Advances in Computational Mathematics (2023) 49:59
https://doi.org/10.1007/s10444-023-10064-w

Adaptive finite element approximation of optimal control problems with the integral fractional Laplacian

Zhaojie Zhou[1] · **Qiming Wang**[2]

Received: 10 July 2022 / Accepted: 28 June 2023 / Published online: 24 July 2023

Abstract
In this paper, we study an adaptive finite element approximation of optimal control problems with integral fractional Laplacian and pointwise control constraints. The state variable is approximated by piecewise linear polynomials, and the control variable is implicitly discretized. Upper and lower bounds of a posteriori error estimates for finite element approximation of the optimal control problem are derived. An h-adaptive algorithm driven by the a posterior error estimator is presented with Dörfler's marking criterion. We prove that the adaptive algorithm yields a sequence of approximations that converge at the optimal algebraic rate. Numerical examples are given to illustrate the theoretical findings.

Keywords Adaptive finite element · Optimal control · Fractional Laplacian · A posteriori error estimate

1 Introduction

Adaptive finite element method (AFEM) has attracted lots of attentions in the past decades as a powerful tool to solve different PDEs with nonsmooth solutions. A great deal of effort was devoted to the design of a posteriori error estimators, following the pioneering work of Babuška and Rheinboldt [1]. We refer to [2] for an overview of AFEM in the applications of solving partial differential equations (PDEs). Besides a posteriori error estimators, convergence and optimality are another two important issues in AFEM. The convergence analysis was started with Dörfler [3] and further

Communicated by: Peter Benner

✉ Zhaojie Zhou
zhouzhaojie@sdnu.edu.cn

1 School of Mathematics and Statistics, Shandong Normal University, 250014 Jinan, China

2 School of Mathematical Sciences, Beijing Normal University, 519087 Zhuhai, China

studied in [4–8]. The optimality was firstly addressed by Binev et al. [4] and further studied by Stevenson [9, 10].

In the last two decades, AFEM also has successful applications in PDE-constrained optimization. The initial works are attributed to Liu and Yan [11] and Becker et al. [12], where a residual type a posteriori error estimate and dual-weighted goal-oriented adaptivity for optimal control problems were investigated, respectively. For more details of AFEM approximation of PDE-constrained optimization, we can refer to [13–19]. Among these works, there are two remarkable works. In [20] Kohls, Rösch and Siebert derived an error equivalence property which enables one to derive reliable and efficient a posteriori error estimators for optimal control problems. In [21], Gong and Yan rigorously prove the convergence and quasi-optimality of AFEM for optimal control problems with respect to the state and adjoint state variables.

Fractional differential operators such as the fractional Laplacian are an increasingly important modeling tool in, e.g., fluids or image denoising ([22, 23]). Compared with integer order differential operators, these fractional operators are nonlocal, which makes both the mathematical analysis of physical models and numerical analysis of numerical methods challenging. In recent years, optimal control problems governed by fractional PDEs have also received lots of attentions. Many literatures are devoted to developing numerical methods or algorithms for optimal control problems governed by fractional PDEs. We refer to [24–30] for the finite element method, [31–33] for the spectral method, and [34, 35] for fast algorithms. Among these literatures, only few work is devoted to optimal control problems with the integral fractional Laplacian. In [30], a priori error estimates of finite element approximation of the control constrained optimal control problems with the fractional Laplacian are discussed.

Note that solutions to fractional differential equations typically have singularities even for smooth data input, which naturally call for using local refined meshes. Therefore, in this paper, we aim to develop AFEM approximation of optimal control problems with the integral fractional Laplacian. Upper and lower bounds of a posteriori error estimates for finite element approximation of optimal control problems are derived. An h adaptive algorithm driven by the a posterior error estimator is presented with Dörfler's marking criterion. Using the abstract, general framework of [36], we show in Theorem 10 that it yields a sequence of approximations that converge at the optimal algebraic rate (with respect to an appropriate nonlinear approximation class) under the assumption on the initial mesh size h_0, i.e., $h_0 \ll 1$. Finally, numerical examples are given to illustrate the theoretical findings.

The paper is organized as follows: Some well-known results on the adaptive finite element approximation to the problem with the fractional Laplacian are introduced in Sect. 2. In Sect. 3, the finite element discrete scheme of the optimal control problem is constructed, and a posteriori error estimates of the state, adjoint state, and control variables are derived. The adaptive algorithm and its optimal convergence rate are presented in Sect. 4. A numerical algorithm and numerical examples are presented to verify the theoretical findings in Sect. 5. Finally, we give a conclusion.

Throughout this paper, we denote by C a generic positive constant independent of the mesh size, which may stand for different values at its different occurrences. We use the symbol $A \lesssim B$ to denote $A \leq CB$ for some constant C that is independent of mesh size.

2 Preliminaries

In this section, we begin with recalling some well-known results on the adaptive finite element approximation to the problem with the fractional Laplacian, which are then used for the convergence analysis of AFEM for optimal control problems with the fractional Laplacian.

Consider the following problem

$$\begin{cases} (-\Delta)^s u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \Omega^c. \end{cases} \tag{2.1}$$

Here, $\Omega \subset R^n$ is a bounded domain, $\Omega^c := R^n \backslash \overline{\Omega}$, and $s \in (0, 1)$. The fractional Laplace operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) := C(n, s) \text{ P.V.} \int_{R^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,$$

where

$$C(n, s) = \frac{2^{2s} s \Gamma(s + \frac{n}{2})}{\pi^{n/2} \Gamma(1 - s)}.$$

Let $\widetilde{\mathbb{H}}^s(\Omega) = \{v \in \mathbb{H}^s(R^n) : v = 0 \text{ in } \Omega^c\}$, and $\mathbb{H}^{-s}(\Omega)$ denote the dual space. We denote by $(\cdot, \cdot)$ the $L^2(\Omega)$ scalar-product. The weak formulation of (2.1) reads: Find $u \in \widetilde{\mathbb{H}}^s(\Omega)$ such that

$$A(u, v) = (f, v), \quad \forall v \in \widetilde{\mathbb{H}}^s(\Omega). \tag{2.2}$$

Here,

$$A(u, v) = \frac{C(n, s)}{2} \iint_{R^n \times R^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dxdy.$$

It follows from [37, Proposition 2.4] that on $\widetilde{\mathbb{H}}^s(\Omega)$ the $\mathbb{H}^s(R^n)$ seminorm is equivalent to the $\mathbb{H}^s(R^n)$ norm. Let us define $\|u\|_{\widetilde{\mathbb{H}}^s(\Omega)} := \sqrt{A(u, u)} = \sqrt{\frac{C(n,s)}{2}} |u|_{\mathbb{H}^s(R^n)}$. Then, we have $A(u, u) \geq \|u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}$. Therefore, by the Lax-Milgram theorem, the Eq. (2.1) admits a unique solution $u \in \widetilde{\mathbb{H}}^s(\Omega)$ for $f \in \mathbb{H}^{-s}(\Omega)$. Since the equation (2.1) is linear with respect to the right-hand side f, we can define a linear and bounded solution operator $\mathcal{S} : L^2(\Omega) \longrightarrow \widetilde{\mathbb{H}}^s(\Omega)$ such that $u = \mathcal{S}f$.

If the domain $\Omega \subset R^n$ is a bounded Lipschitz domain, we have the following regularity result for the problem (2.1); see [38, Theorem 2.1] for more details.

Lemma 1 *Suppose that $u \in \widetilde{\mathbb{H}}^s(\Omega)$ is the solution of the Eq. (2.1) with right hand term $f \in L^2(\Omega)$. Then, the solution $u \in \mathbb{H}^{\theta+s-\epsilon}(\Omega)$ and satisfies*

$$\|u\|_{\mathbb{H}^{\theta+s-\epsilon}(\Omega)} \leq \frac{C(\Omega, n, s)}{\epsilon^{\xi}} \|f\|_{L^2(\Omega)}, \forall 0 < \epsilon < s.$$

Here, $\theta = \min\{s, \frac{1}{2}\}$, $\xi = \frac{1}{2}$ *for* $1/2 < s < 1$ *and* $\xi = \frac{1}{2} + \zeta$ *for* $0 < s < 1/2$ *with a constant* ζ *depending on* Ω *and* n.

For the discretization of the problem (2.1), we consider a γ-shape regular mesh $\mathcal{T}_h$ in the sense of $\max_{T\in\mathcal{T}_h}\left(\mathrm{diam}(T)/|T|^{\frac{1}{n}}\right)$, which partitions the computational domain Ω into n-simplices. To ease notation, we introduce the piecewise constant mesh size function $h_{\mathcal{T}_h} \in L^\infty(\Omega)$ by $h_{\mathcal{T}_h}|_T := h_T := |T|^{\frac{1}{n}}$. Set $h = \max_{T\in\mathcal{T}_h} h_T$. Let $\mathbb{V}_{\mathcal{T}_h}$ be the finite element space consisting of continuous piecewise linear functions over the triangulation $\mathcal{T}_h$

$$\mathbb{V}_{\mathcal{T}_h} = \{v_{\mathcal{T}_h} \in C(\bar{\Omega}) \cap \mathbb{H}_0^1(\Omega);\ v_{\mathcal{T}_h}|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\}.$$

Then, the finite element approximation of problem (2.1) can be characterized as: Find $u_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}$ such that

$$A(u_{\mathcal{T}_h}, v_{\mathcal{T}_h}) = (f, v_{\mathcal{T}_h}), \quad \forall v_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}. \tag{2.3}$$

Similar to the continuous case, we introduce a discrete operator $\mathcal{S}_{\mathcal{T}_h} : L^2(\Omega) \longrightarrow \mathbb{V}_{\mathcal{T}_h}$ such that $u_{\mathcal{T}_h} = \mathcal{S}_{\mathcal{T}_h} f$. Set

$$\eta(h) = \sup_{f\in L^2(\Omega), \|f\|=1} \inf_{\varphi_{\mathcal{T}_h}\in\mathbb{V}_{\mathcal{T}_h}} \|\mathcal{S}f - \varphi_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}.$$

Here $\|\cdot\|$ denotes the norm of $L^2(\Omega)$ space. Let h_0 be the mesh size of the initial mesh $\mathcal{T}_{h_0}$. Set

$$\eta(h_0) = \sup_{h\in(0,h_0]} \eta(h).$$

It is obvious that $\eta(h_0) \ll 1$, if $h_0 \ll 1$. Then, we can prove the following results:

Lemma 2 *For* $f \in L^2(\Omega)$ *we have*

$$\|\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f\|_{\widetilde{\mathbb{H}}^s(\Omega)} \lesssim \eta(h)\|f\|_{L^2(\Omega)} \tag{2.4}$$

and

$$\|\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f\| \lesssim \eta(h)\|\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f\|_{\widetilde{\mathbb{H}}^s(\Omega)}. \tag{2.5}$$

Proof Note that

$$A(\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f, w_{\mathcal{T}_h}) = 0, \forall w_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}.$$

Then, we have for $\varphi_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}$

$$\begin{aligned}\|\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} &\lesssim A(\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f, \mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f)\\ &= A(\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f, \mathcal{S}f - \varphi_{\mathcal{T}_h})\\ &\lesssim \|\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h} f\|_{\widetilde{\mathbb{H}}^s(\Omega)}\|\mathcal{S}f - \varphi_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}.\end{aligned}$$

This implies the first result.

To derive the second estimate, we resort to the duality arguments. Let w be the solution of the following problem with $\psi \in L^2(\Omega)$

$$\begin{cases} (-\Delta)^s w(x) = \psi(x), & x \in \Omega, \\ w(x) = 0, & x \in \Omega^c. \end{cases}$$

Then, we have

$$\|\mathcal{S}\psi - \mathcal{S}_{\mathcal{T}_h}\psi\|_{\widetilde{\mathbb{H}}^s(\Omega)} \lesssim \eta(h)\|\psi\|_{L^2(\Omega)}.$$

Furthermore, we can derive

$$\begin{aligned}(\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h}f, \psi) &= A(\mathcal{S}\psi, \mathcal{S}f - \mathcal{S}_{\mathcal{T}_h}f) \\ &= A(\mathcal{S}\psi - \mathcal{S}_{\mathcal{T}_h}\psi, \mathcal{S}f - \mathcal{S}_{\mathcal{T}_h}f) \\ &\lesssim \|\mathcal{S}\psi - \mathcal{S}_{\mathcal{T}_h}\psi\|_{\widetilde{\mathbb{H}}^s(\Omega)}\|\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h}f\|_{\widetilde{\mathbb{H}}^s(\Omega)} \\ &\lesssim \eta(h)\|\psi\|_{L^2(\Omega)}\|\mathcal{S}f - S_{\mathcal{T}_h}f\|_{\widetilde{\mathbb{H}}^s(\Omega)}.\end{aligned}$$

This yields

$$\|\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h}f\| \lesssim \eta(h)\|\mathcal{S}f - \mathcal{S}_{\mathcal{T}_h}f\|_{\widetilde{\mathbb{H}}^s(\Omega)}.$$

Remark 1 The quantity $\eta(h)$ is determined by the regularity of $\mathcal{S}f$. If the domain Ω is Lipschitz continuous, according to Lemma 1, we have $u = \mathcal{S}f \in \mathbb{H}^{\theta+s-\epsilon}(\Omega)$ for $f \in L^2(\Omega)$. Then, according to [38, Theorem 3.5] we have $\eta(h) = \mathcal{O}(h^\theta|\log(h)|^\kappa)$ for a uniform mesh partition, where $\theta = \min\{s, \frac{1}{2}\}$ and $\kappa = \xi$ if $s \neq 0.5$ and $\kappa = 1+\xi$ if $s = 0.5$ with $\xi > 1/2$ being a constant depending on Ω and n.

In the following, we are going to review the residual type a posteriori error estimator for the finite element approximation of problem (2.1).

According to [39], the function $(-\Delta)^s v_{\mathcal{T}_h}$ is (generally) no longer in $L^2(\Omega)$ for $v_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}$, $\frac{3}{4} < s < 1$, as it has singularities at the mesh skeleton. Therefore, the following local weighted residual error indicators were introduced for $\forall T \in \mathcal{T}_h$

$$\widetilde{\mathcal{R}}_u(u_{\mathcal{T}_h}, T) := \|\widetilde{h}_T^s(f - (-\Delta)^s u_{\mathcal{T}_h})\|_{L^2(T)}, \quad \text{where}$$

$$\widetilde{h}_T^s = \begin{cases} h_T^s, & s \in (0, \frac{1}{2}], \\ h_T^{s-\beta}\omega_{\mathcal{T}_h}^\beta, & s \in (\frac{1}{2}, 1), \ \beta = s - \frac{1}{2}. \end{cases}$$

Here, the function $\omega_{\mathcal{T}_h}(x)$ is defined as follows:

$$\omega_{\mathcal{T}_h}(x) := \inf_{T\in\mathcal{T}_h} \inf_{y\in\partial T} |x - y|.$$

Then, we write the error estimator as the sum of the local error indicators

$$\widetilde{\mathcal{R}}_u(u_{\mathcal{T}_h}, \mathcal{T}_h) := \Big(\sum_{T \in \mathcal{T}_h} \widetilde{\mathcal{R}}_u^2(u_{\mathcal{T}_h}, T) \Big)^{\frac{1}{2}}.$$

For all elements $T \in \mathcal{T}_h$ and $k \in \mathbb{N}_0$, we introduce the k-th order element patch inductively by

$$\Omega_h^0(T) := T, \mathcal{T}_h^0(T) := \{T\}$$
$$\Omega_h^k(T) := \text{interior}\Big(\bigcup_{T' \in \mathcal{T}_h^k(T)} \overline{T'} \Big),$$

where

$$\mathcal{T}_h^k(T) := \{T' \in \mathcal{T}_h : \overline{T'} \cap \overline{\Omega_h^{k-1}(T)} \neq \emptyset\}.$$

According to [39, Theorem 2.3], we have the following upper and lower bounds of a posterior error estimate of the state equation.

Lemma 3 *For* $0 < s < 1$ *and* $f \in L^2(\Omega)$ *the weighted residual error estimator is reliable:*

$$\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \leq C_{\text{rel}} \widetilde{\mathcal{R}}_u(u_{\mathcal{T}_h}, \mathcal{T}_h).$$

Moreover, for $0 < s \leq \frac{1}{2}$ *and* $u \in \mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega) \cap \widetilde{\mathbb{H}}^s(\Omega)$, $0 \leq \epsilon < \min\{s, \frac{1}{2} - s\}$, *the estimator is also efficient*

$$\widetilde{\mathcal{R}}_u^2(u_{\mathcal{T}_h}, \mathcal{T}_h) \leq C_{\text{eff}}^2 \Big(\|u - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \sum_{T \in \mathcal{T}_h} h_T^{1-2\epsilon} \|u - u_{\mathcal{T}_h}\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))} \Big).$$

Remark 2 The previous theorem gives only a weak efficiency result for the case $0 < s \leq 1/2$. However, the numerical results show that the proposed error estimator is observed to be efficient for all $0 < s < 1$.

In the following analysis of optimal control problems, we also need to consider the adjoint state equation. For this purpose, we introduce the adjoint equation of (2.1). For $g \in L^2(\Omega)$, let $z \in \widetilde{\mathbb{H}}^s(\Omega)$ be the solution of the following adjoint equation:

$$A(w, z) = (g, w), \quad \forall w \in \widetilde{\mathbb{H}}^s(\Omega). \tag{2.6}$$

The corresponding finite element approximation is defined as follows: Find $z_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}$ such that

$$A(w_{\mathcal{T}_h}, z_{\mathcal{T}_h}) = (g, w_{\mathcal{T}_h}), \quad \forall w_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}. \tag{2.7}$$

Similarly to the state equation, we introduce the following local weighted residual error indicators for $\forall T \in \mathcal{T}_h$

$$\widetilde{\mathcal{R}}_z(z_{\mathcal{T}_h}, T) := \|\widetilde{h}_T^s (g - (-\Delta)^s z_{\mathcal{T}_h})\|_{L^2(T)}$$

and we write the error estimator as the sum of the local error indicators

$$\widetilde{\mathcal{R}}_z(z_{\mathcal{T}_h}, \mathcal{T}_h) := \Big(\sum_{T \in \mathcal{T}_h} \widetilde{\mathcal{R}}_z^2(z_{\mathcal{T}_h}, T) \Big)^{\frac{1}{2}}.$$

Then, in an analogous way to (2.1), the a posterior error estimate for the adjoint equation is given below [39]

$$\|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \leq C_{\text{rel}} \widetilde{\mathcal{R}}_z(z_{\mathcal{T}_h}, \mathcal{T}_h), \, s \in (0, 1).$$

Moreover, for $0 < s \leq \frac{1}{2}$ and $z \in \mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega) \cap \widetilde{\mathbb{H}}^s(\Omega)$, $0 \leq \epsilon < \min\{s, \frac{1}{2} - s\}$, the estimator is also efficient

$$\widetilde{\mathcal{R}}_z^2(z_{\mathcal{T}_h}, \mathcal{T}_h) \leq C_{\text{eff}} \Big(\|z - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \sum_{T \in \mathcal{T}_h} h_T^{1-2\epsilon} \|z - z_{\mathcal{T}_h}\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))} \Big).$$

3 A posteriori error estimate for optimal control problems

We consider the following fractional optimal control problem:

$$\min_{q \in U_{ad}} J(u, q) := \frac{1}{2} \int_\Omega (u(x) - u_d(x))^2 dx + \frac{\alpha}{2} \int_\Omega q^2(x) dx \tag{3.1}$$

subject to

$$\begin{cases} (-\Delta)^s u(x) = f(x) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \Omega^c. \end{cases} \tag{3.2}$$

The admissible set is given by

$$U_{ad} = \Big\{ v \in L^\infty(\Omega) \Big| a \leq v \leq b, \; a.e., \text{ in } \Omega \Big\}.$$

Here, $a, b \in R$ and $a < b$. The function $u_d \in L^2(\Omega)$ is the desired state, and $\alpha > 0$ is the regularization parameter.

The weak formulation of optimal control problem reads

$$\min_{u \in \widetilde{\mathbb{H}}^s(\Omega), \; q \in U_{ad}} J(u, q) \tag{3.3}$$

subject to

$$A(u, v) = (f + q, v), \quad \forall v \in \widetilde{\mathbb{H}}^s(\Omega). \tag{3.4}$$

For above optimal control problems, we have the following first order optimality conditions.

Lemma 4 *([30, Theorem 3.5]) Let (u, q) be the solution of the optimal control problem (3.1)–(3.2). Then, there exists an adjoint state z such that*

$$\begin{cases} (-\Delta)^s u(x) = f(x) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \Omega^c, \end{cases} \tag{3.5}$$

$$\begin{cases} (-\Delta)^s z(x) = u(x) - u_d(x), & x \in \Omega, \\ z(x) = 0, & x \in \Omega^c, \end{cases} \tag{3.6}$$

and

$$\int_\Omega (\alpha q + z)(v - q) \geq 0, \forall v \in U_{ad}. \tag{3.7}$$

Let

$$P_{U_{ad}}(v) = \max\{a, \min\{v, b\}\}$$

denote the pointwise projection onto the admissible set U_{ad}. The variational inequality (3.7) is equivalent to

$$q = P_{U_{ad}}(-\frac{1}{\alpha} z).$$

The finite element approximation of the optimal control problem (3.1)-(3.2) can be characterized as

$$\min_{(u_{\mathcal{T}_h}, q_{\mathcal{T}_h}) \in \mathbb{V}_{\mathcal{T}_h} \times U_{ad}} J(u_{\mathcal{T}_h}, q_{\mathcal{T}_h}) \tag{3.8}$$

subject to

$$A(u_{\mathcal{T}_h}, v_{\mathcal{T}_h}) = (f + q_{\mathcal{T}_h}, v_{\mathcal{T}_h}), \quad \forall v_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}. \tag{3.9}$$

Here, the control variable is implicitly discretized by variational discretization approach ([41]), i.e., $q_{\mathcal{T}_h} \in U_{ad}$. In general $q_{\mathcal{T}_h}$ is not a finite element function. Similarly to the continuous case, we can derive the discrete first order optimality condition

$$A(u_{\mathcal{T}_h}, v_{\mathcal{T}_h}) = (f + q_{\mathcal{T}_h}, v_{\mathcal{T}_h}), \forall v_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}, \tag{3.10}$$

$$A(w_{\mathcal{T}_h}, z_{\mathcal{T}_h}) = (u_{\mathcal{T}_h} - u_d, w_{\mathcal{T}_h}), \forall w_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h} \tag{3.11}$$

and

$$\int_{\Omega}(\alpha q_{\mathcal{T}_h} + z_{\mathcal{T}_h})(w_{\mathcal{T}_h} - q_{\mathcal{T}_h}) \geq 0, \forall w_{\mathcal{T}_h} \in U_{ad}. \tag{3.12}$$

Again, (3.12) is equivalent to

$$q_{\mathcal{T}_h} = P_{U_{ad}}(-\frac{1}{\alpha} z_{\mathcal{T}_h}).$$

In the following analysis, we are going to derive a posterior error estimates for the optimal control problem. For this purpose, we introduce two auxiliary problems: Find $(u(q_{\mathcal{T}_h}), z(u_{\mathcal{T}_h})) \in \widetilde{\mathbb{H}}^s(\Omega) \times \widetilde{\mathbb{H}}^s(\Omega)$ satisfying

$$\begin{aligned} A(u(q_{\mathcal{T}_h}), v) &= (f + q_{\mathcal{T}_h}, v), \qquad \forall v \in \widetilde{\mathbb{H}}^s(\Omega), \\ A(w, z(u_{\mathcal{T}_h})) &= (u_{\mathcal{T}_h} - u_d, w), \qquad \forall w \in \widetilde{\mathbb{H}}^s(\Omega). \end{aligned} \tag{3.13}$$

Theorem 5 *Let $(u, z, q) \in \widetilde{\mathbb{H}}^s(\Omega) \times \widetilde{\mathbb{H}}^s(\Omega) \times U_{ad}$ and $(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, q_{\mathcal{T}_h}) \in \mathbb{V}_{\mathcal{T}_h} \times \mathbb{V}_{\mathcal{T}_h} \times U_{ad}$ be the solutions of problems (3.5)-(3.7) and (3.10)-(3.12), respectively. Then, the following estimates hold:*

$$\begin{aligned} &\|q - q_{\mathcal{T}_h}\| + \|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \lesssim \\ &\qquad \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} \end{aligned}$$

and

$$\begin{aligned} &\|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} \lesssim \\ &\qquad \|q - q_{\mathcal{T}_h}\| + \|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}. \end{aligned}$$

Proof Setting $v = q_{\mathcal{T}_h}$ in (3.7) and $w_{\mathcal{T}_h} = q$ in (3.12), we are led to

$$(\alpha q + z, q_{\mathcal{T}_h} - q) \geq 0$$

and

$$(\alpha q_{\mathcal{T}_h} + z_{\mathcal{T}_h}, q - q_{\mathcal{T}_h}) \geq 0.$$

Adding the above two inequalities, we have

$$\begin{aligned} \alpha\|q - q_{\mathcal{T}_h}\|^2 &\leq (z_{\mathcal{T}_h} - z, q - q_{\mathcal{T}_h}) \\ &= (z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h}), q - q_{\mathcal{T}_h}) + (z(u_{\mathcal{T}_h}) - z, q - q_{\mathcal{T}_h}). \end{aligned} \tag{3.14}$$

Note that

$$\begin{aligned} A(u - u(q_{\mathcal{T}_h}), w) &= (q - q_{\mathcal{T}_h}, w) \quad \forall w \in \widetilde{\mathbb{H}}^s(\Omega), \\ A(v, z(u_{\mathcal{T}_h}) - z) &= (u_{\mathcal{T}_h} - u, v) \quad \forall v \in \widetilde{\mathbb{H}}^s(\Omega). \end{aligned} \tag{3.15}$$

Setting $w = z(u_{\mathcal{T}_h}) - z$ in (3.15) and $v = u - u(q_{\mathcal{T}_h})$ in (3.15) yields

$$(q - q_{\mathcal{T}_h}, z(u_{\mathcal{T}_h}) - z) = (u_{\mathcal{T}_h} - u, u - u(q_{\mathcal{T}_h})).$$

From (3.14) and the above equation, we have

$$\begin{aligned}
&\alpha\|q - q_{\mathcal{T}_h}\|^2 \\
&\le (z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h}), q - q_{\mathcal{T}_h}) + (u_{\mathcal{T}_h} - u, u - u_{\mathcal{T}_h}) \\
&\quad + (u_{\mathcal{T}_h} - u, u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})) \\
&\le (z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h}), q - q_{\mathcal{T}_h}) + 0 + (u_{\mathcal{T}_h} - u, u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})) \\
&= (z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h}), q - q_{\mathcal{T}_h}) + (u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h}) + u(q_{\mathcal{T}_h}) - u, u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})) \\
&\le \|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\| \cdot \|q - q_{\mathcal{T}_h}\| + \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|^2 + \|u(q_{\mathcal{T}_h}) - u\| \cdot \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|.
\end{aligned} \tag{3.16}$$

By the governing equations of u and $u(q_{\mathcal{T}_h})$ as well as the coercivity of the bilinear form $A(\cdot,\cdot)$, we derive

$$\|u(q_{\mathcal{T}_h}) - u\|_{\widetilde{\mathbb{H}}^s(\Omega)} \le \|q - q_{\mathcal{T}_h}\|.$$

Inserting above estimate into (3.16) and using Young's inequality($ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2$) leads to

$$\begin{aligned}
&\alpha\|q - q_{\mathcal{T}_h}\|^2 \\
&\le \|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\| \cdot \|q - q_{\mathcal{T}_h}\| + \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|^2 + \|q - q_{\mathcal{T}_h}\| \cdot \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\| \\
&\le \frac{\alpha}{2}\|q - q_{\mathcal{T}_h}\|^2 + \frac{1}{\alpha}(\|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\|^2 + \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|^2) + \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|^2.
\end{aligned}$$

This implies

$$\|q - q_{\mathcal{T}_h}\|^2 \lesssim \|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\|^2 + \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|^2. \tag{3.17}$$

Moreover, by the coercivity of the bilinear form $A(\cdot,\cdot)$, we can derive

$$\begin{aligned}
\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} &\le \|u - u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u(q_{\mathcal{T}_h}) - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \\
&\lesssim \|q - q_{\mathcal{T}_h}\| + \|u(q_{\mathcal{T}_h}) - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&\|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \\
&\le \|z - z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z(u_{\mathcal{T}_h}) - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \\
&\lesssim \|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z(u_{\mathcal{T}_h}) - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \\
&\lesssim \|q - q_{\mathcal{T}_h}\| + \|u(q_{\mathcal{T}_h}) - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z(u_{\mathcal{T}_h}) - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}.
\end{aligned} \tag{3.19}$$

Combining (3.17)-(3.19), we deduce

$$\begin{aligned}&\|q-q_{\mathcal{T}_h}\|+\|u-u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|z-z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}\\ \lesssim&\|u_{\mathcal{T}_h}-u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|z_{\mathcal{T}_h}-z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}.\end{aligned}$$

Now, we are at the position to prove a lower bound of $\|q-q_{\mathcal{T}_h}\|+\|u-u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|z-z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}$. Note that

$$\begin{aligned}\|u_{\mathcal{T}_h}-u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}\le&\|u_{\mathcal{T}_h}-u\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|u-u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}\\ \lesssim&\|u_{\mathcal{T}_h}-u\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|q_{\mathcal{T}_h}-q\|.\end{aligned}$$

Similarly, we can derive that

$$\begin{aligned}\|z_{\mathcal{T}_h}-z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}\le&\|z_{\mathcal{T}_h}-z\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|z-z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}\\ \lesssim&\|z_{\mathcal{T}_h}-z\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|u-u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\|u_{\mathcal{T}_h}-u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|z_{\mathcal{T}_h}-z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}\\ \lesssim&\|q-q_{\mathcal{T}_h}\|+\|u-u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|z-z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}.\end{aligned}$$

This completes the proof.

Next, we will prove a compact equivalence property which shows the certain relationship between the finite element optimal control approximation and the associated finite element boundary value approximation.

Theorem 6 *Let (u,z,q) and $(u_{\mathcal{T}_h},z_{\mathcal{T}_h},q_{\mathcal{T}_h})$ be the solutions of (3.5)-(3.7) and the discrete counterpart, respectively. Then, the following estimates hold for $h<h_0\ll 1$*

$$\|u-u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}\le\|u_{\mathcal{T}_h}-u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}+C\eta(h)(\|u-u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|z-z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)})$$

and

$$\|z-z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}\le\|z_{\mathcal{T}_h}-z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}+C\eta(h)(\|u-u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}+\|z-z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}).$$

Proof By the coercivity of the bilinear form $A(\cdot,\cdot)$, we can derive

$$\|z(u_{\mathcal{T}_h})-z\|_{\widetilde{\mathbb{H}}^s(\Omega)}\lesssim\|u_{\mathcal{T}_h}-u\| \tag{3.20}$$

and

$$\|u-u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}\lesssim\|q-q_{\mathcal{T}_h}\|. \tag{3.21}$$

Then, we have

$$\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \leq \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + C\|q - q_{\mathcal{T}_h}\| \tag{3.22}$$

and

$$\|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \leq \|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + C\|u_{\mathcal{T}_h} - u\|. \tag{3.23}$$

By (3.7) and (3.12), we have

$$\begin{aligned}
&\alpha\|q - q_{\mathcal{T}_h}\|^2 \\
&= (\alpha q - \alpha q_{\mathcal{T}_h}, q - q_{\mathcal{T}_h}) \\
&\leq \int_\Omega (z_{\mathcal{T}_h} - z)(q - q_{\mathcal{T}_h})dx \\
&= \int_\Omega (z_{\mathcal{T}_h}(q) - z)(q - q_{\mathcal{T}_h})dx - \int_\Omega (z_{\mathcal{T}_h}(q) - z_{\mathcal{T}_h})(q - q_{\mathcal{T}_h})dx.
\end{aligned} \tag{3.24}$$

Here, $z_{\mathcal{T}_h}(q)$ and $u_{\mathcal{T}_h}(q)$ are defined by

$$\begin{aligned}
A(u_{\mathcal{T}_h}(q), v_{\mathcal{T}_h}) &= (f + q, v_{\mathcal{T}_h}), \qquad \forall v_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}, \\
A(w_{\mathcal{T}_h}, z_{\mathcal{T}_h}(q)) &= (u_{\mathcal{T}_h}(q) - u_d, w_{\mathcal{T}_h}), \qquad \forall w_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}.
\end{aligned} \tag{3.25}$$

Combining (3.10) and (3.25) and choosing $v_{\mathcal{T}_h} = z_{\mathcal{T}_h}(q) - z_{\mathcal{T}_h}$ lead to

$$A(u_{\mathcal{T}_h}(q) - u_{\mathcal{T}_h}, z_{\mathcal{T}_h}(q) - z_{\mathcal{T}_h}) = (q - q_{\mathcal{T}_h}, z_{\mathcal{T}_h}(q) - z_{\mathcal{T}_h}). \tag{3.26}$$

Similarly, by setting $w_{\mathcal{T}_h} = u_{\mathcal{T}_h}(q) - u_{\mathcal{T}_h}$ we have from (3.11) and (3.25)

$$A(u_{\mathcal{T}_h}(q) - u_{\mathcal{T}_h}, z_{\mathcal{T}_h}(q) - z_{\mathcal{T}_h}) = (u_{\mathcal{T}_h}(q) - u_{\mathcal{T}_h}, u_{\mathcal{T}_h}(q) - u_{\mathcal{T}_h}). \tag{3.27}$$

By (3.26) and (3.27), we have

$$(q - q_{\mathcal{T}_h}, z_{\mathcal{T}_h}(q) - z_{\mathcal{T}_h}) = (u_{\mathcal{T}_h}(q) - u_{\mathcal{T}_h}, u_{\mathcal{T}_h}(q) - u_{\mathcal{T}_h}) \geq 0. \tag{3.28}$$

Inserting (3.28) into (3.24) yields

$$\begin{aligned}
&\alpha\|q - q_{\mathcal{T}_h}\|^2 \\
&\leq \int_\Omega (z_{\mathcal{T}_h}(q) - z)(q - q_{\mathcal{T}_h})dx \\
&\leq \|z_{\mathcal{T}_h}(q) - z\| \cdot \|q - q_{\mathcal{T}_h}\|.
\end{aligned}$$

Furthermore, we have

$$\| q - q_{\mathcal{T}_h} \| \lesssim \|z_{\mathcal{T}_h}(q) - z\|. \tag{3.29}$$

To bound $\|z_{\mathcal{T}_h}(q) - z\|$, we introduce the following problem

$$\begin{cases} (-\Delta)^s \phi(x) = z_{\mathcal{T}_h}(q) - z, & x \in \Omega, \\ \phi(x) = 0, & x \in \Omega^c. \end{cases}$$

Let $\phi_{\mathcal{T}_h}$ be the finite element approximation of ϕ. Then, by Lemma 2, we can derive

$$\|\phi - \phi_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \lesssim \eta(h)\|z_{\mathcal{T}_h}(q) - z\|_{L^2(\Omega)}$$

and

$$\|\phi - \phi_{\mathcal{T}_h}\| \lesssim \eta^2(h)\|z_{\mathcal{T}_h}(q) - z\|_{L^2(\Omega)}.$$

Note that $u_{\mathcal{T}_h}(q)$ is the finite element approximation of u. Then, by Lemma 2, we have

$$\|u_{\mathcal{T}_h}(q) - u\| \lesssim \eta(h)\|u_{\mathcal{T}_h}(q) - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}.$$

Then, we can obtain

$$\begin{aligned}
&\|z_{\mathcal{T}_h}(q) - z\|^2 \\
&= ((-\Delta)^s \phi(x), z_{\mathcal{T}_h}(q) - z) \\
&= A(\phi, z_{\mathcal{T}_h}(q) - z) \\
&= A(\phi - \phi_{\mathcal{T}_h}, z_{\mathcal{T}_h}(q) - z) + A(\phi_{\mathcal{T}_h}, z_{\mathcal{T}_h}(q) - z) \\
&= A(\phi - \phi_{\mathcal{T}_h}, z_{\mathcal{T}_h}(q) - z) + (\phi_{\mathcal{T}_h} - \phi, u_{\mathcal{T}_h}(q) - u) + (\phi, u_{\mathcal{T}_h}(q) - u) \\
&\le \|\phi - \phi_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \cdot \|z_{\mathcal{T}_h}(q) - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|\phi_{\mathcal{T}_h} - \phi\| \cdot \|u_{\mathcal{T}_h}(q) - u\| + \|\phi\| \cdot \|u_{\mathcal{T}_h}(q) - u\| \\
&\lesssim \eta(h)\|z_{\mathcal{T}_h}(q) - z\|_{\widetilde{\mathbb{H}}^s(\Omega)}\|z_{\mathcal{T}_h}(q) - z\|_{L^2(\Omega)} + \eta^2(h)\|z_{\mathcal{T}_h}(q) - z\| \cdot \|u_{\mathcal{T}_h}(q) - u\|_{\widetilde{\mathbb{H}}^s(\Omega)} \\
&\quad + \eta(h)\|u_{\mathcal{T}_h}(q) - u\|_{\widetilde{\mathbb{H}}^s(\Omega)} \cdot \|z_{\mathcal{T}_h}(q) - z\|. \\
&\lesssim \eta(h)(\|z_{\mathcal{T}_h}(q) - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h}(q) - u\|_{\widetilde{\mathbb{H}}^s(\Omega)})\|z_{\mathcal{T}_h}(q) - z\| \\
&\quad + \eta^2(h)\|u_{\mathcal{T}_h}(q) - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}\|z_{\mathcal{T}_h}(q) - z\|.
\end{aligned}$$

This yields

$$\|z_{\mathcal{T}_h}(q) - z\| \lesssim \eta(h)(\|z_{\mathcal{T}_h}(q) - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h}(q) - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}). \tag{3.30}$$

Furthermore, we derive by (3.29), (3.30) and triangle inequality

$$\begin{aligned}
&\|q - q_{\mathcal{T}_h}\| \\
&\lesssim \eta(h)(\|z_{\mathcal{T}_h}(q) - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z_{\mathcal{T}_h} - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h}(q) - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \\
&\quad + \|u_{\mathcal{T}_h} - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}) \\
&\lesssim \eta(h)(\|q - q_{\mathcal{T}_h}\| + \|z_{\mathcal{T}_h} - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}).
\end{aligned}$$

Using the fact that $\eta(h) \ll 1$ for $h < h_0 \ll 1$ implies

$$\|q - q_{\mathcal{T}_h}\| \lesssim \eta(h)(\|z_{\mathcal{T}_h} - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}). \tag{3.31}$$

Then, we have

$$\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \leq \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + C\eta(h)(\|z_{\mathcal{T}_h} - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}).$$

In the following analysis, we use a duality argument to estimate $\|u_{\mathcal{T}_h} - u\|$. Let ψ be the solution of the following problem

$$\begin{cases} (-\Delta)^s \psi(x) = u - u_{\mathcal{T}_h}, & x \in \Omega, \\ \psi(x) = 0, & x \in \Omega^c. \end{cases}$$

Let $\psi_{\mathcal{T}_h}$ be the finite element approximation of ψ. Then, by Lemma 2 and in an analogous way to (3.30), we can obtain

$$\begin{aligned}
&\|u - u_{\mathcal{T}_h}\|^2 \\
&= ((-\Delta)^s \psi(x), u - u_{\mathcal{T}_h}) \\
&= A(\psi, u - u_{\mathcal{T}_h}) \\
&= A(\psi - \psi_{\mathcal{T}_h}, u - u_{\mathcal{T}_h}) + A(\psi_{\mathcal{T}_h}, u - u_{\mathcal{T}_h}) \\
&= A(\psi - \psi_{\mathcal{T}_h}, u - u_{\mathcal{T}_h}) + (\psi_{\mathcal{T}_h} - \psi, q - q_{\mathcal{T}_h}) + (\psi, q - q_{\mathcal{T}_h}) \\
&\leq \|\psi - \psi_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \cdot \|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|\psi_{\mathcal{T}_h} - \psi\| \cdot \|q - q_{\mathcal{T}_h}\| + \|\psi\| \cdot \|q - q_{\mathcal{T}_h}\| \\
&\lesssim \eta(h)\|u - u_{\mathcal{T}_h}\| \cdot \|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \eta(h)\|u - u_{\mathcal{T}_h}\| \cdot \|q - q_{\mathcal{T}_h}\| + \|u - u_{\mathcal{T}_h}\| \cdot \|q - q_{\mathcal{T}_h}\|.
\end{aligned}$$

This leads with (3.31) to

$$\begin{aligned}
\| u - u_{\mathcal{T}_h} \| &\lesssim \eta(h)\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|q - q_{\mathcal{T}_h}\| \\
&\lesssim \eta(h)(\|z_{\mathcal{T}_h} - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}).
\end{aligned} \tag{3.32}$$

Inserting the above inequality into (3.23) yields

$$\|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \lesssim \|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \eta(h)(\|z_{\mathcal{T}_h} - z\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|_{\widetilde{\mathbb{H}}^s(\Omega)}).$$

Next, we turn to deriving a posterior error estimates for the optimal control problem. Define

$$\begin{aligned}
\mathcal{R}_u(u_{\mathcal{T}_h}, T) &:= \|\widetilde{h}_T^s(f + q_{\mathcal{T}_h} - (-\Delta)^s u_{\mathcal{T}_h})\|_{L^2(T)}, \\
\mathcal{R}_z(z_{\mathcal{T}_h}, T) &:= \|\widetilde{h}_T^s(u_{\mathcal{T}_h} - u_d - (-\Delta)^s z_{\mathcal{T}_h})\|_{L^2(T)}.
\end{aligned}$$

Then, on a subset $\omega \subset \Omega$, we define the error estimators of the state and adjoint state by

$$\mathcal{R}_u(u_{\mathcal{T}_h}, \omega) := \Big(\sum_{T \in \mathcal{T}_h, T \subset \omega} \mathcal{R}_u^2(u_{\mathcal{T}_h}, T) \Big)^{\frac{1}{2}},$$
$$\mathcal{R}_z(z_{\mathcal{T}_h}, \omega) := \Big(\sum_{T \in \mathcal{T}_h, T \subset \omega} \mathcal{R}_z^2(z_{\mathcal{T}_h}, T) \Big)^{\frac{1}{2}}.$$

Thus, $\mathcal{R}_u(u_{\mathcal{T}_h}, \mathcal{T}_h)$ and $\mathcal{R}_z(z_{\mathcal{T}_h}, \mathcal{T}_h)$ constitute the error estimators for the state equation and the adjoint state equation on Ω with respect to $\mathcal{T}_h$ as follows

$$\mathcal{R}_u(u_{\mathcal{T}_h}, \mathcal{T}_h) := \Big(\sum_{T \in \mathcal{T}_h} \mathcal{R}_u^2(u_{\mathcal{T}_h}, T) \Big)^{\frac{1}{2}}$$

and

$$\mathcal{R}_z(z_{\mathcal{T}_h}, \mathcal{T}_h) := \Big(\sum_{T \in \mathcal{T}_h} \mathcal{R}_z^2(z_{\mathcal{T}_h}, T) \Big)^{\frac{1}{2}}.$$

For ease of exposition, we also define the following quantity:

$$\mathcal{R}_T^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T) := \mathcal{R}_u^2(u_{\mathcal{T}_h}, T) + \mathcal{R}_z^2(z_{\mathcal{T}_h}, T)$$

and the straightforward modification for $\mathcal{R}(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \omega)$ and $\mathcal{R}(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h)$. Set

$$\|(u, z)\|^2_{\widetilde{\mathbb{H}}^s(\omega)} = \|u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}.$$

Note that $u_{\mathcal{T}_h}$ and $z_{\mathcal{T}_h}$ are finite element approximations of $u(q_{\mathcal{T}_h})$ and $z(u_{\mathcal{T}_h})$. Then, by Lemma 3, we can derive the following upper and lower bounds:

Lemma 7 *For $0 < s < 1$ and $f, u_d \in L^2(\Omega)$ the following a posteriori error upper bounds hold*

$$\|u(q_{\mathcal{T}_h}) - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \le C_{\text{rel}} \mathcal{R}_u(u_{\mathcal{T}_h}, \mathcal{T}_h),$$
$$\|z(u_{\mathcal{T}_h}) - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} \le C_{\text{rel}} \mathcal{R}_z(z_{\mathcal{T}_h}, \mathcal{T}_h).$$

Moreover, for $0 < s \le \frac{1}{2}$ and $u(q_{\mathcal{T}_h}), z(u_{\mathcal{T}_h}) \in \mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega) \cap \widetilde{\mathbb{H}}^s(\Omega), 0 \le \epsilon <$ $\min\{s, \frac{1}{2} - s\}$, the following lower a posterior error bounds hold

$$\mathcal{R}_u^2(u_{\mathcal{T}_h}, \mathcal{T}_h) \le C_{\text{eff}}^2 \Big(\|u(q_{\mathcal{T}_h}) - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \sum_{T \in \mathcal{T}_h} h_T^{1-2\epsilon} \|u(q_{\mathcal{T}_h}) - u_{\mathcal{T}_h}\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))} \Big),$$
$$\mathcal{R}_z^2(z_{\mathcal{T}_h}, \mathcal{T}_h) \le C_{\text{eff}}^2 \Big(\|z(u_{\mathcal{T}_h}) - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \sum_{T \in \mathcal{T}_h} h_T^{1-2\epsilon} \|z(u_{\mathcal{T}_h}) - z_{\mathcal{T}_h}\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))} \Big).$$

Theorem 8 *Let* $(u, z, q) \in \widetilde{\mathbb{H}}^s(\Omega) \times \widetilde{\mathbb{H}}^s(\Omega) \times U_{ad}$ *and* $(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, q_{\mathcal{T}_h}) \in \mathbb{V}_{\mathcal{T}_h} \times U_{ad}$ *be the solutions of problems* (3.5)-(3.7) *and* (3.10)-(3.12)*, respectively. Then, the following upper a posteriori error bound holds for* $h < h_0 \ll 1$

$$\|(u - u_{\mathcal{T}_h}, z - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} \le C_1 \mathcal{R}^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h).$$

Proof From Theorem 6 and Lemma 7, we have

$$\begin{aligned}\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} &\le \|u_{\mathcal{T}_h} - u(q_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + C\eta(h)(\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)})\\ &\le C_{\text{rel}} \mathcal{R}_u(u_{\mathcal{T}_h}, \mathcal{T}_h) + C\eta(h)(\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)})\end{aligned}$$

and

$$\begin{aligned}\|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} &\le \|z_{\mathcal{T}_h} - z(u_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)} + C\eta(h)(\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)})\\ &\le C_{\text{rel}} \mathcal{R}_z(z_{\mathcal{T}_h}, \mathcal{T}_h) + C\eta(h)(\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}).\end{aligned}$$

Then, we can derive by the inequality $(a + b)^2 \le 2(a^2 + b^2)$

$$\begin{aligned}&\|(u - u_{\mathcal{T}_h}, z - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\ &\le 2C^2_{\text{rel}} \mathcal{R}^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h) + 4C\eta^2(h)\|(u - u_{\mathcal{T}_h}, z - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\ &\le 2C^2_{\text{rel}} \mathcal{R}^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h) + 4C\eta^2(h_0)\|(u - u_{\mathcal{T}_h}, z - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}.\end{aligned}$$

Note that $\eta(h_0) \ll 1$, if $h_0 \ll 1$. This leads to

$$\|(u - u_{\mathcal{T}_h}, z - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} \le C_1 \mathcal{R}^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h).$$

Here,

$$C_1 = \frac{2\widetilde{C}^2_{\text{rel}}}{1 - 8C\eta^2(h_0)}.$$

Remark 3 : In the following, we try to prove the lower bound a posteriori error estimate. To this end, we need to assume that $u, z, u(q_{\mathcal{T}_h}) \in \mathbb{H}^{\frac{1}{2}+s-\epsilon}(\Omega) \cap \widetilde{\mathbb{H}}^s(\Omega)$ and

$$\begin{aligned}\|u(q_{\mathcal{T}_h}) - u\|_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)} &\lesssim \|q - q_{\mathcal{T}_h}\|,\\ \|z(u_{\mathcal{T}_h}) - z\|_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)} &\lesssim \|u - u_{\mathcal{T}_h}\|\end{aligned} \tag{3.33}$$

hold for some $0 \le \epsilon < \min\{s, \frac{1}{2} - s\}$. From Theorem 6 and Lemma 7, we have

$$\begin{aligned}\mathcal{R}^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h) \le C^2_{\text{eff}}\Big(&\|(u(q_{\mathcal{T}_h}) - u_{\mathcal{T}_h}, z(u_{\mathcal{T}_h}) - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\ &+ \sum_{T \in \mathcal{T}_h} h_T^{1-2\epsilon}\|(u(q_{\mathcal{T}_h}) - u_{\mathcal{T}_h}, z(u_{\mathcal{T}_h}) - z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\Big).\end{aligned}$$

By (3.20), (3.21), (3.31) as well as the inequality $(a+b)^2 \le 2(a^2+b^2)$, we can derive

$$
\begin{aligned}
&\|(u(q_{\mathcal{T}_h})-u_{\mathcal{T}_h}, z(u_{\mathcal{T}_h})-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&= \|u(q_{\mathcal{T}_h})-u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z(u_{\mathcal{T}_h})-z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\le 2\|u(q_{\mathcal{T}_h})-u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + 2\|u-u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + 2\|z(u_{\mathcal{T}_h})-z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + 2\|z-z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\lesssim 2\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + 2\|q_{\mathcal{T}_h}-q\|^2 + 2\|u-u_{\mathcal{T}_h}\|^2\\
&\lesssim 4\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + 8\eta^2(h)\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\lesssim \|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}.
\end{aligned}
$$

We can deal with the second term in the similar manner. Note that

$$
\begin{aligned}
&\|(u(q_{\mathcal{T}_h})-u_{\mathcal{T}_h}, z(u_{\mathcal{T}_h})-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\\
&\le 2\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))} + 2\|(u(q_{\mathcal{T}_h})-u, z(u_{\mathcal{T}_h})-z)\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}.
\end{aligned}
$$

Then, we derive

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} h_T^{1-2\epsilon}\|(u(q_{\mathcal{T}_h})-u_{\mathcal{T}_h}, z(u_{\mathcal{T}_h})-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\\
&\le 2\sum_{T\in\mathcal{T}_h} h_T^{1-2\epsilon}\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\\
&\quad + 2\sum_{T\in\mathcal{T}_h} h_T^{1-2\epsilon}\|(u(q_{\mathcal{T}_h})-u, z(u_{\mathcal{T}_h})-z)\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\\
&\le 2\sum_{T\in\mathcal{T}_h} h_T^{1-2\epsilon}\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\\
&\quad + 2\mathcal{M}h^{1-2\epsilon}\|(u(q_{\mathcal{T}_h})-u, z(u_{\mathcal{T}_h})-z)\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)},
\end{aligned}
\tag{3.34}
$$

where $\mathcal{M}$ denotes the maximum times of an element T appearing in all element patch $\Omega_h^3(T)$. Above estimates combined with (3.31) and (3.32) as well as the inequalities (3.33) lead to

$$
\begin{aligned}
&\mathcal{R}^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h)\\
&\le C\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + C\sum_{T\in\mathcal{T}_h} h_T^{1-2\epsilon}\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\\
&\quad + C\mathcal{M}h^{1-2\epsilon}\|(u(q_{\mathcal{T}_h})-u, z(u_{\mathcal{T}_h})-z)\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)}\\
&\le C\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + C\sum_{T\in\mathcal{T}_h} h_T^{1-2\epsilon}\|(u-u_{\mathcal{T}_h}, z-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}
\end{aligned}
$$

$$
\begin{aligned}
&+C\mathcal{M}h^{1-2\epsilon}(\|u(q_{\mathcal{T}_h})-u\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)}+\|z(u_{\mathcal{T}_h})-z\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)})\\
&\leq C\|(u-u_{\mathcal{T}_h},z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}+C\sum_{T\in\mathcal{T}_h}h_T^{1-2\epsilon}\|(u-u_{\mathcal{T}_h},z-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\\
&+C\mathcal{M}h^{1-2\epsilon}(\|q_{\mathcal{T}_h}-q\|^2+\|u_{\mathcal{T}_h}-u\|^2)\\
&\leq C\|(u-u_{\mathcal{T}_h},z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}+C\sum_{T\in\mathcal{T}_h}h_T^{1-2\epsilon}\|(u-u_{\mathcal{T}_h},z-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\\
&+C\mathcal{M}h^{1-2\epsilon}\eta^2(h)\|(u-u_{\mathcal{T}_h},z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}.
\end{aligned}
$$

Suppose that the initial mesh size satisfies $\mathcal{M}h_0^{1-2\epsilon}\eta^2(h_0)\lesssim 1$. Then, we deduce the following lower bound

$$
\begin{aligned}
&\mathcal{R}^2(u_{\mathcal{T}_h},z_{\mathcal{T}_h},\mathcal{T}_h)\\
&\leq C_2\Big(\|(u-u_{\mathcal{T}_h},z-z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}+\sum_{T\in\mathcal{T}_h}h_T^{1-2\epsilon}\|(u-u_{\mathcal{T}_h},z-z_{\mathcal{T}_h})\|^2_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega_h^3(T))}\Big).
\end{aligned}
$$

According to [30], we have $u,z\in\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)$ for $s\in(0,1)$ for smooth domain. Moreover, from Lemma 1, we have $u,u(q_{\mathcal{T}_h}),z\in\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega), s>1/2$ for a bounded Lipschitz domain and $\|u(q_{\mathcal{T}_h})-u\|_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)}\leq\|q-q_{\mathcal{T}_h}\|$ and $\|z(u_{\mathcal{T}_h})-z\|_{\mathbb{H}^{s+\frac{1}{2}-\epsilon}(\Omega)}\lesssim\|u-u_{\mathcal{T}_h}\|$. From Section 4, we can see that the proof of optimal convergence rate for the adaptive algorithm of the optimal control problem is independent of weak efficiency. Hence, we think additional regularity assumption for the weak efficiency does not conflict with the optimal convergence rate for the adaptive algorithm.

4 Adaptive algorithm and its convergence

In this section, we consider the optimality of AFEM for solving optimal control problems (3.1)–(3.2). Although the convergence and quasi-optimality of AFEM for solving elliptic optimal control problems with pointwise control constraints have been studied in [21], the convergence and quasi-optimality of AFEM for solving fractional optimal control problems are not reported. In the current paper, we will prove that the corresponding adaptive algorithm for the fractional optimal control problem is rate optimal.

Based on the local contribution of the residual error estimator $\mathcal{R}(u_{\mathcal{T}_h},z_{\mathcal{T}_h},\mathcal{T}_h)$, we consider the following standard approach for adaptive mesh refinement of the type $\mathbb{SOLVE}--\mathbb{ESTIMATE}--\mathbb{MARK}--\mathbb{REFINE}$, where the following Dörfler's marking criterion is used to select elements for refinement.

Algorithm 1 Marking strategy

1. Given a parameter $0 < \theta < 1$;
2. Construct a minimal subset $\mathcal{M}_k \subset \mathcal{T}_{h_k}$ such that

$$\sum_{T \in \mathcal{M}_k} \mathcal{R}_T^2(u_{\mathcal{T}_{h_k}}, z_{\mathcal{T}_{h_k}}, T) \geq \theta \mathcal{R}^2(u_{\mathcal{T}_{h_k}}, z_{\mathcal{T}_{h_k}}, \mathcal{T}_{h_k}).$$

3. Mark all the elements in $\mathcal{M}_k$.

The corresponding AFEM algorithm is characterized as follows:

Algorithm 2 Adaptive FEM algorithm

1. Given an initial mesh $\mathcal{T}_{h_0}$ with mesh size h_0 and a tolerance $\text{Tol}_{\text{space}} > 0$.
2. Set $k = 0$ and solve (3.10)-(3.12) to obtain $(u_{\mathcal{T}_{h_k}}, z_{\mathcal{T}_{h_k}}, q_{\mathcal{T}_{h_k}}) \in \mathbb{V}_{\mathcal{T}_{h_k}} \times \mathbb{V}_{\mathcal{T}_{hk}} \times U_{ad}$.
3. Compute the local error indicator $\mathcal{R}_T(u_{\mathcal{T}_{h_k}}, z_{\mathcal{T}_{h_k}}, T), \forall T \in \mathcal{T}_{h_k}$.
4. Construct $\mathcal{M}_k \subset \mathcal{T}_{h_k}$ by the marking Algorithm 1.
5. Refine $\mathcal{M}_k$ to get a new mesh $\mathcal{T}_{h_{k+1}}$ by procedure $\mathbb{REFINE}$.
6. Construct the finite element space $\mathbb{V}_{\mathcal{T}_{h_{k+1}}}$ and solve (3.10)-(3.12) to obtain $(u_{\mathcal{T}_{h_{k+1}}}, z_{\mathcal{T}_{h_{k+1}}}, q_{\mathcal{T}_{h_{k+1}}}) \in \mathbb{V}_{\mathcal{T}_{h_{k+1}}} \times \mathbb{V}_{\mathcal{T}_{h_{k+1}}} \times U_{ad}$.
7. End loop if $\mathcal{R}(u_{\mathcal{T}_{h_k}}, z_{\mathcal{T}_{h_k}}, \mathcal{T}_{h_k}) \leq \text{Tol}_{\text{space}}$, otherwise, set $k = k + 1$ and go to Step 3.

Let $\mathbb{T}$ be the set of all regular triangulations generated by iterated newest vertex bisections of the initial mesh $\mathcal{T}_{h_0}$. Set

$$\mathbb{A}_t(u) = \sup_{N \in \mathbb{N}_0} \min_{\substack{\mathcal{T}_h \in \mathbb{T} \\ \#\mathcal{T}_h - \#\mathcal{T}_{h_0} \leq N}} \mathcal{R}(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h)(N+1)^t.$$

If there exist positive constants c_{opt}, C_{opt} such that

$$c_{opt}\mathbb{A}_t(u) \leq \sup_{\ell \in \mathbb{N}_0} \mathcal{R}(u_{\mathcal{T}_{h_l}}, z_{\mathcal{T}_{h_l}}, \mathcal{T}_{h_l})(\#\mathcal{T}_{h_l})^t \leq C_{opt}\mathbb{A}_t(u), \tag{4.1}$$

we say that the adaptive Algorithm 2 is rate optimal with respect to the error estimator.

To prove the quasi-optimality of the adaptive algorithm, we use the framework of [36]. Roughly speaking, we need four requirements on the error estimator and the problem under investigation, which will be stated below and verified in a series of lemmas.

Assumption 4.1 Let $\mathcal{T}_h \in \mathbb{T}$ be a triangulation of Ω and $\hat{\mathcal{T}}_h \in \mathbb{T}$ be any of its refinements. Denote by $\mathbb{V}_{\mathcal{T}_h}$ and $\mathbb{V}_{\hat{\mathcal{T}}_h}$ the finite element spaces associated with $\mathcal{T}_h$ and $\hat{\mathcal{T}}_h$, respectively. Let $\phi_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}$ and $\phi_{\hat{\mathcal{T}}_h} \in \mathbb{V}_{\hat{\mathcal{T}}_h}$. Suppose that $\Xi(\phi_{\mathcal{T}_h}, \mathcal{T}_h) := \sum_{T \in \mathcal{T}_h} \Xi_T^2(\phi_{\mathcal{T}_h}, T)$ is the error estimator associated with the triangulation $\mathcal{T}_h$. We make the following four assumptions:

(A1) Stability on the non-refined elements: For any subsets $\mathcal{S} \subset \mathcal{T}_h \cap \hat{\mathcal{T}}_h$, there holds

$$\Big|\Big(\sum_{T\in\mathcal{S}} \Xi_T^2(\phi_{\hat{\mathcal{T}}_h}, T)\Big)^{\frac{1}{2}} - \Big(\sum_{T\in\mathcal{S}} \Xi_T^2(\phi_{\mathcal{T}_h}, T)\Big)^{\frac{1}{2}}\Big| \le C_{\text{stab}}\|\phi_{\mathcal{T}_h} - \phi_{\hat{\mathcal{T}}_h}\|_{\mathcal{X}}.$$

(A2) Reduction property on the refined elements: There exist constants $q_{\text{red}} \in (0, 1)$ and $C_{\text{red}} > 0$ such that

$$\sum_{T\in\hat{\mathcal{T}}_h\backslash\mathcal{T}_h} \Xi_T^2(\phi_{\hat{\mathcal{T}}_h}, T) \le q_{\text{red}} \sum_{T\in\mathcal{T}_h\backslash\hat{\mathcal{T}}_h} \Xi_T^2(\phi_{\mathcal{T}_h}, T) + C_{\text{red}}\|\phi_{\mathcal{T}_h} - \phi_{\hat{\mathcal{T}}_h}\|_{\mathcal{X}}^2.$$

(A3) Quasi-orthogonality: There exist constants $C_0 \ge 0$, $C_{\text{orth}} > 0$ such that for all $l, N \in \mathbb{N}_0$, there holds

$$\sum_{k=\ell}^{N}\Big(\|\phi_{\mathcal{T}_{h_{k+1}}} - \phi_{\mathcal{T}_{h_k}}\|_{\mathcal{X}}^2 - C_0\|\phi - \phi_{\mathcal{T}_{h_k}}\|_{\mathcal{X}}^2\Big) \le C_{\text{orth}}\,\Xi^2(\phi_{\mathcal{T}_{h_l}}, \mathcal{T}_{h_l}).$$

(A4) Discrete reliability: Let $\phi_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}$ and $\phi_{\hat{\mathcal{T}}_h} \in \mathbb{V}_{\hat{\mathcal{T}}_h}$. The following estimate holds

$$\|\phi_{\mathcal{T}_h} - \phi_{\hat{\mathcal{T}}_h}\|_{\mathcal{X}}^2 \le C_{\text{rel}}^2 \sum_{T\in\mathcal{T}_h\backslash\hat{\mathcal{T}}_h} \Xi_T^2(\phi_{\mathcal{T}_h}, T).$$

In the following, we are going to verify above assumptions for the error estimator $\Xi_T(\phi_{\mathcal{T}_h}, T) = \mathcal{R}_T(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T)$ and the space $\mathcal{X} = \widetilde{\mathbb{H}}^s(\Omega)$.

4.1 Verification of A_1

Note that A_1 takes the form

$$\Big|\Big(\sum_{T\in\mathcal{S}} \mathcal{R}_T^2(u_{\hat{\mathcal{T}}_h}, z_{\hat{\mathcal{T}}_h}, T)\Big)^{\frac{1}{2}} - \Big(\sum_{T\in\mathcal{S}} \mathcal{R}_T^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T)\Big)^{\frac{1}{2}}\Big|$$
$$\le C_{\text{stab}}\|(u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h}, z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}.$$

Let

$$\omega := \text{interior}\Big(\bigcup_{T\in\mathcal{S}} \overline{T}\Big).$$

From the definition of the estimators, we derive

$$\begin{aligned}
&\Big|\Big(\sum_{T\in\mathcal{S}} \mathcal{R}_T^2(u_{\hat{\mathcal{T}}_h}, z_{\hat{\mathcal{T}}_h}, T)\Big)^{\frac{1}{2}} - \Big(\sum_{T\in\mathcal{S}} \mathcal{R}_T^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T)\Big)^{\frac{1}{2}}\Big| \\
&\lesssim \Big|\|\widetilde{h}_{\mathcal{T}_h}^s(f + q_{\hat{\mathcal{T}}_h} - (-\Delta)^s u_{\hat{\mathcal{T}}_h})\|_{L^2(\omega)} + \|\widetilde{h}_{\mathcal{T}_h}^s(u_{\hat{\mathcal{T}}_h} - u_d - (-\Delta)^s z_{\hat{\mathcal{T}}_h})\|_{L^2(\omega)} \\
&\quad - \|\widetilde{h}_{\mathcal{T}_h}^s(f + q_{\mathcal{T}_h} - (-\Delta)^s u_{\mathcal{T}_h})\|_{L^2(\omega)} - \|\widetilde{h}_{\mathcal{T}_h}^s(u_{\mathcal{T}_h} - u_d - (-\Delta)^s z_{\mathcal{T}_h})\|_{L^2(\omega)}\Big|
\end{aligned}$$

$$\lesssim \|\widetilde{h}_{\mathcal{T}_h}^s(-\Delta)^s(u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h})\|_{L^2(\omega)} + \|\widetilde{h}_{\mathcal{T}_h}^s(-\Delta)^s(z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h})\|_{L^2(\omega)}$$
$$+\|\widetilde{h}_{\mathcal{T}_h}^s(q_{\hat{\mathcal{T}}_h} - q_{\mathcal{T}_h})\|_{L^2(\omega)} + \|\widetilde{h}_{\mathcal{T}_h}^s(u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h})\|_{L^2(\omega)}.$$

Note that $q_{\mathcal{T}_h} = P_{U_{ad}}(-\frac{1}{\alpha}z_{\mathcal{T}_h})$ and $q_{\hat{\mathcal{T}}_h} = P_{U_{ad}}(-\frac{1}{\alpha}z_{\hat{\mathcal{T}}_h})$. Then, we have

$$\|P_{U_{ad}}(-\frac{1}{\alpha}z_{\mathcal{T}_h}) - P_{U_{ad}}(-\frac{1}{\alpha}z_{\hat{\mathcal{T}}_h})\| \leq \frac{1}{\alpha}\|z_{\mathcal{T}_h} - z_{\hat{\mathcal{T}}_h}\|.$$

Furthermore, by the inverse estimate for the fractional Laplacian ([39, Theorem 2.7])

$$\|\widetilde{h}_{\mathcal{T}_h}^s(-\Delta)^s v_{\mathcal{T}_h}\|_{L^2(\Omega)} \leq C_{\text{inv}}\|v_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}, \forall v_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h},$$

where the constant $C_{\text{inv}} > 0$ depends only on $\Omega, n, s,$ and the γ-shape regularity of $\mathcal{T}_h$, and the inverse triangle estimate, we have

$$\Big|\Big(\sum_{T\in\mathcal{S}}\mathcal{R}_T^2(u_{\hat{\mathcal{T}}_h}, z_{\hat{\mathcal{T}}_h}, T)\Big)^{\frac{1}{2}} - \Big(\sum_{T\in\mathcal{S}}\mathcal{R}_T^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T)\Big)^{\frac{1}{2}}\Big|$$
$$\lesssim \|\widetilde{h}_{\mathcal{T}_h}^s(-\Delta)^s(u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h})\|_{L^2(\omega)} + \|\widetilde{h}_{\mathcal{T}_h}^s(-\Delta)^s(z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h})\|_{L^2(\omega)}$$
$$+\|\widetilde{h}_{\mathcal{T}_h}^s(P_{U_{ad}}(-\frac{1}{\alpha}z_{\mathcal{T}_h}) - P_{U_{ad}}(-\frac{1}{\alpha}z_{\hat{\mathcal{T}}_h}))\|_{L^2(\omega)} + \|\widetilde{h}_{\mathcal{T}_h}^s(u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h})\|_{L^2(\omega)}$$
$$\lesssim \|u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}$$
$$\lesssim C_{\text{stab}}\|(u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h}, z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}.$$

4.2 Verification of A_2

Note that A_2 has the form

$$\sum_{T\in\hat{\mathcal{T}}_h\backslash\mathcal{T}_h}\mathcal{R}_T^2(u_{\hat{\mathcal{T}}_h}, z_{\hat{\mathcal{T}}_h}, T)$$
$$\leq q_{\text{red}}\sum_{T\in\mathcal{T}_h\backslash\hat{\mathcal{T}}_h}\mathcal{R}_T^2(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T) + C_{\text{red}}\|(u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h}, z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}^2.$$

For any $T \in \mathcal{T}_h\backslash\hat{\mathcal{T}}_h$, we define $\hat{T} := \{T' \in \hat{\mathcal{T}}_h : T' \subset T\}$. Note that

$$\widetilde{h}_{T'}^s = (|T'|^{\frac{1}{n}})^s \leq (2^{-b}|T|)^{\frac{s}{n}} = 2^{-\frac{bs}{n}}\widetilde{h}_T^s, \quad \text{for } 0 < s \leq 1/2 \tag{4.2}$$

and

$$\widetilde{h}_{T'}^s = (|T'|^{\frac{1}{n}})^{s-\beta}\omega_{\hat{\mathcal{T}}_h}^\beta \leq (2^{-b}|T|)^{\frac{s-\beta}{n}}\omega_{\mathcal{T}_h}^\beta = 2^{-\frac{b(s-\beta)}{n}}\widetilde{h}_T^s, \quad \text{for } 1/2 < s < 1, \tag{4.3}$$

where b denotes the number of bisections of every element $T \in \mathcal{T}_h$ in the refinement. Then, it is clear that

$$\Big(\sum_{T' \in \hat{\mathcal{T}}_h \backslash \mathcal{T}_h,\ T' \in \hat{T}} \mathcal{R}^2_{T'}(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T')\Big)^{\frac{1}{2}} \leq 2^{-\frac{b\zeta}{n}} \Big(\mathcal{R}^2_T(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T)\Big)^{\frac{1}{2}}. \tag{4.4}$$

Here, $\zeta = s$, for $0 < s \leq 1/2$ and $\zeta = s - \beta$, for $1/2 < s < 1$. Therefore, by definition of the estimator and similar to the proof of A_1, we can deduce

$$\begin{aligned}
&\sum_{T \in \hat{\mathcal{T}}_h \backslash \mathcal{T}_h} \mathcal{R}^2_T(u_{\hat{\mathcal{T}}_h}, z_{\hat{\mathcal{T}}_h}, T) \\
&\leq \sum_{T \in \hat{\mathcal{T}}_h \backslash \mathcal{T}_h} \mathcal{R}^2_T(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T) \\
&\quad + \sum_{T \in \hat{\mathcal{T}}_h \backslash \mathcal{T}_h} \Big(\|\widetilde{h}^s_T (q_{\hat{\mathcal{T}}_h} - q_{\mathcal{T}_h})\|^2_{L^2(T)} + \|\widetilde{h}^s_T (u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h})\|^2_{L^2(T)} \\
&\quad + \|\widetilde{h}^s_T (-\Delta)^s (u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h})\|^2_{L^2(T)} + \|\widetilde{h}^s_T (-\Delta)^s (z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h})\|^2_{L^2(T)} \Big) \\
&\leq \sum_{T \in \mathcal{T}_h \backslash \hat{\mathcal{T}}_h} 2^{-2\frac{b\zeta}{n}} \mathcal{R}^2_T(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, T) \\
&\quad + \sum_{T \in \hat{\mathcal{T}}_h \backslash \mathcal{T}_h} \Big(\|\widetilde{h}^s_T (q_{\hat{\mathcal{T}}_h} - q_{\mathcal{T}_h})\|^2_{L^2(T)} + \|\widetilde{h}^s_T (u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h})\|^2_{L^2(T)} \\
&\quad + \|\widetilde{h}^s_T (-\Delta)^s (u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h})\|^2_{L^2(T)} + \|\widetilde{h}^s_T (-\Delta)^s (z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h})\|^2_{L^2(T)} \Big) \\
&\leq q_{\mathrm{red}} \sum_{T \in \mathcal{T}_h \backslash \hat{\mathcal{T}}_h} \mathcal{R}^2_T(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h) + C_{\mathrm{red}} \|(u_{\hat{\mathcal{T}}_h} - u_{\mathcal{T}_h}, z_{\hat{\mathcal{T}}_h} - z_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}.
\end{aligned}$$

Then, we prove A_2 with $q_{\mathrm{red}} = 2^{-2\frac{b\zeta}{n}}$.

4.3 Verification of A_3

Compared to the convergence of adaptive algorithms for symmetric elliptic boundary value problems, the main difficulty is that the optimal control problem lacks the orthogonality. Instead, we turn to prove (quasi)-orthogonality following [6].

Let $\mathcal{T}_H$ denote a coarse shape-regular mesh of Ω with mesh size function $H_{\mathcal{T}_H} \in L^\infty(\Omega)$ by $H_{\mathcal{T}_H}|_T := H_T := |T|^{\frac{1}{n}}$. Set $H = \max_{T \in \mathcal{T}_H} H_T$. Set $u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) := \mathcal{S}_{\mathcal{T}_h}(f + q_{\mathcal{T}_H})$ and $z_{\mathcal{T}_h}(u_{\mathcal{T}_H}) := \mathcal{S}^*_{\mathcal{T}_h}(\mathcal{S}_{\mathcal{T}_H}(f + q_{\mathcal{T}_H}) - u_d)$. Here $\mathcal{S}^*_{\mathcal{T}_H}$ is the ajoint operator of $\mathcal{S}_{\mathcal{T}_H}$. To make the following proof clearly visible, we split it into four steps.

Step 1: Note that

$$\|u - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} = \|u - u_{\mathcal{T}_H}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} - \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + 2A(u - u_{\mathcal{T}_h}, u_{\mathcal{T}_H} - u_{\mathcal{T}_h}),$$

$$\|z - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} = \|z - z_{\mathcal{T}_H}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} - \|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + 2A(z - z_{\mathcal{T}_h}, z_{\mathcal{T}_H} - z_{\mathcal{T}_h}).$$

Now, we go to estimate $A(u - u_{\mathcal{T}_h}, u_{\mathcal{T}_H} - u_{\mathcal{T}_h})$ and $A(z - z_{\mathcal{T}_h}, z_{\mathcal{T}_H} - z_{\mathcal{T}_h})$. Since $u_{\mathcal{T}_H} - u_{\mathcal{T}_h} \in \mathbb{V}_{\mathcal{T}_h}$, we have

$$\begin{aligned} A(u - u_{\mathcal{T}_h}, u_{\mathcal{T}_H} - u_{\mathcal{T}_h}) &= (q - q_{\mathcal{T}_h}, u_{\mathcal{T}_H} - u_{\mathcal{T}_h}) \\ &\le \|q - q_{\mathcal{T}_h}\| \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\| \\ &\lesssim \|q - q_{\mathcal{T}_h}\|^2 + \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2. \end{aligned}$$

From the definitions of $u_{\mathcal{T}_H}$ and $u_{\mathcal{T}_h}(q_{\mathcal{T}_H})$, we can view $u_{\mathcal{T}_H}$ as the finite element approximation of $u_{\mathcal{T}_h}(q_{\mathcal{T}_H})$ in $\mathbb{V}_{\mathcal{T}_H}$. Then, by a duality argument similar to Lemma 2, we can derive

$$\|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2 \lesssim \eta^2(H) \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}. \tag{4.5}$$

In an analogous way, we can derive

$$\|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}(u_{\mathcal{T}_H})\|^2 \lesssim \eta^2(H) \|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}(u_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}. \tag{4.6}$$

To obtain the estimate (4.5) and (4.6), we need to prove the following Lemma.

Lemma 9 *Let $\mathcal{S} = \mathcal{T}_H \backslash \mathcal{T}_h$ denote the set of refined elements from $\mathcal{T}_H$ to $\mathcal{T}_h$. Then, the following estimates hold*

$$\begin{aligned} \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} &\lesssim \sum_{T \in \mathcal{S}} \mathcal{R}_u^2(u_{\mathcal{T}_H}, T), \\ \|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}(u_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} &\lesssim \sum_{T \in \mathcal{S}} \mathcal{R}_z^2(z_{\mathcal{T}_H}, T). \end{aligned}$$

Proof Let $I_{\mathcal{T}_H}$ denote the Scott-Zhang operator([39, Lemma 3.2]) onto $\mathbb{V}_{\mathcal{T}_H}$ which satisfies

$$\|\widetilde{H}^{-s}_{\mathcal{T}_H}(I - I_{\mathcal{T}_H})v\| \le C\|v\|_{\widetilde{\mathbb{H}}^s(\Omega)}, \forall v \in \widetilde{\mathbb{H}}^s(\Omega).$$

Here, the definition of $\widetilde{H}^{-s}_{\mathcal{T}_H}$ is similar to $\widetilde{h}^s_{\mathcal{T}_h}$, i.e., $\widetilde{H}^{-s}_{\mathcal{T}_H} = H^{-s}_{\mathcal{T}_H}, 0 < s \le \frac{1}{2}$ and $\widetilde{H}^{-s}_{\mathcal{T}_H} = H_T^{\beta - s} \omega^{-\beta}_{\mathcal{T}_H}, \frac{1}{2} < s < 1$. Then, by the coercivity and Galerkin orthogonality of the bilinear form $A(\cdot, \cdot)$, we derive

$$\begin{aligned} \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} &\lesssim A(u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_H}, u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_H}) \\ &\lesssim A(u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_H}, (I - I_{\mathcal{T}_H})(u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_H})) \\ &= \langle f + q_{\mathcal{T}_H} - (-\Delta)^s u_{\mathcal{T}_H}, (I - I_{\mathcal{T}_H})(u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_H}) \rangle. \end{aligned}$$

Here, $\langle\cdot,\cdot\rangle$ is the duality pairing that extends the $L^2(\Omega)$ inner product. Let

$$\omega := \text{interior}(\bigcup_{T\in\mathcal{T}_H\cap\mathcal{T}_h} \bar{T}).$$

According to [39, Lemma 3.2], the Scott-Zhang operator can be chosen to satisfy the following condition:

$$(I - I_{\mathcal{T}_H})v = 0, \text{ on } \omega, \text{ for } \forall v \in \mathbb{V}_{\mathcal{T}_h}.$$

Then, we further derive

$$\begin{aligned}
&\|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\lesssim \|\widetilde{H}^s_{\mathcal{T}_H}(f + q_{\mathcal{T}_H} - (-\Delta)^s u_{\mathcal{T}_H})\|_{L^2(\Omega\backslash\omega)}\|\widetilde{H}^{-s}_{\mathcal{T}_H}(I - I_{\mathcal{T}_H})(u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_H})\|_{L^2(\Omega\backslash\omega)}\\
&= \Big(\sum_{T\in\mathcal{S}} \mathcal{R}^2_u(u_{\mathcal{T}_H}, T)\Big)^{\frac{1}{2}} \|\widetilde{H}^{-s}_{\mathcal{T}_H}(I - I_{\mathcal{T}_H})(u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_H})\|_{L^2(\Omega\backslash\omega)}\\
&\lesssim \Big(\sum_{T\in\mathcal{S}} \mathcal{R}^2_u(u_{\mathcal{T}_H}, T)\Big)^{\frac{1}{2}} \|u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_H}\|_{\widetilde{\mathbb{H}}^s(\Omega)}.
\end{aligned}$$

This yields the first result. The second result can be derived in an analogous way.

Step 2: By (3.31), Lemma 9, and (4.5), we have

$$\begin{aligned}
&\|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2\\
&\lesssim \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2 + \|u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_h}\|^2\\
&\lesssim \eta^2(H)\|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|q_{\mathcal{T}_H} - q_{\mathcal{T}_h}\|^2\\
&\lesssim \eta^2(H)\sum_{T\in\mathcal{S}} \mathcal{R}^2_u(u_{\mathcal{T}_H}, T) + \|q_{\mathcal{T}_H} - q\|^2 + \|q - q_{\mathcal{T}_h}\|^2\\
&\lesssim \eta^2(H)\sum_{T\in\mathcal{S}} \mathcal{R}^2_u(u_{\mathcal{T}_H}, T) + \eta^2(h)(\|z_{\mathcal{T}_h} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)})\\
&\qquad + \eta^2(H)(\|z_{\mathcal{T}_H} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}).
\end{aligned}$$

This implies

$$\begin{aligned}
&A(u - u_{\mathcal{T}_h}, u_{\mathcal{T}_H} - u_{\mathcal{T}_h})\\
&\lesssim \eta^2(H)\sum_{T\in\mathcal{S}} \mathcal{R}^2_u(u_{\mathcal{T}_H}, T) + \eta^2(h)(\|z_{\mathcal{T}_h} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)})\\
&\qquad + \eta^2(H)(\|z_{\mathcal{T}_H} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}).
\end{aligned}$$

In an analogous way, we can derive the following result by the estimate (3.30), Lemma 9, and (4.6):

$$\begin{aligned}
&A(z - z_{\mathcal{T}_h}, z_{\mathcal{T}_H} - z_{\mathcal{T}_h})\\
&= (u - u_{\mathcal{T}_h}, z_{\mathcal{T}_H} - z_{\mathcal{T}_h})\\
&\lesssim \|u - u_{\mathcal{T}_h}\|^2 + \|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}\|^2\\
&\lesssim \|u - u_{\mathcal{T}_h}\|^2 + \|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}(u_{\mathcal{T}_H})\|^2 + \|z_{\mathcal{T}_h}(u_{\mathcal{T}_H}) - z_{\mathcal{T}_h}\|^2\\
&\lesssim \|u - u_{\mathcal{T}_h}\|^2 + \eta^2(H)\sum_{T\in\mathcal{S}}\mathcal{R}_z^2(z_{\mathcal{T}_H}, T) + \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2\\
&\lesssim \eta^2(h)(\|z_{\mathcal{T}_h} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)})\\
&\quad + \eta^2(H)\sum_{T\in\mathcal{S}}\mathcal{R}_z^2(z_{\mathcal{T}_H}, T) + \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2
\end{aligned}$$

Combining above estimates leads to

$$\begin{aligned}
&\|u - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} - \|u - u_{\mathcal{T}_H}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\le C\eta^2(h_0)\sum_{T\in\mathcal{S}}\mathcal{R}_u^2(u_{\mathcal{T}_H}, T) + C\eta^2(h_0)(\|z_{\mathcal{T}_h} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)})\\
&\quad + C\eta^2(h_0)(\|z_{\mathcal{T}_H} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)})
\end{aligned}$$

and

$$\begin{aligned}
&\|z - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} - \|z - z_{\mathcal{T}_H}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\le C\eta^2(h_0)\sum_{T\in\mathcal{S}}\mathcal{R}_T^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T) + C\eta^2(h_0)(\|z_{\mathcal{T}_h} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)})\\
&\quad + C\eta^2(h_0)(\|z_{\mathcal{T}_H} - z\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}).
\end{aligned}$$

Step 3: To achieve the final results we further need to estimate the term involved with $\mathcal{R}_T$. For $T \in \mathcal{T}_H \cap \mathcal{T}_h$, we have $\mathcal{R}_T(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T) = \mathcal{R}_T(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T)$. This leads to

$$\begin{aligned}
&\mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_H) - \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_h)\\
&= \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_H\backslash\mathcal{T}_h) - \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_h\backslash\mathcal{T}_H).
\end{aligned}$$

From (4.4), we have

$$\sum_{T'\in\mathcal{T}_h\backslash\mathcal{T}_H}\mathcal{R}_T^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T') \le 2^{-2\frac{b\zeta}{n}}\sum_{T\in\mathcal{T}_H\backslash\mathcal{T}_h}\mathcal{R}_T^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T),$$

i.e.,

$$\mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_h \backslash \mathcal{T}_H) \le 2^{-2\frac{b\zeta}{n}} \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_H \backslash \mathcal{T}_h).$$

Furthermore, we can derive

$$\begin{aligned}(1-2^{-2\frac{b\zeta}{n}})\mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_H \backslash \mathcal{T}_h) &\le \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_H \backslash \mathcal{T}_h) - \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_h \backslash \mathcal{T}_H)\\ &= \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_H) - \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_h).\end{aligned}$$

This implies

$$\sum_{T \in \mathcal{S}} \mathcal{R}_T^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T) \le \frac{1}{1-2^{-2\frac{b\zeta}{n}}} (\mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_H) - \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_h)) \tag{4.7}$$

Step 4: Finally, by (4.2), (4.3) and (4.7) we have

$$\begin{aligned}&\|(u_{\mathcal{T}_H} - u_{\mathcal{T}_h}, z_{\mathcal{T}_H} - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\ &\le (1 + C\eta^2(h_0))\|(u - u_{\mathcal{T}_H}, z - z_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\ &\quad -(1 - C\eta^2(h_0))\|(u - u_{\mathcal{T}_h}, z - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + C\eta^2(h_0) \sum_{T \in \mathcal{S}} \mathcal{R}_T^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T)\\ &\le (1 + C\eta^2(h_0))\|(u - u_{\mathcal{T}_H}, z - z_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\ &\quad -(1 - C\eta^2(h_0))\|(u - u_{\mathcal{T}_h}, z - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\ &\quad + \frac{C\eta^2(h_0)}{1 - 2^{-2\frac{b\zeta}{n}}} (\mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_H) - \mathcal{R}^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, \mathcal{T}_h)).\end{aligned}$$

Then, according to [36, Lemma 3.7], above estimate combined with reliability leads to the general quasi-orthogonality A_3 as follows:

$$\begin{aligned}&\sum_{k=l}^{N} \Big(\|(u_{\mathcal{T}_{h_{k+1}}} - u_{\mathcal{T}_{h_k}}, z_{\mathcal{T}_{h_{k+1}}} - z_{\mathcal{T}_{h_k}})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} - 2C\eta^2(h_0)\|(u - u_{\mathcal{T}_{h_k}}, z - z_{\mathcal{T}_{h_k}})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} \Big)\\ &\le \sum_{k=l}^{N} \Big((1 - C\eta^2(h_0))(\|(u - u_{\mathcal{T}_{h_k}}, z - z_{\mathcal{T}_{h_k}})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} - \|(u - u_{\mathcal{T}_{h_{k+1}}}, z - z_{\mathcal{T}_{h_{k+1}}})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)})\\ &\qquad + \frac{C\eta^2(h_0)}{1 - 2^{-2\frac{b\zeta}{n}}} (\mathcal{R}^2(u_{\mathcal{T}_{h_k}}, z_{\mathcal{T}_{h_k}}, \mathcal{T}_{h_k}) - \mathcal{R}^2(u_{\mathcal{T}_{h_k}}, z_{\mathcal{T}_{h_k}}, \mathcal{T}_{h_{k+1}})) \Big)\\ &\le (1 - C\eta^2(h_0))\|(u - u_{\mathcal{T}_{h_l}}, z - z_{\mathcal{T}_{h_l}})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \frac{C\eta^2(h_0)}{1 - 2^{-2\frac{b\zeta}{n}}} \mathcal{R}^2(u_{\mathcal{T}_{h_l}}, z_{\mathcal{T}_{h_l}}, \mathcal{T}_{h_l})\\ &\le C_{\text{orth}} \mathcal{R}^2(u_{\mathcal{T}_{h_l}}, z_{\mathcal{T}_{h_l}}, \mathcal{T}_{h_l}),\end{aligned}$$

provided $h_0 \ll 1$.

4.4 Verification of A_4

Viewing $q_{\mathcal{T}_H}$ as the continuous solution and $q_{\mathcal{T}_h}$ as its approximation, it follows from (3.31) that

$$\|q_{\mathcal{T}_H} - q_{\mathcal{T}_h}\| \lesssim \eta(H)(\|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}).$$

Then, by the coercivity of $A(\cdot,\cdot)$, Galerkin orthogonality, and Lemma 9, we derive

$$\begin{aligned}
&\|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\lesssim \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_h}(q_{\mathcal{T}_H}) - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\lesssim \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}(q_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|q_{\mathcal{T}_H} - q_{\mathcal{T}_h}\|^2\\
&\lesssim \sum_{T\in\mathcal{T}_H\setminus\mathcal{T}_h} \mathcal{R}_u^2(u_{\mathcal{T}_H}, T) + C\eta^2(H)(\|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)})
\end{aligned}$$

and

$$\begin{aligned}
&\|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\lesssim \|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}(u_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|z_{\mathcal{T}_h} - z_{\mathcal{T}_h}(u_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}\\
&\lesssim \|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}(u_{\mathcal{T}_H})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2\\
&\lesssim \sum_{T\in\mathcal{T}_H\setminus\mathcal{T}_h} \mathcal{R}_T^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T) + C\eta^2(H)(\|z_{\mathcal{T}_H} - z_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} + \|u_{\mathcal{T}_H} - u_{\mathcal{T}_h}\|^2_{\widetilde{\mathbb{H}}^s(\Omega)}).
\end{aligned}$$

Combining the above two estimates and using the fact that $\eta(H) \ll 1$ for $H < h_0 \ll 1$ yields

$$\|(u_{\mathcal{T}_H} - u_{\mathcal{T}_h}, z_{\mathcal{T}_H} - z_{\mathcal{T}_h})\|^2_{\widetilde{\mathbb{H}}^s(\Omega)} \lesssim \sum_{T\in\mathcal{T}_H\setminus\mathcal{T}_h} \mathcal{R}_T^2(u_{\mathcal{T}_H}, z_{\mathcal{T}_H}, T).$$

4.5 Main result

Theorem 10 *Assume that $h \in (0, h_0)$, $0 < h_0 \ll 1$. Under the assumptions $(A_1) - (A_4)$ of Assumption 4.1 and $0 < \theta \ll 1$, the Algorithm 2 is rate optimal in the sense* (4.1).

Remark 4 We remark that (4.1) implies $\mathcal{R}(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h) = \mathcal{O}(N^{-t})$ for the optimal triangulations $\mathcal{T}_h \in \mathbb{T}$ with cardinality N. However, according to [39], the characterization of $\mathbb{A}_t(u) < \infty$ in terms of u, z and the data f, u_d is still open due to nonlocal operator and the strong efficiency of the error estimator unavailable.

5 Numerical experiments

In this section, we use the adaptive finite element algorithm to solve the optimal control problem (3.1)-(3.2) and verify the a posteriori error analysis and the convergence rate of the adaptive algorithm. We use the code developed in [40] for solving the state and the adjoint state equations with finite element methods. Before that, we define the following effectivity

$$\text{effectivity} := \frac{\mathcal{R}(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h)}{\|(u - u_{\mathcal{T}_h}, z - z_{\mathcal{T}_h})\|_{\widetilde{\mathbb{H}}^s(\Omega)}}.$$

Example 1 In this example, we consider problem (3.1)-(3.2) on a unit circle $\Omega = B_1(0)$, and the exact solutions are as follows:

$$\begin{aligned} u &= \frac{2^{-2s}(1-|x|^2)^s}{\Gamma(1+s)^2}, \\ z &= \kappa u, \\ q &= \max\{-1, \min\{0, -\frac{1}{\alpha} z\}\}, \end{aligned}$$

where $\kappa = 3$ and $\alpha = 1$. The functions f and u_d can be determined from the exact solutions.

Two cases of $s = 0.25$ and $s = 0.75$ are considered. We know that the exact solutions of the state and adjoint are smooth inside the unit circle but have singularities at $\partial\Omega$. Therefore, the expected refinement should be carried out at the boundary. Figure 1 shows the final refinement mesh with $s = 0.25$, $\theta = 0.7$ and the profiles of numerical solutions of state and control. The final refinement mesh with $s = 0.75$, $\theta = 0.5$ and the profiles of numerical solutions of the state and control are also shown in Fig. 2. We can find that the meshes obtained by the adaptive finite element algorithms are mainly refined in regions close to the boundary, where the solutions are singular. The reliability of the a posteriori error estimate is verified.

In the left plot of Fig. 3, we see the convergece behavior of the estimator $\mathcal{R}(u_{\mathcal{T}_h}, z_{\mathcal{T}_h}, \mathcal{T}_h)$, the state estimator $\mathcal{R}_u(u_{\mathcal{T}_h}, \mathcal{T}_h)$, the adjoint estimator $\mathcal{R}_z(z_{\mathcal{T}_h}, \mathcal{T}_h)$

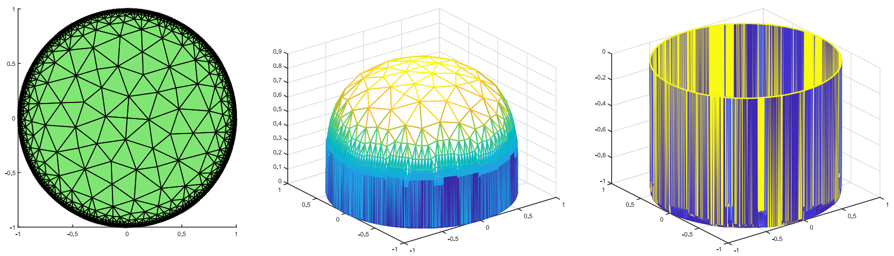


Fig. 1 Adaptively generated mesh (left) as well as the profiles of the numerically computed state (middle) and control (right) with $s = 0.25$, $\theta = 0.7$ on the circle

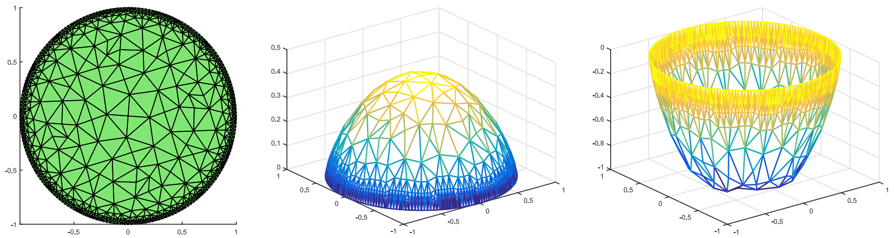


Fig. 2 Adaptively generated mesh (left) as well as the profiles of the numerically computed state (middle) and control (right) with $s = 0.75, \theta = 0.5$ on the circle

as well as the state error $\|u - u_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}$ and adjoint state error $\|z - z_{\mathcal{T}_h}\|_{\widetilde{\mathbb{H}}^s(\Omega)}$ in the energy norm for uniform and adaptive meshes with $s = 0.25, \theta = 0.7$. The same contents are shown in the right of Fig. 3 with $s = 0.75, \theta = 0.5$. It can be observed that the reduced rates ($N^{-1/4}$) of the estimators and errors are obtained for uniformly refined meshes due to the singularities of the exact solutions at the boundary of the circle. The rates of the estimators and errors are restored to $N^{-1/2}$ for the adaptively refined meshes, which is the optimal convergence rate.

Finally, we vary the parameter $\theta \in \{0.1, 0.4, 0.7, 1\}$ for fixed $s = 0.25$. In the left plot of Fig. 4, the values of the estimators, the state, and adjoint estimators are shown. In the right plot of Fig. 4, the values of the state and adjoint errors are shown. The non-optimal convergence is obtained when the marking parameter $\theta = 1$, because this is uniform refinement. When $\theta < 1$, it is clear that the rates of the indicators, estimators, and errors are restored to $N^{-1/2}$.

Example 2 In the second example, we consider an optimal control problem without explicit solutions. We set $\Omega = (-1, 1)^2$, $\alpha = 1$, $a = 0$, $b = 0.3$, respectively.

In Fig. 5, we show the final refinement mesh with $s = 0.25, \theta = 0.7$ and the profiles of the numerical solutions of state and control. Figure 6 shows the final mesh and numerical solutions with $s = 0.75, \theta = 0.5$. We can observe that the

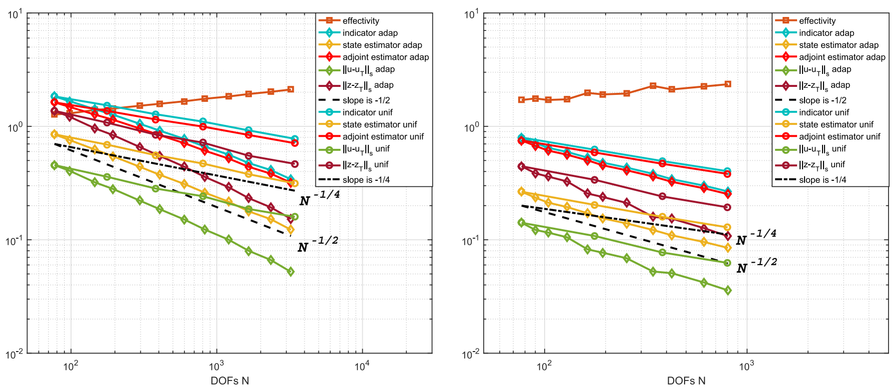


Fig. 3 The convergence behavior of the estimators and errors with $s = 0.25, \theta = 0.7$ (left) and $s = 0.75, \theta = 0.5$ (right) for uniform and adaptive refinement on the circle

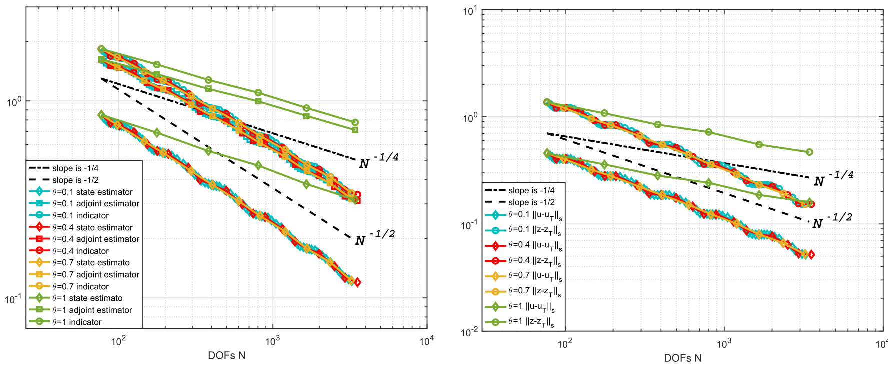
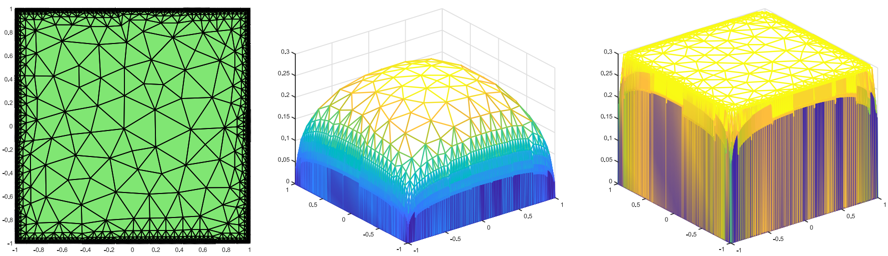
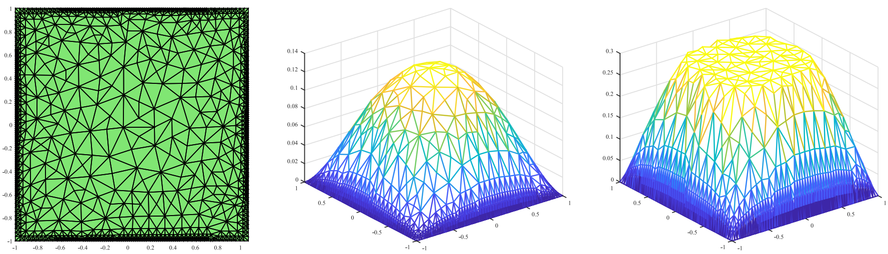


Fig. 4 The convergent behaviors of the estimators and errors for fixed $s = 0.25$ and $\theta = 0.1, 0.4, 0.7, 1$ respectively on the circle



(a) (b) (c)

Fig. 5 Adaptively generated mesh (left) as well as the profiles of the numerically computed state (middle) and control (right) with $s = 0.25, \theta = 0.7$ on the square



(a) (b) (c)

Fig. 6 Adaptively generated mesh (left) as well as the profiles of the numerically computed state (middle) and control (right) with $s = 0.75, \theta = 0.5$ on the square

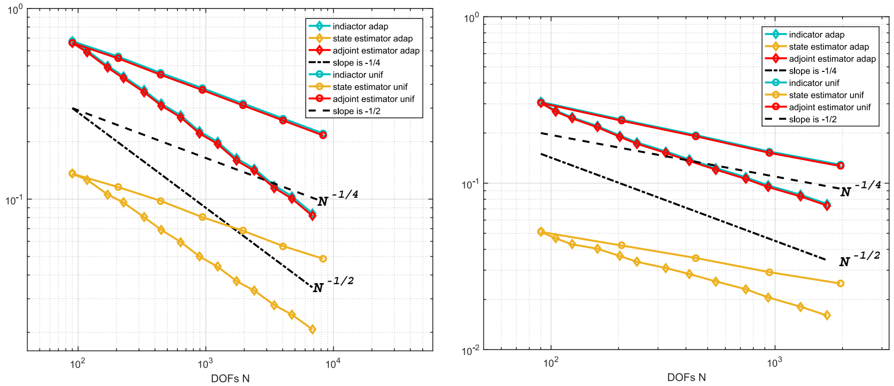


Fig. 7 The convergent behavior of the estimators with $s = 0.25, \theta = 0.7$ (left) and $s = 0.75, \theta = 0.5$ (right) for uniform and adaptive refinement on the square

main refinement behavior is carried out at the whole boundary of the square, which shows that the estimators accurately capture the singularities of the exact solutions at the whole boundary and then guide the mesh refinement. The refined results are consistent with our expectations.

The convergence behaviors of the indicators and estimators with $s = 0.25, \theta = 0.7$ and $s = 0.75, \theta = 0.5$ for uniform and adaptive refinement on the square are shown in Fig. 7. The empirical results are the same as for the previous example. The convergence rates of the estimators and indicators are only $N^{-1/4}$ for uniform refinement, while optimal convergence rates $N^{-1/2}$ is obtained for the adaptive refinement.

Funding The work was supported by the National Natural Science Foundation of China under Grant No. 11971276 and 12171287.

Data Availability All relevant data are within the paper.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. Babuška, I., Rheinboldt, W.C.: Error estimates for adaptive finite element computations. SIAM J. Numer. Anal. **15**, 736–754 (1978)
2. Verfürth, R.: A review of a posteriori error estimates and adaptive mesh refinement techniques. Wiley-Teubner, New York (1996)
3. Dörfler, W.: A convergent adaptive algorithm for Poisson equation. SIAM J. Numer. Anal. **33**, 1106–1124 (1996)
4. Binev, P., Dahmen, W., DeVore, R.: Adaptive finite element methods with convergence rates. Numer. Math. **97**, 219–268 (2004)
5. Cascon, J.M., Kreuzer, C., Nochetto, R.H., Siebert, K.G.: Quasi-optimal convergence rate for an adaptive finite element method. SIAM J. Numer. Anal. **46**(5), 2524–2550 (2008)

6. Mekchay, K., Nochetto, R.H.: Convergence of adaptive finite element methods for general second order linear elliptic PDEs. SIAM J. Numer. Anal. **43**, 1803–1827 (2005)
7. Morin, P., Nochetto, R.H., Siebert, K.G.: Data oscillation and convergence of adaptive FEM. SIAM J. Numer. Anal. **38**, 466–488 (2000)
8. Morin, P., Nochetto, R.H., Siebert, K.G.: Convergence of adaptive finite element methods. SIAM Rev. **44**, 631–658 (2002)
9. Stevenson, R.: Optimality of a standard adaptive finite element method. Found. Comput. Math. **7**, 245–269 (2007)
10. Stevenson, R.: The completion of locally refined simplicial partitions created by bisection. Math. Comput. **77**, 227–241 (2008)
11. Liu, W.B., Yan, N.N.: A posteriori error analysis for convex distributed optimal control problems. Adv. Comput. Math. **15**(1–4), 285–309 (2001)
12. Becker, R., Kapp, H., Rannacher, R.: Adaptive finite element methods for optimal control of partial differential equations: basic concept. SIAM J. Control Optim. **39**, 113–132 (2000)
13. Hintermüller, M., Hoppe, R.H.W.: Goal-oriented adaptivity in control constrained optimal control of partial differential equations. SIAM J. Control Optim. **47**(4), 1721–1743 (2008)
14. Hintermüller, M., Hoppe, R.H.W., Iliash, Y., Kieweg, M.: An a posteriori error analysis of adaptive finite element methods for distributed elliptic control problems with control constraints. ESAIM: Control Optim. Calc. Var. **14**, 540–560 (2008)
15. Li, R., Liu, W.B., Ma, H.P., Tang, T.: Adaptive finite element approximation for distributed elliptic optimal control problems. SIAM J. Control Optim. **41**(5), 1321–1349 (2002)
16. Liu, W.B., Yan, N.N.: A posteriori error estimates for convex boundary control problems. SIAM J. Numer. Anal. **39**(1), 73–99 (2001)
17. Liu, W.B., Yan, N.N.: A posteriori error estimates for optimal problems governed by Stokes equations. SIAM J. Numer. Anal. **40**, 1850–1869 (2003)
18. Liu, W.B., Yan, N.N.: A posteriori error estimates for optimal control problems governed by parabolic equations. Numer. Math. **93**, 497–521 (2003)
19. Liu, W.B., Yan, N.N.: Adaptive finite element methods for optimal control governed by PDEs. Science Press, Beijing (2008)
20. Kohls, K., Rösch, A., Siebert, K.G.: A posteriori error analysis of optimal control problems with control constraints. SIAM J. Control Optim. **52**, 1832–1861 (2014)
21. Gong, W., Yan, N.N.: Adaptive finite element method for elliptic optimal control problems: convergence and optimality. Numer. Math. **135**, 1121–1170 (2017)
22. Constantin, P., Wu, J.: Behavior of solutions of 2D quasi-geostrophic equations. SIAM J. Math. Anal. **30**, 937–948 (1999)
23. Gatto, P., Hesthaven, J.S.: Numerical approximation of the fractional Laplacian via hp-finite elements, with an application to image denoising. J. Sci. Comput. **65**(1), 249–270 (2015)
24. Antil, H., Otárola, E.: A FEM for an optimal control problem of fractional powers of elliptic operators. SIAM J. Control Optim. **53**(6), 3432–3456 (2015)
25. Antil, H., Otárola, E., Salgado, A.J.: A space-time fractional optimal control problem: analysis and discretization. SIAM J. Control Optim. **54**(3), 1295–1328 (2016)
26. Zhou, Z.J., Gong, W.: Finite element approximation of optimal control problems governed by time fractional diffusion equation. Comput. Math. Appl. **71**, 301–318 (2016)
27. Jin, B., Li, B., Zhou, Z.(2020) Pointwise-in-time error estimates for an optimal control problem with subdiffusion constraint. IMA J. Numer. Anal. **40**377–404
28. Gunzburger, M., Wang, J.: Error analysis of fully discrete finite element approximations to an optimal control problem governed by a time-fractional PDE. SIAM J. Control Optim. **57**, 241–263 (2019)
29. Dohr, S., Kahle, C., Rogovs, S., Swierczynski, P.(2019) A FEM for an optimal control problem subject to the fractional Laplace equation. Calcolo **56**(37)
30. D'Elia, M., Glusa, C., Otárola, E.: A priori error estimates for the optimal control of the integral fractional Laplacian. SIAM J. Control Optim. **57**(4), 2775–2798 (2019)
31. Ye, X., Xu, C.: A spectral method for optimal control problem governed by the abnormal diffusion equation with integral constraint on the state. Sci. China Math. **46**, 1053–1070 (2016)
32. Zhang, L., Zhou, Z.J.: Spectral Galerkin approximation of optimal control problem governed by Riesz fractional differential equation. Appl. Numer. Math. **143**, 247–262 (2019)
33. Wang, F.Y., Zhang, Z.Q., Zhou, Z.J.: A spectral Galerkin approximation of optimal control problem governed by fractional advection-diffusion-reaction equations, p. 386. J. Comput. Appl, Math (2021)

34. Du, N., Wang, H., Liu, W.B.: A fast gradient projection method for a constrained fractional optimal control. J. Sci. Comput. **68**, 1–20 (2016)
35. Zhang, C.Y., Liu, H.P., Zhou, Z.J.: A priori error analysis for time-stepping discontinuous Galerkin finite element approximation of time fractional optimal control problem. J. Sci. Comput. **80**, 993–1018 (2019)
36. Carstensen, C., Feischl, M., Praetorius, D.: Axioms of adaptivity. Comput. Math. Appl. **67**(6), 1195–1253 (2014)
37. Acosta, G., Pablo Borthagaray, J.: A fractional Laplace equation: regularity of solutions and finite element approximations. SIAM J. Numer. Anal. **55**(2), 472–495 (2017)
38. Borthagaray, J.P., Leykekhman, D., Nochetto, R.H.: Local energy estimates for the fractional Laplacian. SIAM J. Numer. Anal. **59**(4), 1918–1947 (2021)
39. Faustmann, M., Melenk, J.M., Praetorius, D.: Quasi-optimal convergence rate for an adaptive method for the integral fractional Laplacian. Math. Comp. **90**(330), 1557–1587 (2021)
40. Acosta, G., Bersetche, F., Borthagaray, J.: A short FE implementation for a 2D homogeneous Dirichlet problem of a fractional Laplacian. Comput. Math. Appl. **74**(2017), 784–816 (2017)
41. Hinze, M.: A variational discretization concept in control constrained optimization: the linear-quadratic case. Comput. Optim. Appl. **30**(1), 45–61 (2005)