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The improvement of the truncated Euler-Maruyama method for non-Lipschitz stochastic differential equations

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Abstract
This paper is concerned with the numerical approximations for stochastic differential equations with non-Lipschitz drift or diffusion coefficients. A modified truncated Euler-Maruyama discretization scheme is developed. Moreover, by establishing the criteria on stochastic C-stability and B-consistency of the truncated Euler-Maruyama method, we obtain the strong convergence and the convergence rate of the numerical method. Finally, numerical examples are given to illustrate our theoretical results.

Keywords Truncated Euler-Maruyama · non-Lipschitz · C-stability · B-consistency · Convergence rate

Mathematics Subject Classification (2010) 65C30 · 60H10

1 Introduction

Due to the wide applications in many fields, such as finance, biology and engineering stochastic differential equations (SDEs) have been extensively studied in the past decades. Lots of existing results, especially the early ones, were established under the global or local Lipschitz conditions and the linear growth assumptions. In 2002, replacing the linear growth condition with the Khasminskii-type condition, Mao [1] developed the existence and uniqueness result of solution for a class of super-linear SDEs. As we know, the local Lipschitz condition is necessary for the local maximum solutions. However, there are a large number of models, such as most mathematical financial models including Ait-Sahalia[2],Cox-Ingersoll-Ross[3],3/2 models [4] and

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SIS epidemic model [5], whose coefficients fail to satisfy the local Lipschitz condition. Thus it is meaningful and necessary to consider the SDEs with non-Lipschitz condition.

Most SDEs can not be solved analytically, so study on numerical solutions for SDEs has received widespread attention. During the past decades, various numerical methods have been proposed for SDEs under the global or local Lipschitz conditions and the linear-growth-type conditions. But Hutzenthaler et al. [6] have proved that the classical Euler-Maruyama (EM) method is divergent for SDEs whose drift or diffusion coefficients are super-linear. Therefore, much attention has been devoted to modifying the classical EM method such that the modified method is effective to deal with the super-linear SDEs. For example, Hutzenthaler et al. established the tamed EM method [7], Liu et al. proved the stopping EM method [8], Mao proposed the truncated EM method [9, 10], Guo modified the truncated EM method to obtain the partially truncated EM method [11, 12] and Beyn discussed the projected EM method [13, 14].

In the framework of Higham [15], the boundness of the numerical approximation plays a key role in order to get the convergence rate of the numerical method. Whereas it is often difficult to prove the boundness of numerical method for some mathematical financial models. So Beyn et al. [13, 14] investigated the C-stability and B-consistency of numerical approximations, which can help avoid studying higher moment estimates of the numerical scheme. In this way, the convergence analysis of numerical methods can be simplified significantly. Meanwhile, Chassagneux et al. [16] proposed convergence analysis by Lamperti transform for SDEs with non-Lipschitz diffusion. But the Lamperti transform method may be failure when the diffusion is not continuously differentiable or its inverse function is not well defined. Moreover, the Lamperti transform method is a little cumbersome. We always need to calculate the Lamperti transformation and verify the boundness of the transformed numerical solution. In this paper, we will thus carry out the convergence analysis directly rather than use the Lamperti transform and obtain the convergence rate by the C-stability and B-consistency technique.

The remainder of this paper is organized as follows. Notations, assumptions, structure numerical methods and some useful lemmas will be presented in Section 2. In Section 3, we will establish our main results that the modified EM method is stochastically C-stable and B-consistent and hence will be strongly convergent. Section 4 covers an application to the famous Ait-Sahalia model to illustrate our theoretical result. Finally, the article is concluded in Section 5.

2 Preliminaries and useful lemmas

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. $\mathbb{E}$ denotes the probability expectation with respect to $\mathbb{P}$. $\mathbb{R}^+$ stands for the set of all positive real number. We write $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively for the Euclidean inner product and the Euclidean norm on $\mathbb{R}$. Let $L^2(\Omega, \mathbb{R})$ denote the family of $\mathbb{R}$-valued random variables ξ with $\mathbb{E}|\xi|^2 < \infty$ and for simplicity we set $L^2(\Omega, \mathbb{R}) = L^2$. Let $W(t)$ be a 1-dimensional $\mathcal{F}_t$-adapted Brownian motion defined on the probability space. For two real number a and b, we use $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. For a set G, its indicator function is denoted by I_G, namely $I_G(x) = 1$ if $x \in G$ and $I_G(x) = 0$ otherwise. We denote by $(Id - \mathbb{E}[\cdot|\mathcal{F}_t])Y = Y - \mathbb{E}[Y|\mathcal{F}_t]$ the projection of an $\mathbb{R}$-value random variable orthogonal to the conditional expectation $\mathbb{E}[Y|\mathcal{F}_t]$.

Consider a 1-dimensional nonlinear SDEs

$$dX(t) = f(X(t))dt + g(X(t))dW(t) \tag{2.1}$$

with the initial value $X(0)$, where $f, g : \mathbb{R} \to \mathbb{R}$ and W is the standard Brownian motion.

Throughout this paper, we impose the following standing hypotheses.

Assumption 2.1 *There are constants $\alpha, \beta, \lambda > 0$ and $K > 0$ such that*

$$(x-y)(f(x)-f(y)) + \lambda|g(x)-g(y)|^2 \le K|x-y|^2,$$

$$|f(x)-f(y)| \vee |g(x)-g(y)| \le K\Big(1+|x|^\alpha+|y|^\alpha+\frac{1}{|x|^\beta}+\frac{1}{|y|^\beta}\Big)|x-y|$$

and

$$|f(x)| \vee |g(x)| \le K(1+|x|^{\alpha+1}+\frac{1}{|x|^{\beta-1}})$$

for all $(x, y) \in \mathbb{R}^+$.

Remark 2.2 *The stochastic differential systems satisfying* Assumption 2.1 *exist widely in the real world, such as the Ait-Sahalia interest rate model in finance and the SIS epidemic model in biomedicine. In* Section 4*, we analyse these two models in detail.*

Assumption 2.3 *Assume for any $p \in [2, \infty)$, there exists a pair of constants $q' > p(\alpha+1)$ and $q > p\beta$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|X(t)|^{q'} + |X(t)|^{-q}\Big] < \infty.$$

Remark 2.4 Assumption 2.3 *is in fact the property of moment boundness for the analytical solution $X(t)$ of SDE* Eq. 2.1*, which has been investigated in many works, such as* [6–10, 15–17] *and references therein. Moreover, we can know from these works that the Ait-Sahalia, Cox-Ingersoll-Ross,* 3/2 *and SIS models all belong to this class.*

Assumption 2.5 *The strictly positive constants k and k' satisfy $2\beta k \le 1$ and $2\alpha k' \le 1$.*

Before giving the numerical method, we define the truncated mapping $\pi_\Delta : \mathbb{R}^+ \to \mathbb{R}^+$ for a given step size $\Delta \in (0, 1)$ by

$$\pi_\Delta(x) = \big((\Delta^k \vee x) \wedge \Delta^{-k'}\big).$$

Now let us present the truncated EM method Ψ, which is a one-step scheme

$$\Psi(x, \Delta) := \pi_\Delta(x) + f_\Delta(x)\Delta + g_\Delta(x)\Delta W_n, \tag{2.2}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$, $f_\Delta(x) = f(\pi_\Delta(x))$ and $g_\Delta(x) = g(\pi_\Delta(x))$.

Next we establish two important lemmas for later use.

Lemma 2.6 *Let* Assumptions 2.1 and 2.5 *hold. Then for all* $\Delta \in (0, 1)$*, we have*

$$\Big|\pi_\Delta(x)-\pi_\Delta(y)+(f_\Delta(x)-f_\Delta(y))\Delta\Big|^2+2\lambda\Delta|g_\Delta(x)-g_\Delta(y)|^2 \le (1+C\Delta)|x-y|^2. \tag{2.3}$$

where C *stands for generic positive real constants and its values may change between occurrences, but independent of the step size* Δ.

Proof We can derive from Assumption 2.1 that

$$\begin{aligned}
&|\pi_\Delta(x) - \pi_\Delta(y) + (f_\Delta(x) - f_\Delta(y))\Delta|^2 \\
=\ & |\pi_\Delta(x) - \pi_\Delta(y)|^2 + 2\Big\langle\pi_\Delta(x) - \pi_\Delta(y), f_\Delta(x) - f_\Delta(y)\Big\rangle\Delta + |f_\Delta(x) - f_\Delta(y)|^2\Delta^2 \\
\le\ & |\pi_\Delta(x) - \pi_\Delta(y)|^2 + 2K|\pi_\Delta(x) - \pi_\Delta(y)|^2\Delta - 2\lambda\Delta|g_\Delta(x) - g_\Delta(y)|^2 + |f_\Delta(x) - f_\Delta(y)|^2\Delta^2.
\end{aligned}$$

According to Assumptions 2.1 and 2.5 ($2\beta k \le 1$ and $2\alpha k' \le 1$), we get

$$\begin{aligned}
&|f_\Delta(x) - f_\Delta(y)| \\
\le\ & K(1 + |\pi_\Delta(x)|^\alpha + |\pi_\Delta(y)|^\alpha + \frac{1}{|\pi_\Delta(x)|^\beta} + \frac{1}{|\pi_\Delta(y)|^\beta})|\pi_\Delta(x) - \pi_\Delta(y)| \\
\le\ & K(1 + 2\Delta^{-\alpha k'} + 2\Delta^{-\beta k})|\pi_\Delta(x) - \pi_\Delta(y)| \\
\le\ & K(1 + 4\Delta^{-1/2})|\pi_\Delta(x) - \pi_\Delta(y)|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|\pi_\Delta(x) - \pi_\Delta(y) + (f_\Delta(x) - f_\Delta(y))\Delta|^2 + 2\lambda\Delta|g_\Delta(x) - g_\Delta(y)|^2 \\
\le\ & |\pi_\Delta(x) - \pi_\Delta(y)|^2 + 2K\Delta|\pi_\Delta(x) - \pi_\Delta(y)|^2 + K^2(1 + 4\Delta^{-1/2})^2\Delta^2|\pi_\Delta(x) - \pi_\Delta(y)|^2 \\
\le\ & (1 + C\Delta)|\pi_\Delta(x) - \pi_\Delta(y)|^2.
\end{aligned}$$

It is easy to see we just need to prove $|\pi_\Delta(x) - \pi_\Delta(y)| \le |x - y|$ to get the inequality Eq. 2.3. According to the symmetry of x and y, we may as well let $x < y$, so we have *Case 1*. when $x < y < \Delta^k < \Delta^{-k'}$, $\pi_\Delta(x) = \Delta^k$ and $\pi_\Delta(y) = \Delta^k$. Then the inequality

$$|\pi_\Delta(x) - \pi_\Delta(y)| = 0 \le |x - y|$$

holds clearly.

Case 2. when $x < \Delta^k < y < \Delta^{-k'}$, $\pi_\Delta(x) = \Delta^k$ and $\pi_\Delta(y) = y$. Thus

$$|\pi_\Delta(x) - \pi_\Delta(y)| = |\Delta^k - y| \le |x - y|.$$

Case 3. when $\Delta^k < x < y < \Delta^{-k'}$, $\pi_\Delta(x) = x$ and $\pi_\Delta(y) = y$. Then the required expression holds obviously.

Case 4. when $\Delta^k < x < \Delta^{-k'} < y$, then $\pi_\Delta(x) = x$ and $\pi_\Delta(y) = \Delta^{-k'}$. Therefore,

$$|\pi_\Delta(x) - \pi_\Delta(y)| = |x - \Delta^{-k'}| \le |x - y|.$$

Case 5. when $\Delta^k < \Delta^{-k'} < x < y$, $\pi_\Delta(x) = \Delta^{-k'}$ and $\pi_\Delta(y) = \Delta^{-k'}$. The case is same as case 1, so the inequality also holds.

In conclusion, the inequality Eq. 2.3 holds. The proof is complete. □

Lemma 2.7 *Under* Assumptions 2.1 and 2.3*, for any* $p \in [2, \infty)$ *and* $t_1, t_2 \in [0, T]$*, there exists a constant* C *such that*

$$\mathbb{E}|X(t_1) - X(t_2)|^p \le C\Delta^{p/2}.$$

Proof Without loss of generality, let $0 \le t_1 < t_2 \le T$. Integrating both sides of Eq. 2.1, by the Hölder inequality and Itô isometry, we get

$$\begin{aligned}\mathbb{E}|X(t_1) - X(t_2)|^p &= \mathbb{E}\Big|\int_{t_1}^{t_2} f(X(s))ds + \int_{t_1}^{t_2} g(X(s))dW(s)\Big|^p \\ &\le C\Big(\Delta^{p-1}\int_{t_1}^{t_2}\mathbb{E}|f(X(s))|^p ds + \Delta^{\frac{p-2}{2}}\int_{t_1}^{t_2}\mathbb{E}|g(X(s))|^p ds\Big).\end{aligned}$$

It follows from Assumptions 2.1 and 2.3 that

$$\mathbb{E}|f(x)|^p \le K\mathbb{E}(1 + |x|^{\alpha+1} + |x|^{-(\beta-1)})^p \le C(1 + \mathbb{E}|x|^{p(\alpha+1)} + |x|^{-p(\beta-1)}) \le C,$$

where C is a positive constant whose value may change back and forth. Similarly, we have $\mathbb{E}|g(x)|^p \le C$. So we can further obtain

$$\mathbb{E}|X(t_1) - X(t_2)|^p \le C\Delta^{p/2}.$$

The proof is complete. □

For the convenience of the reader, we list the definitions on C-stability and B-consistency. We also refer the reader to [13] and [14] for more details.

Definition 2.8 (see Definition 3.2 in [13]) *A stochastic one-step method* (Ψ, Δ) *is called stochastically* C*-stable if there exists a constant* C_{stab} *and a parameter value* $\lambda \in (1, \infty)$ *such that for all* $t \in [0, T]$ *and all random variable* $Y, Z \in L^2$*, it holds that*

$$\begin{aligned}&\|\mathbb{E}[\Psi(Y, \Delta) - \Psi(Z, \Delta)|\mathcal{F}_t]\|_{L^2}^2 + \lambda\|(Id - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi(Y, \Delta) - \Psi(Z, \Delta))\|_{L^2}^2 \\ &\le (1 + C_{stab}\Delta)\|Y - Z\|_{L^2}^2.\end{aligned} \tag{2.4}$$

Definition 2.9 (see Definition 3.3 in [13]) *A stochastic one-step method* (Ψ, Δ) *is called stochastically* B*-consistent of order* $\gamma > 0$ *to* Eq. 2.1 *if there exists a constant* C_{cons} *such that for every* $t \in [0, T]$*, it holds that*

$$\|\mathbb{E}[X(t+\Delta) - \Psi(X(t), \Delta)|\mathcal{F}_t]\|_{L^2} \leq C_{cons}\Delta^{\gamma+1} \tag{2.5}$$

and

$$\|(Id - \mathbb{E}[\cdot|\mathcal{F}_t])(X(t+\Delta) - \Psi(X(t), \Delta))\|_{L^2} \leq C_{cons}\Delta^{\gamma+\frac{1}{2}}, \tag{2.6}$$

where $X(t)$ *denotes the exact solution to* Eq. 2.1.

3 Main results

We can know from Theorem 3.7 in [13] that in order to obtain the convergence rate of the one-step numerical method, we only need to prove the numerical method is C-stable and B-consistent. Thus, let us establish the results on stochastic C-stability and B-consistency respectively.

Theorem 3.1 *Under* Assumptions 2.1, 2.3 and 2.5*, for any time step* $\Delta \in (0, 1)$*, the truncated EM method is stochastically* C*-stable.*

Proof Recalling the definition of the truncated EM method $\Psi(x, \Delta)$ in Eq. 2.2, we have

$$\begin{aligned}
&\mathbb{E}[\Psi(Y, \Delta) - \Psi(Z, \Delta)|\mathcal{F}_t] \\
&= \mathbb{E}[\pi_\Delta(Y) + f_\Delta(Y)\Delta + g_\Delta(Y)\Delta W_k - \pi_\Delta(Z) - f_\Delta(Z)\Delta - g_\Delta(Z)\Delta W_k|\mathcal{F}_t] \\
&= \pi_\Delta(Y) - \pi_\Delta(Z) + (f_\Delta(Y) - f_\Delta(Z))\Delta
\end{aligned}$$

and

$$\begin{aligned}
&(Id - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi(Y, \Delta) - \Psi(Z, \Delta)) \\
&= \Psi(Y, \Delta) - \Psi(Z, \Delta) - \mathbb{E}[\Psi(Y, \Delta) - \Psi(Z, \Delta)|\mathcal{F}_t] \\
&= (g_\Delta(Y) - g_\Delta(Z))\Delta W_k.
\end{aligned}$$

Then from the Itô isometry and Lemma 2.6, it follows that

$$\begin{aligned}
&||\pi_\Delta(Y) - \pi_\Delta(Z) + (f_\Delta(Y) - f_\Delta(Z))\Delta||^2_{L^2} + 2\eta||(g_\Delta(Y) - g_\Delta(Z))\Delta W_k||^2_{L^2} \\
&= \mathbb{E}\Big[|\pi_\Delta(Y) - \pi_\Delta(Z) + (f_\Delta(Y) - f_\Delta(Z))\Delta|^2 + 2\eta\Delta|(g_\Delta(Y) - g_\Delta(Z))|^2\Big] \\
&\leq (1 + C\Delta)\mathbb{E}|Y - Z|^2 = (1 + C\Delta)||Y - Z||^2_{L^2},
\end{aligned}$$

which implies condition Eq. 2.4 holds. The proof is therefore complete. □

Theorem 3.2 *Let* Assumptions 2.1, 2.3 and 2.5 *hold. Then for any time-step* $\Delta \in (0, 1)$*, the truncated EM method is stochastically* B*-consistent of order* $\gamma = \frac{1}{2}$.

Proof It follows from Eqs. 2.1 and 2.2 that

$$
\begin{aligned}
& X(t_{n+1}) - \Psi(X(t_n), \Delta) \\
= & X(t_n) + \int_{t_n}^{t_{n+1}} f(X(s))ds + \int_{t_n}^{t_{n+1}} g(X(s))dW(s) \\
& -\pi_\Delta(X(t_n)) - f_\Delta(X(t_n))\Delta - g_\Delta(X(t_n))\Delta W_n \\
= & X(t_n) - \pi_\Delta(X(t_n)) + \int_{t_n}^{t_{n+1}} \Big(f(X(s)) - f(X(t_n))\Big)ds + \Big(f(X(t_n)) - f_\Delta(X(t_n))\Big)\Delta \\
& + \int_{t_n}^{t_{n+1}} \Big(g(X(s)) - g(X(t_n))\Big)dW(s) + \Big(g(X(t_n)) - g_\Delta(X(t_n))\Big)\Delta W_n.
\end{aligned}
$$

Taking condition expectation on both sides yields

$$
\begin{aligned}
& \Big\| \mathbb{E}[X(t_{n+1}) - \Psi(X(t_n), \Delta)|\mathcal{F}_t] \Big\|_{L^2} \\
\leq & \Big\| X(t_n) - \pi_\Delta(X(t_n)) \Big\|_{L^2} + \Big\| f(X(t_n)) - f_\Delta(X(t_n)) \Big\|_{L^2} \Delta \\
& + \int_{t_n}^{t_{n+1}} \big\| \mathbb{E}[f(X(s)) - f(X(t_n))|\mathcal{F}_t] \big\|_{L^2} ds \\
:= & I_1 + I_2 + I_3.
\end{aligned} \tag{3.1}
$$

For the first term I_1, we have

$$
\begin{aligned}
& \mathbb{E}\Big[|X(t_n) - \pi_\Delta(X(t_n))|^2\Big] \\
= & \mathbb{E}\Big[|X(t_n) - \pi_\Delta(X(t_n))|^2 I_{\{X(t_n)<\Delta^k\}}\Big] + 0 + \mathbb{E}\Big[|X(t_n) - \pi_\Delta(X(t_n))|^2 I_{\{X(t_n)\geq\Delta^{-k'}\}}\Big] \\
= & \mathbb{E}\Big[|X(t_n) - \Delta^k|^2 I_{\{X(t_n)<\Delta^k\}}\Big] + \mathbb{E}\Big[|X(t_n) - \Delta^{-k'}|^2 I_{\{X(t_n)\geq\Delta^{-k'}\}}\Big] \\
\leq & \Delta^{2k}\mathbb{P}\{X(t_n) < \Delta^k\} + \mathbb{E}\Big[|X(t_n)|^2 I_{\{X(t_n)\geq\Delta^{-k'}\}}\Big].
\end{aligned}
$$

Set $\eta = \frac{q'}{2}$ and $\theta = \frac{q'}{q'-2}$ as the conjugate exponent. Then by the Hölder inequality, the Chebyshev inequality and Assumption 2.3, we get

$$
\begin{aligned}
\mathbb{E}\Big[|X(t_n)|^2 I_{\{X(t_n)>\Delta^{-k'}\}}\Big] & \leq \mathbb{E}[|X(t_n)|^{q'}]^{1/\eta}\mathbb{P}\{X(t_n) > \Delta^{-k'}\}^{1/\theta} \\
& \leq C\mathbb{P}\{X(t_n) > \Delta^{-k'}\}^{1/\theta} \leq C\Big(\frac{\mathbb{E}|X(t_n)|^{q'}}{\Delta^{-k'q'}}\Big)^{1/\theta} \\
& \leq C\Delta^{\frac{k'q'}{\theta}} = C\Delta^{k'(q'-2)}.
\end{aligned}
$$

Similarly, we have

$$
\mathbb{P}\{X(t_n) < \Delta^k\} = \mathbb{P}\{X(t_n)^{-q} > \Delta^{-kq}\} \leq \frac{\mathbb{E}|X(t_n)|^{-q}}{\Delta^{-kq}} \leq C\Delta^{kq}.
$$

Then

$$\Delta^{2k}\mathbb{P}\{X(t_n) < \Delta^k\} \leq C\Delta^{k(q+2)}.$$

Therefore

$$\mathbb{E}|X(t_n) - \pi_\Delta(X(t_n))|^2 \leq C\Big(\Delta^{k'(q'-2)} + \Delta^{k(q+2)}\Big).$$

Choosing $k' = \frac{1}{2\alpha}, q' = 6\alpha + 2$ and $k = \frac{1}{2\beta}, q = 6\beta - 2$, we can get $k'(q'-2) = 3$, $k(q+2) = 3$, so

$$\mathbb{E}|X(t_n) - \pi_\Delta(X(t_n))|^2 \leq C\Delta^3.$$

Consequently,

$$\begin{aligned} I_1 &= \Big\|X(t_n) - \pi_\Delta(X(t_n))\Big\|_{L^2} = \Big(\mathbb{E}|X(t_n) - \pi_\Delta(X(t_n))|^2\Big)^{1/2} \\ &\leq C\Delta^{\frac{3}{2}}. \end{aligned} \tag{3.2}$$

For the second term I_2, from Assumption 2.1, we can obtain

$$\begin{aligned} &|f(X(t_n)) - f_\Delta(X(t_n))|^2 \\ &\leq K\Big(1 + |X(t_n)|^\alpha + |\pi_\Delta(X(t_n))|^\alpha + \frac{1}{|X(t_n)|^\beta} + \frac{1}{|\pi_\Delta(X(t_n))|^\beta}\Big)^2 |X(t_n) - \pi_\Delta(X(t_n))|^2 \\ &\leq K\Big(2 + |X(t_n)|^{2\alpha} + |\pi_\Delta(X(t_n))|^{2\alpha} + \frac{1}{|X(t_n)|^{2\beta}} + \frac{1}{|\pi_\Delta(X(t_n))|^{2\beta}}\Big)|X(t_n) - \pi_\Delta(X(t_n))|^2 \\ &\leq C(1 + |X(t_n)|^{-2\beta})\Delta^{2k} I_{\{X(t_n) < \Delta^k\}} + C(1 + |X(t_n)|^{2\alpha})|X(t_n)|^2 I_{\{X(t_n) \geq \Delta^{-k'}\}}. \end{aligned}$$

Set $\eta = \frac{q}{2\beta}$ and $\theta = \frac{q}{q-2\beta}$ as the conjugate exponent, we get

$$\begin{aligned} &\mathbb{E}\Big[(1 + |X(t_n)|^{-2\beta})\Delta^{2k} I_{\{X(t_n) < \Delta^k\}}\Big] \\ &= \mathbb{E}\Big[\Delta^{2k} I_{\{X(t_n) < \Delta^k\}}\Big] + \mathbb{E}\Big[|X(t_n)|^{-2\beta}\Delta^{2k} I_{\{X(t_n) < \Delta^k\}}\Big] \\ &\leq \Delta^{2k}\mathbb{P}\{X(t_n) < \Delta^k\} + \Delta^{2k}(\mathbb{E}|X(t_n)|^{-q})^{1/\eta}\mathbb{P}\{X(t_n) < \Delta^k\}^{1/\theta} \\ &\leq \Delta^{2k}\mathbb{P}\{X(t_n) < \Delta^k\} + C\Delta^{2k}\mathbb{P}\{X(t_n) < \Delta^k\}^{1/\theta} \\ &\leq C\Delta^{2k}\Delta^{kq} + C\Delta^{2k}\Delta^{k(q-2\beta+2)} \\ &\leq C\Delta^{k(q-2\beta+2)}. \end{aligned}$$

Similarly,

$$\mathbb{E}\Big[(1 + |X(t_n)|^{2\alpha})|X(t_n)|^2 I_{\{X(t_n) \geq \Delta^{-k'}\}}\Big] \leq C\Delta^{k'(q'-2\alpha-2)}.$$

Then

$$\mathbb{E}|f(X(t_n)) - f_\Delta(X(t_n))|^2 \le C\Big(\Delta^{k(q-2\beta+2)} + \Delta^{k'(q'-2\alpha-2)}\Big).$$

Here we choose $k = \frac{1}{2\beta}$, $q = 4\beta - 2$, $k' = \frac{1}{2\alpha}$ and $q' = 4\alpha + 2$ such that $k(q - 2\beta + 2) = 1$ and $k'(q' - 2\alpha - 2) = 1$. Therefore,

$$\begin{aligned} I_2 &= \Delta\Big\|f(X(t_n)) - f_\Delta(X(t_n))\Big\|_{L^2} = \Delta\Big(\mathbb{E}|f(X(t_n)) - f_\Delta(X(t_n))|^2\Big)^{1/2} \\ &\le C\Delta^{\frac{3}{2}}. \end{aligned} \tag{3.3}$$

For the last term I_3, since $||\mathbb{E}[Y|\mathcal{F}_t]||_{L^2} \le ||Y||_{L^2}$, thus

$$\begin{aligned} I_3 &= \int_{t_n}^{t_{n+1}} \Big\|\mathbb{E}[f(X(s)) - f(X(t_n))|\mathcal{F}_t]\Big\|_{L^2} ds \\ &\le \int_{t_n}^{t_{n+1}} \Big\|f(X(s)) - f(X(t_n))\Big\|_{L^2} ds. \end{aligned}$$

However, due to Assumption 2.1, we have

$$\begin{aligned} &\mathbb{E}|f(X(s)) - f(X(t_n))|^2 \\ \le\ & K\mathbb{E}\Big(1 + |X(s)|^a + |X(t_n)|^\alpha + \frac{1}{|X(s)|^\beta} + \frac{1}{|X(t_n)|^\beta}\Big)^2 |X(s) - X(t_n)|^2 \\ \le\ & K\mathbb{E}\Big(1 + |X(s)|^{2\alpha} + |X(t_n)|^{2\alpha}\Big)|X(s) - X(t_n)|^2 \\ &+ K\mathbb{E}\Big(1 + \frac{1}{|X(s)|^{2\beta}} + \frac{1}{|X(t_n)|^{2\beta}}\Big)|X(s) - X(t_n)|^2 \\ :=\ & I_{31} + I_{32}. \end{aligned}$$

For the term I_{31}, we choose $\eta = \frac{q'}{2\alpha}$ and $\theta = \frac{q'}{q'-2\alpha}$ as the conjugate exponent, then recalling Assumption 2.3, we have

$$\begin{aligned} &\mathbb{E}(1 + |X(s)|^{2\alpha} + |X(t_n)|^{2\alpha})|X(s) - X(t_n)|^2 \\ &\le \Big[\mathbb{E}(1 + |X(s)|^{q'} + |X(t_n)|^{q'})\Big]^{\frac{2\alpha}{q'}} \Big[\mathbb{E}|X(s) - X(t_n)|^{\frac{2q'}{q'-2\alpha}}\Big]^{\frac{q'-2\alpha}{q'}} \\ &\le C\Big[\mathbb{E}|X(s) - X(t_n)|^{\frac{2q'}{q'-2\alpha}}\Big]^{\frac{q'-2\alpha}{q'}}. \end{aligned}$$

Finally, by setting $p = \frac{q'}{q'-2\alpha}$ in Lemma 2.7, we can obtain

$$\mathbb{E}(1 + |X(s)|^{2\alpha} + |X(t_n)|^{2\alpha})|X(s) - X(t_n)|^2 \le C\Delta.$$

For the term I_{32}, by the same technique, we can also have

$$\mathbb{E}\Big(1+\frac{1}{|X(s)|^{2\beta}}+\frac{1}{|X(t_n)|^{2\beta}}\Big)|X(s)-X(t_n)|^2 \le C\Delta.$$

So

$$\mathbb{E}|f(X(s))-f(X(t_n))|^2 \le C\Delta.$$

Hence, we can obtain

$$\begin{aligned} I_3 &\le \int_{t_n}^{t_{n+1}} \Big\| f(X(s))-f(X(t_n))\Big\|_{L^2} ds \\ &= \int_{t_n}^{t_{n+1}} \Big(\mathbb{E}|f(X(s))-f(X(t_n))|^2\Big)^{1/2} ds \\ &\le C\Delta^{3/2}. \end{aligned} \tag{3.4}$$

Substituting Eqs. 3.2, 3.3 and 3.4 into Eq. 3.1, we have

$$\Big\|\mathbb{E}[X(t_{n+1})-\Psi(X(t_n),\Delta)|\mathcal{F}_t]\Big\|_{L^2} \le C\Delta^{3/2}.$$

That is to say, Eq. 2.5 holds with $\gamma = \frac{1}{2}$.
Let us continue to prove Eq. 2.6. It is easy to get

$$\begin{aligned} &\Big\|(Id-\mathbb{E}[\cdot|\mathcal{F}_t])(X(t_{n+1})-\Psi(X(t_n),\Delta))\Big\|_{L^2} \\ \le& \int_{t_n}^{t_{n+1}} \Big\|(Id-\mathbb{E}[\cdot|\mathcal{F}_t])(f(X(s))-f(X(t_n)))\Big\|_{L^2} ds \\ &+\Big\|\int_{t_n}^{t_{n+1}} (g(X(s))-g(X(t_n)))dW(s)\Big\|_{L^2} \\ &+\Big\|(g(X(t_n))-g_\Delta(X(t_n)))\Delta W\Big\|_{L^2} \\ :=& J_1+J_2+J_3. \end{aligned}$$

For the term J_1, considering the inequality $\|(Id-\mathbb{E}[\cdot|\mathcal{F}_t])Y\|_{L^2} \le \|Y\|_{L^2}$ for all $Y \in L^2$ and the estimation of I_3, we have

$$\begin{aligned} J_1 &= \int_{t_n}^{t_{n+1}} \Big\|(Id-\mathbb{E}[\cdot|\mathcal{F}_t])(f(X(s))-f(X(t_n)))\Big\|_{L^2} ds \\ &\le \int_{t_n}^{t_{n+1}} \Big\|f(X(s))-f(X(t_n))\Big\|_{L^2} ds \\ &\le C\Delta^{\frac{3}{2}}. \end{aligned} \tag{3.5}$$

By the same technique as that used in the estimation of I_3, we can get

$$\mathbb{E}|g(X(s)) - g(X(t_n))|^2 \leq C\Delta.$$

Then by the Itô isometry, we have

$$\begin{aligned} J_2 &= \left\| \int_{t_n}^{t_{n+1}} (g(X(s)) - g(X(t_n)))dW(s) \right\|_{L^2} \\ &\leq \Big(\int_{t_n}^{t_{n+1}} \|g(X(s)) - g(X(t_n))\|_{L^2}^2 ds \Big)^{\frac{1}{2}} \\ &= \Big(\int_{t_n}^{t_{n+1}} \mathbb{E}|g(X(s)) - g(X(t_n))|^2 ds \Big)^{\frac{1}{2}} \\ &\leq C\Delta. \end{aligned} \tag{3.6}$$

By the Itô isometry and in a similar way to estimating I_2, we can obtain the estimation for J_3 as follows

$$\begin{aligned} J_3 &= \left\| (g(X(t_n)) - g_\Delta(X(t_n)))\Delta W \right\|_{L^2} \\ &= \Delta \left\| (g(X(t_n)) - g_\Delta(X(t_n))) \right\|_{L^2} \\ &\leq C\Delta. \end{aligned} \tag{3.7}$$

Combining Eqs. 3.5, 3.6 and 3.7, we find Eq. 2.6 holds with $\gamma = 1/2$. Therefore, the proof is complete. □

We conclude this section by stating the strong convergence result for the truncated EM method, which follows directly from Theorem 3.1 and Theorem 3.2 as well as Theorem 3.7 in [13].

Proposition 3.3 *Let the functions $f(x)$ and $g(x)$ satisfy* Assumptions 2.1, 2.3 and 2.5. *Then for any $\Delta \in (0, 1)$, the truncated EM method has $1/2$ strong convergent rate. Moreover,*

$$\mathbb{E}|X(T) - \Psi(X(T), \Delta)|^2 \leq C\Delta.$$

4 Numerical examples

Example 4.1 *(Ait-Sahalia model) Let us consider the well-know Ait-Sahalia interest rate model. According to* [17]*, there exists a strong solution $X(t)$ to*

$$dX(t) = \Big(\frac{a_{-1}}{X_t} - a_0 + a_1 X_t - a_2 X_t^{\varrho}\Big)dt + \gamma X_t^{\rho} dW(t),$$

on $(0, \infty)$ with the initial value $x_0 = 1$ and the time $T = 1$. Here $W(t)$ is a scalar Brownian motion and all constant parameters are nonnegative, especially, $\rho, \varrho > 1$.

We may as well choose the parameters $(a_{-1}, a_0, a_1, a_2, \gamma) = (1, 1, 1, 1, 1)$ and $(\varrho, \rho) = (5, 2)$. Thus, for any $x, y \in \mathbb{R}^+$,

$$\begin{aligned}
&(x-y)(f(x)-f(y)) + \lambda|g(x)-g(y)|^2 \\
&= (x-y)(\frac{1}{x} - \frac{1}{y} + x - y + y^5 - x^5) + \lambda|x^2 - y^2|^2 \\
&= [-\frac{1}{xy} + 1 - (x^4 + x^3y + x^2y^2 + xy^3 + y^4) + \lambda(x+y)^2]|x-y|^2 \\
&\leq [1 - (x^4 + x^3y + x^2y^2 + xy^3 + y^4) + \lambda(x+y)^2]|x-y|^2.
\end{aligned}$$

However

$$-(x^3y + xy^3) = -xy(x^2+y^2) \leq 0.5(x^2+y^2)^2 = 0.5(x^4+y^4) + x^2y^2.$$

Hence

$$\begin{aligned}
&(x-y)(f(x)-f(y)) + \lambda|g(x)-g(y)|^2 \\
&\leq [1 - 0.5(x^4+y^4) + \lambda(x^2+y^2)]|x-y|^2 \\
&\leq [2 + \lambda^2]|x-y|^2.
\end{aligned}$$

Moreover, let $\alpha = 4$, $\beta = 2$, then the Assumption 2.1 is satisfied.

Based on [17 Lemma 2.1] , there exists a strong solution on $(0, \infty)$ to the model, and we recall that if $2\rho < \varrho + 1$, then $\sup_{0\leq t\leq T} \mathbb{E}|X(t)|^p$ and $\sup_{0\leq t\leq T} \mathbb{E}|X(t)|^{-p}$ are finite for all $p \neq 0$. In other words, Assumption 2.3 holds.

Let us fix these parameters and recall that $\alpha = 4$ and $\beta = 2$. We choose $k = 1/4$ and $k' = 1/8$ such that $2\beta k = 1$ and $2\alpha k' = 1$. That is, Assumption 2.5 holds.

We use 2^{-12} as the step size of the reference solution and choose $2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}$ and 2^{-6} respectively as the step size of the truncated EM. For each step size, 500 paths are simulated. Finally, we get the result presented in the figure, which clearly indicates that our numerical method has 1/2 strong convergence rate. Please note that the red dashed is reference line (Fig. 1).

Example 4.2 *(SIS model) Let us consider the following stochastic SIS epidemic model* [5].

$$\begin{cases} dS(t) = [\mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t)] - \sigma S(t)I(t)dW(t), \\ dI(t) = [\beta S(t)I(t) - (\mu+\gamma)I(t)]dt + \sigma S(t)I(t)dW(t), \end{cases}$$

with initial values satisfying $I(0) + S(0) = S_0 + I_0 = N$ *where* $N, \mu, \gamma, \beta, \sigma$ *are non-negative numbers. Since* $d[S(t) + I(t)] = [\mu N - \mu(S(t) + I(t))]dt$*, it is easy to obtain* $I(t) + S(t) = N$*. Then we only need to consider the SIS model for* $I(t)$ *as follows*

$$dI(t) = [\eta I(t) - \beta I^2(t)]dt + \sigma I(t)(N - I(t))dW(t)$$

where $\eta = \beta N - \mu - \gamma$.

We can choose the $(\eta, \beta, \sigma, N) = (5, 0.5, 0.035, 100)$ (see e.g. [5 Example 4.2]) (Fig. 2).

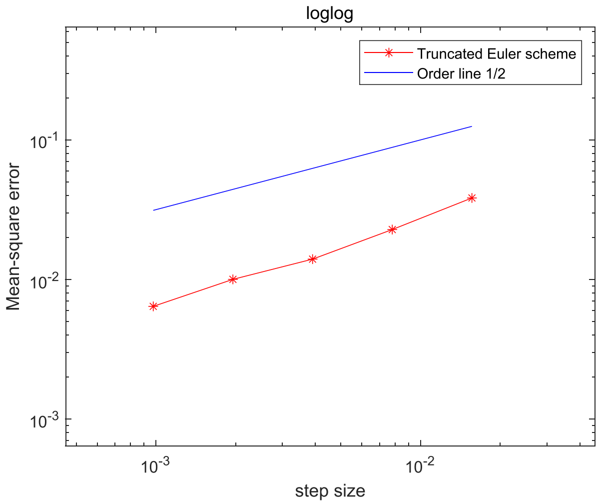

Fig. 1 Ait-Sahalia model

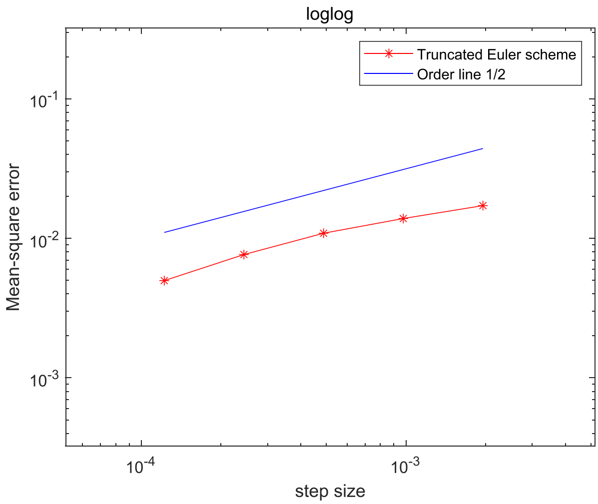

Fig. 2 SIS epidemic model

5 Conclusion

In this paper, we have studied the numerical approximations for stochastic differential equations with non-Lipschitz coefficients. A modified truncated Euler-Maruyama discretization scheme has been proposed and its convergence and the convergence rate have been obtained by establishing the criteria on stochastic C-stability and B-consistency of the numerical method. We have also verified the theoretical results with a famous financial example.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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