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Order two superconvergence of the CDG finite elements for non-self adjoint and indefinite elliptic equations

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Abstract

A conforming discontinuous Galerkin (CDG) finite element method is designed for solving second order non-self adjoint and indefinite elliptic equations. Unlike other discontinuous Galerkin (DG) methods, the numerical trace on the edge/triangle between two elements is not the average of two discontinuous P_k functions, but a lifted P_{k+2} function from four (eight in 3D) nearby P_k functions. While all existing DG methods have the optimal order of convergence, this CDG method has a superconvergence of order two above the optimal order when solving general second order elliptic equations. Due to the superconvergence, a post-process lifts a P_k CDG solution to a quasi-optimal P_{k+2} solution on each element. Numerical tests in 2D and 3D are provided confirming the theory.

Keywords Finite element · Conforming discontinuous Galerkin method · Second order elliptic equation · Triangular mesh · Tetrahedral mesh

Mathematics Subject Classification (2010) Primary · 65N15 · 65N30

1 Introduction

We solve the following second order elliptic problem:

$$-\nabla \cdot (a\nabla u) + \mathbf{b} \cdot \nabla u + cu = f \quad \text{in } \Omega, \tag{1.1}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

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where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded polytopal domain with a Lipschitz boundary, $a = (a_{ij}(x))_{d\times d}$ is a symmetric, uniformly positive definite matrix of coefficients, i.e., there is a positive constant α such that

$$\xi^T a\xi \geq \alpha\, \xi^T \xi \quad \forall \xi \in \mathbb{R}^d, \tag{1.3}$$

and function c satisfies

$$\inf_{\mathbf{x}\in\Omega} c > \frac{1}{2\alpha}\|\mathbf{b}\|_{L^\infty(\Omega)}.$$

The continuous Galerkin finite element method approximates the solution of (1.1) by continuous piecewise P_k polynomials on a triangular or tetrahedral mesh. That is, finding $u_h \in V_h \subset H_0^1(\Omega)$ such that

$$(a\nabla u_h, \nabla v_h) + ((\mathbf{b}\cdot\nabla + c)u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{1.4}$$

Such a method is called a conforming finite element method. The nonconforming finite element method employs piecewise P_k polynomials which are continuous weakly between elements at order P_{k-1}. The weak form (1.4) remains.

A third class of finite element methods is the discontinuous Galerkin (DG) methods, where the finite element space consists of totally discontinuous piecewise P_k polynomials on a triangular or tetrahedral mesh. In all DG methods, inter-element integral terms and a penalty (stabilizer) term are added to the weak form (1.4) in order to keep consistency and to obtain convergent solutions, cf. [2]. But a conforming discontinuous Galerkin (CDG) method is introduced in [4, 9–16] which keeps the weak form (1.4) of the conforming finite element method, unlike rest DG methods.

In this work, we extend the CDG method of [16] to general second order elliptic equations. In the CDG finite element method, the inter-element trace v_b of discontinuous functions is no longer the simple average of two functions v_h on the two sides. It is defined by two steps. First, on an edge e, we define a lifted $P_{k+2}(U_e)$ polynomial (where U_e is a patch of triangles) from four discontinuous P_k functions nearby, $\{v_h \ v_h|_{T_i} \in P_k(T_i),\ i = 1, \ldots, 4\}$, or eight P_k functions in 3D. In the second step, we define the trace v_b to be the L^2-projection of this lifted P_{k+2} polynomial into $P_{k+1}(e)$. We show that such a CDG solution converges two orders above the optimal order. That is, the error between the local L^2 projection of the true solution and the CDG P_k solution converges at $O(h^{k+3})$ in L^2-norm, and at $O(h^{k+2})$ in H^1-like norm. Because of this superconvergence, we show that such a P_k CDG solution can be postprocessed to a quasi-optimal P_{k+2} solution locally on each element. Numerical examples are computed in 2D and 3D, confirming the theory.

2 Preliminary

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of quasi-uniform triangles in 2D or tetrahedra in 3D. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and by

$h = \max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$. Denote by $\mathcal{E}_h$ the set of all edges or face-triangles in $\mathcal{T}_h$, and by $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ the set of all interior edges s or face-triangles.

For the purpose of error analysis, we define a WG (weak Galerkin) finite element space as follows: cf. [3, 5–8, 17], for $k \geq 1$,

$$\tilde{V}_h = \{v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_{k+1}(e), \quad e \subset \partial T,\ T \in \mathcal{T}_h,\ v_b|_{\partial\Omega} = 0\}. \tag{2.1}$$

Please note that any function $v \in \tilde{V}_h$ has a single value v_b on each edge $e \in \mathcal{E}_h$.

For $v = \{v_0, v_b\} \in \tilde{V}_h$, a weak gradient $\nabla_w v$ is a piecewise vector valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w v|_T \in [P_{k+1}(T)]^d$ satisfies

$$(\nabla_w v, \mathbf{q})_T = (\nabla v_0, \mathbf{q})_T + \langle v_b - v_0, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \qquad \forall \mathbf{q} \in [P_{k+1}(T)]^d. \tag{2.2}$$

Lemma 2.1 *([1]) For $v = \{v_0, v_b\} \in \tilde{V}_h$, we have*

$$C_1\|v\|_{1,h} \leq \|\nabla_w v\| \leq C_2\|v\|_{1,h}, \tag{2.3}$$

where

$$\|v\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} (\|\nabla v_0\|_T^2 + h_T^{-1}\|v_0 - v_b\|_{\partial T}^2). \tag{2.4}$$

Let Π_k and Π_k^b be the generic local L^2 projections onto $[P_k(T)]^j$ for $T \in \mathcal{T}_h$ and $[P_k(e)]^j$ for $e \in \mathcal{E}_h$, respectively, where $j = 1, \cdots, d$. Define $Q_h u = \{\Pi_k u, \Pi_{k+1}^b u\} \in \tilde{V}_h$.

Lemma 2.2 *([1]) For $u \in H^1(\Omega)$, then*

$$\nabla_w Q_h u = \Pi_{k+1}\nabla u. \tag{2.5}$$

3 CDG finite element scheme

For a given integer $k \geq 1$, let V_h be the CDG finite element space associated with $\mathcal{T}_h$ by

$$V_h = \{v \in L^2(\Omega) :\ v|_T \in P_k(T),\ T \in \mathcal{T}_h\}. \tag{3.1}$$

To connect the vector spaces V_h and $\tilde{V}_h$, we define an embedding operator $E_h : V_h \to \tilde{V}_h$ such that for $v \in V_h$

$$E_h v = \{v, \Pi_{k+1} E_e v\} \in \tilde{V}_h, \tag{3.2}$$

where Π_{k+1} is the L^2 projection on edge e and $E_e \in P_{k+2}(U_e)$ is defined by

$$\begin{cases} E_e v = 0, & \text{if } e \subset \partial\Omega, \\ (E_e v, \Pi_k w)_{S_e} = (v, \Pi_k w)_{S_e} \ \forall w \in P_{k+2}(U_e), & \text{if } e \in \mathcal{E}_h^0. \end{cases} \tag{3.3}$$

Here, Π_k is a local L^2-projection on to space $\prod_{i=1}^4 P_k(S_i)$, S_e is a union of 4 aligned squares $\{S_i\}$ inside U_e,

$$\begin{aligned} S_1 &= [x_c - \frac{5}{4}\gamma_0 h, x_c - \frac{3}{4}\gamma_0 h] \times [y_c - \frac{5}{4}\gamma_0 h, y_c - \frac{3}{4}\gamma_0 h], \\ S_2 &= [x_c + \frac{3}{4}\gamma_0 h, x_c + \frac{5}{4}\gamma_0 h] \times [y_c - \frac{5}{4}\gamma_0 h, y_c - \frac{3}{4}\gamma_0 h], \\ S_3 &= [x_c + \frac{3}{4}\gamma_0 h, x_c + \frac{5}{4}\gamma_0 h] \times [y_c + \frac{3}{4}\gamma_0 h, y_c + \frac{5}{4}\gamma_0 h], \\ S_4 &= [x_c - \frac{5}{4}\gamma_0 h, x_c - \frac{3}{4}\gamma_0 h] \times [y_c + \frac{3}{4}\gamma_0 h, y_c + \frac{5}{4}\gamma_0 h], \end{aligned}$$

for some fixed $\gamma_0 > 0$, and U_e is a union of triangles containing the four aligned squares, cf. Fig. 1. One would choose the four triangles as close to e as possible, in order to reduce the constant in the error bound. But they do not have to include the two triangles which have e as an edge. Here, four aligned squares may rotate together. [16] proves that (3.3) defines an $E_e v$. [16] shows also that it preserves $P_{k+2}(U_e)$ polynomials in the sense $E_e v = w$ if $v|_{S_i} = \Pi_{k,S_e} w$ for all $w \in P_{k+2}(U_e)$. In 3D, the set $\{S_i\}$ in (3.3) contains eight aligned cubes, two in each direction.

Lemma 3.1 *([16]) For $k \geq 1$, the lifting operator $E_e : V_h \to P_{k+2}(U_e)$, defined in (3.3), has an order $k+2$ accuracy, i.e., for any $u \in H^{k+3}(\Omega)$,*

$$\|E_e \Pi_k u - u\|_{0,U_e} + h\|\nabla(E_e \Pi_k u - u)\|_{0,U_e} \leq C h^{k+3} |u|_{k+3,U_e}. \tag{3.4}$$

Since $E_h v \in \tilde{V}_h$, $\nabla_w(E_h v)$ can be calculated by (2.2). For $v \in V_h$, its weak gradient $\nabla_w v$ is defined as

$$\nabla_w v = \nabla_w E_h v. \tag{3.5}$$

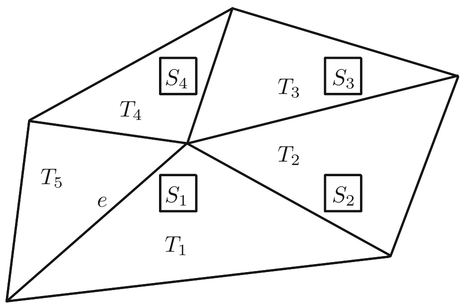

Fig. 1 A closed polygon $U_e = \cup_{i=1}^{n_e} \overline{T_i}$ contains 4 aligned squares, for an edge e, where $n_e = 5$ and $\overline{T_i}$ is the closure of T_i

The CDG finite element method is to find $u_h \in V_h$ such that

$$A(u_h, v) = (f, v) \quad \forall v \in V_h. \tag{3.6}$$

where

$$A(u_h, v) = (a\nabla_w u_h, \nabla_w v) + (\mathbf{b} \cdot \nabla_w u_h, v) + (cu_h, v). \tag{3.7}$$

Defining a norm as follows for $v = \{v_0, v_b\} \in \tilde{V}_h$,

$$|||v|||^2 = \|\nabla_w v\|^2 + \|v_0\|^2. \tag{3.8}$$

For $v \in V_h$, $|||v|||$ is defined as

$$|||v||| = |||E_h v|||. \tag{3.9}$$

Lemma 3.2 *Assume $\kappa = \beta - \frac{\|\mathbf{b}\|^2_{L^\infty(\Omega)}}{2\alpha} > 0$. Then, we have for $v = \{v_0, v_b\} \in \tilde{V}_h$*

$$A(v, v) \geq \gamma |||v|||^2 \tag{3.10}$$

$$A(v, w) \leq C|||v||| \, |||w|||. \tag{3.11}$$

Proof By (1.3) and $\beta = \text{ess inf}\{c(x) : x \in \Omega\}$, we have

$$\begin{aligned}
A(v, v) &\geq \alpha\|\nabla_w v\|^2 + (\mathbf{b} \cdot \nabla_w v, v_0) + \beta\|v_0\|^2 \\
&\geq \alpha\|\nabla_w v\|^2 - \|\mathbf{b}\|_{L^\infty(\Omega)}\|\nabla_w v\|\|v_0\| + \beta\|v_0\|^2 \\
&\geq \frac{\alpha}{2}\|\nabla_w v\|^2 + (\beta - \frac{\|\mathbf{b}\|^2_{L^\infty(\Omega)}}{2\alpha})\|v_0\|^2 \\
&\geq \gamma|||v|||^2
\end{aligned}$$

where $\gamma = \min\{\frac{\alpha}{2}, \kappa\}$. (3.11) is obtained by assuming bounded coefficients. This completes the proof. □

The well posedness of the CDG finite element method is a direct result of the lemma above.

Lemma 3.3 *The CDG finite element method* (3.6) *has a unique solution.*

4 Superconvergence in energy norm

In this section, we will obtain order two superconvergence for the CDG finite element solution in (3.6). The superconvergence of the corresponding WG method [1] will be used to achieve such a goal.

Let $\tilde{u}_h \in \tilde{V}_h$ be the solution of the WG method such that

$$A(\tilde{u}_h, v) = (f, v_0) \quad \forall v = \{v_0, v_b\} \in \tilde{V}_h. \tag{4.1}$$

The superconvergence of the WG finite element solution $\tilde{u}_h$ is derived in [17] described by the following lemma.

Lemma 4.1 *([17]) Let $\tilde{u}_h = \{\tilde{u}_0, \tilde{u}_b\} \in \tilde{V}_h$ be the WG finite element solution of (4.1). Then,*

$$h\|\nabla_w(Q_h u - \tilde{u}_h)\| + \|\Pi_k u - \tilde{u}_0\| \le Ch^{k+3}|u|_{k+3}, \tag{4.2}$$

where $\Pi_k u$ is the L^2 projection, $Q_h u = \{\Pi_k u, \Pi_{k+1}^b u\} \in \tilde{V}_h$ and $\Pi_{k+1}^b u$ is the L^2 projection on an edge.

For any function $\varphi \in H^1(T)$, the following trace inequality holds true:

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{4.3}$$

Lemma 4.2 *Let $u \in H^{k+3}(\Omega)$. Then, we have*

$$|||Q_h u - \Pi_k u||| \le Ch^{k+2}|u|_{k+3}, \tag{4.4}$$
$$|||\Pi_k u - \tilde{u}_h||| \le Ch^{k+2}|u|_{k+3}. \tag{4.5}$$

Proof Recall $Q_h u = \{\Pi_k u, \Pi_{k+1}^b u\}$ and $E_h\Pi_k u = \{\Pi_k u, \Pi_{k+1}^b E_e\Pi_k u\}$. Using (3.5), (4.3), inverse inequality and (3.4), we have with $q = \nabla_w(Q_h u - E_h\Pi_k u)$,

$$\begin{aligned}
&\|\nabla_w(Q_h u - \Pi_k u)\|^2 = \|\nabla_w(Q_h u - E_h\Pi_k u)\|^2 \\
&= \sum_{T\in T_h} \langle \Pi_{k+1}^b u - \Pi_{k+1}^b E_e\Pi_k u, q\rangle_{\partial T} \\
&= \sum_{T\in T_h} \langle u - E_e\Pi_k u, q\rangle_{\partial T} \\
&\le \Big(\sum_{T\in T_h} h_T^{-1}\|u - E_e\Pi_k u\|_{0,\partial T}^2\Big)^{1/2}\Big(\sum_{T\in T_h} h_T\|q\|_{0,\partial T}^2\Big)^{1/2} \\
&\le C\Big(\sum_{T\in T_h} h_T^{-2}\|u - E_e\Pi_k u\|_{0,T}^2 + \|\nabla(u - E_e\Pi_k u)\|_{0,T}^2\Big)^{1/2}\|q\| \\
&\le Ch^{k+2}|u|_{k+3}\|\nabla_w(Q_h u - \Pi_k u)\|.
\end{aligned} \tag{4.6}$$

By (2.4), $|||Q_h u - \Pi_k u||| = |||Q_h u - E_h\Pi_k u|||$. It follows from the definition of $Q_h u$ and $E_h\Pi_k u$,

$$Q_h u - E_h\Pi_k u = \{\Pi_k u - \Pi_k u, Q_b u - \Pi_{k+1}^b E_b\Pi_k u\} = \{0,\ Q_b u - \Pi_{k+1}^b E_b\Pi_k u\}.$$

Thus, we have

$$\|(Q_h u - E_h \Pi_k u)_0\| = 0. \tag{4.7}$$

Combining (4.6), (4.7) and (4.6), we have

$$\begin{aligned}
|\!|\!| Q_h u - \Pi_k u |\!|\!|^2 &= |\!|\!| Q_h u - E_h \Pi_k u |\!|\!|^2 \\
&= \|\nabla_w (Q_h u - E_h \Pi_k u)\|^2 + \|(Q_h u - E_h \Pi_k u)_0\|^2 \\
&= \|\nabla_w (Q_h u - E_h \Pi_k u)\|^2 \\
&\leq Ch^{2(k+2)} |u|_{k+3}^2,
\end{aligned}$$

which proves (4.4). It follows from (4.2) and (4.4),

$$|\!|\!| \Pi_k u - \tilde{u}_h |\!|\!| \leq |\!|\!| \Pi_k u - Q_h u |\!|\!| + |\!|\!| Q_h u - \tilde{u}_h |\!|\!| \leq Ch^{k+2} |u|_{k+3}.$$

This completes the proof of the lemma. □

Subtracting (3.6) from (4.1) implies

$$A(\tilde{u}_h - u_h, v) = 0 \quad \forall v \in V_h. \tag{4.8}$$

The following lemma provides the error bound for $\tilde{u}_h - u_h$.

Lemma 4.3 *Let $u \in H^{k+3}(\Omega)$. Then, we have*

$$|\!|\!| \tilde{u}_h - u_h |\!|\!| \leq Ch^{k+2} |u|_{k+3}. \tag{4.9}$$

Proof By (3.10), (4.8), and (4.5),

$$\begin{aligned}
\gamma |\!|\!| \tilde{u}_h - u_h |\!|\!|^2 &\leq A(\tilde{u}_h - u_h, \tilde{u}_h - u_h) \\
&= A(\tilde{u}_h - u_h, \tilde{u}_h - \Pi_k u) \\
&\leq C |\!|\!| \tilde{u}_h - u_h |\!|\!| |\!|\!| \tilde{u}_h - \Pi_k u |\!|\!|.
\end{aligned}$$

Combining the inequality above with (4.5), we have

$$|\!|\!| \tilde{u}_h - u_h |\!|\!| \leq Ch^{k+2} |u|_{k+3}. \tag{4.10}$$

The proof is completed.

The order two superconvergence of the CDG solution in an energy norm is obtained in the following theorem.

Theorem 4.1 *Let $u \in H^{k+3}(\Omega) \cap H_0^1(\Omega)$ be the exact solution of (1.1). Let $u_h \in V_h$ be the CDG solution of (3.6). Then*

$$|\!|\!| \Pi_k u - u_h |\!|\!| \leq Ch^{k+2} |u|_{k+3}. \tag{4.11}$$

Proof By (4.2), (4.4), and (4.9), we have

$$\begin{aligned}
&|||\Pi_k u - u_h|||\\
&\le |||\Pi_k u - Q_h u||| + |||Q_h u - \tilde{u}_h||| + |||\tilde{u}_h - u_h|||\\
&\le Ch^{k+2}|u|_{k+3},
\end{aligned}$$

which finishes the proof of the theorem. □

5 Superconvergence in L2 norm

Let $e_h = \{e_0, e_b\} = \tilde{u}_h - E_h u_h = \{\tilde{u}_0 - u_h, \tilde{u}_b - u_b\} \in \tilde{V}_h$ with u_b defined in (3.3). We consider the corresponding dual problem: seek $w \in H_0^1(\Omega)$ satisfying

$$-\nabla \cdot (a\nabla w) - \nabla \cdot (\mathbf{b}w)w + cw = e_0 = \tilde{u}_0 - u_h \quad \text{in } \Omega. \tag{5.1}$$

Recall that $\tilde{u}_h = \{\tilde{u}_0, \tilde{u}_b\}$ and u_h are the solutions of the WG method (4.1) and the CDG method (3.6), respectively. Assume that the following H^2-regularity holds

$$\|w\|_2 \le C\|e_0\|. \tag{5.2}$$

Lemma 5.1 *For $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and $e_h \in \tilde{V}_h$, we have*

$$\begin{aligned}
&(-\nabla \cdot (a\nabla w) + \nabla \cdot (\mathbf{b}w)w + cw, e_0)\\
&= A(e_h, Q_h w) + E_1(w, e_h) + E_2(w, e_h)\\
&\quad + E_3(w, e_h) + E_4(w, e_h) + E_5(w, e_h),
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
E_1(w, e_h) &= (a(\nabla w - \nabla_w Q_h w, \nabla_w e_h),\\
E_2(w, e_h) &= \langle (a\nabla w - \Pi_{k+1}(a\nabla w)) \cdot \mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h}\\
E_3(w, e_h) &= \langle (\Pi_{k+1}(\mathbf{b}w) - \mathbf{b}w) \cdot \mathbf{n}, e_b - e_0\rangle_{\partial\mathcal{T}_h}\\
E_4(w, e_h) &= (\mathbf{b} \cdot \nabla_w e_h, w - \Pi_k w)\\
E_5(w, e_h) &= (w - \Pi_k w, ce_0).
\end{aligned}$$

Proof Using the integration by parts and the fact that $\langle a\nabla w \cdot \mathbf{n}, e_b\rangle_{\partial\mathcal{T}_h} = 0$, we have

$$(-\nabla \cdot (a\nabla w, e_0) = (a\nabla w, \nabla e_0)_{\mathcal{T}_h} - \langle a\nabla u \cdot \mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h}$$

It follows from integration by parts and (2.2) that

$$\begin{aligned}
(a\nabla w, \nabla e_0)_{\mathcal{T}_h} &= (\Pi_{k+1}(a\nabla w), \nabla e_0)_{\mathcal{T}_h}\\
&= -(e_0, \nabla \cdot \Pi_{k+1}(a\nabla w))_{\mathcal{T}_h} + \langle e_0, \Pi_{k+1}(a\nabla w) \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}
\end{aligned}$$

$$
\begin{aligned}
&= (\Pi_{k+1}(a\nabla w), \nabla_w e_h)_{\mathcal{T}_h} + \langle e_0 - e_b, \Pi_{k+1}(a\nabla w)\cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (a\nabla w, \nabla_w e_h) + \langle e_0 - e_b, \Pi_{k+1}(a\nabla w)\cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (a\nabla_w Q_h w, \nabla_w e_h) + (a(\nabla u - \nabla_w Q_h w), \nabla_w e_h)\\
&+ \langle e_0 - e_b, \Pi_{k+1}(a\nabla w)\cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

It follows from the two equations above,

$$
-(\nabla\cdot(a\nabla w, e_0) = (a\nabla_w Q_h w, \nabla_w e_h) + E_1(w, e_h) + E_2(w, e_h) \tag{5.4}
$$

Similarly, we have

$$
\begin{aligned}
-(\nabla\cdot(\mathbf{b}w), e_0) &= (\mathbf{b}w, \nabla e_0)_{\mathcal{T}_h} - \langle \mathbf{b}w\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h}\\
&= (\Pi_{k+1}(\mathbf{b}w), \nabla e_0)_{\mathcal{T}_h} + \langle \mathbf{b}w\cdot\mathbf{n}, e_b - e_0\rangle_{\partial\mathcal{T}_h}\\
&= (\Pi_{k+1}\mathbf{b}w, \nabla e_0)_{\mathcal{T}_h} + \langle \Pi_{k+1}(\mathbf{b}w)\cdot\mathbf{n}, e_b - e_0\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle(\Pi_{k+1}(\mathbf{b}w) - \mathbf{b}w)\cdot\mathbf{n}, e_b - e_0\rangle_{\partial\mathcal{T}_h}\\
&= (\mathbf{b}\cdot\nabla_w e_h, w) + E_3(w, e_h)\\
&= (\mathbf{b}\cdot\nabla_w e_h, \Pi_k w) + (\mathbf{b}\cdot\nabla_w e_h, w - \Pi_k w) + E_3(w, e_h)\\
&= (\mathbf{b}\cdot\nabla_w e_h, \Pi_k w) + E_4(w, e_h) + E_3(w, e_h),
\end{aligned}
$$

which implies

$$
-(\nabla\cdot(\mathbf{b}w), e_0) = (\mathbf{b}\cdot\nabla_w e_h, \Pi_k w) + E_4(w, e_h) + E_3(w, e_h). \tag{5.5}
$$

It is straightforward to have

$$
(cw, e_0) = (c\Pi_k w, e_0) + (w - \Pi_k w, ce_0) = (c\Pi_k w, e_0) + E_5(w, e_h). \tag{5.6}
$$

Combining (5.4)–(5.6) implies (5.3). □

Lemma 5.2 *For $w \in H^2(\Omega)$ and $e_h \in \tilde{V}_h$, we have*

$$
E_1(w, e_h) \le Ch^{k+3}|u|_{k+3}\|w\|_2, \tag{5.7}
$$
$$
E_2(w, e_h) \le Ch^{k+3}|u|_{k+3}\|w\|_2, \tag{5.8}
$$
$$
E_3(w, e_h) \le Ch^{k+3}|u|_{k+3}\|w\|_2, \tag{5.9}
$$
$$
E_4(w, e_h) \le Ch^{k+3}|u|_{k+3}\|w\|_2, \tag{5.10}
$$
$$
E_5(w, e_h) \le Ch^{k+3}|u|_{k+3}\|w\|_2. \tag{5.11}
$$

Proof It follows from (3.11) and (4.11)

$$
\begin{aligned}
E_1(w, e_h) &\le |(a(\nabla w - \nabla_w Q_h w, \nabla_w e_h)|\\
&= |(a(\nabla w - \Pi_{k+1}\nabla w), \nabla_w e_h)|\\
&\le C\|\nabla w - \Pi_{k+1}\nabla w\|\|\nabla_w e_h\|
\end{aligned}
$$

$$\leq Ch^{k+3}|u|_{k+3}\|w\|_2.$$

Using the Cauchy-Schwarz inequality, (4.3), (2.3), and (4.9), we have

$$\begin{aligned}
|E_2(w,e_h)| &= \left|\sum_{T\in\mathcal{T}_h}\langle(a\nabla w-\Pi_{k+1}(a\nabla w))\cdot\mathbf{n}, e_0-e_b\rangle_{\partial T}\right|\\
&\leq C\sum_{T\in\mathcal{T}_h}\|a\nabla w-\Pi_{k+1}(a\nabla w)\|_{\partial T}\|e_0-e_b\|_{\partial T}\\
&\leq C\left(\sum_{T\in\mathcal{T}_h}h_T\|a\nabla w-\Pi_{k+1}(a\nabla w)\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|e_0-e_b\|_{\partial T}^2\right)^{\frac{1}{2}}\\
&\leq Ch|w|_2{|\kern-.25ex|\kern-.25ex|} e_h{|\kern-.25ex|\kern-.25ex|}\\
&\leq Ch^{k+3}|u|_{k+3}\|w\|_2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
E_3(w,e_h) &= \langle(\Pi_{k+1}(\mathbf{b}w)-\mathbf{b}w)\cdot\mathbf{n}, e_b-e_0\rangle_{\partial\mathcal{T}_h}\\
&\leq Ch^{k+3}|u|_{k+3}\|w\|_2.
\end{aligned}$$

By the Cauchy-Schwarz inequality and (4.9), we obtain

$$\begin{aligned}
E_4(w,e_h) &= (\mathbf{b}\cdot\nabla_w e_h, w-\Pi_k w)\leq Ch^{k+3}|u|_{k+3}\|w\|_2\\
E_5(w,e_h) &= (w-\Pi_k w, ce_0)\leq Ch^{k+3}|u|_{k+3}\|w\|_2.
\end{aligned}$$

This completes the proof. □

In the next theorem, we will prove the order two superconvergence of the CDG solution in the L^2-norm.

Theorem 5.1 *Let $u\in H^{k+3}(\Omega)\cap H_0^1(\Omega)$ be the exact solution of (1.1). Let $u_h\in V_h$ be the CDG solution of (3.6). Then,*

$$\|\Pi_k u-u_h\|\leq Ch^{k+3}|u|_{k+3}. \tag{5.12}$$

Proof Testing (5.1) by e_0 and using (5.3) and (4.8) give

$$\begin{aligned}
\|e_0\|^2 &= (-\nabla\cdot(a\nabla w)+\nabla\cdot(\mathbf{b}w)w+cw, e_0) \qquad (5.13)\\
&= A(e_h, Q_h w-\Pi_k w)+E_1(w,e_h)+E_2(w,e_h)+E_3(w,e_h)\\
&\quad+E_4(w,e_h)+E_5(w,e_h)
\end{aligned}$$

It follows from (3.11) and (4.9)

$$A(e_h, Q_h w-\Pi_k w)\leq {|\kern-.25ex|\kern-.25ex|} e_h{|\kern-.25ex|\kern-.25ex|}\,{|\kern-.25ex|\kern-.25ex|} Q_h w-\Pi_k w{|\kern-.25ex|\kern-.25ex|} \tag{5.14}$$

$$\leq Ch^{k+3}|u|_{k+3}\|w\|_2.$$

By (5.13), (5.7)-(5.11) and (5.14), we have

$$\|e_0\|^2 \leq Ch^{k+3}|u|_{k+3}\|w\|_2. \tag{5.15}$$

It follows from (5.15) and (5.2),

$$\|\tilde{u}_0 - u_h\| \leq Ch^{k+3}|u|_{k+3}. \tag{5.16}$$

Using (4.2) and (5.16), we have

$$\|\Pi_k u - u_h\| \leq \|\Pi_k u - \tilde{u}_0\| + \|\tilde{u}_0 - u_h\| \tag{5.17}$$

$$\leq Ch^{k+3}|u|_{k+3}. \tag{5.18}$$

We have proved the theorem. □

6 A locally lifted P_{k+2} solution

We proved that the P_k CDG solution is order two superconvergent in both L^2 norm and H^1-like norm. We can use the superconvergent solution (to the L^2-projection of u) and its superconvergent weak gradient to reconstruct a quasi-optimal P_{k+2} solution on each triangle/tetrahedron.

On each element T, we solve a local problem that finds $\hat{u}_h \in P_{k+2}(T)$ by

$$(\nabla \hat{u}_h - \nabla_w u_h, \nabla v)_T = 0 \quad \forall v \in P_{k+2}(T)\backslash P_0(T), \tag{6.1}$$

$$(\hat{u}_h - u_h, v)_T = 0 \quad \forall v \in P_0(T). \tag{6.2}$$

Theorem 6.1 *Let $u \in H_0^1(\Omega) \cap H^{k+3}(\Omega)$ be the exact solution of (1.1)–(1.2). Let $\hat{u}_h \in \Pi_{T\in\mathcal{T}_h} P_{k+2}(T)$ be the locally lifted solution of (6.1)–(6.2). Then, there exists a constant C such that*

$$\|u - \hat{u}_h\|_0 \leq Ch^{k+3}|u|_{k+3}. \tag{6.3}$$

Proof It is straightforward to show that (6.1)–(6.2) has a unique solution, cf. [16].

As the estimate (6.3) is equation independent, by (4.11) and (5.12), the theorem is proved exactly the same way as in [16]. □

7 Numerical tests

In the first example, we solve the 2nd order elliptic problem (1.1)–(1.2) on the unit square domain $\Omega = (0, 1) \times (0, 1)$, where

$$a = 2 + x + y, \qquad \mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad c = 4 - x - y.$$

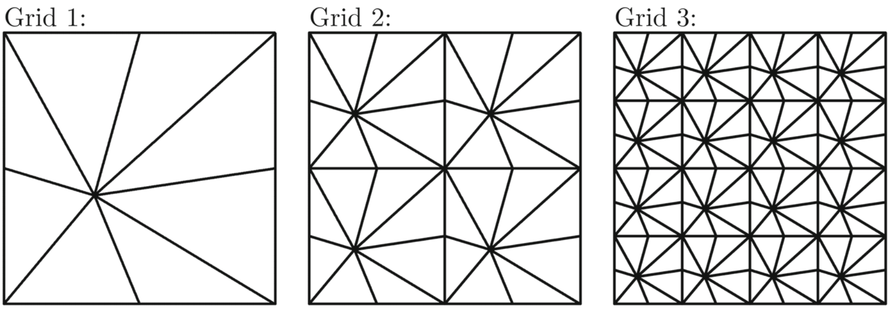

Fig. 2 The first three levels of triangular grids for the computation in Tables 1, 2, and 3

The function f is chosen so that the exact solution is

$$u(x, y) = x^3(1-x)y(1-y)^3. \tag{7.1}$$

We use perturbed triangular grids shown as in Fig. 2. We compute this example by three CDG finite elements. The results are listed in Tables 1, 2, and 3. The proved orders of superconvergence are achieved in all cases. That is, for example, for P_1 finite element method, the optimal orders of convergence are 2 and 1 in L^2 and H^1-like norms, respectively. The order two superconvergence means the order 4 and order 3 convergence in L^2 and H^1-like norms, respectively. The locally postprocessed P_3 solution converges at the optimal order 4 in L^2-norms.

In the second example, we test a case of an L-shape domain with corner singularity. It is known that the H^1-convergence is independent of such domain singularity, but dependent of the smoothness of the true solution. The L^2-convergence is only $2/3$ order, instead of 1 full order, higher than that of H^1-convergence, in theory. As we choose a smooth solution, which and its first derivatives vanish at the singular corner, in addition, we do still get two order superconvergence in both H^1 and L^2 norms.

Table 1 Error profile for (7.1), on grids as shown in Fig. 2

Grid	$\Vert\Pi_k u - u_h\Vert_0$	rate	$\vert\vert\vert\Pi_k u - u_h\vert\vert\vert$	rate
	By the P_1 CDG finite element (3.1).			
4	0.1551E-05	4.1	0.1539E-03	2.9
5	0.9326E-07	4.1	0.2028E-04	2.9
6	0.5749E-08	4.0	0.2610E-05	3.0
	By the P_1 CDG solution and its P_3 lift.			
	$\Vert u - u_0\Vert_h$	rate	$\Vert u - \hat{u}_h\Vert_0$	rate
4	0.2579E-04	2.1	0.1700E-05	4.0
5	0.6348E-05	2.0	0.1063E-06	4.0
6	0.1579E-05	2.0	0.6685E-08	4.0

Table 2 Error profile for (7.1), on grids as shown in Fig. 2

Grid	$\|\Pi_k u - u_h\|_0$	rate	$\|\|\|\Pi_k u - u_h\|\|\|$	rate
	By the P_2 CDG finite element (3.1).			
3	0.2924E-05	4.5	0.1533E-03	3.6
4	0.1087E-06	4.7	0.1092E-04	3.8
5	0.3712E-08	4.9	0.7298E-06	3.9
	By the P_2 CDG solution and its P_4 lift.			
	$\|u - u_0\|_h$	rate	$\|u - \hat{u}_h\|_0$	rate
3	0.1389E-04	3.1	0.3058E-05	4.5
4	0.1604E-05	3.1	0.1130E-06	4.8
5	0.1929E-06	3.1	0.3844E-08	4.9

We solve the 2nd order elliptic problem (1.1)–(1.2) on the L-shape domain $\Omega = (-1,1)^2 \setminus (0,1)^2$, where

$$a = 2 + x + y, \qquad \mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad c = 4 - x - y.$$

The function f is chosen so that the exact solution is

$$u(x,y) = (1-x)x^2(1-x)(1+y)y^2(1-y). \tag{7.2}$$

We use perturbed triangular grids shown as in Fig. 3. We compute this example by three CDG finite elements. The results are listed in Tables 4, 5, and 6. The proved orders of superconvergence are still achieved in all cases, even when the domain has a singular corner.

Table 3 Error profile for (7.1), on grids as shown in Fig. 2

Grid	$\|\Pi_k u - u_h\|_0$	rate	$\|\|\|\Pi_k u - u_h\|\|\|$	rate
	By the P_3 CDG finite element (3.1).			
2	0.1189E-04	4.9	0.3860E-03	3.8
3	0.2526E-06	5.6	0.1515E-04	4.7
4	0.4630E-08	5.8	0.5366E-06	4.8
	By the P_3 CDG solution and its P_5 lift.			
	$\|u - u_0\|_h$	rate	$\|u - \hat{u}_h\|_0$	rate
2	0.3394E-04	3.7	0.1214E-04	4.9
3	0.1838E-05	4.2	0.2573E-06	5.6
4	0.1041E-06	4.1	0.4714E-08	5.8

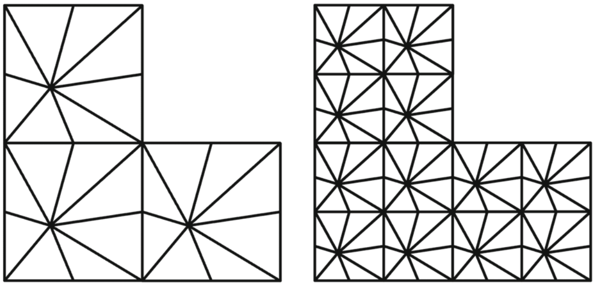

Fig. 3 The first three levels of triangular grids for the computation in Tables 4, 5, and 6

Table 4 Error profile for (7.2), on grids as shown in Fig. 3

Grid	$\|\Pi_k u - u_h\|_0$	rate	$\|\|\|\Pi_k u - u_h\|\|\|$	rate
	By the P_1 CDG finite element (3.1).			
3	0.1769E-03	4.1	0.8136E-02	2.8
4	0.9958E-05	4.2	0.1035E-02	3.0
5	0.5859E-06	4.1	0.1294E-03	3.0
	By the P_1 CDG solution and its P_3 lift.			
	$\|u - u_0\|_h$	rate	$\|u - \hat{u}_h\|_0$	rate
3	0.8709E-03	2.6	0.1894E-03	4.0
4	0.2073E-03	2.1	0.1116E-04	4.1
5	0.5184E-04	2.0	0.6760E-06	4.0

Table 5 Error profile for (7.2), on grids as shown in Fig. 3

Grid	$\|\Pi_k u - u_h\|_0$	rate	$\|\|\|\Pi_k u - u_h\|\|\|$	rate
	By the P_2 CDG finite element (3.1).			
2	0.4884E-03	3.2	0.1302E-01	2.4
3	0.1806E-04	4.8	0.9195E-03	3.8
4	0.6042E-06	4.9	0.5901E-04	4.0
	By the P_2 CDG solution and its P_4 lift.			
	$\|u - u_0\|_h$	rate	$\|u - \hat{u}_h\|_0$	rate
2	0.8966E-03	2.8	0.5079E-03	3.2
3	0.9257E-04	3.3	0.1873E-04	4.8
4	0.1176E-04	3.0	0.6246E-06	4.9

Table 6 Error profile for (7.2), on grids as shown in Fig. 3

Grid	$\|\Pi_k u - u_h\|_0$	rate	$\|\|\|\Pi_k u - u_h\|\|\|$	rate
	By the P_3 CDG finite element (3.1).			
2	0.6050E-04	5.3	0.1903E-02	4.4
3	0.1135E-05	5.7	0.6519E-04	4.9
4	0.1929E-07	5.9	0.2117E-05	4.9
	By the P_3 CDG solution and its P_5 lift.			
	$\|u - u_0\|_h$	rate	$\|u - \hat{u}_h\|_0$	rate
2	0.1582E-03	4.2	0.6188E-04	5.3
3	0.1029E-04	3.9	0.1156E-05	5.7
4	0.6945E-06	3.9	0.1961E-07	5.9

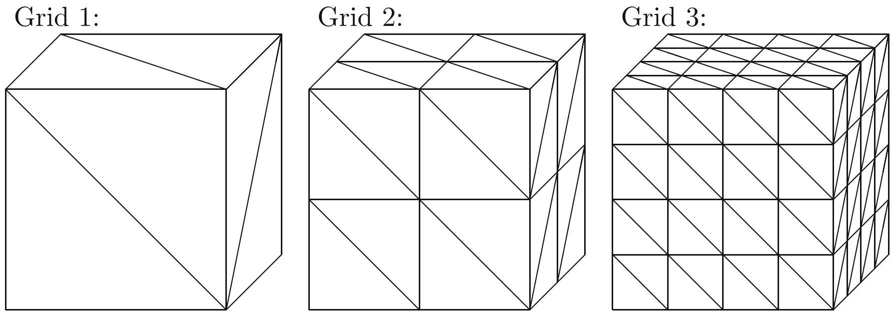

Fig. 4 The first three levels of tetrahedral grids used in Tables 7, 8, and 9

Table 7 Error profile for (7.3), on grids as shown in Fig. 4

Grid	$\|\Pi_k u - u_h\|_0$	rate	$\|\|\Pi_k u - u_h\|\|$	rate
	By the P_1 CDG finite element (3.1).			
3	0.7246E-03	4.17	0.4633E-01	2.77
4	0.4013E-04	4.17	0.5954E-02	2.96
5	0.2418E-05	4.05	0.7529E-03	2.98
	By the P_1 CDG solution and its P_3 lift.			
	$\|u - u_0\|_h$	rate	$\|u - \hat{u}_h\|_0$	rate
3	0.7226E-02	2.36	0.2842E-03	4.28
4	0.1735E-02	2.06	0.2872E-04	3.31
5	0.4312E-03	2.01	0.1980E-05	3.86

Table 8 Error profile for (7.3), on grids as shown in Fig. 4

Grid	$\|\Pi_k u - u_h\|_0$	rate	$\|\|\Pi_k u - u_h\|\|$	rate
	By the P_2 CDG finite element (3.1).			
3	0.8129E-04	4.97	0.8504E-02	3.69
4	0.2482E-05	5.03	0.5560E-03	3.93
5	0.7662E-07	5.02	0.3520E-04	3.98
	By the P_2 CDG solution and its P_4 lift.			
	$\|u - u_0\|_h$	rate	$\|u - \hat{u}_h\|_0$	rate
3	0.1156E-02	3.05	0.9931E-04	4.89
4	0.1451E-03	2.99	0.3120E-05	4.99
5	0.1820E-04	3.00	0.9752E-07	5.00

Table 9 Error profile for (7.3), on grids as shown in Fig. 4

Grid	$\|\Pi_k u - u_h\|_0$	rate	$\|\|\Pi_k u - u_h\|\|$	rate
	By the P_3 CDG finite element (3.1).			
2	0.5301E-03	6.39	0.2650E-01	5.04
3	0.9307E-05	5.83	0.1035E-02	4.68
4	0.1432E-06	6.02	0.3438E-04	4.91
	By the P_3 CDG solution and its P_5 lift.			
	$\|u - u_0\|_h$	rate	$\|u - \hat{u}_h\|_0$	rate
2	0.2106E-02	4.43	0.7341E-03	5.58
3	0.1369E-03	3.94	0.1336E-04	5.78
4	0.8722E-05	3.97	0.1970E-06	6.08

In the third example, we solve a 3D problem (1.1)–(1.2) on the unit cube domain $\Omega = (0, 1)^3$, where

$$a = 4 + x + y + z, \qquad \mathbf{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad c = 4 - x - y - z.$$

The function f is chosen so that the exact solution is

$$u(x, y, z) = \sin \pi x \sin \pi y \sin \pi z. \tag{7.3}$$

We use uniform tetrahedral grids shown as in Fig. 4. We compute this example by three CDG finite elements. The results are listed in Tables 7, 8, and 9. The proved orders of superconvergence are achieved in all cases.

Declarations

Conflict of interest The authors declare no competing interests.

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