Adv Comput Math (2023) 49:12
https://doi.org/10.1007/s10444-023-10010-w

Supercloseness analysis of a stabilizer-free weak Galerkin finite element method for viscoelastic wave equations with variable coefficients

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Received: 3 May 2022 / Accepted: 9 January 2023 / Published online: 20 February 2023

Abstract

In this article, we are concerned about a stabilizer-free weak Galerkin (SFWG) finite element method for approximating a second-order linear viscoelastic wave equation with variable coefficients. For SFWG solutions, both semidiscrete and fully discrete convergence analysis is considered. The second-order Newmark scheme is employed to develop the fully discrete scheme. We obtain supercloseness of order two, which is two orders higher than the optimal convergence rate in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms. In other words, we attain $\mathcal{O}(h^{k+3} + \tau^2)$ in $L^\infty(L^2)$ norm and $\mathcal{O}(h^{k+2} + \tau^2)$ in $L^\infty(H^1)$ norm. Several numerical experiments in a two-dimensional setting are carried out to validate our theoretical convergence findings. These experiments confirm the robustness and accuracy of the proposed method.

Keywords Viscoelastic wave equations · Stabilizer-free weak Galerkin method · Semidiscrete and fully discrete schemes · Supercloseness

Mathematics Subject Classification (2010) 65N15 · 65N30

1 Introduction

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^d$ $(d = 2, 3)$ with boundary $\partial\Omega$. In Ω, we consider a SFWG finite element method for solving a second-order linear viscoelastic wave equation,

$$u_{tt} - \nabla \cdot (\alpha \nabla u) - \nabla \cdot (\beta \nabla u_t) = f(x, t) \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

Communicated by: Long Chen

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with initial and boundary conditions;

$$u(x,0)=u^0,\ u_t(x,0)=v^0 \text{ in } \Omega;\ u(x,t)=0 \text{ on } \partial\Omega\times(0,T], \tag{1.2}$$

where $u(x,t)$ represents the displacement, $\alpha(x)$ is the coefficient of elasticity and $\beta(x)$ is the coefficient of viscoelasticity. We assume that the coefficient matrices $\alpha=(\alpha_{ij}(x))_{2\times 2}$ and $\beta=(\beta_{ij}(x))_{2\times 2}$ are symmetric and uniformly positive definite in Ω. The initial data $\{u^0(x), v^0(x)\}$ and the forcing term f are assumed to be smooth functions in their respective domains of definition, and T is the finite terminal observation time. Additional regularity assumptions were made throughout the paper to carry out the convergence analysis.

The model problems describe the wave propagation processes of

actual vibration in a viscoelastic medium (1.1) and (1.2) and [14, 34] provides some details on how the PDE was derived from physical principles. The viscoelastic wave equation can be used to describe the attenuation of seismic waves in fluid-saturated systems. In seismic wave simulation, two types of wave equations are used: pure elastic wave equations and viscoelastic wave equations. This study describes a generalized wave equation that incorporates viscoelastic and pure elastic phenomena into a single wave equation. The viscoelasticity in the generalized wave equation is defined by a controlling parameter β. There are numerous physical backgrounds for the viscoelastic wave equations, such as during the heat conduction in memory materials [29], gas diffusion [61], propagation of sound through viscous media [42], and so on. We aim to develop a SFWG method to effectively simulate the wave propagation in viscoelastic media. We are well aware that, due to the complexities of the data and the geometry of the computational domain in real-world applications, the model problems (1.1) and (1.2) may not have a classical solution. The rate of convergence of numerical approximations is highly dependent on the smoothness of a solution. For the related regularity results of the model problems (1.1) and (1.2), we refer to [42, 49].

Many efforts are devoted to find the accurate and efficient numerical solution for the associated second-order viscoelastic wave equations, like, conforming finite element methods for the model problems (1.1) and (1.2) in the case $\beta(x)=\gamma_1\alpha(x)$, where γ_1 is a constant, have been considered in [26, 32, 46]. The mixed finite element method and discontinuous Galerkin methods for the viscoelastic wave equation have also been considered in the existing literature (see [31, 39, 44, 45]). Existing literature dictates a diverse collection of finite element algorithms for equation (1.1) without first-order derivatives in time. Several numerical methods have been developed to solve the wave equation via finite elements, such as conforming finite element methods [10, 27, 41], Mixed FEMs [18, 25], and Discontinuous Galerkin (DG) methods. DG methods for solving the wave equation have appeared in the literature, e.g., the penalty DG method (PDG) (cf. [2, 28]), the local DG (LDG) (cf. [8, 17]), the hybrid DG (HDG) (cf. [11, 15, 16]), and hybrid high-order (HHO) method (cf. [12, 13]). In the same context, more recently, the weak Galerkin finite element methods have gained attention in the field of numerical partial differential equations. The WG-FEMs refers to the numerical algorithms for differential equations which are derived from weak formulations of the problems by replacing the involved differential operators by its weak forms and adding parameter free stabilizers [51]. The WG method in

[51] has many new features including symmetric positive definite formulation, higher order of convergence and, more importantly, allowing the use of general meshes such as hybrid meshes, polygonal and polyhedral meshes, and meshes with hanging nodes. There is abundant existing literature on such PDEs; e.g., for elliptic equations [36, 37, 50, 52], parabolic equations [20, 33, 60, 62], hyperbolic equations [4, 30, 53, 59]. One close relative of the WG finite element method is the hybrid high-order (HHO) method [12, 22]. The reconstruction operator in the HHO method corresponds to the weak gradient in the WG methods. Hence, HHO and WG methods differ only in the choice of the discrete unknowns and in the stabilization design. A comparative study on the weak Galerkin finite element methods (WGFEMs) with the HHO methods for biharmonic equation can be found in [23]. Although the close connections between HHO and WG methods should be mutually beneficial, the author believes that these connections are not sufficiently explicit in the literature, and the connection between hybrid high-order methods and weak Galerkin methods is also meant to draw the community's attention to this opportunity. Furthermore, a comparative study on WGFEMs with the widely accepted discontinuous Galerkin finite element methods (DGFEMs) and the classical mixed finite element methods (MFEMs) can be found in [35].

The stabilization term is one of the most challenging aspects of WG-FEMs and other discontinuous Galerkin finite element methods. This term is frequently employed in finite element formulations with discontinuous approximations to perform the connection of discontinuous functions across element boundaries, which complicates their implementation and analysis. Ye and Zhang [54] proposed a stabilizer-free weak Galerkin (SFWG) finite element method for solving a second-order elliptic equation on polytopal meshes without compromising the order of accuracy. This method is much simpler than the standard WG-FEMs, with fewer coefficients and high orders of accuracy on polytopal/polyhedral meshes. The reported method has been developed by [5, 6, 54–57] to solve the elliptic equations and [3] for the parabolic equation. However, the stabilizer-free weak Galerkin method is still in its stage of development and research is underway to develop robust higher order methods with a comparable number of unknowns to weak Galerkin methods when implemented appropriately.

The main contribution of this paper is to develop and analyze an SFWG finite element method for the second-order linear viscoelastic wave equation with variable coefficients. The main aspect of our proof is using a non-standard elliptic-type projection operator instead of the usual elliptic projection. The error estimates in the triple-bar norm and L^2 norm are new (see Theorems 3.2, 3.3, and 4.1). More precisely, we have obtained convergence rates two orders higher than the optimal convergence rates in the triple bar norm and L^2 norm. That means we have achieved supercloseness property for the SFWG space $(\mathcal{P}_k(K), \mathcal{P}_{k+1}(\partial K), [\mathcal{P}_{k+1}(K)]^2)$ in the triple-bar norm and L^2 norm. The semidiscrete and fully discrete algorithms and error estimates are discussed for variable coefficients, while most existing work (except [19, 50]) assumed piecewise constant coefficients. The mathematical analysis for variable coefficients adds more challenges than one would imagine, and this work fills a gap in the existing literature. Furthermore, semidiscrete error analysis has been extended for a fully discrete scheme. The fully discrete space-time finite element discretization can be reinterpreted as the Crank-Nicolson discretization of the reformulation of the governing equation in the first-order system, as in Baker [9]. Several numerical experiments have been reported to establish the efficiency of the SFWG method in scientific computing. It is noteworthy that the earlier work on viscoelastic wave equation with constant coefficients via standard WG-FEMs have considered the backward Euler scheme (see Theorem 3.2, [53]). This article is only concerned with the order of convergence in the triple-bar norm. The optimal error estimates in the L^2 norm for viscoelastic wave equation using standard WG-FEMs are still unexplored. Therefore, this study is motivated by removing stabilizers from WG-FEMs, simplifying the formulation, and reducing programming complexity for solving the viscoelastic wave equation. Furthermore, it reduces programming complexity and saves CPU time in numerical calculations while maintaining the high accuracy of numerical solutions. We have obtained accuracy rates two orders higher than the optimal convergence rates for the corresponding SFWG solution in the L^2 and triple-bar norms. Our obtained results intend to enhance the numerical analysis technique of the second-order viscoelastic wave equation. To my best knowledge, the SFWG finite element method for the viscoelastic wave equation with variable coefficients have not been illustrated yet.

To conclude the introduction, we describe the lineup of this paper. Section 2 of the paper introduces some commonly used notations and the SFWG discretization. The error estimates of the semidiscrete SFWG algorithm are discussed in Section 3. In addition, Section 4 proposes a Newmark second-order scheme and establishes supercloseness a priori error bounds in the $L^\infty(H^1)$ and $L^\infty(L^2)$ norms. In contrast, Section 5 examines various numerical examples demonstrating the SFWG algorithm's robustness. In Section 6, a summary of this work's contributions is described. Finally, in the "Appendix," there is a description of the semidiscrete solution's stability. Throughout this paper, C is a positive generic constant independent of the mesh parameters h (defined in Section 2.2) and time step τ (defined in Section 5) and not necessarily the same at each occurrence.

2 Preliminaries and SFWG discretization

2.1 Basic notations

We will follow the usual notation for Sobolev spaces and norms throughout the work. For any domain $\mathcal{M} \subset \Omega \subset \mathbb{R}^2$, we use $\|\cdot\|_{s,\mathcal{M}}$ and $|\cdot|_{s,\mathcal{M}}$ to denote the norm and semi-norm in the Sobolev space $H^s(\mathcal{M})$ for any $s \geq 1$, respectively. The inner product in $H^s(\mathcal{M})$ is denoted by $(\cdot,\cdot)_{s,\mathcal{M}}$. The space $H^0(\mathcal{M})$ coincides with $L^2(\mathcal{M})$, for which the norm and the inner product are denoted by $\|\cdot\|_{\mathcal{M}}$ and $(\cdot,\cdot)_{\mathcal{M}}$, respectively. For simplicity of notation, we skip the subscript $\mathcal{M}$ in the norm and inner product notation when $\mathcal{M} = \Omega$. $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, which is also the closure of $C_0^\infty(\Omega)$ (the set of all C^∞ functions with compact support) with respect to the norm of $H^s(\Omega)$ (cf. [1]). Furthermore, for a given Banach space $(\mathcal{B}, \|.\|_{\mathcal{B}})$, we will make use of the Bôchner spaces

$$L^p(0,T;\mathcal{B}) = \left(\phi(t) \in \mathcal{B} \text{ for a.e. } t \in [0,T] \text{ and } \int_0^T \|\phi(t)\|_{\mathcal{B}}^2 dt < \infty\right),$$

endowed with the norm

$$\|\phi\|_{L^p(0,T;\mathcal{B})} = \begin{cases} \left(\int_0^T \|\phi(t)\|_{\mathcal{B}}^2 dt\right)^{\frac{1}{2}}, & \text{for } p = 2, \\ \operatorname{ess\,sup}_{t\in[0,T]} \|\phi(t)\|_{\mathcal{B}}, & \text{for } p = \infty. \end{cases}$$

We denote by $H^m(0,T;\mathcal{B})$, $1 \le m < \infty$, the space of all measurable functions $\phi : [0,T] \to \mathcal{B}$ for which

$$\|\phi\|_{H^m(0,T;\mathcal{B})} = \left(\sum_{j=0}^{m} \int_0^T \left\|\frac{\partial^j \phi(t)}{\partial t^j}\right\|_{\mathcal{B}}^2 dt\right)^{\frac{1}{2}} < \infty.$$

For our notational convenience, we will be using $\frac{\partial \phi}{\partial t}$, ϕ_t or ϕ' interchangeably to denote time differentiation of ϕ. Similar remarks hold for other higher-order time derivatives. When no risk of confusion exists, we shall write $L^2(\mathcal{B})$ for $L^2(0,T;\mathcal{B})$, $L^\infty(\mathcal{B})$ for $L^\infty(0,T;\mathcal{B})$ and $H^m(\mathcal{B})$ for $H^m(0,T;\mathcal{B})$.

2.2 Stabilizer-free weak Galerkin discretization

In this section, we shall describe the SFWG finite element discretization for the model problems (1.1) and (1.2) and review the definition of the weak gradient operator.

For some $h_0 > 0$ and $h \in (0, h_0]$, let $\mathcal{T}_h$ be a partition of the domain Ω consisting of triangles in two-dimensional space satisfying a set of conditions specified in [52]. Denote by $\mathcal{F}_h$ the set of all edges in $\mathcal{T}_h$ and let $\mathcal{F}_h^0$ be the set of all interior edges. For every element $K \in \mathcal{T}_h$, we denote by $|K|$ the measure of K and by h_K its diameter and mesh size $h = \max_{K\in\mathcal{T}_h} h_K$ for $\mathcal{T}_h$.

Let K be any triangle with interior K^0 and boundary ∂K. For any given integer $k \ge 1$, denote $\mathcal{P}_k(K)$ the space of polynomials of total degree k or less on the element $K \in \mathcal{T}_h$. Analogously, $\mathcal{P}_{k+1}(e)$ denotes the space of polynomials of total degree $k+1$ or less on edge $e \in \mathcal{F}_h$. On each element $K \in \mathcal{T}_h$, define the following local SFWG finite element space

$$\mathcal{V}(K) = \{v = \{v_0, v_b\} : v_0 \in \mathcal{P}_k(K),\ v_b \in \mathcal{P}_{k+1}(e)\}.$$

The global weak finite element space $\mathcal{V}_h$ associated with $\mathcal{T}_h$ is defined as

$$\mathcal{V}_h = \{v = \{v_0, v_b\} :\ v|_K \in \mathcal{V}(K),\ [v_b]_e = 0,\ \forall e \in \mathcal{F}_h^0\}. \tag{2.1}$$

Here, $[v_b]_e := v_b|_{K_i} - v_b|_{K_j}, i \neq j$ represent the jump across an interior edge $e \in \mathcal{F}_h^0$.

Denote by $\mathcal{V}_h^0$ the subspace of $\mathcal{V}_h$ consisting of all finite element functions with vanishing boundary value

$$\mathcal{V}_h^0 = \{v \in \mathcal{V}_h : v_b|_{\partial\Omega} = 0\}. \tag{2.2}$$

For each $v_h = \{v_0, v_b\} \in \mathcal{V}_h$, the weak gradient of it, denoted by ∇_{w}, is defined as the unique polynomial $(\nabla_{\mathrm{w}} v_h) \in [\mathcal{P}_{k+1}(K)]^2$ that satisfies the following equation

$$(\nabla_{\mathrm{w}} v_h, \mathbf{q})_K = -\int_K v_0(\nabla \cdot \mathbf{q}) dK + \int_{\partial K} v_b(\mathbf{q} \cdot \mathbf{n}) ds \quad \forall\, \mathbf{q} \in [\mathcal{P}_{k+1}(K)]^2. \tag{2.3}$$

The usual L^2 inner product can be written locally on each element as follows

$$(\nabla_{\mathrm{w}} v_h, \nabla_{\mathrm{w}} w_h) := \sum_{K \in \mathcal{T}_h} (\nabla_{\mathrm{w}} v_h, \nabla_{\mathrm{w}} w_h)_K, \quad v_h,\ w_h \in \mathcal{V}_h. \tag{2.4}$$

For each element $K \in \mathcal{T}_h$, denote by $\mathcal{Q}_0$ the usual L^2 projection operator from $L^2(K)$ onto $\mathcal{P}_k(K)$ and by $\mathcal{Q}_b$ the L^2 projection from $L^2(e)$ onto $\mathcal{P}_{k+1}(e)$ for any $e \in \mathcal{F}_h$. We shall combine $\mathcal{Q}_0$ with $\mathcal{Q}_b$ by writing $\mathcal{Q}_h = \{\mathcal{Q}_0, \mathcal{Q}_b\}$. In addition to $\mathcal{Q}_h$, let $\mathbb{Q}_h$ be another local L^2 projection from $[L^2(K)]^2$ onto $[\mathcal{P}_{k+1}(K)]^2$.

We recall the following crucial approximation properties for local projections $\mathcal{Q}_0$ and $\mathbb{Q}_h$.

Lemma 2.1 (Lemma 4.1, [52]) *Let $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity assumption as specified in* [52]. *Then, for $\varphi \in H^{s+1}(\Omega)$, we have*

$$\sum_{K \in \mathcal{T}_h} \left(\|\varphi - \mathcal{Q}_0\varphi\|_K^2 + h_K^2 \|\nabla(\varphi - \mathcal{Q}_0\varphi)\|_K^2 \right) \le C h^{2(s+1)} \|\varphi\|_{s+1}^2, \ 0 \le s \le k.$$

Lemma 2.2 (Lemma 4.1, [52]) *Let $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity assumption as specified in* [52]. *Then, for $v \in H^{s+1}(\Omega)$, we have*

$$\sum_{K \in \mathcal{T}_h} \left(\|\nabla v - \mathbb{Q}_h(\nabla v)\|_K^2 + h_K^2 \|\nabla(\nabla v - \mathbb{Q}_h(\nabla v))\|_K^2 \right) \le C h^{2s} \|v\|_{s+1}^2,$$
$$0 \le s \le k+2.$$

For ease of use, we adopt the notations listed below

$$(w_h, \varphi_0) := \sum_{K \in \mathcal{T}_h} (w_h, \varphi_0)_K = \sum_{K \in \mathcal{T}_h} \int_K w_0 \varphi_0 dK, \ w_h,\ \varphi_0 \in \mathcal{V}_h. \tag{2.5}$$

$$\langle w_h, \varphi_0 \rangle := \sum_{K \in \mathcal{T}_h} \langle w_h, \varphi_0 \rangle_{\partial K} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} w_0 \varphi_0 ds, \ w_h,\ \varphi_0 \in \mathcal{V}_h. \tag{2.6}$$

$$(u_{htt}, v_0) := \sum_{K \in \mathcal{T}_h} \left(\frac{\partial^2 u_0}{\partial t^2}, v_0 \right)_K = \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^2 u_0}{\partial t^2} \varphi_0 dK \ \ u_{htt},\ \varphi_0 \in \mathcal{V}_h. \tag{2.7}$$

Next, we recall the following identity for our later analysis (cf. Lemma 2.1, [6])

$$\nabla_{\mathrm{w}}(\mathcal{Q}_h v) = \mathbb{Q}_h(\nabla v), \ \forall\, v \in H^1(K). \tag{2.8}$$

Next, we define the bilinear maps $\mathcal{A}_{r,w} : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ $(r = 1, 2)$, involving discrete weak gradient operator ∇_{w}, by

$$\mathcal{A}_{1,w}(w_h, v_h) := \sum_{K \in \mathcal{T}_h} \left(\alpha \nabla_{\mathrm{w}} w_h, \nabla_{\mathrm{w}} v_h \right)_K, \ \forall\, w_h, v_h \in \mathcal{V}_h, \tag{2.9}$$

and

$$\mathcal{A}_{2,w}(w_h, v_h) := \sum_{K\in\mathcal{T}_h} \left(\beta\nabla_{\rm w} w_h, \nabla_{\rm w} v_h\right)_K, \ \forall\, w_h, v_h \in \mathcal{V}_h. \tag{2.10}$$

For any $v_h \in \mathcal{V}_h$, the triple-bar norm is defined as (cf. [6])

$$|||v_h|||^2 := \sum_{K\in\mathcal{T}_h} \|\nabla_{\rm w} v_h\|_K^2. \tag{2.11}$$

It follows from uniformly positive-definite of the coefficient matrices α and β in which there exist two positive constants σ_* and σ^* such that

$$\sigma_*|||v_h|||^2 \le \mathcal{A}_{r,w}(v_h, v_h) \le \sigma^*|||v_h|||^2 \ \text{ for r} = 1, 2. \tag{2.12}$$

The triple-bar norm is coercive with respect to the semi-norm $\|\cdot\|_{1,h}$ defined by

$$\|v_h\|_{1,h} := \Big(\sum_{K\in\mathcal{T}_h} (\|\nabla v_0\|_K^2 + h_K^{-1}\|v_0 - v_b\|_{\partial K}^2)\Big)^{\frac{1}{2}}, \ v_h = \{v_0, v_b\} \in \mathcal{V}_h. \tag{2.13}$$

In other words, for any $v_h = \{v_0, v_b\} \in \mathcal{V}_h$, there exist two constant $C_1, \ C_2 > 0$ such that the following inequalities hold true (cf. Lemma 3.2, [6])

$$C_1\|v_h\|_{1,h} \le |||v_h||| \le C_2\|v_h\|_{1,h}. \tag{2.14}$$

It is easy to see that $\|v_h\|_{1,h}$ and $|||v_h|||$ define norms in $\mathcal{V}_h^0$.

For any $\varphi \in H^1(K)$, the following trace inequality holds (cf. [52])

$$\|\varphi\|_e^2 \le C(h_K^{-1}\|\varphi\|_K^2 + h_K\|\nabla\varphi\|_K^2). \tag{2.15}$$

In addition, the following Poincaŕe-type inequality also holds (cf. Lemma 7.4, [51])

$$\|v_0\| \le C|||v_h|||, \ v_h = \{v_0, v_b\} \in \mathcal{V}_h^0. \tag{2.16}$$

3 Error analysis for the semidiscrete scheme

This section deals with the error analysis for the spatially discrete SFWG scheme. Supercloseness convergence rate in $L^\infty(H^1)$ norm and $L^\infty(L^2)$ norm is derived under the appropriate regularity assumptions on the solution.

The continuous-time SFWG finite element approximation to the model problems (1.1) and (1.2) is defined as follows: Find $u_h = \{u_0, u_b\} : [0, T] \to \mathcal{V}_h^0$ satisfying

$$(u_h'', \phi_0) + \mathcal{A}_{1,w}(u_h, \phi_h) + \mathcal{A}_{2,w}(u_h', \phi_h) = (f, \phi_0), \ \forall\, \phi_h = \{\phi_0, \phi_b\} \in \mathcal{V}_h^0, \tag{3.1}$$

with $u_h(0) = \mathcal{Q}_h u^0$, and $u_h'(0) = \mathcal{Q}_h v^0$ are approximations of the initial functions u^0 and v^0.

The following result deals with the existence and uniqueness of the SFWG solution u_h. We borrowed the basic technique from [38].

Theorem 3.1 *For each $h \in (0, h_0]$, there exists a function $u_h \in C^2(0, T; \mathcal{V}_h^0)$ satisfying (3.1).*

Proof For a given element $K \in \{\mathcal{T}_h\}_{0<h\leq h_0}$, let $\{\phi_{0,i} : i = 1, 2, \ldots, N_0\}$ be a set of basis functions for $\mathcal{P}_k(K)$, where $N_0 = \dim\big(\mathcal{P}_k(K)\big)$, and $\{\phi_{b,i} : i = 1, 2, \ldots, N_b\}$ be a set of basis function for $\mathcal{P}_{k+1}(e)$, where $N_b = \dim\big(\mathcal{P}_{k+1}(e)\big)$. Then, every $v_h = \{v_0, v_b\} \in \{\mathcal{V}_h^0\}_{0<h\leq h_0}$ can be written as

$$v_h|_K = \left\{ \sum_{i=1}^{N_0} d_{0,i}(t)\phi_{0,i}, \sum_{j=1}^{N_b} d_{b,j}(t)\phi_{b,j} \right\},$$

where $d_{0,i},\ d_{b,j} : (0, T] \to \mathbb{R}$ are the coefficient functions for $1 \leq i \leq N_0$ and $1 \leq j \leq N_b$. For $1 \leq i \leq N_0 + N_b$, we write $\hat{\phi}_{i,h} = \{\hat{\phi}_{0,i}, \hat{\phi}_{b,i}\}$ with

$$\begin{aligned}\hat{\phi}_{0,i} &= \phi_{0,i} \ \text{ for } 1 \leq i \leq N_0 \quad \& \quad \hat{\phi}_{0,i} = 0 \ \text{ for } \ N_0 + 1 \leq i \leq N_0 + N_b,\\ \hat{\phi}_{b,i} &= 0 \ \text{ for } \ 1 \leq i \leq N_0 \quad \& \quad \hat{\phi}_{b,i} = \phi_{b,i-N_0} \ \text{ for } \ N_0 + 1 \leq i \leq N_0 + N_b,\end{aligned}$$

and similarly to capture the unknown coefficient functions, we define

$$\hat{d}_{i,h} = d_{0,i} \ \text{ for } 1 \leq i \leq N_0 \ \ \& \ \ \hat{d}_{i,h} = d_{b,i-N_0} \ \text{ for } N_0 + 1 \leq i \leq N_0 + N_b.$$

Then, we seek our semidiscrete solution $u_h = \{u_0, u_b\} \in \mathcal{V}_h^0$ such that

$$u_h|_K = \sum_{i=1}^{N_0+N_b} \hat{d}_{i,h}(t)\hat{\phi}_{i,h} = \left\{ \sum_{i=1}^{N_0+N_b} \hat{d}_{i,h}(t)\hat{\phi}_{0,i}, \sum_{j=1}^{N_0+N_b} \hat{d}_{j,h}(t)\hat{\phi}_{b,j} \right\}, \quad K \in \mathcal{T}_h.$$

Now, set $v_h = \hat{\phi}_{j,h}$, $j = 1, 2, \ldots, N_0 + N_b$ in (3.1) to obtain

$$\begin{aligned}&\left(\sum_{i=1}^{N_0+N_b} \hat{d}''_{i,h}(t)\hat{\phi}_{i,h}, \hat{\phi}_{j,h} \right) + \mathcal{A}_{1,w}\left(\sum_{i=1}^{N_0+N_b} \hat{d}_{i,h}(t)\hat{\phi}_{i,h}, \hat{\phi}_{j,h} \right)\\ &\quad + \mathcal{A}_{2,w}\left(\sum_{i=1}^{N_0+N_b} \hat{d}'_{i,h}(t)\hat{\phi}_{i,h}, \hat{\phi}_{j,h} \right) = (f, \hat{\phi}_{j,h}), \quad j = 1, 2, \ldots, N_0 + N_b.\end{aligned}$$

We can rearrange the above equation as

$$\begin{aligned}&\sum_{i=1}^{N_0+N_b} \hat{d}''_{i,h}(t)\big(\hat{\phi}_{i,h}, \hat{\phi}_{j,h}\big) + \sum_{i=1}^{N_0+N_b} \hat{d}_{i,h}(t)\mathcal{A}_{1,w}\big(\hat{\phi}_{i,h}, \hat{\phi}_{j,h}\big)\\ &\quad + \sum_{i=1}^{N_0+N_b} \hat{d}'_{i,h}(t)\mathcal{A}_{2,w}\big(\hat{\phi}_{i,h}, \hat{\phi}_{j,h}\big) = (f, \hat{\phi}_{j,h}), \quad j = 1, 2, \ldots, N_0 + N_b.\end{aligned}$$

On each element K, the local stiffness matrix $\mathcal{A}_{r,K}$ $(r = 1,\ 2)$ associated with the bilinear maps $\mathcal{A}_{r,w}(\cdot, \cdot)$ defined by (2.9) and (2.10) can thus be written as a block matrix

$$\mathcal{A}_{r,K} = \begin{bmatrix} \mathcal{A}_{0,0} & \mathcal{A}_{0,b} \\ \mathcal{A}_{b,0} & \mathcal{A}_{b,b} \end{bmatrix}, \tag{3.2}$$

where $\mathcal{A}_{0,0}$ is a $N_0 \times N_0$, $\mathcal{A}_{0,b}$ is a $N_0 \times N_b$, $\mathcal{A}_{b,0}$ is a $N_b \times N_0$, and $\mathcal{A}_{b,b}$ is a $N_b \times N_b$ matrices. More precisely, these matrices are given by

$$\mathcal{A}_{0,0} = [\mathcal{A}_{r,w}(\phi_{0,j}, \phi_{0,i})_K]_{i,j}, \quad \mathcal{A}_{0,b} = [\mathcal{A}_{r,w}(\phi_{0,j}, \phi_{b,i})_K]_{i,j}$$

$$\mathcal{A}_{b,0} = [\mathcal{A}_{r,w}(\phi_{b,j}, \phi_{0,i})_K]_{i,j}, \quad \mathcal{A}_{b,b} = [\mathcal{A}_{r,w}(\phi_{b,j}, \phi_{b,i})_K]_{i,j},$$

where i, j are the row and column indices, respectively.

We denote by

$$\hat{d}_{h,0} = [\hat{d}_{1,h}(0), \ldots, \hat{d}_{N_0+N_b,h}(0)]^T$$

and

$$\hat{d}_{h,1} = [\hat{d}'_{1,h}(0), \ldots, \hat{d}'_{N_0+N_b,h}(0)]^T,$$

the components of the given initial approximation $u_h(0)$ and $u'_h(0)$ respectively.

Then, for our semidiscrete solution, we need to find an unknown vector $\hat{d}_h(t) = [\hat{d}_{1,h}(t), \ldots, \hat{d}_{N_0+N_b,h}(t)]^T$ such that

$$\begin{cases} \mathcal{C}_K \hat{d}_h''(t) + \mathcal{A}_{1,K}\hat{d}_h(t) + \mathcal{A}_{2,K}\hat{d}_h'(t) = F_K(t) \\ \hat{d}_h(0) = \hat{d}_{h,0}, \quad \text{and} \quad \hat{d}_h'(0) = \hat{d}_{h,1}, \quad \text{for} \quad t \in (0, T], \end{cases} \tag{3.3}$$

where the coefficient matrices are given by

$$\mathcal{C}_K = [\mathcal{C}_{i,j}], \quad \mathcal{C}_{i,j} = (\hat{\phi}_{0,i}, \hat{\phi}_{0,j}),$$

and the source term is given by

$$F_h = [F_1, \ldots, F_{N_0+N_b}], \quad F_j = (f, \hat{\phi}_{0,j}),$$

with $1 \le i, j \le N_0 + N_b$.

Note that the matrices and right-hand side vectors all are well-defined. Since

$$|(\hat{\phi}_{0,i}, \hat{\phi}_{0,j})| \le \|\hat{\phi}_{0,i}\| \|\hat{\phi}_{0,j}\| \;\; \& \;\; |\mathcal{A}_{r,K}(\hat{\phi}_{i,h}, \hat{\phi}_{j,h})| \le C\|\hat{\phi}_{i,h}\|_{1,h}\|\hat{\phi}_{j,h}\|_{1,h},$$

and

$$|(f, \hat{\phi}_{0,j})| \le \|f\| \|\hat{\phi}_{0,j}\|,$$

for all $t \in (0, T]$ and $(r = 1, 2)$.

Furthermore, for any $v \in \mathbb{R}^{N_0+N_b} \setminus \{0\}$, we have

$$v^T C_K v = (\hat{v}, \hat{v})_K > 0, \quad \hat{v} = \sum_{i=1}^{N_0+N_b} v_i \hat{\phi}_{0,i}.$$

Hence, the matrix C_K is invertible for all $t \in (0, T]$ and the equation (3.3) can be restated as:

$$\begin{cases} \hat{d}_h''(t) + \mathcal{C}_K^{-1}\mathcal{A}_{1,K}\hat{d}_h(t) + \mathcal{C}_K^{-1}\mathcal{A}_{2,K}\hat{d}_h'(t) = \mathcal{C}_K^{-1}F_K(t) \\ \hat{d}_h(0) = \hat{d}_{h,0}, \quad \text{and} \quad \hat{d}_h'(0) = \hat{d}_{h,1}. \end{cases}$$

Now, the existence of the solution $u_h \in C^2(0, T; \mathcal{V}_h^0)$ follows from the standard ODE theory. □

Regarding the stability of u_h, we have the following result. We can find the proof of this Lemma in the Appendix.

Lemma 3.1 *For any $t \in (0, T]$, let u_h satisfy the SFWG scheme (3.1). Then, we have*

$$\int_0^t \|u_h''''(t)\|^2 ds + \int_0^t \left|\left\|u_h'''(t)\right\|\right|^2 ds \le C\Big(\|u^0\|^2_{H^6(\Omega)} + \|v^0\|^2_{H^6(\Omega)} + \|f\|^2_{H^3(J;H^2(\Omega))}\Big).$$

Now, we define the error as:

$$e_h = \{e_0, e_b\} := \{u_0 - \mathcal{Q}_0 u, u_b - \mathcal{Q}_b u\} = u_h - \mathcal{Q}_h u.$$

Error e_h is characterized in the following result, which is vital for our convergence analysis.

Lemma 3.2 *For all $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{V}_h^0$, we have*

$$(e_h'', \phi_0) + \mathcal{A}_{1,w}(e_h, \phi_h) + \mathcal{A}_{2,w}(e_h', \phi_h) = \ell_1(u, \phi_h) + \ell_2(u, \phi_h) + \ell_3(u', \phi_h) + \ell_4(u', \phi_h), \tag{3.4}$$

where bilinear forms $\ell_i(\cdot,\cdot)$, $i = 1, 2, 3, 4$ are given by

$$\begin{aligned}
\ell_1(u, \phi_h) &= \sum_{K\in\mathcal{T}_h} \big\langle(\alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u))\cdot \boldsymbol{n}, \phi_0 - \phi_b\big\rangle_{\partial K},\\
\ell_2(u, \phi_h) &= \sum_{K\in\mathcal{T}_h} (\alpha\mathbb{Q}_h(\nabla u) - \mathbb{Q}_h(\alpha\nabla u), \nabla_{\rm w}\phi_h)_K,\\
\ell_3(u, \phi_h) &= \sum_{K\in\mathcal{T}_h} \big\langle(\beta\nabla u - \mathbb{Q}_h(\beta\nabla u))\cdot \boldsymbol{n}, \phi_0 - \phi_b\big\rangle_{\partial K},\\
\ell_4(u, \phi_h) &= \sum_{K\in\mathcal{T}_h} (\beta\mathbb{Q}_h(\nabla u) - \mathbb{Q}_h(\beta\nabla u), \nabla_{\rm w}\phi_h)_K.
\end{aligned}$$

Proof On each element $K \in \mathcal{T}_h$, for $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{V}_h^0$, we test equation (1.1) against ϕ_0 to arrive at

$$\begin{aligned}
(f, \phi_0) &= (u'', \phi_0) - \sum_{K\in\mathcal{T}_h} (\nabla\cdot(\alpha\nabla u), \phi_0)_K - \sum_{K\in\mathcal{T}_h} (\nabla\cdot(\beta\nabla u'), \phi_0)_K\\
&= (u'', \phi_0) + \sum_{K\in\mathcal{T}_h} (\alpha\nabla u, \nabla\phi_0)_K - \sum_{K\in\mathcal{T}_h} \langle\alpha\nabla u\cdot\mathbf{n}, \phi_0\rangle_{\partial K}\\
&\quad + \sum_{K\in\mathcal{T}_h} (\beta\nabla u', \nabla\phi_0)_K - \sum_{K\in\mathcal{T}_h} \langle\beta\nabla u\cdot\mathbf{n}, \phi_0\rangle_{\partial K}\\
&= (u'', \phi_0) + \sum_{K\in\mathcal{T}_h} (\alpha\nabla u, \nabla\phi_0)_K - \sum_{K\in\mathcal{T}_h} \langle\alpha\nabla u\cdot\mathbf{n}, \phi_0 - \phi_b\rangle_{\partial K}\\
&\quad + \sum_{K\in\mathcal{T}_h} (\beta\nabla u', \nabla\phi_0)_K - \sum_{K\in\mathcal{T}_h} \langle\beta\nabla u\cdot\mathbf{n}, \phi_0 - \phi_b\rangle_{\partial K}.
\end{aligned}\tag{3.5}$$

Here, we have used the divergence theorem and the fact that

$$\sum_{K\in\mathcal{T}_h}\langle\alpha\nabla u\cdot\mathbf{n},\phi_b\rangle_{\partial K}=0\ \ \text{and}\ \ \sum_{K\in\mathcal{T}_h}\langle\beta\nabla u\cdot\mathbf{n},\phi_b\rangle_{\partial K}=0.$$

Then, integration by part together with definition (2.3) for weak gradient and L^2 projection $\mathbb{Q}_h$ yields

$$\begin{aligned}(\mathbb{Q}_h(\alpha\nabla u),\nabla_{\rm w}\phi_h)_K &= -(\phi_0,\nabla\cdot(\mathbb{Q}_h(\alpha\nabla u)))_K+\langle\phi_b,\mathbb{Q}_h(\alpha\nabla u)\cdot\mathbf{n}\rangle_{\partial K}\\ &=(\nabla\phi_0,\mathbb{Q}_h(\alpha\nabla u))_K-\langle\phi_0-\phi_b,\mathbb{Q}_h(\alpha\nabla u)\cdot\mathbf{n}\rangle_{\partial K}\\ &=(\nabla\phi_0,\alpha\nabla u)_K-\langle\phi_0-\phi_b,\mathbb{Q}_h(\alpha\nabla u)\cdot\mathbf{n}\rangle_{\partial K}.\end{aligned}\tag{3.6}$$

As a consequence of (3.6), we get

$$(\mathbb{Q}_h(\beta\nabla u'),\nabla_{\rm w}\phi_h)_K=(\nabla\phi_0,\beta\nabla u')_K-\langle\phi_0-\phi_b,\mathbb{Q}_h(\beta\nabla u')\cdot\mathbf{n}\rangle_{\partial K}.\tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we have

$$\begin{aligned}(f,\phi_0)&=(u'',\phi_0)+\sum_{K\in\mathcal{T}_h}(\mathbb{Q}_h(\alpha\nabla u),\nabla_{\rm w}\phi_h)_K+\sum_{K\in\mathcal{T}_h}(\mathbb{Q}_h(\beta\nabla u'),\nabla_{\rm w}\phi_h)_K\\ &\quad+\sum_{K\in\mathcal{T}_h}\langle\phi_0-\phi_b,(\mathbb{Q}_h(\alpha\nabla u)-\alpha\nabla u)\cdot\mathbf{n}\rangle_{\partial K}\\ &\quad+\sum_{K\in\mathcal{T}_h}\langle\phi_0-\phi_b,(\mathbb{Q}_h(\beta\nabla u')-\beta\nabla u')\cdot\mathbf{n}\rangle_{\partial K}\\ &=(\mathcal{Q}_hu'',\phi_0)+\sum_{K\in\mathcal{T}_h}(\alpha\nabla_{\rm w}\mathcal{Q}_hu,\nabla_{\rm w}\phi_h)_K+\sum_{K\in\mathcal{T}_h}(\beta\nabla_{\rm w}\mathcal{Q}_hu',\nabla_{\rm w}\phi_h)_K\\ &\quad-\ell_1(u,\phi_h)-\ell_2(u,\phi_h)-\ell_3(u',\phi_h)-\ell_4(u',\phi_h)\end{aligned}\tag{3.8}$$

Using the definition of the bilinear map $\mathcal{A}_r(\cdot,\cdot)$ in the above equation leads to

$$\begin{aligned}(\mathcal{Q}_hu'',\phi_0)\ +\ \mathcal{A}_{1,w}(\mathcal{Q}_hu,\phi_h)+\mathcal{A}_{2,w}(\mathcal{Q}_hu',\phi_h)&=(f,\phi_0)\\ +\ \ell_1(u,\phi_h)+\ell_2(u,\phi_h)+\ell_3(u',\phi_h)&+\ell_4(u',\phi_h).\end{aligned}\tag{3.9}$$

Subtracting (3.9) from (3.1) leads us to the desired result. □

We state the following crucial estimates for the bilinear maps $\ell_i,\ i=1,2,3,4$.

Lemma 3.3 *Assume that $\mathcal{T}_h$ is shaped regular discretization of computational domain Ω. Then, for $u\in H^{k+3}(\Omega)$ and $\phi_h\in\mathcal{V}_h^0$, we have*

$$|\ell_1(u,\phi_h)|\le C(\|\alpha\|_{k+2,\infty})h^{k+2}\|u\|_{k+3}|||\phi_h|||,\tag{3.10}$$
$$|\ell_2(u,\phi_h)|\le C(\|\alpha\|_{1,\infty})h^{k+3}\|u\|_{k+3}|||\phi_h|||,\tag{3.11}$$
$$|\ell_3(u,\phi_h)|\le C(\|\beta\|_{k+2,\infty})h^{k+2}\|u\|_{k+3}|||\phi_h|||,\tag{3.12}$$
$$|\ell_4(u,\phi_h)|\le C(\|\beta\|_{1,\infty})h^{k+3}\|u\|_{k+3}|||\phi_h|||,\tag{3.13}$$

where $C(\|\alpha\|_{k+2,\infty})$ and $C(\|\beta\|_{k+2,\infty})$ are a positive constant depending on $\|\cdot\|_{k+2,\infty}$ the element-wise $W^{k+2,\infty}$ norm of the coefficient matrix α and β.

Proof For ℓ_1 estimate, apply trace inequality (2.15) and Lemma 2.2, and we get

$$
\begin{aligned}
|\ell_1(u,\phi_h)| &\le \sum_{K\in\mathcal{T}_h} \|\alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u)\|_{\partial K}\|\phi_0-\phi_b\|_{\partial K} \\
&\le C\Big(\sum_{K\in\mathcal{T}_h} h_K\|\alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u)\|^2_{\partial K}\Big)^{\frac{1}{2}}\Big(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|\phi_0-\phi_b\|^2_{\partial K}\Big)^{\frac{1}{2}} \\
&\le C\Big(\sum_{K\in\mathcal{T}_h}\Big(\|\alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u)\|^2_K + h_K^2\|\nabla(\alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u))\|^2_K\Big)\Big)^{\frac{1}{2}} \\
&\quad\times\Big(\sum_{K\in\mathcal{T}_h}\Big(\|\nabla\phi_0\|^2_K + h_K^{-1}\|\phi_0-\phi_b\|^2_{\partial K}\Big)\Big)^{\frac{1}{2}} \\
&\le C(\|\alpha\|_{k+2,\infty})h^{k+2}\|u\|_{k+3}\|\phi_h\|_{1,h} \\
&\le C(\|\alpha\|_{k+2,\infty})h^{k+2}\|u\|_{k+3}|||\phi_h|||. \qquad (3.14)
\end{aligned}
$$

Let $\bar{\alpha}$ be the average of α on each element $K\in\mathcal{T}_h$. Then, we have (see page 2118 in [52])

$$\|\alpha-\bar{\alpha}\|_{L^\infty(\Omega)} \le Ch\|\nabla\alpha\|_{L^\infty(\Omega)}. \qquad (3.15)$$

Then, using the definition of $\mathbb{Q}_h$ operator, Lemma 2.2, and estimate (3.15), we obtain

$$
\begin{aligned}
|l_2(u,\phi_h)| &= \Big|\sum_{K\in\mathcal{T}_h}(\alpha\mathbb{Q}_h(\nabla u) - \alpha\nabla u, \nabla_{\mathrm{w}}\phi_h)_K\Big| \\
&= \Big|\sum_{K\in\mathcal{T}_h}(\mathbb{Q}_h(\nabla u) - \nabla u, (\alpha-\bar{\alpha})\nabla_{\mathrm{w}}\phi_h)_K\Big| \\
&\le \sum_{K\in\mathcal{T}_h}\Big|(\mathbb{Q}_h(\nabla u) - \nabla u, (\alpha-\bar{\alpha})\nabla_{\mathrm{w}}\phi_h)_K\Big| \\
&\le Ch\|\alpha\|_{1,\infty}\sum_{K\in\mathcal{T}_h}\Big|(\mathbb{Q}_h(\nabla u) - \nabla u, \nabla_{\mathrm{w}}\phi_h)_K\Big| \\
&\le C(\|\alpha\|_{1,\infty})h^{k+3}\|u\|_{k+3}|||\phi_h|||. \qquad (3.16)
\end{aligned}
$$

Similar arguments yield

$$
\begin{aligned}
|\ell_3(u,\phi_h)| &\le C(\|\beta\|_{k+2,\infty})h^{k+2}\|u\|_{k+3}|||\phi_h|||, \\
|\ell_4(u,\phi_h)| &\le C(\|\beta\|_{1,\infty})h^{k+3}\|u\|_{k+3}|||\phi_h|||,
\end{aligned}
$$

The proof is completed. □

By letting $v_h = e_h'$ in (3.4), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|e_0'\|^2 + \frac{\sigma_*}{2}\frac{d}{dt}|||e_h|||^2 + \sigma_*\big|\big|\big|e_h'\big|\big|\big|^2 &\le \ell_1(u,e_h') + \ell_2(u,e_h') \\
&+ \ell_3(u',e_h') + \ell_4(u',e_h').
\end{aligned}
$$

By integration over the time interval $[0, t]$, we get

$$\frac{1}{2}\|e_0'\|^2 + \sigma_* \int_0^t |||e_h'|||^2 ds + \frac{\sigma_*}{2} |||e_h|||^2 \le \frac{1}{2}\|e_0'(0)\|^2 + \frac{\sigma_*}{2} |||e_h(0)|||^2$$
$$+ \int_0^t |\ell_1(u, e_h')| ds + \int_0^t |\ell_2(u, e_h')| ds + \int_0^t |\ell_3(u', e_h')| ds + \int_0^t |\ell_4(u', e_h')| ds$$

It then follows from the estimates (3.10)–(3.13) together with Young's inequality for some appropriate $\nu > 0$ lead to

$$\frac{1}{2}\|e_0'\|^2 + \sigma_* \int_0^t |||e_h'|||^2 ds + \frac{\sigma_*}{2} |||e_h|||^2 \le C_\nu \int_0^t |||e_h'|||^2 ds$$
$$+ C(\nu) h^{2(k+2)} \int_0^t \left(\|u\|_{k+3}^2 + \|u'\|_{k+3}^2 \right) ds.$$

Here, we have used the fact that $e_h(0) = u_h(0) - \mathcal{Q}_h u^0 = 0$. As a consequence, we have $e_h'(0) = 0$.

We can rearrange the above inequality as

$$|||e_h|||^2 \le C h^{2(k+2)} \int_0^t (\|u\|_{k+3}^2 + \|u'\|_{k+3}^2) ds. \tag{3.17}$$

Finally, the estimate (3.17) leads us to the following point-wise $L^\infty(0, T; H^1(\Omega))$ convergence result.

Theorem 3.2 *Let $u \in H^1(0, T; H^{k+3}(\Omega))$ be the solution of the model problems (1.1) and (1.2) and $u_h \in C^2(0, T; \mathcal{V}_h^0)$ be the solution of SFWG scheme (3.1). Then, we have*

$$|||\mathcal{Q}_h u - u_h||| \le C h^{k+2} \left(\int_0^T \left\{ \|u\|_{k+3}^2 + \|u'\|_{k+3}^2 \right\} ds \right)^{\frac{1}{2}}.$$

Remark 3.1 Theorem 3.2 has been established with constant coefficients in (see Theorem 3.2 in [53]) with the stabilizer-based WG-FEM using weak Galerkin space

$$(\mathcal{P}_{k+2}(K), \mathcal{P}_{k+1}(\partial K), [\mathcal{P}_{k+1}(K)]^2).$$

In the present study, we have derived the same order of convergence using the SFWG method with variable coefficients with weak Galerkin space

$$(\mathcal{P}_k(K), \mathcal{P}_{k+1}(\partial K), [\mathcal{P}_{k+1}(K)]^2).$$

To get a supercloseness order of error estimate in the L^2 norm, we define a non-standard elliptic type projection operator, which is crucial for our later error analysis. For $z \in H^1(0, T; \mathcal{X})$, where $\mathcal{X} = H_0^1(\Omega) \cap H^2(\Omega)$, define $\mathcal{E}_h : \mathcal{X} \to \mathcal{V}_h^0$ by

$$\mathcal{A}_{1,w}(\mathcal{E}_h z, \phi_h) + \mathcal{A}_{2,w}((\mathcal{E}_h z)', \phi_h) = (f_z, \phi_0), \ \forall \ \phi_h = \{\phi_0, \phi_b\} \in \mathcal{V}_h^0, \tag{3.18}$$

with $(\mathcal{E}_h z)(0) = \mathcal{Q}_h z^0 \in \mathcal{V}_h^0$ and

$$f_z = -\nabla \cdot \left((\alpha \nabla z) + (\beta \nabla z') \right).$$

Note that $\mathcal{E}_h z$ can be recognized as the SFWG finite element approximation solution applied to the following initial boundary value problem

$$-\nabla\cdot(\alpha\nabla z)-\nabla\cdot(\beta\nabla z')=f_z \ \text{ in } \ \Omega\times(0,T], \tag{3.19}$$

with boundary condition $z(x,t)=0$ on $\partial\Omega\times(0,T]$ and initial condition $z(x,0)=z^0$ in Ω.

Remark 3.2 Note that in the H^1 and L^2 norms, error bounds for the projection operator $\mathcal{E}_h$ satisfying equation (3.18) with constant coefficients have been proved using the standard WG methods (see cf. [21]). A limitation of this article is the use of constant coefficients, which do not occur in some applications. In the next lemma, we use the SFWG algorithm with variable coefficients to bound the error $\zeta_u=\{(\zeta_u)_0,(\zeta_u)_b\}:=\mathcal{Q}_h u-\mathcal{E}_h u$ in both triple-bar and L^2 norms. Here, the weak function $\{(\zeta_u)_0,(\zeta_u)_b\}$ in our weak finite element space $\mathcal{V}_h$. For simplicity, we continue to denote $\big((\zeta_u)_0,\varphi\big)_K$ by $\big(\zeta_u,\varphi\big)_K$, where $K\in\mathcal{T}_h$.

Lemma 3.4 *Let $u\in H^1(0,T;H^{k+3}(\Omega))$ be the solution of the model problems (1.1) and (1.2). Let $\mathcal{E}_h u\in C^1(0,T;\mathcal{V}_h^0)$ be the SFWG solution of the (3.19). Then, there exists a constant C such that*

$$|||\mathcal{Q}_h u-\mathcal{E}_h u|||\le Ch^{k+2}\|u\|_{H^1(0,T;H^{k+3}(\Omega))}, \tag{3.20}$$

$$\|\mathcal{Q}_h u-\mathcal{E}_h u\|\le Ch^{k+3}\|u\|_{H^1(0,T;H^{k+3}(\Omega))}. \tag{3.21}$$

Proof The following analysis used to derive (3.9), we obtain

$$\begin{aligned}\mathcal{A}_{1,w}(\mathcal{Q}_h u,\phi_h)\ +\ \mathcal{A}_{2,w}((\mathcal{Q}_h u)_t,\phi_h)&=(f_u,\phi_0)+\ell_1(u,\phi_h)+\ell_2(u,\phi_h)\\&\quad+\ \ell_3(u',\phi_h)+\ell_4(u',\phi_h),\ \forall\,\phi_h=\{\phi_0,\phi_b\}\in\mathcal{V}_h^0.\end{aligned} \tag{3.22}$$

Now, subtracting (3.18) from the above equation, we arrive at the following error relation for ζ_u

$$\begin{aligned}\mathcal{A}_{1,w}(\zeta_u(t),\phi_h)+\mathcal{A}_{2,w}(\zeta_u'(t),\phi_h)&=\ell_1(u,\phi_h)+\ell_2(u,\phi_h)+\ell_3(u',\phi_h)\\&\quad+\ell_4(u',\phi_h),\ \ t\in(0,T].\end{aligned} \tag{3.23}$$

Finally, setting $\phi_h=\zeta_u$ in (3.23) and then standard analysis leads to the following estimate

$$\begin{aligned}|||\zeta_u(t)|||^2&\le C\big(|||\zeta_u(0)|||^2+h^{2(k+2)}\|u\|^2_{H^1(0,T;H^{k+3}(\Omega))}\big)\\&\le Ch^{2(k+2)}\|u\|^2_{H^1(0,T;H^{k+3}(\Omega))}.\end{aligned} \tag{3.24}$$

Here, we have used the fact that $\zeta_u(0)=0$.

For the estimate (3.21), we define a dual problem that seeks a solution $w\in H^1(J;H^2(\Omega))$ such that

$$-\nabla\cdot\big((\alpha\nabla w)-(\beta\nabla w')\big)=\zeta_u \ \text{ in } \ \Omega\times J, \tag{3.25}$$

and $w(\tau) = 0$ for some $\tau \in J = [0, T]$. Assume that there exists a unique solution $w \in H^1(J; H^2(\Omega))$ such that (cf. [7])

$$\|w\|_{H^1(J;H^2(\Omega))} \le C\|\zeta_u\|_{L^2(J;L^2(\Omega))}. \tag{3.26}$$

Multiply the equation (3.25) by ζ_u', we get

$$(\zeta_u, \zeta_u') = \big(- \nabla \cdot \big((\alpha \nabla w) - (\beta \nabla w')\big), \zeta_u'\big). \tag{3.27}$$

Next, arguing as in (3.8), we obtain

$$\begin{aligned}(\zeta_u, \zeta_u') = &\sum_{K\in\mathcal{T}_h} \Big\{(\alpha\nabla_{\rm w}\mathcal{Q}_h w, \nabla_{\rm w}\zeta_u')_K - (\beta\nabla_{\rm w}\mathcal{Q}_h w', \nabla_{\rm w}\zeta_u')_K\Big\} \\ &-\ell_1(w, \zeta_u') - \ell_2(w, \zeta_u') + \ell_3(w', \zeta_u') + \ell_4(w', \zeta_u'),\end{aligned} \tag{3.28}$$

where the bilinear maps $\ell_i(\cdot, \cdot)$ are as defined in Lemma 3.2.

Now, integrate equation (3.28) in $[0, \tau]$ to obtain

$$\begin{aligned}\frac{1}{2}\|\zeta_u(\tau)\|^2 &= \sum_{K\in\mathcal{T}_h}\int_0^\tau \big\{(\alpha\nabla_{\rm w}\mathcal{Q}_h w, \nabla_{\rm w}\zeta_u')_K - (\beta\nabla_{\rm w}\mathcal{Q}_h w', \nabla_{\rm w}\zeta_u')_K\big\}ds \\ &- \int_0^\tau \ell_1(w, \zeta_u')ds + \int_0^\tau \ell_3(w', \zeta_u')ds - \int_0^\tau \ell_2(w, \zeta_u')ds + \int_0^\tau \ell_4(w', \zeta_u')ds \\ &= \sum_{K\in\mathcal{T}_h}\int_0^\tau \big\{(-\alpha\nabla_{\rm w}\mathcal{Q}_h w', \nabla_{\rm w}\zeta_u)_K - (\beta\nabla_{\rm w}\mathcal{Q}_h w', \nabla_{\rm w}\zeta_u')_K\big\}ds \\ &+ \sum_{K\in\mathcal{T}_h}(\alpha\nabla_{\rm w}\mathcal{Q}_h w, \nabla_{\rm w}\zeta_u)_K(0) - \int_0^\tau \ell_1(w, \zeta_u')ds + \int_0^\tau \ell_3(w', \zeta_u')ds. \\ &- \int_0^\tau \ell_2(w, \zeta_u')ds + \int_0^\tau \ell_4(w', \zeta_u')ds.\end{aligned}$$

In the above, we have applied the facts that $(\nabla_{\rm w} v_h(t))' = \nabla_{\rm w} v_h'(t)$ and $\nabla_{\rm w} v_h(t)|_{t=0}$ $= \nabla_{\rm w} v_h(0)$ for $v_h \in \mathcal{V}_h^0$, and $\zeta_u(0) = 0$. Hence the factor $(\alpha\nabla_{\rm w}\mathcal{Q}_h w, \nabla_{\rm w}\zeta_u)_K(0) =$ $(\alpha\nabla_{\rm w}\mathcal{Q}_h w(0), \nabla_{\rm w}\zeta_u(0))_K = 0$.

Furthermore, using bilinear maps (2.9) and (2.10), we can rearrange the above equation as follows

$$\begin{aligned}\frac{1}{2}\|\zeta_u(\tau)\|^2 &= \int_0^\tau \big\{-\mathcal{A}_{1,w}(\zeta_u, \mathcal{Q}_h w') - \mathcal{A}_{2,w}(\zeta_u', \mathcal{Q}_h w')\big\}ds \\ &- \int_0^\tau \ell_1(w, \zeta_u')ds + \int_0^\tau \ell_3(w', \zeta_u')ds - \int_0^\tau \ell_2(w, \zeta_u')ds + \int_0^\tau \ell_4(w', \zeta_u')ds.\end{aligned}$$

Next, using the error equation (3.23), we obtain

$$\begin{aligned}\frac{1}{2}\|\zeta_u(\tau)\|^2 &= -\int_0^\tau \ell_1(u, \mathcal{Q}_h w')ds - \int_0^\tau \ell_3(u', \mathcal{Q}_h w')ds - \int_0^\tau \ell_2(u, \mathcal{Q}_h w')ds \\ &- \int_0^\tau \ell_4(u', \mathcal{Q}_h w')ds - \int_0^\tau \ell_1(w, \zeta_u')ds + \int_0^\tau \ell_3(w', \zeta_u')ds - \int_0^\tau \ell_2(w, \zeta_u')ds \\ &+ \int_0^\tau \ell_4(w', \zeta_u')ds.\end{aligned} \tag{3.29}$$

The right hand side of the above equation needs to be estimated now. To do this, first, we consider the term $\ell_1(w, \zeta_u')$ and employ the same procedure as in (3.14) along with Lemma 2.2 ($s = 1$), we get

$$\begin{aligned} |\ell_1(w, \zeta_u')| &\leq C\Big(\sum_{K\in\mathcal{T}_h} \big(\|\alpha\nabla w - \mathbb{Q}_h(\alpha\nabla w)\|_K^2 \\ &\qquad + h_K^2 \|\nabla(\alpha\nabla w - \mathbb{Q}_h(\alpha\nabla w))\|_K^2\big)\Big)^{\frac{1}{2}} \times |||\zeta_u'||| \\ &\leq C(\|\alpha\|_{1,\infty}) h \|w\|_2 |||\zeta_u'|||. \end{aligned} \tag{3.30}$$

Likewise, for the term $\ell_3(w', \zeta_u')$, we obtain

$$|\ell_3(w', \zeta_u')| \leq C(\|\beta\|_{1,\infty}) h \|w'\|_2 |||\zeta_u'|||. \tag{3.31}$$

Combining the estimates (3.30) and (3.31), we obtain

$$|\ell_1(w, \zeta_u')| + |\ell_3(w', \zeta_u')| \leq Ch(\|w\|_2 + \|w'\|_2)|||\zeta_u'|||, \tag{3.32}$$

which together with estimate (3.20) and (3.26) yields

$$\int_0^\tau \ell_1(w, \zeta_u')ds + \int_0^\tau \ell_3(w', \zeta_u')ds \leq Ch^{k+3}\|u\|_{H^1(H^{k+3})}\|\zeta_u\|_{L^2(L^2)}. \tag{3.33}$$

Next, we consider the term $\ell_2(w, \zeta_u')$ and follows the same procedure as in (3.16) along with Lemma 2.2 ($s = 0$), we get

$$\begin{aligned} |l_2(w, \zeta_u')| &\leq Ch\|\alpha\|_{1,\infty} \sum_{K\in\mathcal{T}_h} \Big|(\mathbb{Q}_h(\nabla w) - \nabla w, \nabla_{\mathrm{w}}\zeta_u')_K\Big| \\ &\leq C(\|\alpha\|_{1,\infty}) h \|w\|_1 |||\zeta_u'|||. \end{aligned} \tag{3.34}$$

Similarly, for the term $\ell_4(w', \zeta_u')$, we obtain

$$|\ell_4(w', \zeta_u')| \leq C(\|\beta\|_{1,\infty}) h \|w'\|_1 |||\zeta_u'|||. \tag{3.35}$$

Combining the estimates (3.34) and (3.35), we obtain

$$\begin{aligned} |\ell_2(w, \zeta_u')| + |\ell_4(w', \zeta_u')| &\leq Ch(\|w\|_1 + \|w'\|_1)|||\zeta_u'||| \\ &\leq Ch(\|w\|_2 + \|w'\|_2)|||\zeta_u'|||. \end{aligned} \tag{3.36}$$

As a consequence of (3.33), we get

$$\int_0^\tau \ell_2(w, \zeta_u')ds + \int_0^\tau \ell_4(w', \zeta_u')ds \leq Ch^{k+3}\|u\|_{H^1(H^{k+3})}\|\zeta_u\|_{L^2(L^2)}. \tag{3.37}$$

Next, we use trace inequality (2.15), and Lemma 2.1 to get the bound for the term $\ell_1(u, \mathcal{Q}_h w')$ as

$$
\begin{aligned}
|l_1(u, \mathcal{Q}_h w')| &= \Big| \sum_{K\in\mathcal{T}_h} \langle (\alpha\nabla u - \mathbb{Q}_h(\alpha\nabla u))\cdot n, \mathcal{Q}_0 w' - \mathcal{Q}_b w'\rangle_{\partial K} \Big| \\
&\le C\Big(\sum_{K\in\mathcal{T}_h} \|\nabla u - \mathbb{Q}_h(\nabla u)\|^2_{\partial K}\Big)^{\frac{1}{2}} \Big(\sum_{K\in\mathcal{T}_h} \|\mathcal{Q}_0 w' - \mathcal{Q}_b w'\|^2_{\partial K}\Big)^{\frac{1}{2}} \\
&\le C\Big(\sum_{K\in\mathcal{T}_h} \|\nabla u - \mathbb{Q}_h(\nabla u)\|^2_K + h_K^2 \|\nabla(\nabla u - \mathbb{Q}_h(\nabla u))\|^2_K\Big)^{\frac{1}{2}} \\
&\quad \times\Big(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|\mathcal{Q}_0 w' - \mathcal{Q}_b w'\|^2_{\partial K}\Big)^{\frac{1}{2}} \\
&\le Ch^{k+2}\|u\|_{k+3}\Big(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|\mathcal{Q}_0 w' - w'\|^2_{\partial K}\Big)^{1/2} \\
&\le Ch^{k+2}\|u\|_{k+3} \\
&\quad \times\Big(\sum_{K\in\mathcal{T}_h} h_K^{-2}\|\mathcal{Q}_0 w' - w'\|^2_K + \|\nabla(\mathcal{Q}_0 w' - w')\|^2_K\Big)^{1/2} \\
&\le Ch^{k+2}\|u\|_{k+3} h\|w'\|_2. \qquad (3.38)
\end{aligned}
$$

Similar arguments yield

$$
|l_3(u', \mathcal{Q}_h w')| \le Ch^{k+3}\|u'\|_{k+3}\|w'\|_2. \qquad (3.39)
$$

Now, for the estimates of remaining terms $\ell_2(u, \mathcal{Q}_h w')$ & $\ell_4(u', \mathcal{Q}_h w')$, we apply the estimates (3.11), and (3.13) which leads us to

$$
\begin{aligned}
|l_2(u, \mathcal{Q}_h w')| + |l_4(u', \mathcal{Q}_h w')| &\le h^{k+3}\|u\|_{k+3} |||\mathcal{Q}_h w'||| \\
&\le h^{k+3}\|u\|_{k+3}\|\zeta_u\|_{L^2(L^2)}. \qquad (3.40)
\end{aligned}
$$

In the last inequality, we have used the estimate (6.19) (see Appendix, Remark 6.1).

Now, apply the estimates (3.38), (3.39), and (3.40) on RHS of (3.29), we get

$$
\begin{aligned}
&\int_0^\tau \ell_1(u, \mathcal{Q}_h w')ds + \int_0^\tau \ell_3(u', \mathcal{Q}_h w')ds + \int_0^\tau \ell_2(u, \mathcal{Q}_h w')ds \\
&\quad + \int_0^\tau \ell_4(u', \mathcal{Q}_h w')ds \le Ch^{k+3}\|u\|_{H^1(H^{k+3})}\big(\|w\|_{H^1(H^2)} + \|\zeta_u\|_{L^2(L^2)}\big) \\
&\qquad \le Ch^{k+3}\|u\|_{H^1(H^{k+3})}\|\zeta_u\|_{L^2(L^2)}. \qquad (3.41)
\end{aligned}
$$

In the last inequality, we used the estimate (3.26).

Combining the estimates (3.29) and (3.41) and then using Young's inequality, we obtain

$$
\frac{1}{2}\|\zeta_u(\tau)\|^2 \le C_\nu\|\zeta_u\|^2_{L^2(L^2)} + C(\nu)h^{2(k+3)}\|u\|^2_{H^1(H^{k+3})}, \qquad (3.42)
$$

for some appropriate $\nu > 0$.

Now, we select τ such that $\|\zeta_u(\tau)\| = \max_{t\in[0,T]} \|\zeta_u(t)\|$ so that $\|\zeta_u\|^2_{L^2(L^2)} \leq T\|\zeta_u(\tau)\|^2$ and subsequently estimate (3.42) reduces to

$$\frac{1}{2}\|\zeta_u(\tau)\|^2 \leq C_\nu T\|\zeta_u(\tau)\|^2 + C(\nu)h^{2(k+3)}\|u\|^2_{H^1(H^{k+3})}.$$

Thus, for suitable $\nu > 0$, we obtain

$$\|\zeta_u(\tau)\|^2 \leq Ch^{2(k+3)}\|u\|^2_{H^1(H^{k+3})}.$$

The proof is completed. □

Remark 3.3 A modification in the dual problem that seeks a solution $w \in H^1(0, T; H^2(\Omega))$ such that

$$-\nabla\cdot\big((\alpha\nabla w) - (\beta\nabla w')\big) = \zeta_u' \text{ in } \Omega,$$

with $w(T) = 0$ leads to the following estimate

$$\begin{aligned}\|\zeta_u'\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \|(\mathcal{Q}_h u - \mathcal{E}_h u)'\|^2 dt \\ &\leq Ch^{2(k+3)}\|u\|^2_{H^1(0,T;H^{k+3}(\Omega))}.\end{aligned} \tag{3.43}$$

We omit the details.

Next, to obtain supercloseness error estimate in the L^2 norm, we split our error $e_h = u_h - \mathcal{Q}_h u$ into two standard components ζ_u and θ using the following relation

$$e_h(t) = u_h(t) - \mathcal{Q}_h u(t) := \theta(t) + \zeta_u(t),$$

where $\theta = u_h - \mathcal{E}_h u$. From Lemma 3.4, we already have a bound for $\zeta_u(t)$. We only need to bound $\theta(t)$.

Next, using the definitions of projection operators $\mathcal{E}_h$ and $\mathcal{Q}_h$, we arrive at the following important identity

$$\begin{aligned}&(\theta'', \phi_0) + \mathcal{A}_{1,w}(\theta, \phi_h) + \mathcal{A}_{2,w}(\theta', \phi_h) \\ &= \left(\frac{\partial^2 u_h}{\partial t^2}, \phi_0\right) + \mathcal{A}_{1,w}(u_h, \phi_h) + \mathcal{A}_{2,w}(u_h', \phi_h) - \left(\frac{\partial^2 \mathcal{E}_h u}{\partial t^2}, \phi_0\right) \\ &\quad - \mathcal{A}_{1,w}(\mathcal{E}_h u, \phi_h) - \mathcal{A}_{2,w}((\mathcal{E}_h u)', \phi_h) \\ &= (f, \phi_0) - \left(\frac{\partial^2 \mathcal{E}_h u}{\partial t^2}, \phi_0\right) - (-\nabla\cdot(\alpha\nabla u), \phi_h) - (-\nabla\cdot(\beta\nabla u'), \phi_h) \\ &= \left(\frac{\partial^2 u}{\partial t^2}, \phi_0\right) - \left(\frac{\partial^2 \mathcal{E}_h u}{\partial t^2}, \phi_0\right) = \left(\frac{\partial^2}{\partial t^2}(\mathcal{Q}_h u - \mathcal{E}_h u)\right., \phi_0) \\ &= \left(\frac{\partial^2}{\partial t^2}(-\zeta_u), \phi_0\right), \ \forall\, \phi_h = \{\phi_0, \phi_b\} \in \mathcal{V}_h^0,\end{aligned}$$

which can be rearranged as

$$\frac{d}{dt}(\theta', \phi_0) - (\theta', \phi_0') + \frac{d}{dt}\mathcal{A}_{2,w}(\theta, \phi_h) - \mathcal{A}_{2,w}(\theta, \phi_h') + \mathcal{A}_{1,w}(\theta, \phi_h)$$
$$= \frac{d}{dt}(-\zeta_u', \phi_0) + (\zeta_u', \phi_0'), \ \forall\, \phi_h = \{\phi_0, \phi_b\} \in \mathcal{V}_h^0, \quad t > 0. \tag{3.44}$$

Now, for $0 < \xi \le T$, we define

$$\hat{\theta}(\cdot, t) := \int_t^{\xi} \theta(\cdot, s) ds, \quad 0 \le t \le T.$$

Then, clearly $\hat{\theta}(\xi) = 0$ and $\frac{\partial \hat{\theta}}{\partial t} = -\theta(\cdot, t)$, $0 \le t \le T$. Now, substituting $\phi_h = \hat{\theta}(t) \in \mathcal{V}_h^0$ in (3.44) and making some rearrangements, we get

$$\frac{1}{2}\frac{d}{dt}(\theta, \theta) + \frac{d}{dt}\mathcal{A}_{2,w}(\theta, \hat{\theta}) + \mathcal{A}_{2,w}(\theta, \theta) - \frac{1}{2}\frac{d}{dt}\mathcal{A}_{1,w}(\hat{\theta}, \hat{\theta})$$
$$= \frac{d}{dt}(-e_h', \hat{\theta}) - (\zeta_u', \theta). \tag{3.45}$$

Integrating (3.45) from 0 to ξ, and using the fact that $\hat{\theta}(\xi) = 0$ and $\theta(0) = 0$, we derive

$$\frac{1}{2}\|\theta(\xi)\|^2 + \frac{1}{2}\mathcal{A}_{1,w}(\hat{\theta}(0), \hat{\theta}(0)) + \sigma_* \int_0^{\xi} |||\theta|||^2 \mathrm{ds} \le (e_h'(0), \hat{\theta}(0)) - \int_0^{\xi} (\zeta_u', \theta)\mathrm{ds}$$

Applied Cauchy-Schwarz inequality, Lemma 3.4 together with the fact that $e_h(0) = 0$, $e_h'(0) = 0$ and $\mathcal{A}_{1,w}(\hat{\theta}(0), \hat{\theta}(0)) > 0$, yields

$$\frac{1}{2}\|\theta(\xi)\|^2 + \sigma_* \int_0^{\xi} |||\theta|||^2 \mathrm{ds} \le \int_0^{\xi} \|\zeta_u'\| \|\theta)\|\mathrm{ds}. \tag{3.46}$$

Since θ is continuous in the time variable, we select ξ such that $\|\theta(\xi)\| = \max_{0 \le t \le T} \|\theta(t)\|$. We observe that $\|\hat{\theta}(0)\| \le \xi\|\theta(\xi)\|$, which together with Young's inequality for some appropriate $\nu > 0$, leads us to

$$\frac{1}{2}\|\theta(\xi)\|^2 + \sigma_* \int_0^{\xi} |||\theta|||^2 \mathrm{ds} \le \max_{0 \le t \le T} \|\theta(t)\| \int_0^{\xi} \|\zeta_u'\| dt$$
$$\le \|\theta(\xi)\| \sqrt{T} \|\zeta_u'\|_{L^2(L^2)}$$
$$\le C_\nu \|\theta(\xi)\|^2 + C(\nu) T \|\zeta_u'\|^2_{L^2(L^2)}.$$

We can restate the above inequality

$$\|\theta(\xi)\|^2_{L^\infty(L^2)} \le C\|\zeta_u'\|^2_{L^2(L^2)} \tag{3.47}$$

Then, from the estimate (3.43), we recall

$$\|\zeta_u'\|_{L^2(L^2)} \le Ch^{k+3}\|u\|_{H^1(0,T;H^{k+3}(\Omega))}. \tag{3.48}$$

Finally, combining the estimates (3.47) and (3.48) together with Lemma 3.4 leads us to the following error estimate for the semidiscrete scheme (3.1).

Theorem 3.3 *Let u and u_h be the solutions of the model problems (1.1) and (1.2) and SFWG scheme (3.1), respectively. Assume that $u \in L^2(0,T;H^{k+3}(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0,T;H^{k+3}(\Omega))$. Then, there is a constant C such that*

$$\|\mathcal{Q}_h u - u_h\| \leq Ch^{k+3}\Big(\|u\|_{L^2(H^{k+3})} + \|\frac{\partial u}{\partial t}\|_{L^2(H^{k+3})}\Big).$$

Remark 3.4 Recall that $\mathcal{Q}_h u|_{K^0} = \mathcal{Q}_0 u$ and $u_h|_{K^0} = u_0$, where K^0 is the interior of the element K. Then, Theorem 3.3 leads to the following estimate

$$\|\mathcal{Q}_0 u - u_0\| \leq Ch^{k+3}\Big(\|u\|_{L^2(H^{k+3})} + \|\frac{\partial u}{\partial t}\|_{L^2(H^{k+3})}\Big).$$

4 Fully discrete scheme

In this section, we describe an implicit second-order Newmark scheme for time discretization that we use to approximate the solution of the model problems (1.1) and (1.2).

We first divide the time interval $J = [0, T]$ into N equally spaced subintervals by the following points

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

with $t_n = n\tau,\ \tau = \frac{T}{N}$ being the time step. Let $I_n = (t_{n-1}, t_n]$ be the nth sub-interval. For a sequence $\{\gamma^n\}_{n=0}^N \subset L^2(\Omega)$, we define

$$\partial_\tau \gamma^n = \frac{\gamma^{n+1} - \gamma^n}{\tau} \quad \text{and} \quad \gamma^{n+\frac{1}{2}} = \frac{1}{2}(\gamma^{n+1} + \gamma^n), \quad n = 0, 1, \ldots, N-1.$$

Also, for a continuous mapping $\psi : [0, T] \to L^2(\Omega)$, we define $\psi^n := \psi(\cdot, t_n)$, $0 \leq n \leq N$. Then, the fully discrete SFWG finite element approximation to the model problems (1.1) and (1.2) is defined as follows: Find $U^n = \{U_0^n, U_b^n\} \in \mathcal{V}_h^0$ such that

$$\partial_\tau U^n = p^{n+\frac{1}{2}} \ \text{ for } \ n = 0,\ 1, 2, \ldots, N-1, \tag{4.1}$$

and

$$(\partial_\tau p^n, \phi_0) + \mathcal{A}_{1,w}(U^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_{2,w}(p^{n+\frac{1}{2}}, \phi_h) = (f^{n+\frac{1}{2}}, \phi_0), \ \forall\, \phi_h \in \mathcal{V}_h^0, \tag{4.2}$$

with $U^0 = \mathcal{Q}_h u^0$ and $p^0 = \mathcal{Q}_h v^0$.

For the well-posedness of the fully discrete schemes (4.1) and (4.2), we have the following result in terms of the auxiliary variable p^n.

Lemma 4.1 *There exists a unique sequence $\{U^n\}_{n=0}^N \subset \mathcal{V}_h^0$ and a corresponding unique sequence $\{p^n\}_{n=0}^N \subset \mathcal{V}_h^0$ satisfying the fully discrete schemes (4.1) and (4.2).*

Proof From (4.1), we have

$$U^{n+1} = \frac{\tau}{2}(p^{n+1} + p^n) + U^n. \tag{4.3}$$

Using (4.3) in (4.2), we get

$$(p^{n+1},\phi_0)+\frac{\tau^2}{4}\mathcal{A}_{1,w}(p^{n+1},\phi_h)+\frac{\tau}{2}\mathcal{A}_{2,w}(p^{n+1},\phi_h)=\mathcal{F}^n(\phi_h),\ \forall\ \phi_h\in\mathcal{V}_h^0, \quad (4.4)$$

where $\mathcal{F}^n$ is the linear functional given by

$$\begin{aligned}\mathcal{F}^n(\phi_h) &= (p^n,\phi_0)-\frac{\tau^2}{4}\mathcal{A}_{1,w}(p^n,\phi_h)-\frac{\tau}{2}\mathcal{A}_{2,w}(p^n,\phi_h)\\ &\quad -\tau\mathcal{A}_{1,w}(U^n,\phi_h)+\tau(f^{n+\frac{1}{2}},\phi_0),\ \forall\ \phi_h=\{\phi_0,\phi_b\}\in\mathcal{V}_h^0. \quad (4.5)\end{aligned}$$

Due to the positivity of bilinear forms $\mathcal{A}_{1,w}(\cdot,\cdot)$ & $\mathcal{A}_{2,w}(\cdot,\cdot)$, there exists uniquely defined $p^{n+1}\in\mathcal{V}_h^0$ satisfying equation (4.4) and subsequently U^{n+1} exists uniquely for $n=0,1,\ldots,N-1$. □

Later on, we will need the following results. The proofs involve using Taylor's series and standard arguments; therefore, details are omitted.

Lemma 4.2 *For any $v\in H^3(J;L^2(\Omega))$, we have*

$$\|\partial_\tau v^n - v_t^{n+\frac{1}{2}}\|^2 \le \frac{\tau^3}{120}\int_{t_n}^{t_{n+1}}\|v_{ttt}\|^2dt. \quad (4.6)$$

In order to compute the error between U^n and $\mathcal{Q}_h u^n$, we first established the error for $\xi^n := u_h^n - U^n$, for $1\le n\le N$. To do so, we have the following result.

Lemma 4.3 *Let u and U^n be the solutions of the model problems (1.1) and (1.2) and the fully discrete schemes (4.1) and (4.2), respectively. Then, we have*

$$\max_{1\le n\le N}\|\xi^n\|^2 \le C\tau^4\Big(\int_0^T\|u_h''''\|^2dt+\int_0^T|||u_h'''|||^2dt\Big).$$

Proof Substitute $t=t_n$ and $t=t_{n+1}$ in (3.1) and then add to have

$$\begin{aligned}(\partial_\tau u_{ht}^n,\phi_0)\ +\ \mathcal{A}_{1,w}(u_h^{n+\frac{1}{2}},\phi_h)+\mathcal{A}_{2,w}(u_{ht}^{n+\frac{1}{2}},\phi_h) &= (f^{n+\frac{1}{2}},\phi_0)\\ +\ (\omega^n,\phi_0),\ \forall\ \phi_h=\{\phi_0,\phi_b\}\in\mathcal{V}_h^0, & \quad (4.7)\end{aligned}$$

where $\omega^n := \partial_\tau u_{ht}^n - u_{htt}^{n+\frac{1}{2}}$.

Now, subtracting (4.2) from (4.7), we have

$$(\partial_\tau\mathcal{P}^n,\phi_0)+\mathcal{A}_{1,w}(\xi^{n+\frac{1}{2}},\phi_h)+\mathcal{A}_{2,w}(\mathcal{P}^{n+\frac{1}{2}},\phi_h)=(\omega^n,\phi_0),\ \forall\ \phi_h\in\mathcal{V}_h^0, \quad (4.8)$$

with $\mathcal{P}^n := u_{ht}^n - p^n$.

From (4.1), it is easy to observe that

$$\partial_\tau\xi^n=\mathcal{P}^{n+\frac{1}{2}}+\partial_\tau u_h^n-u_{ht}^{n+\frac{1}{2}}=\mathcal{P}^{n+\frac{1}{2}}+\alpha^n,\ \alpha^n:=\partial_\tau u_h^n-u_{ht}^{n+\frac{1}{2}}, \quad (4.9)$$

so that

$$\xi^n = \tau \sum_{k=0}^{n-1} \partial_\tau \xi^k = \tau \sum_{k=0}^{n-1} \mathcal{P}^{k+\frac{1}{2}} + \tau \sum_{k=0}^{n-1} \alpha^k \;\; \& \;\; \mathcal{P}^n = \tau \sum_{k=0}^{n-1} \partial_\tau \mathcal{P}^k.$$

Here, we have used the fact that $\xi^0 = u_h^0 - U^0 = \mathcal{Q}_h u^0 - \mathcal{Q}_h u^0 = 0$ and $\mathcal{P}^0 = u_{ht}^0 - p^0 = \mathcal{Q}_h v^0 - \mathcal{Q}_h v^0 = 0$.

Hence, applying the above relations, it follows that

$$\partial_\tau \xi^n = \frac{\tau}{2}\Bigg(\sum_{k=0}^{n} \partial_\tau \mathcal{P}^k + \sum_{k=0}^{n-1} \partial_\tau \mathcal{P}^k \Bigg) + \alpha^n, \tag{4.10}$$

$$\xi^{n+\frac{1}{2}} = \frac{\tau}{2}\Bigg(\sum_{k=0}^{n} \mathcal{P}^{k+\frac{1}{2}} + \sum_{k=0}^{n-1} \mathcal{P}^{k+\frac{1}{2}} \Bigg) + \frac{\tau}{2}\Bigg(\sum_{k=0}^{n} \alpha^k + \sum_{k=0}^{n-1} \alpha^k \Bigg). \tag{4.11}$$

Now, we define a sequence $\{s^n\}_{n=0}^N$ such that $s^0 = 0$ and

$$s^n = \tau \sum_{k=0}^{n-1} \xi^{k+\frac{1}{2}}, \;\; n = 1, 2, \ldots, N-1,$$

so that

$$s^{n+\frac{1}{2}} = \frac{\tau}{2}\Bigg(\sum_{k=0}^{n} \xi^{k+\frac{1}{2}} + \sum_{k=0}^{n-1} \xi^{k+\frac{1}{2}} \Bigg). \tag{4.12}$$

Hence, applying the identities (4.10)-(4.12) leads us to

$$\begin{aligned} &(\partial_\tau \xi^n, \phi_0) + \mathcal{A}_{1,w}(s^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_{2,w}(\xi^{n+\frac{1}{2}}, \phi_h) \\ &= \frac{\tau}{2} \sum_{k=0}^{n} \Big\{ (\partial_\tau \mathcal{P}^k, \phi_0) + \mathcal{A}_{2,w}(\mathcal{P}^{k+\frac{1}{2}}, \phi_h) + \mathcal{A}_{1,w}(\xi^{k+\frac{1}{2}}, \phi_h) \Big\} \\ &+ \frac{\tau}{2} \sum_{k=0}^{n-1} \Big\{ (\partial_\tau \mathcal{P}^k, \phi_0) + \mathcal{A}_{2,w}(\mathcal{P}^{k+\frac{1}{2}}, \phi_h) + \mathcal{A}_{1,w}(\xi^{k+\frac{1}{2}}, \phi_h) \Big\} \\ &+ (\alpha^n, \phi_0) + \frac{\tau}{2} \mathcal{A}_{2,w}\Bigg(\sum_{k=0}^{n} \alpha^k + \sum_{k=0}^{n-1} \alpha^k, \phi_h \Bigg), \end{aligned}$$

for any $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{V}_h^0$.

Using (4.8), for $1 \le n \le N-1$, we derive

$$\begin{aligned} (\partial_\tau \xi^n, \phi_0) + \mathcal{A}_{2,w}(\xi^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_{1,w}(s^{n+\frac{1}{2}}, \phi_h) \\ = (\Re_1^n, \phi_0) + \mathcal{A}_{2,w}(\Re_2^n, \phi_h), \end{aligned} \tag{4.13}$$

where

$$\Re_1^n := \frac{\tau}{2}\omega^n + \tau \sum_{k=0}^{n-1} \omega^k + \alpha^n \;\; \& \;\; \Re_2^n := \frac{\tau}{2}\alpha^n + \tau \sum_{k=0}^{n-1} \alpha^k.$$

Substituting $\phi_h = \xi^{n+\frac{1}{2}} = \partial_\tau s^n$ in (4.13) and making some rearrangements, we arrive at

$$(\xi^{n+1}, \xi^{n+1}) + 2\tau \mathcal{A}_{2,w}(\xi^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}) + \mathcal{A}_{1,w}(s^{n+1}, s^{n+1})$$
$$= (\xi^n, \xi^n) + \mathcal{A}_{1,w}(s^n, s^n) + 2\tau(\mathfrak{R}_1^n, \xi^{n+\frac{1}{2}}) + 2\tau \mathcal{A}_{2,w}(\mathfrak{R}_2^n, \xi^{n+\frac{1}{2}}).$$

Next, using Cauchy-Schwarz inequality, together with coercivity (2.12), we obtain

$$(\xi^{n+1}, \xi^{n+1}) + 2\tau \|\xi^{n+\frac{1}{2}}\|^2 + 2\tau \left|\left|\left|\xi^{n+\frac{1}{2}}\right|\right|\right|^2 + \mathcal{A}_{1,w}(s^{n+1}, s^{n+1})$$
$$\leq (\xi^n, \xi^n) + \mathcal{A}_{1,w}(s^n, s^n) + 2\tau \|\mathfrak{R}_1^n\| \|\xi^{n+\frac{1}{2}}\| + 2\tau \|\mathfrak{R}_2^n\| \|\xi^{n+\frac{1}{2}}\|$$
$$+2\tau |||\mathfrak{R}_2^n||| \left|\left|\left|\xi^{n+\frac{1}{2}}\right|\right|\right|.$$

Now, applying Young's inequality by appropriately selecting $\nu > 0$ in the above relation leads us to

$$(\xi^{n+1}, \xi^{n+1}) + \mathcal{A}_{1,w}(s^{n+1}, s^{n+1}) \leq (\xi^n, \xi^n) + \mathcal{A}_{1,w}(s^n, s^n)$$
$$+ C(\nu)\tau\Big(\|\mathfrak{R}_1^n\|^2 + \|\mathfrak{R}_2^n\|^2 + |||\mathfrak{R}_2^n|||^2\Big). \tag{4.14}$$

Summing (4.14) from $n = 1$ to $n = l - 1$ with $2 \leq l \leq N$, we obtain

$$\max_{2\leq n\leq N} \|\xi^n\|^2 \leq \|\xi^1\|^2 + \left|\left|\left|s^1\right|\right|\right|^2 + C(\nu)\tau \sum_{n=0}^{l-1} \left(\|\mathfrak{R}_1^n\|^2 + |||\mathfrak{R}_2^n|||^2\right). \tag{4.15}$$

For estimation of the terms ξ^1 and s^1, we note that

$$s^1 = \tau\xi^{\frac{1}{2}} = \frac{\tau}{2}\xi^1 \;\;\&\;\; \mathcal{P}^{\frac{1}{2}} = \frac{\mathcal{P}^1}{2} = \frac{\xi^1}{\tau} - \alpha^0.$$

Now, putting $n = 0$ in the error equation (4.8) and using the above identities, we have

$$\frac{2}{\tau^2}(\xi^1, \phi_0) + \frac{1}{\tau}\mathcal{A}_{2,w}(\xi^1, \phi_h) + \frac{1}{\tau}\mathcal{A}_{1,w}(s^1, \phi_h)$$
$$= (\omega^0, \phi_0) + \frac{2}{\tau}(\alpha^0, \phi_0) + \mathcal{A}_{2,w}(\alpha^0, \phi_h), \;\; \forall\, \phi_h \in \mathcal{V}_h^0. \tag{4.16}$$

Substituting $\phi_h = \xi^1 = \frac{2}{\tau}s_1$ in (4.16) together with coercivity (2.12), we obtain

$$\|\xi^1\|^2 + \left|\left|\left|s^1\right|\right|\right|^2 \leq \frac{\tau^2}{2}(\omega^0, \xi^1) + \tau(\alpha^0, \xi^1) + \tau\mathcal{A}_{2,w}(\alpha^0, s^1).$$

Next, use Cauchy-Schwarz and Young's inequalities to have

$$\|\xi^1\|^2 + \left|\left|\left|s^1\right|\right|\right|^2 \leq \frac{\tau^4}{4}\|\omega^0\|^2 + C_\nu\|\xi^1\|^2 + \left(\tau^2 + \frac{\tau^4}{4}\right)\|\alpha^0\|^2$$
$$+C_\nu\|\xi^1\|^2 + \tau^2\left|\left|\left|\alpha^0\right|\right|\right|^2 + C(\nu)\left|\left|\left|s^1\right|\right|\right|^2.$$

Now, selecting $\nu > 0$ appropriately leads us to

$$\|\xi^1\|^2 + \left|\left|\left|s^1\right|\right|\right|^2 \leq C\left(\tau^4\|\omega^0\|^2 + \tau^2\left|\left|\left|\alpha^0\right|\right|\right|^2\right). \tag{4.17}$$

Combining (4.15) and (4.17), we have

$$\max_{1\le n\le N}\|\xi^n\|^2 \le C\Big(\tau^4\|\omega^0\|^2+\tau^2\Big|\Big|\Big|\alpha^0\Big|\Big|\Big|^2+\tau\sum_{n=0}^{l-1}\big(\|\Re_1^n\|^2+|||\Re_2^n|||^2\big)\Big). \tag{4.18}$$

Now, we shall estimate both terms $\Re_1^n$ and $\Re_2^n$. For the estimation of $\Re_1^n$, use triangle inequality and Cauchy-Schwarz inequality to have

$$\begin{aligned}\|\Re_1^n\|^2 &\le C\Big(\frac{\tau^2}{4}\|\omega^n\|^2+\tau^2\|\sum_{k=0}^{n-1}\omega^k\|^2+\|\alpha^n\|^2\Big)\\ &\le C\Big(\frac{\tau^2}{4}\|\omega^n\|^2+\tau^2 N\sum_{k=0}^{n-1}\|\omega^k\|^2+\|\alpha^n\|^2\Big)\\ &\le C\Big(\frac{\tau^2}{4}\|\omega^n\|^2+\tau\sum_{k=0}^{n-1}\|\omega^k\|^2+\|\alpha^n\|^2\Big).\end{aligned}$$

Then, using Lemma 4.2, we obtain

$$\begin{aligned}\|\Re_1^n\|^2 \le C\Big(&\tau^5\int_{t_n}^{t_{n+1}}\|u_h''''\|^2dt+\tau^4\int_0^T\|u_h''''\|^2dt\\ &+\tau^3\int_{t_n}^{t_{n+1}}\|u_h'''\|^2dt\Big).\end{aligned} \tag{4.19}$$

The following estimate for $\Re_2^n$ is achieved using the same technique as used for deriving $\Re_1^n$

$$|||\Re_2^n|||^2 \le C\Big(\tau^5\int_{t_n}^{t_{n+1}}|||u_h'''|||^2dt+\tau^4\int_0^T|||u_h'''|||^2dt\Big). \tag{4.20}$$

Finally, using (4.19) and (4.20) in (4.18), we obtain

$$\max_{1\le n\le N}\|\xi^n\|^2 \le C\tau^4\Big(\int_0^T\|u_h''''\|^2dt+\int_0^T|||u_h'''|||^2dt\Big).$$

□

The following Theorem states the optimal $L^\infty(L^2)$ error result.

Theorem 4.1 *Suppose that the model problems (1.1) and (1.2) and the fully discrete schemes (4.1) and (4.2) has a unique solution u and U^n, respectively. Assume that the initial data $u^0\in H^6(\Omega)\cap H_0^1(\Omega)$ and $v^0\in H^6(\Omega)\cap H_0^1(\Omega)$. Furthermore, suppose that $u\in H^1(0,T;H^{k+3}(\Omega))$, then, we have*

$$\max_{0\le n\le N}\|\mathcal{Q}_h u^n-U^n\| \le C(u)\big(h^{k+3}+\tau^2\big),$$

where $C(u) := C\Big\{\|u^0\|^2_{H^6(\Omega)}+\|v^0\|^2_{H^6(\Omega)}+\|f\|^2_{H^3(H^2(\Omega))}+\|u\|^2_{H^1(H^{k+3}(\Omega))}\Big\}^{\frac{1}{2}}.$

Proof We split the error $\mathcal{Q}_h u^n - U^n$ as

$$\mathcal{Q}_h u^n - U^n = \mathcal{Q}_h u^n - u_h^n + u_h^n - U^n.$$

Applying the triangle inequality in the above relation, we get

$$\|\mathcal{Q}_h u^n - U^n\| \le \|\mathcal{Q}_h u^n - u_h^n\| + \|u_h^n - U^n\|.$$

Using Theorem 3.3, Lemmas 3.1, and 4.3 in the above inequality gives the desired result. □

Remark 4.1 For the fully discrete solution $U^n = \{U_0^n, U_b^n\}$, we observe that

$$\|u^n - U^n\|_K = \|u^n - U_0^n\|_K \le \|u^n - \mathcal{Q}_0 u^n\|_K + \|\mathcal{Q}_0 u^n - U_0^n\|_K, \quad K \in \mathcal{T}_h.$$

Now, using the standard approximation property for the L^2 projection $\mathcal{Q}_0 : L^2(K) \to \mathcal{P}_k(K),\ K \in \mathcal{T}_h$, (see Lemma 4.1 in [52]), we obtain

$$\sum_{K \in \mathcal{T}_h} \left(\|u^n - \mathcal{Q}_0 u^n\|_K^2 + h^2 \|\nabla(u^n - \mathcal{Q}_0 u^n)\|_K^2 \right) \le C h^{2(k+1)} \|u^n\|_{H^{k+1}(\Omega)}^2.$$

Therefore, for the fully discrete schemes (4.1) and (4.2), we obtain

$$\|u^n - U^n\| \le \mathcal{O}(h^{k+1} + \tau^2), \tag{4.21}$$

with weak Galerkin space $(\mathcal{P}_k(K), \mathcal{P}_{k+1}(\partial K), [\mathcal{P}_{k+1}(K)]^2)$. For the possible extension of the estimate (4.21), we refer to achieving superconvergence by one-dimensional discontinuous finite elements [58]. We are working on improving the estimate (4.21) by bridging the HHO and the weak Galerkin methods.

5 Numerical section

In this section, we shall illustrate various types of numerical examples to validate the theoretical convergence results for the second-order viscoelastic wave equations (1.1) and (1.2) in $\Omega \times J$, where $\Omega \subset \mathbb{R}^2$ and $J = [0, 1]$. The model problems (1.1) and (1.2) are solved on finite element partitions of various configurations, including triangular and rectangular meshes, to demonstrate the flexibility and efficiency of the SFWG algorithm.

Let U^n be the SFWG solution defined by (4.1) and (4.2). We calculated the following error to illustrate the convergence history of the SFWG approach in terms of discretization error:

$$e_h := \mathcal{Q}_h u(x, t_n) - U^n$$

at final time $t_n = T$ with respect to the triple-bar norm and L^2 norm. More precisely, the errors are reported with respect to the triple-bar norm and L^2 norm through tables for the SFWG space

$$(\mathcal{P}_k(K), \mathcal{P}_{k+1}(\partial K), [\mathcal{P}_{k+1}(K)]^2)$$

with a time step $\tau = O(h^{k+1})$.

Let e_i be the error corresponding to the $L^\infty(H^1)$ norm or $L^\infty(L^2)$ norm on the ith iteration for a given finite number of successive iterations (indexed by i), and h_i

be the corresponding mesh size. The expected order of convergence (EOC) is then defined as

$$EOC(e_i) = \log\left(\frac{e_{i+1}}{e_i}\right) / \log\left(\frac{h_{i+1}}{h_i}\right).$$

Example 5.1 (Smooth solution with smooth coefficient) In the first numerical example, we consider the model problems (1.1) and (1.2) in $\Omega \times J$, where $\Omega = (0, 1) \times (0, 1)$. Numerical solution is compared with the following exact solution

$$u = t^2 \exp(-t) \sin(\pi x) \sin(\pi y),$$

with coefficients selected as $\beta = I = \alpha$

The choice for u determines the source term f, initial data, and boundary data. In this case, we have uniformly partitioned the domain into $n \times n$ sub rectangles, then divided each rectangular element by a diagonal line with a negative slope, with the mesh size set to $h = 1/n$. Figure 1 depicts the initial mesh and its refinement. The $L^\infty(H^1)$ norm and $L^\infty(L^2)$ norm errors for linear, quadratic, and cubic WG spaces at final time $T = 1$ are reported in Table 1, demonstrating that the rate of convergence $\mathcal{O}(h^{k+2})$ in the triple-bar norm and $\mathcal{O}(h^{k+3})$ in the L^2 norm. Table 2 demonstrating the rate of convergence in time direction, which shown the second order accuracy.

Example 5.2 (Smooth solution with smooth coefficient) In this example, we consider the model problems (1.1) and (1.2) in $\Omega \times J$, where $\Omega = (0, 1) \times (0, 1)$. Numerical solution is compared with the following exact solution

$$u(x, y, t) = \frac{t^2}{4} \exp(-t) \sin(\pi x) \sin(\pi y).$$

I took the following data from the true solution $u(x, y, t)$, which has the coefficients $\alpha = I = \beta$. As illustrated in Fig. 2, we have uniformly partitioned the domain into $n \times n$ sub rectangles with a mesh size of $h = 1/n$. Table 3 show the sequence of convergence for the WG spaces $k = 1, k = 2$, and $k = 3$. It demonstrated that we had obtained the optimal order of accuracy in the triple-bar and L^2 norms.

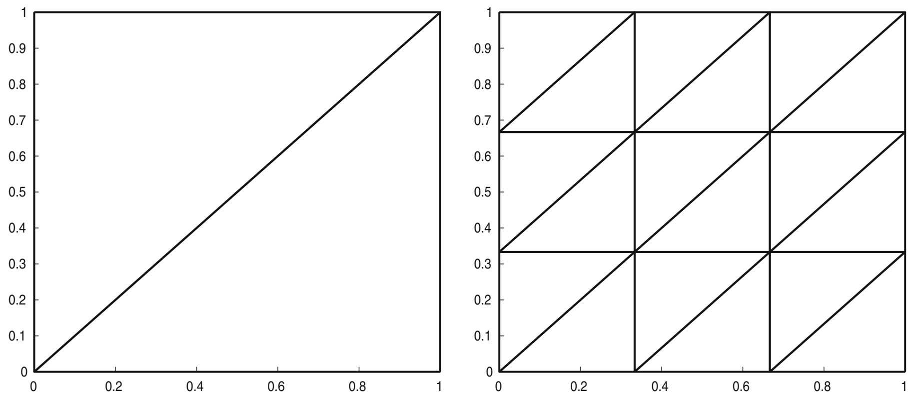

Fig. 1 An initial triangular mesh for $h = 1/2$ (left), and its refinement for $h = 1/8$ (right) in Example 5.1

Table 1 The history of convergence with $\tau = h^{k+1}$ in Example 5.1

$k = 1$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	1/2	6.468691e-03	-	3.724247e-02	-
	1/4	4.635769e-04	3.802593e+00	4.352819e-03	3.096927e+00
	1/8	3.009466e-05	3.945229e+00	5.362389e-04	3.021002e+00
	1/16	1.900158e-06	3.985316e+00	6.701159e-05	3.000393e+00
	1/32	1.190808e-07	3.996106e+00	8.389944e-06	2.997678e+00
	1/64	7.355213e-09	4.017029e+00	1.022743e-06	3.036217e+00
$k = 2$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	1/2	9.129500e-04	-	6.101349e-03	-
	1/4	1.644990e-05	5.794385e+00	3.185192e-04	4.259676e+00
	1/8	3.447039e-07	5.576578e+00	1.974468e-05	4.011844e+00
	1/16	8.922583e-09	5.271752e+00	1.236310e-06	3.997352e+00
	1/32	2.623241e-10	5.088039e+00	7.737803e-08	3.997973e+00
	1/64	7.520471e-12	5.124383e+00	4.706211e-09	4.039286e+00
$k = 3$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	1/2	1.856813e-04	-	1.208604e-03	-
	1/4	8.503071e-07	7.770629e+00	3.003950e-05	5.330337e+00
	1/8	7.365781e-09	6.851002e+00	9.534857e-07	4.977506e+00
	1/16	1.047106e-10	6.136359e+00	2.999974e-08	4.990190e+00
	1/32	1.526153e-12	6.100364e+00	9.739272e-10	4.944992e+00

Example 5.3 (Smooth solution with discontinuous coefficients) In this example, we will describe the SFWG algorithm for discontinuous coefficients, which is concerned with the higher order of convergence. To do so, we modify this example from [24]. We consider the following IBVP

$$\gamma u_{tt} - \alpha \Delta u - \beta \Delta u_t = f(x, t) \ \text{ in } \Omega \times J, \tag{5.1}$$

where $\Omega = (-1, 1) \times (-1, 1)$ and $J = [0, 1]$.

The exact solution $u(x, t)$ is the same as in Example 5.2. For this case, finite element partitioning is referred to as triangulation. Then, to emphasize the significance

Table 2 The history of convergence in time in Example 5.1

$k = 1$	τ	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	1/2	1.497415e-02	0	7.115828e-02	0
	1/4	3.134150e-03	2.256328e+00	1.445307e-02	2.299656e+00
	1/8	7.298239e-04	2.102454e+00	3.284000e-03	2.137849e+00
	1/16	1.788235e-04	2.029012e+00	7.972708e-04	2.042312e+00
	1/32	4.447484e-05	2.007475e+00	1.977739e-04	2.011218e+00
	1/64	1.110421e-05	2.001883e+00	4.934586e-05	2.002851e+00

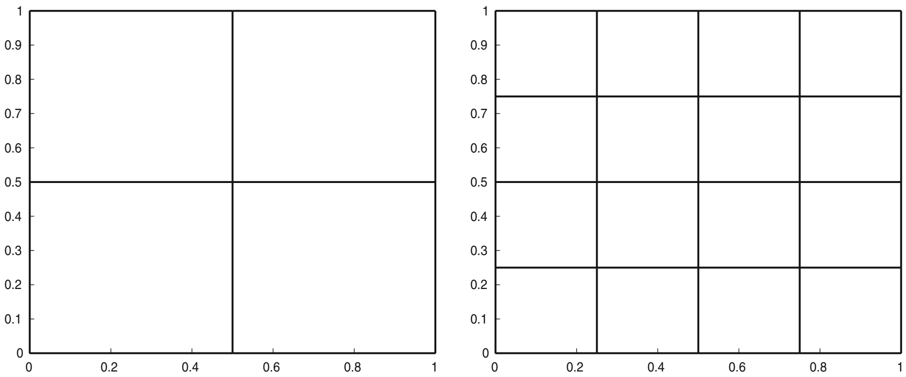

Fig. 2 An initial rectangular mesh for $h = 1/2$ (left), and its refinement $h = 1/4$ (right) in Example 5.2

Table 3 The history of convergence in Example 5.2

$k = 1$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	$1/2$	2.356697e-03	-	1.069616e-02	-
	$1/4$	1.776841e-04	3.729380e+00	1.538992e-03	2.797035e+00
	$1/8$	1.155518e-05	3.942703e+00	1.981903e-04	2.957027e+00
	$1/16$	7.290864e-07	3.986306e+00	2.495079e-05	2.989729e+00
	$1/32$	4.567492e-08	3.996616e+00	3.124340e-06	2.997462e+00
	$1/64$	2.683221e-09	4.089364e+00	3.750312e-07	3.058468e+00
$k = 2$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	$1/2$	3.144880e-04	-	1.577914e-03	-
	$1/4$	5.395449e-06	5.865118e+00	6.104106e-05	4.692094e+00
	$1/8$	8.788066e-08	5.940053e+00	2.265242e-06	4.752043e+00
	$1/16$	1.490727e-09	5.881458e+00	8.683974e-08	4.705166e+00
	$1/32$	2.917810e-11	5.674987e+00	3.732893e-09	4.539989e+00
	$1/64$	6.430722e-13	5.503761e+00	1.740321e-10	4.422868e+00
$k = 3$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	$1/2$	5.087126e-05	-	2.642776e-04	-
	$1/4$	2.530311e-07	7.651392e+00	3.181595e-06	6.3762e+00
	$1/8$	2.116147e-09	6.90173e+00	5.905929e-08	5.7514e+00
	$1/16$	2.780496e-11	6.25007e+00	1.597479e-09	5.2083e+00
	$1/32$	3.917809e-13	6.14926e+00	4.732893e-11	5.0769e+00

Table 4 The history of convergence in Example 5.3

$k = 1$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	1/2	7.805155e-04	-	1.209341e-04	-
	1/4	5.126943e-05	3.928256e+00	1.703776e-05	2.827413e+00
	1/8	3.214451e-06	3.995454e+00	2.305186e-06	2.885781e+00
	1/16	2.009431e-07	3.999714e+00	2.981483e-07	2.950781e+00
	1/32	1.255910e-08	3.999982e+00	3.782245e-08	2.978715e+00
	1/64	7.820511e-10	4.005326e+00	4.708344e-09	3.005951e+00
[6pt] $k = 2$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	1/2	2.035232e-04	-	2.396299e-05	-
	1/4	3.214578e-06	5.984420e+00	9.834507e-07	4.606811e+00
	1/8	5.023629e-08	5.999755e+00	6.174704e-08	3.993410e+00
	1/16	7.849282e-10	6.000025e+00	3.956745e-09	3.963983e+00
	1/32	1.502711e-11	5.706921e+00	2.382245e-10	4.053920e+00
	1/64	4.563261e-13	5.041358e+00	1.564672e-11	3.928389e+00

of our model problem, we choose the data appearing in the above problem from the u with the following physical coefficients, which we select from [47].

$$(\gamma, \beta, \alpha) = \left(\gamma, \alpha_e, C_E^2\right)$$
$$= \begin{cases} \left(1, 1.2 \times 10^{-4}, 1.44 \times 10^8\right) & \text{if } x^2 + y^2 \leq 1/4, \\ \left(1, 1.6 \times 10^{-4}, 1.96 \times 10^8\right) & \text{if } x^2 + y^2 > 1/4. \end{cases}$$

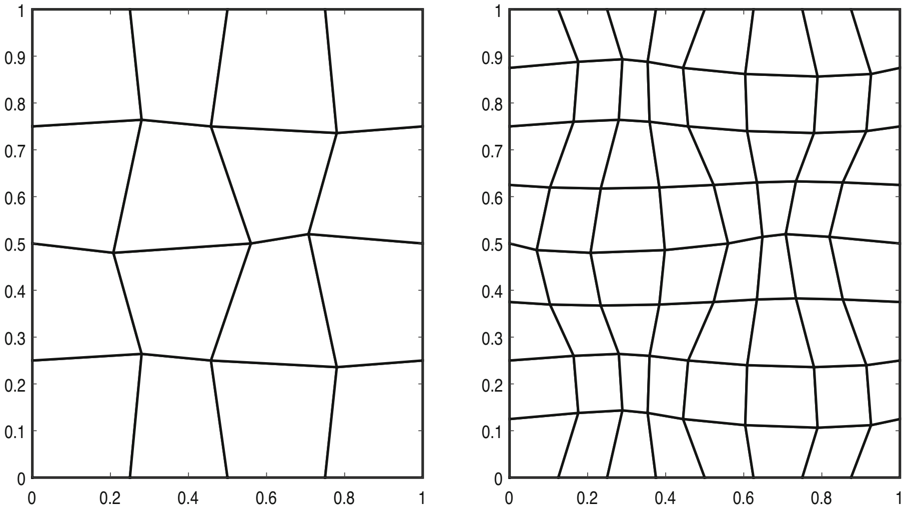

Fig. 3 An initial quadrilateral mesh (left), and its refinement (right)

Table 5 The history of convergence in Example 5.4

$k = 1$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	1/2	7.195140e-02	–	1.389247e-01	–
	1/4	4.254137e-03	4.080084e+00	1.665745e-02	3.060063e+00
	1/8	2.664639e-04	3.996854e+00	2.085589e-03	2.997641e+00
	1/16	1.655666e-05	4.008456e+00	2.595551e-04	3.006342e+00
	1/32	1.032863e-06	4.002692e+00	3.239902e-05	3.002019e+00
	1/64	6.452204e-08	4.000712e+00	4.048378e-06	3.000534e+00
$k = 2$	h	$\|e_h\|$	EOC	$\|\|\|e_h\|\|\|$	EOC
	1/2	8.207043e-04	–	4.848864e-03	–
	1/4	2.010429e-04	2.029359e+00	1.688366e-03	1.522019e+00
	1/8	7.028182e-06	4.838208e+00	1.364999e-04	3.628656e+00
	1/16	2.251476e-07	4.964209e+00	1.033595e-05	3.723156e+00
	1/32	7.077973e-09	4.991391e+00	7.716701e-07	3.743543e+00
	1/64	2.214920e-10	4.998010e+00	5.741410e-08	3.748507e+00

The Dual Phase-Lag (DPL) bioheat transport model on multi-layered material describes the above mentioned problem. Because the thermal properties of biological media differ between layers, heterogeneity in the underlying media is natural. The DPL model is widely used to investigate heat transport in metallic films during ultra-fast laser heating [40, 48]. C_E represents the equivalent thermal wave speed, and the electron thermal diffusivity of the material is represented by α_e. Our numerical results are based on uniform triangular meshes with $k = 1$, and $k = 2$ at final time $T = 1$. We have obtained optimal order of convergence in both L^2 and H^1 norms, as shown in Table 4, which confirms our theoretical conclusions in Theorem 4.1.

Example 5.4 (Smooth solution on quadrilateral mesh) In this example, we will glance at the SFWG algorithm on a quadrilateral mesh, emphasizing a higher order of convergence. We solve the model problems (1.1) and (1.2) in $\Omega = (0, 1)^2$ with the exact solution

$$u(x, y, t) = t^2 \exp(-t) \sin(2\pi x) \sin(2\pi y).$$

The initial mesh is depicted in Fig. 3 (Left). Figure 3 (right) show the mesh generated by the uniform refinement procedure. Table 5 shows the errors with respect to the triple-bar norm and the $L^\infty(L^2)$ norm for $k = 1$ and $k = 2$ SFWG spaces at final time $T = 1$.

6 Conclusion

In this study, we have described SFWG finite element method for a second-order linear viscoelastic wave equation with variable coefficients. We have applied the SFWG method for space discretization and the implicit second-order Newmark scheme for time discretization and obtained superconvergence in $L^\infty(H^1)$ norm and

$L^\infty(L^2)$ norm, that is, the two orders higher than optimal order. Numerous numerical examples are illustrated using various degrees of polynomials in the SFWG spaces.

The extension of this theory with a high order of accuracy is being considered as future work for the Westervelt's quasi-linear acoustic wave equation

$$c^{-2}u'' - \nabla \cdot \big(\alpha(x)\nabla u(x,t) + \beta(x)\nabla u'\big) = \gamma(u^2)'' \ \text{ in } (0,T] \times \Omega, \tag{6.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded and convex domain. The equation (6.1) is widely used to simulate high-intensity focused ultrasound fields generated by medical ultrasound transducers.

Appendix

Proof of Lemma 3.1. Differentiating (3.1) twice with respect to time t and substitute $\phi_h = u_h'''(t)$, we have

$$(u_h'''', u_h''') + \mathcal{A}_{1,w}(u_h'', u_h''') + \mathcal{A}_{2,w}(u_h''', u_h''') = (f'', u_h''').$$

We can restate the above equation as

$$\frac{1}{2}\frac{d}{dt}\Big(\|u_h'''\|^2 + \mathcal{A}_{1,w}(u_h'', u_h'')\Big) + \mathcal{A}_{2,w}(u''', u''') = (f'', u''').$$

Now, integrate the above equation with respect to time from 0 to t and apply the Cauchy-Schwarz inequality; we get

$$\begin{aligned}
&\frac{1}{2}\|u_h'''\|^2 + \frac{1}{2}|||u_h''|||^2 + \int_0^t \|u_h'''\|^2 ds + \int_0^t |||u_h'''|||^2 ds \\
&\quad \le \frac{1}{2}\|u_h'''(0)\|^2 + \frac{1}{2}\,\big|\big|\big|u_h''(0)\big|\big|\big|^2 + \frac{1}{2}\Big(\int_0^t \|u_h'''(s)\|^2 ds + \int_0^t \|f''\|^2 ds\Big).
\end{aligned}$$

We can rearrange the above equation as

$$\begin{aligned}
\int_0^t \|u_h'''(s)\|^2 ds + \int_0^t |||u_h'''(s)|||^2 ds \le C\Big(&\|u_h'''(0)\|^2 + \big|\big|\big|u_h'''(0)\big|\big|\big|^2 \\
&+ \int_0^t \|f''(s)\|^2 ds\Big).
\end{aligned} \tag{6.2}$$

Now, we need to bound $\|u_h'''(0)\|^2$ and $|||u_h'''(0)|||^2$ in (6.2). To this end, taking $t \to 0^+$ in (1.1), it follow that for $0 \le \lambda \le k$,

$$\|u''(0)\|_\lambda \le C\Big(\|u^0\|_{\lambda+2} + \|u'(0)\|_{\lambda+2} + \|f\|_{H^1(H^\lambda)}\Big) \tag{6.3}$$

and

$$\|u'''(0)\|_\lambda \le C\Big(\|u^0\|_{\lambda+4} + \|u'(0)\|_{\lambda+4} + \|f\|_{H^2(H^\lambda)}\Big) \tag{6.4}$$

Next, we differentiate (3.1) with respect to time t and using the definition of $\mathcal{E}_h$ operator (3.18). Then, setting $t \to 0^+$ to have

$$\begin{aligned}(u_h'''(0), \phi_0) &= -\mathcal{A}_{1,w}(u_h'(0), \phi_h) - \mathcal{A}_{2,w}(u_h''(0), \phi_h) + (f'(0), \phi_0)\\ &= -\mathcal{A}_{1,w}(\mathcal{E}_h u'(0), \phi_h) - \mathcal{A}_{2,w}(\mathcal{E}_h u''(0), \phi_h) + (f'(0), \phi_0)\\ &= (\nabla \cdot (\alpha \nabla u'(0), \phi_h)) + (\nabla \cdot (\beta \nabla u''(0)), \phi_h) + (f'(0), \phi_0)\end{aligned}$$

Now, applying the Cauchy-Schwarz inequality together with the estimate (6.3) in the above equation with $\lambda = 2$, we obtained

$$\|u_h'''(0)\| \leq C\Big(\|u^0\|_{H^4(\Omega)} + \|v^0\|_{H^4(\Omega)} + \|f\|_{H^2(J;H^2(\Omega))}\Big). \tag{6.5}$$

In the previous estimate, we have used the fact that (cf. [43], Proposition 7.1)

$$\sup_{0\leq t\leq T} \|v(t)\|_{\mathcal{B}} \leq C(T)\|v\|_{H^1(J;\mathcal{B})} \; \forall v \in H^1(J; \mathcal{B}), \tag{6.6}$$

for any Banach space $\mathcal{B}$.

As a consequence of estimate (6.5) together with standard inverse inequality, estimate (6.4) with $\lambda = 2$, and the fact that $\|u\|_{L^2(K)} \leq Ch\|u\|_{2,K}$, we obtain

$$|||u_h'''(0)||| \leq Ch^{-1}\|u_h'''(0)\| \leq C\Big(\|u^0\|_{H^6(\Omega)} + \|v^0\|_{H^6(\Omega)} + \|f\|_{H^2(J;H^2)}\Big). \tag{6.7}$$

Again, we are differentiating (3.1) thrice with respect to time t and substitute $\phi_h = u_h''''(t)$, we get

$$(u_h''''', u_h'''') + \mathcal{A}_{1,w}(u_h''', u_h'''') + \mathcal{A}_{2,w}(u_h'''', u_h'''') = (f''', u_h'''').$$

Then, it follows from (6.2) that

$$\begin{aligned}\int_0^t \|u_h''''(s)\|^2 ds + \int_0^t |||u_h''''(s)|||^2 ds \leq C\Big(\|u_h''''(0)\|^2 + |||u_h'''(0)|||^2\\ + \int_0^t \|f'''(s)\|^2 ds\Big).\end{aligned} \tag{6.8}$$

Here, the term $|||u_h'''(0)|||$ can be bound using the estimate (6.7). To the bound $\|u_h''''(0)\|$ and in (6.8), we follow the step from (6.3)–(6.7), and we get

$$\|u_h''''(0)\| \leq C\Big(\|u^0\|_{H^6(\Omega)} + \|v^0\|_{H^6(\Omega)} + \|f\|_{H^3(J;H^2(\Omega))}\Big).$$ □

Lemma 6.1 *Let $w \in H^1(0, T; H^2(\Omega))$ be the solutions of the (3.19) and w_h be its SFWG approximation. Then, there exists a constant C such that*

$$|||\mathcal{Q}_h w - w_h||| \leq Ch\|\zeta_u\|_{L^2(J;L^2(\Omega))}, \tag{6.9}$$

Proof The following analysis used to derive (3.9), we obtain

$$\begin{aligned}\mathcal{A}_{1,w}(\mathcal{Q}_h w, \phi_h) \; + \; \mathcal{A}_{2,w}((\mathcal{Q}_h w)_t, \phi_h) = (f_w, \phi_0) + \ell_1(w, \phi_h) + \ell_2(w, \phi_h)\\ + \; \ell_3(w', \phi_h) + \ell_4(w', \phi_h), \quad \forall\, \phi_h = \{\phi_0, \phi_b\} \in \mathcal{V}_h^0.\end{aligned} \tag{6.10}$$

Next, we may define $w_h \in \mathcal{V}_h^0$ as the solution to the SFWG approximation of the equation (3.19) that follows

$$\mathcal{A}_{1,w}(w_h, \varphi_h) + \mathcal{A}_{2,w}(w_h', \varphi_h) = (f_w, \varphi_0), \quad \forall\, \varphi_h = \{\varphi_0, \varphi_b\} \in \mathcal{V}_h^0, \tag{6.11}$$

with $w_h(\tau) = \mathcal{Q}_h w^0$.

Now, subtracting (6.11) from the equation (6.10), we arrive at the following error relation for $\tilde{e}_h := \mathcal{Q}_h w - w_h$

$$\mathcal{A}_{1,w}(\tilde{e}_h(t), \phi_h) + \mathcal{A}_{2,w}(\tilde{e}_h'(t), \phi_h) = \ell_1(w, \phi_h) + \ell_2(w, \phi_h) + \ell_3(w', \phi_h) \\ + \ell_4(w', \phi_h), \ \forall\, \varphi_h \in \mathcal{V}_h^0, \ t \in (0, T]. \tag{6.12}$$

Finally, putting $\phi_h = \tilde{e}_h$ in (6.12) and then standard analysis as we did in Theorem 3.2 combined with the estimations (3.32) and (3.36) yields the following estimate

$$\begin{aligned} |||\tilde{e}_h(t)|||^2 &\leq C\big(|||\tilde{e}_h(0)|||^2 + h^2\|w\|^2_{H^1(0,T;H^2(\Omega))}\big) \\ &\leq Ch^2\|w\|^2_{H^1(0,T;H^2(\Omega))} \\ &\leq Ch^2\|\zeta_u\|^2_{L^2(J;L^2(\Omega))}. \end{aligned}$$

Here, we have used the estimate (3.26) together with the fact that $\tilde{e}_h(0) = 0$. The proof is completed. □

Remark 6.1 We recall a dual problem that seeks a solution $w \in H^1(J; H^2(\Omega))$ such that

$$-\nabla\cdot\big((\alpha\nabla w) - (\beta\nabla w')\big) = \zeta_u \quad \text{in} \quad \Omega \times J, \tag{6.13}$$

and $w(\tau) = 0$ for some $\tau \in J$.

We may define $w_h \in \mathcal{V}_h^0$ as the solution to the discrete problem of the equation (6.13) that follows

$$\mathcal{A}_{1,w}(w_h, \varphi_h) - \mathcal{A}_{2,w}(w_h', \varphi_h) = (\zeta_u, \varphi_0), \quad \forall\, \varphi_h = \{\varphi_0, \varphi_b\} \in \mathcal{V}_h^0, \tag{6.14}$$

with $w_h(\tau) = 0$.

Setting $\varphi_h = w_h$ in (6.14) and using the coercive property (2.12), we obtain

$$\sigma_*|||w_h|||^2 - \frac{1}{2}\frac{d}{dt}\big(\mathcal{A}_{2,w}(w_h(s), w_h(s))\big) \leq \|\zeta_u\|\|w_0\|.$$

Next, integrate the above equation in $[0, \tau]$ to obtain

$$\sigma_*|||w_h|||^2 + \frac{1}{2}\mathcal{A}_{2,w}(w_h(0), w_h(0))\big) \leq \|\zeta_u\|\|w_0\|.$$

Here, we used the fact that $w_h(\tau) = 0$ and hence, $\mathcal{A}_{2,w}(w_h(\tau), w_h(\tau)) = 0$.

Now, we apply the Poincaŕe-type inequality (2.16) and positive definiteness of $\mathcal{A}_{2,w}(\cdot,\cdot)$ in the above estimate, we get

$$|||w_h||| \leq C\|\zeta_u\|. \tag{6.15}$$

When we set $\varphi_h = w_h'$ in (6.14), we can get

$$\big|||w_h'\big||| \leq C\|\zeta_u\|. \tag{6.16}$$

The following estimates are satisfied by w_h, which is the SFWG approximation to w (see estimate (6.9))

$$|||\mathcal{Q}_h w - w_h||| \leq Ch\|\zeta_u\|_{L^2(L^2)}. \tag{6.17}$$

Now, we combine estimates (6.15) and (6.17) to obtain

$$\begin{aligned}|||\mathcal{Q}_h w||| = |||\mathcal{Q}_h w - w_h + w_h||| &\leq |||\mathcal{Q}_h w - w_h||| + |||w_h||| \\ &\leq C\|\zeta_u\|_{L^2(L^2)}.\end{aligned} \tag{6.18}$$

As a consequence, we can prove that

$$\left|\left|\left|\mathcal{Q}_h w'\right|\right|\right| = \left|\left|\left|\mathcal{Q}_h w' - w_h' + w_h'\right|\right|\right| \leq C\|\zeta_u\|_{L^2(L^2)}. \tag{6.19}$$

Acknowledgements The author are grateful to the anonymous referees for valuable comments and suggestions which greatly improved the presentation of this paper.

Declarations

Conflict of interest The author declares that there is no conflict of interest.

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