G_k(x) = \int_0^x G_{k-1}(t)dt, \quad 2 \le k \le s+2.$$

Then, through $s+2$ times integration by parts, we can get

$$\begin{aligned}
\left|I_{13} - Q_{N_1}^{CCF}[F]\right| &= \left|\int_0^1 x^{s+1}(1-x)^{s+1}\varphi(x) J_m(\omega x^\gamma)\mathrm{d}x\right| \\
&= \left|\int_0^1 \varphi(x)(1-x)^{s+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x t^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right)\mathrm{d}x\right| \\
&= \left|\int_0^1 \varphi(x)(1-x)^{s+1} G_1'(x)\mathrm{d}x\right| \\
&\le \left|\varphi(x)(1-x)^{s+1} G_1(x)|_0^1\right| + \left|\int_0^1 G_1(x)(1-x)^s \varphi_1(x)\mathrm{d}x\right| \\
&= \left|\int_0^1 \varphi_1(x)(1-x)^s G_2'(x)\mathrm{d}x\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\varphi_1(x)(1-x)^s G_2(x)|_0^1\right| + \left|\int_0^1 G_2(x)(1-x)^{s-1}\varphi_2(x)\mathrm{d}x\right| \\
&\vdots \\
&\leq \left|\varphi_{s+1}(x)G_{s+2}(x)|_0^1\right| + \left|\int_0^1 G_{s+2}(x)\varphi'_{s+1}(x)\mathrm{d}x\right|. \qquad (3.18)
\end{aligned}
$$

In addition, it is known from the literature [11, p. 28] that

$$
\int_a^x \mathrm{d}t_n \int_a^{t_n} \mathrm{d}t_{n-1} \cdots \int_a^{t_2} f(t_1)\mathrm{d}t_1 = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\mathrm{d}t. \qquad (3.19)
$$

Then we can know from (3.18) and (3.19)

$$
\begin{aligned}
\left|I_{13} - Q_{N_1}^{CCF}[F]\right| \leq |\varphi_{s+1}(1)| &\left|\int_0^1 t^{s+1}(1-t)^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right| \\
&+ \left|\int_0^1 \varphi'_{s+1}(x)\left[\int_0^x t^{s+1}(x-t)^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right]\mathrm{d}x\right|. \qquad (3.20)
\end{aligned}
$$

For the second line of the formula (3.20), we use the integral mean valve theorem to obtain

$$
\begin{aligned}
\left|I_{13} - Q_{N_1}^{CCF}[F]\right| \leq |\varphi_{s+1}(1)| &\left|\int_0^1 t^{s+1}(1-t)^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right| \\
&+ \left|\varphi'_{s+1}(\varepsilon)\right|\left|\int_0^\varepsilon t^{s+1}(\varepsilon-t)^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right|, \qquad (3.21)
\end{aligned}
$$

where $\varepsilon \in [0, 1]$.

In the following, we divide the discussion into three cases.

(I) If $\varepsilon = 1$, we can achieve

$$
\begin{aligned}
\left|I_{13} - Q_{N_1}^{CCF}[F]\right| &\leq \left[|\varphi_{s+1}(1)| + \left|\varphi'_{s+1}(1)\right|\right]\left|\int_0^1 t^{s+1}(1-t)^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right| \\
&= O\left(\omega^{-\min\left\{s+\frac{5}{2}, \frac{s+2}{\gamma}\right\}}\right), \ \omega \to +\infty. \qquad (3.22)
\end{aligned}
$$

(II) If $\varepsilon = 0$, it follows that

$$
\begin{aligned}
\left|I_{13} - Q_{N_1}^{CCF}[F]\right| &\leq |\varphi_{s+1}(1)|\left|\int_0^1 t^{s+1}(1-t)^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right| \\
&= O\left(\omega^{-\min\left\{s+\frac{5}{2}, \frac{s+2}{\gamma}\right\}}\right), \ \omega \to +\infty. \qquad (3.23)
\end{aligned}
$$

(III) If $0 < \varepsilon < 1$, according to Lemma 3.1, we can obtain

$$\left|\int_0^1 t^{s+1}(1-t)^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right| = O\left(\omega^{-\min\left\{s+\frac{5}{2},\frac{s+2}{\gamma}\right\}}\right),\ \omega \to +\infty, \quad (3.24)$$

$$\left|\int_0^\varepsilon t^{s+1}(\varepsilon-t)^{s+1} J_m(\omega t^\gamma)\mathrm{d}t\right| = O\left(\omega^{-\min\left\{s+\frac{5}{2},\frac{s+2}{\gamma}\right\}}\right),\ \omega \to +\infty. \quad (3.25)$$

Combining (3.21)–(3.25), we have

$$\left|I_{13} - Q_{N_1}^{CCF}[F]\right| = O\left(\omega^{-\min\left\{s+\frac{5}{2},\frac{s+2}{\gamma}\right\}}\right),\ \omega \to +\infty. \quad (3.26)$$

Based on Theorem 3.1 and Theorem 3.2, the error of the quadrature formula (2.34) satisfies

$$O\left(\omega^{-\min\left\{s+\frac{5}{2},\frac{s+2}{\gamma},2n+\frac{3}{2}\right\}}\right),\ \omega \to +\infty. \quad (3.27)$$

In the following, we perform numerical experiments to verify their errors analysis. Experimental results show that the proposed methods are accurate and effective.

4 Numerical experiments

In this section, we provide some figures and tables to validate the efficacy and precision of our presented methods. Matlab 2018a with 64-digit arithmetic is used to compute all examples, and all computation is run on a 2.4 GHz PC with 16 GB of RAM. In addition, for the computation of Q_{12}, we all take $n = 5$.

Example 1 In Fig. 2 we show an example $\int_0^1 \frac{e^x}{x-0.4} J_3(\omega x^2)\mathrm{d}x$ with $N_1 = 3$ by employing the quadrature formula (2.34). The figure on the left is the absolute errors, and the scaled absolute errors by $\omega^{\frac{s+2}{2}}$ are shown on the right.

Example 2 In Fig. 3 we show an example $\int_0^1 \frac{\sin(x)}{x-0.3} J_2(\omega x^{\frac{5}{2}})\mathrm{d}x$ with $N_1 = 2$ by employing the quadrature formula (2.34). The figure on the left is the absolute errors, and the scaled absolute errors by $\omega^{\frac{2(s+2)}{5}}$ are shown on the right.

Example 3 In Fig. 4 we show an example $\int_0^1 \frac{\sin(x)}{x-0.2} J_4(\omega x^3)\mathrm{d}x$ with $N_1 = 3$ by employing the quadrature formula (2.34). The figure on the left is the absolute errors, and the scaled absolute errors by $\omega^{\frac{s+2}{3}}$ are shown on the right.

Example 4 In Fig. 5 we show an example $\int_0^1 \frac{\cos(x)}{x-0.2} J_2(\omega x^2)\mathrm{d}x$ with $N_1 = 2$ by employing the quadrature formula (2.34). The figure on the left is the absolute errors, the scaled absolute errors by $\omega^{\frac{s+2}{2}}$ are shown on the right.

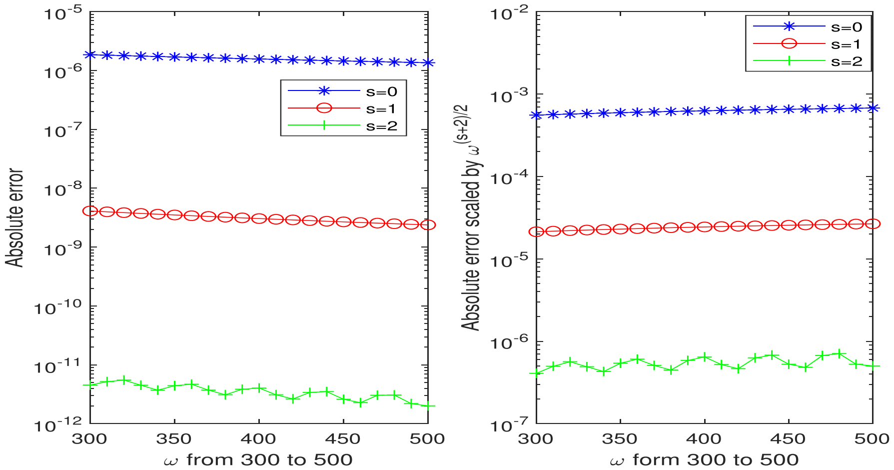

Fig. 2 The absolute errors (left figure) and them multiplied by $\omega^{\frac{s+2}{2}}$ (right figure) of $Q[e^x]$, for the integral $\int_0^1 \frac{e^x}{x-0.4} J_3(\omega x^2)\mathrm{d}x$ are shown. The blue, red and green curves correspond to $s = 0, 1, 2$, respectively. Here, s is the interpolated multiplicity at the endpoints $\{0, 1\}$

Example 5 In Fig. 6 we show an example $\int_0^1 \frac{e^x}{x-0.1} J_1(\omega x^{\frac{5}{2}})\mathrm{d}x$ with $N_1 = 3$ by employing the quadrature formula (2.34). The figure on the left is the absolute errors, and the scaled absolute errors by $\omega^{\frac{2(s+2)}{5}}$ are shown on the right.

Example 6 In Fig. 7 we show an example $\int_0^1 \frac{\cos(x)}{x-0.2} J_5(\omega x^3)\mathrm{d}x$ with $N_1 = 3$ by employing the quadrature formula (2.34). The figure on the left is the absolute errors, and the scaled absolute errors by $\omega^{\frac{s+2}{3}}$ are shown on the right.

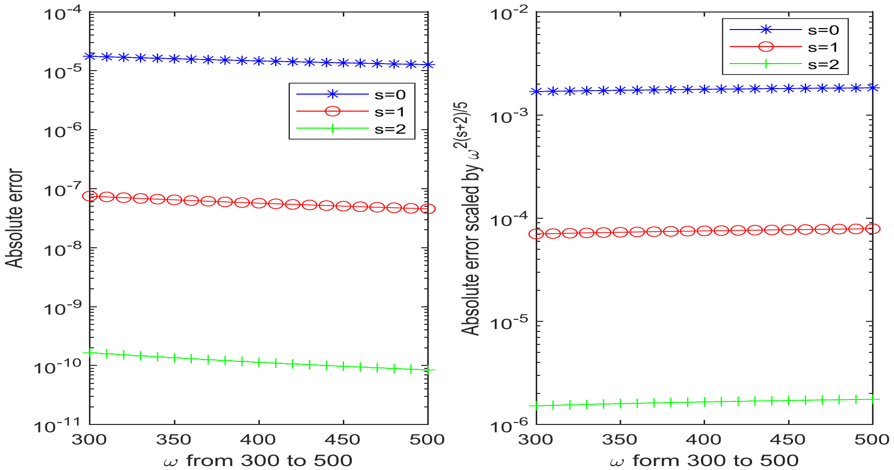

Fig. 3 The absolute errors (left figure) and them multiplied by $\omega^{\frac{2(s+2)}{5}}$ (right figure) of $Q[\sin(x)]$, for the integral $\int_0^1 \frac{\sin(x)}{x-0.3} J_2(\omega x^{\frac{5}{2}})\mathrm{d}x$ are shown. The multiplicities of endpoints $\{0, 1\}$ respectively satisfy $\{s = 0$ (blue), $s = 1$ (red), $s = 2$ (green)$\}$

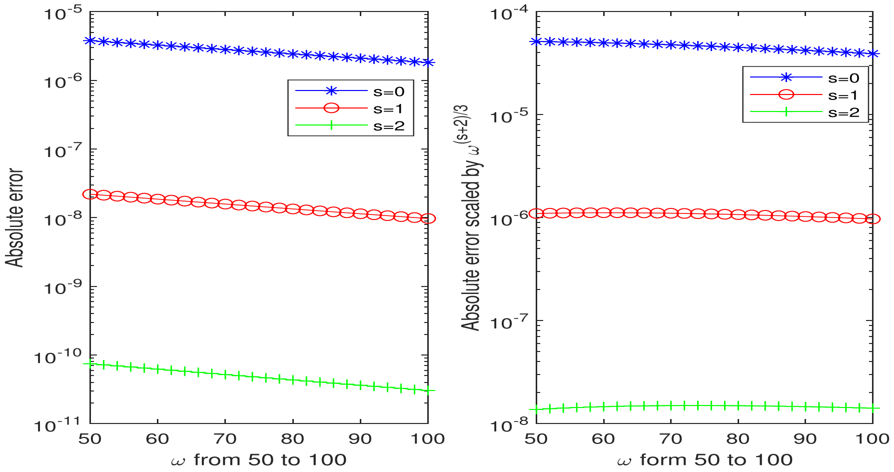

Fig. 4 The absolute errors (left figure) and them multiplied by $\omega^{\frac{s+2}{3}}$ (right figure) of $Q[\sin(x)]$, for the integral $\int_0^1 \frac{\sin(x)}{x-0.2} J_4(\omega x^3)\mathrm{d}x$. The blue, red and green curves correspond to the interpolation multiplicities $s = 0, 1, 2$ at the endpoints $\{0, 1\}$, respectively

From the above Figs. 2-7, we can discover that the rate of convergence is consistent with the error analysis given in (3.24).

In the following, we give four tables of the absolute errors.

From the above Figs. 2-7 and Tables 4, 5, 6 and 7, since the high accuracy can be obtained only through a few interpolation nodes and multiplicities, we easily verify that the proposed method is efficient to compute the highly oscillatory Bessel integral (1.1) with the moderate and large ω. In addition, it is very easy to find that when we fix the

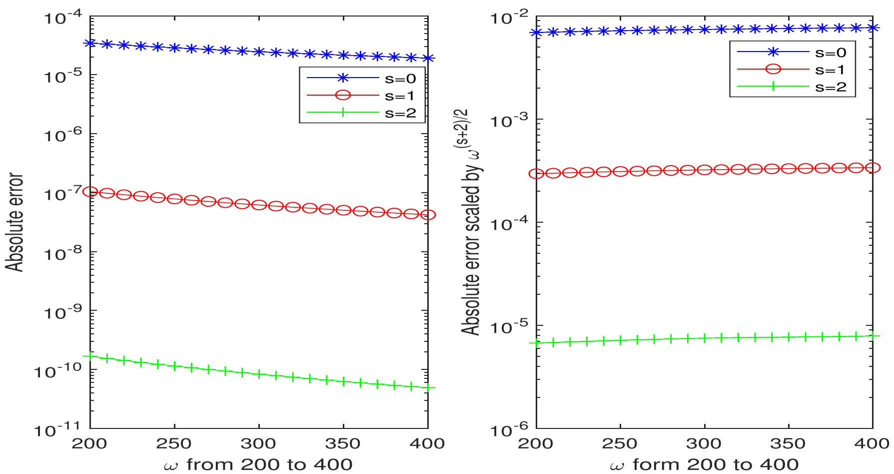

Fig. 5 The absolute errors (left figure) and them multiplied by $\omega^{\frac{s+2}{2}}$ (right figure) of $Q[\cos(x)]$, for the integral $\int_0^1 \frac{\cos(x)}{x-0.2} J_2(\omega x^2)\mathrm{d}x$ are shown. The blue, red and green curves correspond to the interpolation multiplicities $s = 0, 1, 2$ at the endpoints $\{0, 1\}$, respectively

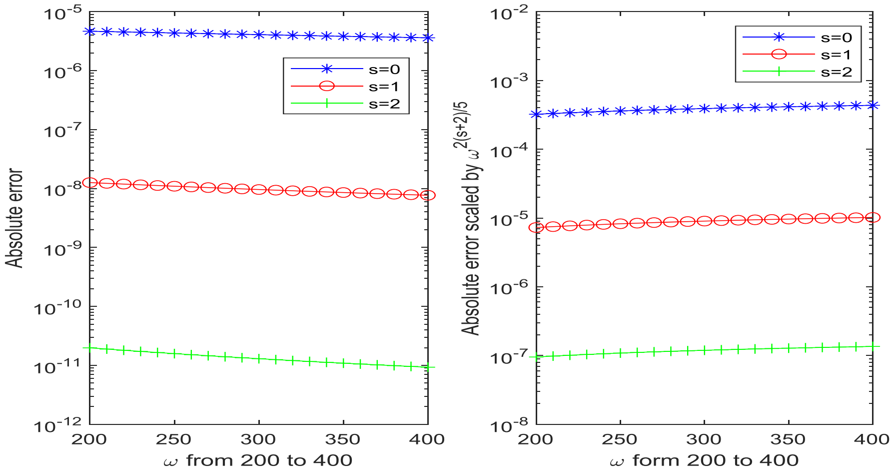

Fig. 6 The absolute errors (left figure) and them multiplied by $\omega^{\frac{2(s+2)}{5}}$ (right figure) of $Q[e^x]$, for the integral $\int_0^1 \frac{e^x}{x-0.1} J_1(\omega x^{\frac{5}{2}})\mathrm{d}x$ are shown. The blue, red and green curves correspond to $s = 0, 1, 2$, respectively. Here, s is the interpolated multiplicity at the endpoints $\{0, 1\}$

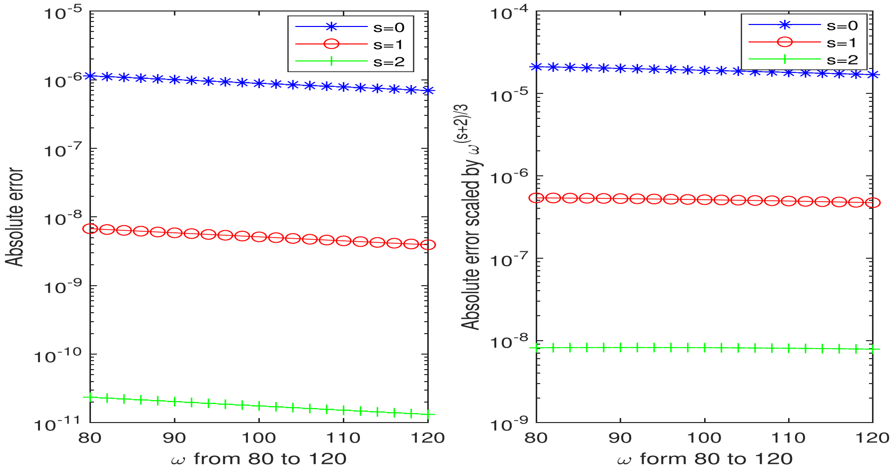

Fig. 7 The absolute errors (left figure) and them multiplied by $\omega^{\frac{s+2}{3}}$ (right figure) of $Q[\cos(x)]$, for the integral $\int_0^1 \frac{\cos(x)}{x-0.2} J_5(\omega x^3)\mathrm{d}x$ are shown. The interpolated multiplicities of endpoints $\{0, 1\}$ respectively satisfy $\{s = 0$ (blue), $s = 1$ (red), $s = 2$ (green)$\}$

Table 4 The absolute errors of calculating the integral $\int_0^1 \frac{\sin(x)}{x-0.8} J_5(\omega x^2)\mathrm{d}x$ with $N_1 = 3$ by employing the quadrature formula (2.34)

s	$\omega = 50$	$\omega = 60$	$\omega = 150$	$\omega = 200$	$\omega = 500$
$s = 0$	1.44×10^{-6}	7.94×10^{-7}	7.46×10^{-7}	8.24×10^{-7}	6.38×10^{-7}
$s = 1$	7.58×10^{-9}	3.96×10^{-9}	2.66×10^{-9}	2.62×10^{-9}	1.37×10^{-9}
$s = 2$	2.28×10^{-11}	1.11×10^{-11}	5.70×10^{-12}	4.91×10^{-12}	1.77×10^{-12}

Table 5 The absolute errors of calculating the integral $\int_0^1 \frac{e^x}{x-0.6} J_3(\omega x^3)\mathrm{d}x$ with $N_1 = 3$ by employing the quadrature formula (2.34)

s	$\omega = 20$	$\omega = 50$	$\omega = 70$	$\omega = 90$	$\omega = 150$
$s = 0$	1.01×10^{-6}	7.76×10^{-7}	5.59×10^{-7}	3.97×10^{-7}	1.12×10^{-7}
$s = 1$	4.36×10^{-9}	3.28×10^{-9}	2.27×10^{-9}	1.56×10^{-9}	4.09×10^{-10}
$s = 2$	2.54×10^{-10}	1.40×10^{-10}	8.61×10^{-11}	5.27×10^{-11}	1.11×10^{-11}

interpolation nodes, the accuracy becomes higher as ω and interpolated multiplicities at the two endpoints increase; on the other hand, the accuracy also improves with the increase of the interpolation nodes when ω and interpolated multiplicities at the two endpoints are fixed. Further, examples 1-6 also confirm that our error analysis is correct.

5 Conclusion

In this paper, we propose an efficient hybrid method to approximate the highly oscillatory Bessel integral (1.1). We convert the integral (1.1) into three integrals, namely I_{11}, I_{12} and I_{13}. For the integral I_{11}, according to the relationship between Meijer G function and Bessel function, we construct the useful explicit formula (2.1) to compute it exactly. For the integral I_{12}, based on the analytical continuation, the integral is converted into two infinite integrals on $[0, +\infty)$, and we can effectively calculate the resulting two infinite integrals through applying the Gauss–Laguerre quadrature rule. For the integral I_{13}, we interpolate $F(x)$ at the Clenshaw-Curtis point to obtain the quadrature formula. Then, the significant recurrence relation (2.24) of the modified moments is derived by employing the Bessel equation and the properties of Chebyshev polynomials, and the values of the initial modified moments are accurately and effectively calculated by applying the explicit formula (2.33). Finally, we provide strict error analysis for the proposed methods and obtain asymptotic order. We discover that the accuracy can be greatly increased whether we interpolate at two endpoints using derivatives or increase the frequency ω. The proposed methods are simple in the process of construction. And, it is easy to implement the given numerical algorithms. Moreover, when 2γ is a positive integer, i.e. $\gamma = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \cdots$, the obtained quadrature formula (2.34) is applicable. In addition, only a few interpolation nodes

Table 6 The absolute errors of calculating the integral $\int_0^1 \frac{\sin(x)}{x-0.4} J_3(\omega x^2)\mathrm{d}x$ with $\omega = 200$ by employing the quadrature formula (2.34)

s	$N_1 = 1$	$N_1 = 2$	$N_1 = 3$	$N_1 = 4$	$N_1 = 5$
$s = 0$	3.34×10^{-4}	1.34×10^{-5}	1.31×10^{-6}	7.79×10^{-9}	1.04×10^{-9}
$s = 1$	1.78×10^{-6}	5.16×10^{-8}	3.66×10^{-9}	1.57×10^{-11}	1.52×10^{-12}
$s = 2$	4.55×10^{-9}	1.03×10^{-10}	5.05×10^{-12}	3.61×10^{-14}	5.47×10^{-14}

Table 7 The absolute errors of calculating the integral $\int_0^1 \frac{e^x}{x-0.4} J_3(\omega x^2)\mathrm{d}x$ with $N_1 = 2$ by employing the quadrature formula (2.34)

s	$\omega = 10^3$	$\omega = 10^4$	$\omega = 10^5$	$\omega = 10^6$	$\omega = 10^7$
$s = 0$	1.31×10^{-5}	1.46×10^{-6}	1.52×10^{-7}	1.54×10^{-8}	1.53×10^{-9}
$s = 1$	2.35×10^{-8}	8.66×10^{-10}	2.86×10^{-11}	9.14×10^{-13}	3.03×10^{-14}
$s = 2$	2.23×10^{-11}	3.94×10^{-13}	4.23×10^{-14}	1.28×10^{-14}	4.06×10^{-15}

are needed to achieve quite high accuracy for moderate and large ω. The numerical experiments validate the correctness of the theoretical analysis and the efficiency of the proposed methods.

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Declarations

Conflict of interest The authors declare no competing interests.

References

1. Ablowitz, M.J., Fokas, A.S.: Complex Variables: Introduction and Applications. Cambridge University Press, New York (2003)
2. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. National Bureau of Standards, Washington, D.C (1964)
3. Arfken, G.: Mathematical Methods for Physicists, third. Academic Press, Orlando, FL (1985)
4. Bao, G., Sun, W.: A fast algorithm for the electromagnetic scattering from a large cavity. SIAM J. Sci. Comput. **27**(2), 553–574 (2005)
5. Bateman, H., Erdélyi, A.: Higher Transcendental Functions, I. Krieger, New York (1981)
6. Brunner, H.: Open problems in the computational solution of Volterra functional equations with highly oscillatory kernels. Effective Computational Methods for Highly Oscillatory Solutions, Isaac Newton Institute, HOP (2007)
7. Brunner, H.: On the numerical solution of first-kind Volterra integral equations with highly oscillatory kernels, Isaac Newton Institute. HOP: Highly Oscillatory Problems: From Theory to Applications, pp 13–17 (2010)
8. Chen, R.: Numerical approximations to integrals with a highly oscillatory Bessel kernel. Appl. Numer. Math. **62**(5), 636–648 (2012)
9. Chen, R.: Fast integration for Cauchy principal value integrals of oscillatory kind. Acta Appl. Math. **123**(1), 21–30 (2013)
10. Chen, R.: On the evaluation of Bessel transformations with the oscillators via asymptotic series of Whittaker functions. J. Comput. Appl. Math. **250**, 107–121 (2013)
11. Davis, P.J., Rabinowitz, P.: Methods of Numerical Integration, second ed., Academic Press (1984)
12. Gautschi, W.: Computational aspects of three-term recurrence relations. SIAM Rev. **9**(1), 24–82 (1967)
13. Hasegawa, T., Sugiura, H.: Uniform approximation to finite Hilbert transform of oscillatory functions and its algorithm. J. Comput. Appl. Math. **358**, 327–342 (2019)

14. He, G., Xiang, S.: An improved algorithm for the evaluation of Cauchy principal value integrals of oscillatory functions and its application. J. Comput. Appl. Math. **280**, 1–13 (2015)
15. https://functions.wolfram.com/HypergeometricFunctions/HypergeometricPFQ/06/02/02/
16. http://functions.wolfram.com/HypergeometricFunctions/MeijerG/21/02/07/0004/
17. http://functions.wolfram.com/07.34.06.0031.02
18. Huybrechs, D., Vandewalle, S.: A sparse discretization for integral equation formulations of high frequency scattering problems. SIAM J. Sci. Comput. **29**(6), 2305–2328 (2007)
19. Kang, H., Zhang, M.: Numerical evaluation and analysis of highly oscillatory singular Bessel transforms with a particular oscillator. J. Comput. Appl. Math. **420**, 114835 (2023)
20. Keller, P.: A practical algorithm for computing Cauchy principal value integrals. Appl. Math. Comput. **218**(9), 4988–5001 (2012)
21. Lozier, D.W.: Numerical Solution of Linear Difference Equations. Report NBSIR 80-1976, National Bureau of Standerds. Washington, D.C (1980)
22. Luke, Y.L.: Integrals of Bessel Functions. McGraw-Hill, New York (1962)
23. Mason, J.C., Handscomb, D.C.: Chebyshev Polynomials. Chapman and Hall/CRC, New York (2003)
24. Okecha, G.E.: Quadrature formulae for Cauchy principal value integrals of oscillatory kind. Math. Comp. **49**(179), 259–268 (1987)
25. Oliver, J.: The numerical solution of linear recurrence relations. Numer. Math. **11**, 349–360 (1968)
26. Oreshkin, B.: http://www.mathworks.com/matlabcentral/fileexchange/31490-meijerg/content/MeijerG/MeijerG.m
27. Piessens, R., Branders, M.: Modified Clenshaw-Curtis method for the computation of Bessel function integrals. BIT Numer. Math. **23**, 370–381 (1983)
28. Wang, Z., Guo, D.: Introduction to Special Functions. Peking University Press, Beijing (2000)
29. Wang, H., Xiang, S.: Uniform approximations to Cauchy principal value integrals of oscillatory functions. Appl. Math. Comput. **215**(5), 1886–1894 (2009)
30. Wang, H., Xiang, S.: On the evaluation of Cauchy principal value integrals of oscillatory functions. J. Comput. Appl. Math. **234**(1), 95–100 (2010)
31. Wang, H.: A unified framework for asymptotic analysis and computation of finite Hankel transform. J. Math. Anal. Appl. **483**, 123640 (2020)
32. Wang, H., Zhang, L., Huybrechs, D.: Asymptotic expansions and fast computation of oscillatory Hilbert transforms. Numer. Math. **123**, 709–743 (2013)
33. Whittaker, E.T., Watson, G.N.: A Course of Modern Analysis. Cambridge University Press, Cambridge (1927)
34. Xiang, S.: Numerical analysis of a fast integration method for highly oscillatory functions. BIT Numer. Math. **47**(2), 469–482 (2007)
35. Xiang, S., Gui, W., Mo, P.: Numerical quadrature for Bessel transformations. Appl. Numer. Math. **58**(9), 1247–1261 (2008)
36. Xiang, S., Wang, H.: Fast integration of highly oscillatory integrals with exotic oscillators. Math. Comp. **79**(270), 829–844 (2010)
37. Xiang, S., Cho, Y.J., Wang, H., Brunner, H.: Clenshaw-Curtis-Filon-type methods for highly oscillatory Bessel transforms and applications. IMA J. Numer. Anal. **31**, 1281–1314 (2011)
38. Xiang, S., Cho, Y.J., Wang, H., Brunner, H.: Erratum to " Clenshaw-Curtis-Filon-type methods for highly oscillatory Bessel transforms and applications " (IMA Journal of Numerical Analysis (2011) 31: 1281–1314). IMA J. Numer. Anal. **33**, 1480–1483 (2013)
39. Xu, Z., Xiang, S., He, G.: Efficient evaluation of oscillatory Bessel Hilbert transforms. J. Comput. Appl. Math. **258**, 57–66 (2014)