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A new linearized fourth-order conservative compact difference scheme for the SRLW equations

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Abstract
In this paper, a novel three-point fourth-order compact operator is considered to construct new linearized conservative compact finite difference scheme for the symmetric regularized long wave (SRLW) equations based on the reduction order method with three-level linearized technique. The discrete conservative laws, boundedness and unique solvability are studied. The convergence order $\mathcal{O}(\tau^2 + h^4)$ in the L^∞-norm and stability of the present compact scheme are proved by the discrete energy method. Numerical examples are given to support the theoretical analysis.

Keywords SRLW equations · Reduction order method · Linearized compact difference scheme · Conservation · Convergence

Mathematics Subject Classification (2010) 65M12 · 65M70 · 65N06

1 Introduction

The (1+1)-dimensional symmetric regularized long wave (SRLW) equations with the initial and L-periodic boundary conditions expressed as the following first-order system [17]:

$$u_t - u_{xxt} + \rho_x + uu_x = 0, \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{1.1}$$

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$$\rho_t + u_x = 0, \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x), \quad x \in \mathbb{R}, \tag{1.3}$$

$$u(x, t) = u(x + L, t), \quad \rho(x, t) = \rho(x + L, t), \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{1.4}$$

where $u(x, t)$ and $\rho(x, t)$ are the fluid velocity and the density, respectively. $u_0(x)$ and $\rho_0(x)$ are two known smooth periodic functions with period L. This system has been shown to describe weakly nonlinear ion-acoustic, space-charge waves and solitary waves with bidirectional propagation [2, 17]. It suffices to model only a single period $\Omega = [0, L]$. The initial-boundary problem (1.1)–(1.4) has the following conservative quantity:

$$\mathscr{M}_1(t) = \int_\Omega u(x, t)\mathrm{d}x = \int_\Omega u_0(x)\mathrm{d}x = \mathscr{M}_1(0), \quad t > 0, \tag{1.5}$$

$$\mathscr{M}_2(t) = \int_\Omega \rho(x, t)\mathrm{d}x = \int_\Omega \rho_0(x)\mathrm{d}x = \mathscr{M}_2(0), \quad t > 0, \tag{1.6}$$

$$\begin{aligned}\mathscr{E}(t) &= \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|\rho\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|u_{0x}\|_{L^2}^2 \\ &\quad + \|\rho_0\|_{L^2}^2 = \mathscr{E}(0), \quad t > 0.\end{aligned} \tag{1.7}$$

The density function ρ can be removed from Eqs. (1.1)–(1.2), then it turn to a single nonlinear equation from the velocity function:

$$u_{tt} - u_{xxtt} - u_{xx} + \frac{1}{2}(u^2)_{xt} = 0, \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{1.8}$$

which is explicitly symmetric in the x and t derivatives and is similar to the regularized long wave (RLW) equation or BBM equation [15].

The research of theoretical analysis and numerical methods for the SRLW equations have attracted wide attention. The travelling wave solutions of the SRLW equations in the form of hyperbolic secant squared were presented and the difference between the solutions of the SRLW and RLW equations were numerically disclosed by Seyler and Fenstermacher [17]. Clarkson [2] discussed similarity reductions and Painleve analysis for the SRLW equations and modified BBM equations. Existence and uniqueness of solutions, existence of global attractors and exact travelling wave solution of the SRLW equations were studied [3, 22]. The spectral method, Fourier pseudo-spectral method and Chebyshev pseudo-spectral method were developed to approximate the nonlinear term of the SRLW equations [5, 8, 10, 16, 27]. Zhao et al. [26] developed explicit two-step method in time direction and the mixed finite element method in spatial direction. The optimal a priori error estimates for fully discrete explicit two-step mixed scheme were derived. Mittal and Tripathi [13] proposed a new collocation of modified cubic B-spline method basis functions over the finite elements. The unconditional stability of the method has been discussed.

In addition to above numerical methods, the finite difference method is one of significantly effective numerical methods. Some conservative difference schemes for solving the SRLW equations were developed [1, 19, 20]. These schemes are convergent with order $\mathscr{O}(\tau^2 + h^2)$ in the L^∞-norm for u and L^2-norm for ρ. Yimnet et al. [23] proposed a four-level average difference scheme for solving Eq. (1.8). The new conservation of mass and convergence order $\mathscr{O}(\tau^2 + h^2)$ in the L^∞-norm for u were proved. A review of all the numerical method reveals that high-order difference methods are still scarce. To best our knowledge, the feasible methods for solving the SRLW equations with high accuracy are to construct five-point non-compact [7, 14] and three-point compact schemes [6, 9]. However, the ghost points or fictitious points are requisite when deal with the points near boundary. In addition, it is difficult to derive the convergence order in the discrete H^2-norm for u and L^∞- and H^1-norms for ρ. To avoid dealing with ghost point, Li et al. [11, 12] proposed compact schemes for SRLW equations by using the inverse compact operators. But the nonlinear term with inverse compact operator becomes more complicated.

Motivated by the above finite difference methods for solving the SRLW equations, we are interested in constructing a new conservative three-point linearized compact difference scheme for solving the SRLW equations to overcome the difficulties of discrete boundaries and nonlinear terms. In this work, the main contribution are three-fold:

- Three three-point fourth-order compact operators are considered to construct new a compact difference scheme for the SRLW equations based on reduction method. The fourth-order compact operators were successfully applied to BBMB equation [24] and viscous Burgers' equation [21, 25]. Unconditionally convergent with order $\mathscr{O}(\tau^2+h^4)$ in discrete L^∞-norm is proved by the discrete energy method [21, 24]. In addition, the new compact method is different from the fourth-order non-compact methods [7, 14] and compact methods in [6, 9, 11, 12]. As such, the theoretical analysis for the present compact scheme can be worked out using the sophisticated discrete energy method, which is different from the matrix analysis method [11, 12].
- Conservations of discrete mass and energy, boundedness and uniquely solvability of the present scheme are derived in detail.
- The convergence order $\mathscr{O}(\tau^2 + h^4)$ in the discrete L^∞-, L^2-, H^1- and H^2-norm for u and in the discrete L^∞-, L^2- and H^1-norm for ρ of the present scheme are proved at length. The main difficulty in the analysis of the approximation of nonlinear terms affecting stability and convergence is completely overcome by using the sophisticated discrete energy method.

The rest of this paper are arranged as follows. In Section 2, Some useful notations and lemmas are introduced and proved in detail. In Section 3, a three-level linearized fourth-order compact difference scheme for solving the initial-boundary problem (1.1)–(1.4) is constructed based on the reduction order method. In Sections 4–6, the discrete conservative laws, boundedness and unique solvability of the present scheme are proved in detail. The convergence order $\mathscr{O}(\tau^2+h^4)$ in the L^∞-, L^2-, H^1- and H^2-norm for u^n and in the L^∞-, L^2- and H^1-norm for ρ^n and stability of the scheme are proved strictly by the discrete energy method in Section 7. Numerical examples are presented to show that computed results support our theoretical analysis in Section 8. Finally, some concluding remarks are given in Section 9.

2 Notations and lemmas

In this section, we introduce some useful notations and lemmas. We first divide the domain $[0, L] \times [0, T]$. Let $h = L/J$ and $\tau = T/N$ be the space-step and time-step, respectively, where J and N are given to be two positive integers. Defined $u = \{u_j^n | 0 \le j \le J,\ 0 \le n \le N\}$ be a discrete grid function on $\Omega_{h,\tau} = \{(x_j, t_n) | x_j = jh,\ t_n = n\tau,\ 0 \le j \le J,\ 0 \le n \le N\}$, we denote the following notations

$$\begin{aligned}
(u_j^n)_x &= \frac{1}{h}(u_{j+1}^n - u_j^n), \quad (u_j^n)_{\bar{x}} = \frac{1}{h}(u_j^n - u_{j-1}^n), \quad (u_j^n)_{\hat{x}} = \frac{1}{2h}(u_{j+1}^n - u_{j-1}^n),\\
(u_j^n)_t &= \frac{1}{\tau}(u_j^{n+1} - u_j^n),\\
(u_j^n)_{\hat{t}} &= \frac{1}{2\tau}(u_j^{n+1} - u_j^{n-1}), \quad u_j^{n+\frac{1}{2}} = \frac{1}{2}(u_j^{n+1} + u_j^n), \quad u_j^{\bar{n}} = \frac{1}{2}(u_j^{n+1} + u_j^{n-1}).
\end{aligned}$$

Denote the discrete space

$$\mathscr{U}_{h,per} = \{u \mid u = (u_j),\ u_{j+J} = u_j,\ j \in Z\}.$$

For any grid functions $u, v \in \mathscr{U}_{h,per}$, we define the discrete inner products and the corresponding norms as follows

$$\langle u, v \rangle = h \sum_{j=1}^{J} u_j v_j, \quad \|u\| = \sqrt{\langle u, u \rangle}, \quad \|u\|_\infty = \max_{1 \le j \le J} |u_j|.$$

and the function notation [4]

$$\psi(u_j, v_j) = \frac{1}{3}[u_j (v_j)_{\hat{x}} + (u_j v_j)_{\hat{x}}], \quad 1 \le j \le J.$$

Throughout this paper, C is a positive real constant, which depends on neither h nor τ, and may have different values for different occurrences.

Lemma 2.1 [21] *For any grid functions $u, v \in \mathscr{U}_{h,per}$, we have*

$$\langle u_{x\bar{x}}, v \rangle = -\langle u_x, v_x \rangle, \quad \langle u_{\hat{x}}, v \rangle = -\langle u, v_{\hat{x}} \rangle.$$

Especially, we have

$$\langle u_{x\bar{x}}, u \rangle = -\|u_x\|^2, \quad \langle u_{x\bar{x}}, u_{x\bar{x}} \rangle = \|u_{x\bar{x}}\|^2, \quad \langle u_{\hat{x}}, u \rangle = 0, \quad \langle u_{x\bar{x}}, u_{\hat{x}} \rangle = 0,$$
$$\langle \psi(u, v), v \rangle = 0.$$

Lemma 2.2 [18] *For any grid function $u \in \mathscr{U}_{h,per}$, we have*

$$\|u\|_\infty \le \frac{\sqrt{L}}{2}\|u_x\|, \quad \|u\| \le \frac{L}{\sqrt{6}}\|u_x\|, \quad \|u_x\| \le \frac{2}{h}\|u\|, \quad \|u_{x\bar{x}}\| \le \frac{2}{h}\|u_x\|,$$
$$\|u_{\hat{x}}\| \le \|u_x\| = \|u_{\bar{x}}\|.$$

Lemma 2.3 [21, 24] *Let $f(x) \in C^5[x_{j-1}, x_{j+1}]$ and denote $F_j = f(x_j)$ and $G_j = f''(x_j)$, $1 \le j \le J$, we have*

$$f'(x_j) = (F_j)_{\hat{x}} - \frac{h^2}{6}(G_j)_{x\bar{x}} + \mathscr{O}(h^4), \quad 1 \le j \le J,$$

$$f''(x_j) = (F_j)_{x\bar{x}} - \frac{h^2}{12}(G_j)_{x\bar{x}} + \mathscr{O}(h^4), \quad 1 \le j \le J,$$

$$f(x_j)f'(x_j) = \psi(F_j, F_j) - \frac{h^2}{2}\psi(G_j, F_j) + \mathscr{O}(h^4), \quad 1 \le j \le J.$$

Lemma 2.4 *For any grid functions $u, v, R \in \mathscr{U}_{h,per}$, satisfying*

$$v_j^n = (u_j^n)_{x\bar{x}} - \frac{h^2}{12}(v_j^n)_{x\bar{x}} + R_j^n, \quad 1 \le j \le J, \quad 0 \le n \le N, \tag{2.1}$$

we have

$$\begin{aligned} \langle v^n, u^n \rangle &= -\|u_x^n\|^2 - \frac{h^2}{12}\|v^n\|^2 + \frac{h^4}{144}\|v_x^n\|^2 + \frac{h^2}{12}\langle R^n, v^n \rangle \\ &\quad + \langle R^n, u^n \rangle, \quad 0 \le n \le N, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \langle v_t^n, 2u^{n+\frac{1}{2}} \rangle &= -\|u_x^n\|_t^2 - \frac{h^2}{12}\|v^n\|_t^2 + \frac{h^4}{144}\|v_x^n\|_t^2 + \frac{h^2}{12}\langle v_t^n, 2R^{n+\frac{1}{2}} \rangle \\ &\quad + \langle R_t^n, 2u^{n+\frac{1}{2}} \rangle, \quad 0 \le n \le N-1, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \langle v_t^n, 2u_{x\bar{x}}^{n+\frac{1}{2}} \rangle &= \|u_{x\bar{x}}^n\|_t^2 + \frac{h^2}{12}\|v_x^n\|_t^2 - \frac{h^4}{144}\|v_{x\bar{x}}^n\|_t^2 + \frac{h^2}{12}\langle v_{x\bar{x}t}^n, 2R^{n+\frac{1}{2}} \rangle \\ &\quad + \langle R_t^n, 2u_{x\bar{x}}^{n+\frac{1}{2}} \rangle, \quad 0 \le n \le N-1, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \langle v_{\hat{t}}^n, 2u^{\bar{n}} \rangle &= -\|u_x^n\|_{\hat{t}}^2 - \frac{h^2}{12}\|v^n\|_{\hat{t}}^2 + \frac{h^4}{144}\|v_x^n\|_{\hat{t}}^2 + \frac{h^2}{12}\langle v_{\hat{t}}^n, 2R^{\bar{n}} \rangle \\ &\quad + \langle R_{\hat{t}}^n, 2u^{\bar{n}} \rangle, \quad 1 \le n \le N-1, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \langle v_{\hat{t}}^n, 2u_{x\bar{x}}^{\bar{n}} \rangle &= \|u_{x\bar{x}}^n\|_{\hat{t}}^2 + \frac{h^2}{12}\|v_x^n\|_{\hat{t}}^2 - \frac{h^4}{144}\|v_{x\bar{x}}^n\|_{\hat{t}}^2 + \frac{h^2}{12}\langle v_{x\bar{x}\hat{t}}^n, 2R^{\bar{n}} \rangle \\ &\quad + \langle R_{\hat{t}}^n, 2u_{x\bar{x}}^{\bar{n}} \rangle, \quad 1 \le n \le N-1, \end{aligned} \tag{2.6}$$

where

$$\|u^n\|_t^2 = \frac{1}{\tau}(\|u^{n+1}\|^2 - \|u^n\|^2), \quad \|u^n\|_{\hat{t}}^2 = \frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^{n-1}\|^2).$$

Proof Equation (2.2) comes from Wang et al. [21]. Considering Eq. (2.1), we have

$$(v_j^n)_t = (u_j^n)_{x\bar{x}t} - \frac{h^2}{12}(v_j^n)_{x\bar{x}t} + (R_j^n)_t, \quad 1 \le j \le J, \quad 0 \le n \le N-1, \tag{2.7}$$

$$v_j^{n+\frac{1}{2}} = (u_j^{n+\frac{1}{2}})_{x\bar{x}} - \frac{h^2}{12}(v_j^{n+\frac{1}{2}})_{x\bar{x}} + R_j^{n+\frac{1}{2}}, \quad 1 \le j \le J, \quad 0 \le n \le N-1. \tag{2.8}$$

Taking the inner product of Eq. (2.7) with $2u^{n+\frac{1}{2}}$ and with $2u_{x\bar{x}}^{n+\frac{1}{2}}$, we have

$$\begin{aligned}
\langle v_t^n, 2u^{n+\frac{1}{2}}\rangle &= \left\langle u_{x\bar{x}t}^n - \frac{h^2}{12}v_{x\bar{x}t}^n + R_t^n, 2u^{n+\frac{1}{2}}\right\rangle \\
&= -\|u_x^n\|_t^2 - \frac{h^2}{12}\langle v_t^n, 2u_{x\bar{x}}^{n+\frac{1}{2}}\rangle + \langle R_t^n, 2u^{n+\frac{1}{2}}\rangle \\
&= -\|u_x^n\|_t^2 - \frac{h^2}{12}\left\langle v_t^n, 2v^{n+\frac{1}{2}} + \frac{h^2}{6}v_{x\bar{x}}^{n+\frac{1}{2}} - 2R^{n+\frac{1}{2}}\right\rangle + \langle R_t^n, 2u^{n+\frac{1}{2}}\rangle \\
&= -\|u_x^n\|_t^2 - \frac{h^2}{12}\|v^n\|_t^2 + \frac{h^4}{144}\|v_x^n\|_t^2 + \frac{h^2}{12}\langle v_t^n, 2R^{n+\frac{1}{2}}\rangle + \langle R_t^n, 2u^{n+\frac{1}{2}}\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\langle v_t^n, 2u_{x\bar{x}}^{n+\frac{1}{2}}\rangle &= \left\langle u_{x\bar{x}t}^n - \frac{h^2}{12}v_{x\bar{x}t}^n + R_t^n, 2u_{x\bar{x}}^{n+\frac{1}{2}}\right\rangle \\
&= \|u_{x\bar{x}}^n\|_t^2 - \frac{h^2}{12}\langle v_{x\bar{x}t}^n, 2u_{x\bar{x}}^{n+\frac{1}{2}}\rangle + \langle R_t^n, 2u_{x\bar{x}}^{n+\frac{1}{2}}\rangle \\
&= \|u_{x\bar{x}}^n\|_t^2 - \frac{h^2}{12}\left\langle v_{x\bar{x}t}^n, 2v^{n+\frac{1}{2}} + \frac{h^2}{6}v_{x\bar{x}}^{n+\frac{1}{2}} - 2R^{n+\frac{1}{2}}\right\rangle + \langle R_t^n, 2u_{x\bar{x}}^{n+\frac{1}{2}}\rangle \\
&= \|u_{x\bar{x}}^n\|_t^2 + \frac{h^2}{12}\|v_x^n\|_t^2 - \frac{h^4}{144}\|v_{x\bar{x}}^n\|_t^2 + \frac{h^2}{12}\langle v_{x\bar{x}t}^n, 2R^{n+\frac{1}{2}}\rangle + \langle R_t^n, 2u_{x\bar{x}}^{n+\frac{1}{2}}\rangle.
\end{aligned}$$

Again considering Eq. (2.1), we have

$$(v_j^n)_{\hat{t}} = (u_j^n)_{x\bar{x}\hat{t}} - \frac{h^2}{12}(v_j^n)_{x\bar{x}\hat{t}} + (R_j^n)_{\hat{t}}, \quad 1 \le j \le J, \quad 1 \le n \le N-1, \tag{2.9}$$

$$v_j^{\bar{n}} = (u_j^{\bar{n}})_{x\bar{x}} - \frac{h^2}{12}(v_j^{\bar{n}})_{x\bar{x}} + R_j^{\bar{n}}, \quad 1 \le j \le J, \quad 1 \le n \le N-1. \tag{2.10}$$

Taking the inner product of Eq. (2.9) with $2u^{\bar{n}}$ and with $2u_{x\bar{x}}^{\bar{n}}$, similar to Eqs. (2.3) and (2.4), we obtain Eqs. (2.5) and (2.6). This completes the proof. □

Lemma 2.5 *For any grid functions* $u, \rho, v, w, R, S \in \mathscr{U}_{h,per}$, *satisfying*

$$v_j = (u_j)_{x\bar{x}} - \frac{h^2}{12}(v_j)_{x\bar{x}} + R_j, \quad 1 \le j \le J, \tag{2.11}$$

$$w_j = (\rho_j)_{x\bar{x}} - \frac{h^2}{12}(w_j)_{x\bar{x}} + S_j, \quad 1 \le j \le J, \tag{2.12}$$

we have

$$\begin{aligned}
&\langle \rho_{\hat{x}}, u\rangle + \langle u_{\hat{x}}, \rho\rangle = 0, \quad \langle \rho_{\hat{x}}, u_{x\bar{x}}\rangle + \langle u_{\hat{x}}, \rho_{x\bar{x}}\rangle = 0, \\
&\langle w_{\hat{x}}, u\rangle + \langle v_{\hat{x}}, \rho\rangle = \frac{h^2}{12}(\langle w_{\hat{x}}, R\rangle + \langle v_{\hat{x}}, S\rangle) - \langle S, u_{\hat{x}}\rangle - \langle R, \rho_{\hat{x}}\rangle, \\
&\langle w_{\hat{x}}, u_{x\bar{x}}\rangle + \langle v_{\hat{x}}, \rho_{x\bar{x}}\rangle = -\langle w_{\hat{x}}, R\rangle - \langle v_{\hat{x}}, S\rangle.
\end{aligned}$$

Proof Using Lemma 2.1, we have

$$\langle \rho_{\hat{x}}, u\rangle + \langle u_{\hat{x}}, \rho\rangle = \langle \rho_{\hat{x}}, u\rangle - \langle u, \rho_{\hat{x}}\rangle = 0, \quad \langle \rho_{\hat{x}}, u_{x\bar{x}}\rangle + \langle u_{\hat{x}}, \rho_{x\bar{x}}\rangle = 0.$$

It follows from Eqs. (2.11) and (2.12) that

$$\begin{aligned}
\langle w_{\hat{x}}, u\rangle + \langle v_{\hat{x}}, \rho\rangle &= -\left\langle \rho_{x\bar{x}} - \frac{h^2}{12} w_{x\bar{x}} + S, u_{\hat{x}}\right\rangle - \left\langle u_{x\bar{x}} - \frac{h^2}{12} v_{x\bar{x}} + R, \rho_{\hat{x}}\right\rangle \\
&= -\frac{h^2}{12}(\langle w_{\hat{x}}, u_{x\bar{x}}\rangle + \langle v_{\hat{x}}, \rho_{x\bar{x}}\rangle) - \langle S, u_{\hat{x}}\rangle - \langle R, \rho_{\hat{x}}\rangle \\
&= -\frac{h^2}{12}\left(\left\langle w_{\hat{x}}, v + \frac{h^2}{12} v_{x\bar{x}} - R\right\rangle + \left\langle v_{\hat{x}}, w + \frac{h^2}{12} w_{x\bar{x}} - S\right\rangle\right) \\
&\quad -\langle S, u_{\hat{x}}\rangle - \langle R, \rho_{\hat{x}}\rangle \\
&= \frac{h^2}{12}(\langle w_{\hat{x}}, R\rangle + \langle v_{\hat{x}}, S\rangle) - \langle S, u_{\hat{x}}\rangle - \langle R, \rho_{\hat{x}}\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\langle w_{\hat{x}}, u_{x\bar{x}}\rangle + \langle v_{\hat{x}}, \rho_{x\bar{x}}\rangle &= \left\langle w_{\hat{x}}, v + \frac{h^2}{12} v_{x\bar{x}} - R\right\rangle + \left\langle v_{\hat{x}}, w + \frac{h^2}{12} w_{x\bar{x}} - S\right\rangle \\
&= \frac{h^2}{12}(\langle w_{\hat{x}}, v_{x\bar{x}}\rangle + \langle v_{\hat{x}}, w_{x\bar{x}}\rangle) - \langle w_{\hat{x}}, R\rangle - \langle v_{\hat{x}}, S\rangle \\
&= -\langle w_{\hat{x}}, R\rangle - \langle v_{\hat{x}}, S\rangle.
\end{aligned}$$

This completes the proof. □

3 Construction of compact difference scheme

Let $v = u_{xx}$, $w = \rho_{xx}$, then the problem (1.1)–(1.4) is equivalent to

$$u_t - v_t + \rho_x + uu_x = 0, \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{3.1}$$

$$\rho_t + u_x = 0, \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{3.2}$$

$$v = u_{xx}, \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{3.3}$$

$$w = \rho_{xx}, \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{3.4}$$

$$u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x), \quad x \in \mathbb{R}, \tag{3.5}$$

$$u(x, t) = u(x + L, t), \quad \rho(x, t) = \rho(x + L, t), \quad x \in \mathbb{R}, \quad t \in (0, T], \tag{3.6}$$

$$v(x, t) = v(x + L, t), \quad w(x, t) = w(x + L, t), \quad x \in \mathbb{R}, \quad t \in (0, T]. \tag{3.7}$$

Define the grid functions

$$u_j^n \approx U_j^n = u(x_j, t_n), \quad \rho_j^n \approx \phi_j^n = \rho(x_j, t_n), \quad v_j^n \approx V_j^n = v(x_j, t_n),$$
$$w_j^n \approx W_j^n = w(x_j, t_n),$$

where $0 \le j \le J$, $0 \le n \le N$. Considering Eqs. (3.1)–(3.2) at the point $(x_j, t_{\frac{1}{2}})$ and (x_j, t_n) and Eqs. (3.3)–(3.4) at the point (x_j, t_n), according to Lemma 2.3, we have

$$(U_j^0)_t - (V_j^0)_t + (\phi_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(W_j^{\frac{1}{2}})_{\hat{x}} + \psi(U_j^0, U_j^{\frac{1}{2}}) - \frac{h^2}{2}\psi(V_j^0, U_j^{\frac{1}{2}}) = P_j^0,$$
$$1 \le j \le J, \tag{3.8}$$

$$(\phi_j^0)_t + (U_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(V_j^{\frac{1}{2}})_{\hat{x}} = Q_j^0, \quad 1 \le j \le J, \tag{3.9}$$

$$(U_j^n)_{\hat{t}} - (V_j^n)_{\hat{t}} + (\phi_j^{\bar{n}})_{\hat{x}} - \frac{h^2}{6}(W_j^{\bar{n}})_{\hat{x}} + \psi(U_j^n, U_j^{\bar{n}}) - \frac{h^2}{2}\psi(V_j^n, U_j^{\bar{n}}) = P_j^n,$$
$$1 \le j \le J, \ 1 \le n \le N-1, \tag{3.10}$$

$$(\phi_j^n)_{\hat{t}} + (U_j^{\bar{n}})_{\hat{x}} - \frac{h^2}{6}(V_j^{\bar{n}})_{\hat{x}} = Q_j^n, \quad 1 \le j \le J, \quad 1 \le n \le N-1, \tag{3.11}$$

$$V_j^n = (U_j^n)_{x\bar{x}} - \frac{h^2}{12}(V_j^n)_{x\bar{x}} + R_j^n, \quad 1 \le j \le J, \quad 0 \le n \le N, \tag{3.12}$$

$$W_j^n = (\phi_j^n)_{x\bar{x}} - \frac{h^2}{12}(W_j^n)_{x\bar{x}} + S_j^n, \quad 1 \le j \le J, \quad 0 \le n \le N, \tag{3.13}$$

$$U_j^0 = u_0(x_j), \quad \phi_j^0 = \rho_0(x_j), \quad 0 \le j \le J, \tag{3.14}$$

$$U_j^n = U_{j+J}^n, \quad \phi_j^n = \phi_{j+J}^n, \quad 0 \le j \le J, \quad 0 \le n \le N, \tag{3.15}$$

$$V_j^n = V_{j+J}^n, \quad W_j^n = W_{j+J}^n, \quad 0 \le j \le J, \quad 0 \le n \le N, \tag{3.16}$$

where $\max_{j,n}\{|(Q_j^0)_x|, \ |(Q_j^n)_x|\} \le C\tau^2$, $\max_{j,n}\{|R_j^n|, \ |S_j^n|\} \le Ch^4$ and

$$\max_{j,n}\{|P_j^0|, \ |Q_j^0|, \ |P_j^n|, \ |Q_j^n|, \ |(R_j^n)_{\hat{t}}|, \ |R_j^{\bar{n}}|, \ |S_j^{\bar{n}}|\} \le C(\tau^2 + h^4).$$

Replacing the grid functions U_j^n, ϕ_j^n, V_j^n, W_j^n by u_j^n, ρ_j^n, v_j^n, w_j^n in Eqs. (3.8)–(3.16), respectively, and ignoring the small terms P_j^0, Q_j^0, P_j^n, Q_j^n, R_j^n, S_j^n, we construct the following linearized compact finite difference scheme for Eqs. (3.1)–(3.7):

$$(u_j^0)_t - (v_j^0)_t + (\rho_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(w_j^{\frac{1}{2}})_{\hat{x}} + \psi(u_j^0, u_j^{\frac{1}{2}}) - \frac{h^2}{2}\psi(v_j^0, u_j^{\frac{1}{2}}) = 0,$$
$$1 \le j \le J, \tag{3.17}$$

$$(\rho_j^0)_t + (u_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(v_j^{\frac{1}{2}})_{\hat{x}} = 0, \quad 1 \le j \le J, \tag{3.18}$$

$$(u_j^n)_{\hat{t}} - (v_j^n)_{\hat{t}} + (\rho_j^{\bar{n}})_{\hat{x}} - \frac{h^2}{6}(w_j^{\bar{n}})_{\hat{x}} + \psi(u_j^n, u_j^{\bar{n}}) - \frac{h^2}{2}\psi(v_j^n, u_j^{\bar{n}}) = 0,$$
$$1 \le j \le J, \quad 1 \le n \le N-1, \tag{3.19}$$

$$(\rho_j^n)_{\hat{t}} + (u_j^{\bar{n}})_{\hat{x}} - \frac{h^2}{6}(v_j^{\bar{n}})_{\hat{x}} = 0, \quad 1 \le j \le J, \quad 1 \le n \le N-1, \tag{3.20}$$

$$v_j^n = (u_j^n)_{x\bar{x}} - \frac{h^2}{12}(v_j^n)_{x\bar{x}}, \quad 1 \le j \le J, \quad 0 \le n \le N, \tag{3.21}$$

$$w_j^n = (\rho_j^n)_{x\bar{x}} - \frac{h^2}{12}(w_j^n)_{x\bar{x}}, \quad 1 \le j \le J, \quad 0 \le n \le N, \tag{3.22}$$

$$u_j^0 = u_0(x_j), \quad \rho_j^0 = \rho_0(x_j), \quad 0 \le j \le J, \tag{3.23}$$

$$u_j^n = u_{j+J}^n, \quad \rho_j^n = \rho_{j+J}^n, \quad 0 \le j \le J, \quad 0 \le n \le N, \tag{3.24}$$

$$v_j^n = v_{j+J}^n, \quad w_j^n = w_{j+J}^n, \quad 0 \le j \le J, \quad 0 \le n \le N. \tag{3.25}$$

4 Conservative laws

Theorem 4.1 *Suppose that* $u_0(x) \in H^2(\Omega)$, $\rho_0(x) \in H^1(\Omega)$ *and* (u^n, ρ^n, v^n, w^n) *is the solution of* Eqs. (3.17)–(3.25)*, then it is discrete conservative in the senses:*

$$\begin{aligned} M_1^{n+1} :=& \frac{h}{2}\sum_{j=1}^{J}(u_j^{n+1} + u_j^n) - \frac{h}{2}\sum_{j=1}^{J}(v_j^{n+1} + v_j^n) + \frac{h\tau}{6}\sum_{j=1}^{J} u_j^n (u_j^{n+1})_{\hat{x}} \\ &-\frac{h^3\tau}{12}\sum_{j=1}^{J}(u_j^n)_{x\bar{x}}(u_j^{n+1})_{\hat{x}} - \frac{h^5\tau}{144}\sum_{j=1}^{J}(v_j^n)_{\hat{x}} v_j^{n+1} - \frac{h^7\tau}{1728}\sum_{j=1}^{J}(v_j^n)_{\hat{x}}(v_j^{n+1})_{x\bar{x}} \\ =& M_1^n = \ldots = M_1^1 = M_1^0, \quad 0 \le n \le N-1, \end{aligned} \tag{4.1}$$

$$M_2^{n+1} := \frac{h}{2}\sum_{j=1}^{J}(\rho_j^{n+1} + \rho_j^n) = M_2^n = \ldots = M_2^1 = M_2^0, \quad 0 \le n \le N-1, \tag{4.2}$$

$$\begin{aligned} E^{n+1} :=& \frac{1}{2}(\|u^{n+1}\|^2 + \|u^n\|^2) + \frac{1}{2}(\|u_x^{n+1}\|^2 + \|u_x^n\|^2) + \frac{h^2}{24}(\|v^{n+1}\|^2 + \|v^n\|^2) \\ &-\frac{h^4}{288}(\|v_x^{n+1}\|^2 + \|v_x^n\|^2) + \frac{1}{2}(\|\rho^{n+1}\|^2 + \|\rho^n\|^2) \\ =& E^n = \ldots = E^1 = E^0, \quad 0 \le n \le N-1, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} M_1^0 =& h\sum_{j=1}^{J}(u_j^0 - v_j^0) + \frac{h\tau}{6}\sum_{j=1}^{J} u_j^0 (u_j^0)_{\hat{x}} - \frac{h^3\tau}{12}\sum_{j=1}^{J}(u_j^0)_{x\bar{x}}(u_j^0)_{\hat{x}} \\ &-\frac{h^5\tau}{144}\sum_{j=1}^{J}(v_j^0)_{\hat{x}} v_j^0 - \frac{h^7\tau}{1728}\sum_{j=1}^{J}(v_j^0)_{\hat{x}}(v_j^0)_{x\bar{x}}, \\ M_2^0 =& h\sum_{J=1}^{J}\rho_j^0, \quad E^0 = \|u^0\|^2 + \|u_x^0\|^2 + \frac{h^2}{12}\|v^0\|^2 - \frac{h^4}{144}\|v_x^0\|^2 + \|\rho^0\|^2. \end{aligned}$$

Proof Multiplying both-hand sides of Eqs. (3.19) and (3.20) with h and summing up j from 1 to J, we have

$$\frac{h}{2\tau}\sum_{j=1}^{J}(u_j^{n+1}-u_j^{n-1})-\frac{h}{2\tau}\sum_{j=1}^{J}(v_j^{n+1}-v_j^{n-1})+h\sum_{j=1}^{J}\psi(u_j^n,u_j^{\bar{n}})$$
$$-\frac{h^3}{2}\sum_{j=1}^{J}\psi(v_j^n,u_j^{\bar{n}})=0,$$
$$\frac{h}{2\tau}\sum_{j=1}^{J}(\rho_j^{n+1}-\rho_j^{n-1})=0,$$

Applying Lemma 2.1 and (3.21), we have

$$h\sum_{j=1}^{J}\psi(u_j^n,u_j^{\bar{n}})=\frac{h}{3}\sum_{j=1}^{J}\Big[u_j^n(u_j^{\bar{n}})_{\hat{x}}+(u_j^nu_j^{\bar{n}})_{\hat{x}}\Big]=\frac{h}{6}\sum_{j=1}^{J}\Big[u_j^n(u_j^{n+1})_{\hat{x}}-u_j^{n-1}(u_j^n)_{\hat{x}}\Big],$$

and

$$\begin{aligned}\frac{h^3}{2}\sum_{j=1}^{J}\psi(v_j^n,u_j^{\bar{n}}) &= \frac{h^3}{6}\sum_{j=1}^{J}\Big[v_j^n(u_j^{\bar{n}})_{\hat{x}}+(v_j^nu_j^{\bar{n}})_{\hat{x}}\Big]\\
&= \frac{h^3}{6}\sum_{j=1}^{J}\Big[(u_j^n)_{x\bar{x}}(u_j^{\bar{n}})_{\hat{x}}-\frac{h^2}{12}(v_j^n)_{x\bar{x}}(u_j^{\bar{n}})_{\hat{x}}\Big]\\
&= \frac{h^3}{6}\sum_{j=1}^{J}(u_j^n)_{x\bar{x}}(u_j^{\bar{n}})_{\hat{x}}+\frac{h^5}{72}\sum_{j=1}^{J}(v_j^n)_{\hat{x}}\Big[v_j^{\bar{n}}+\frac{h^2}{12}(v_j^{\bar{n}})_{x\bar{x}}\Big]\\
&= \frac{h^3}{12}\sum_{j=1}^{J}(u_j^n)_{x\bar{x}}(u_j^{n+1})_{\hat{x}}+\frac{h^5}{144}\sum_{j=1}^{J}(v_j^n)_{\hat{x}}v_j^{n+1}\\
&\quad+\frac{h^7}{1728}\sum_{j=1}^{J}(v_j^n)_{\hat{x}}(v_j^{n+1})_{x\bar{x}}\\
&\quad-\frac{h^3}{12}\sum_{j=1}^{J}(u_j^{n-1})_{x\bar{x}}(u_j^n)_{\hat{x}}-\frac{h^5}{144}\sum_{j=1}^{J}(v_j^{n-1})_{\hat{x}}v_j^n\\
&\quad-\frac{h^7}{1728}\sum_{j=1}^{J}(v_j^{n-1})_{\hat{x}}(v_j^n)_{x\bar{x}},\end{aligned}$$

then we have

$$M_1^{n+1}=M_1^n,\quad M_2^{n+1}=M_2^n,\quad 1\le n\le N-1. \tag{4.4}$$

Multiplying both-hand sides of Eqs. (3.17) and (3.18) with h and summing up j from 1 to J, we have

$$\frac{h}{\tau}\sum_{j=1}^{J}(u_j^1-u_j^0)-\frac{h}{\tau}\sum_{j=1}^{J}(v_j^1-v_j^0)+h\sum_{j=1}^{J}\psi(u_j^0,u_j^{\frac{1}{2}})-\frac{h^3}{2}\sum_{j=1}^{J}\psi(v_j^0,u_j^{\frac{1}{2}})=0,$$

$$\frac{h}{\tau}\sum_{j=1}^{J}(\rho_j^1-\rho_j^0)=0,$$

Similar to Eq. (4.4), we obtain

$$\begin{aligned}
&h\sum_{j=1}^{J}(u_j^1+u_j^0)-h\sum_{j=1}^{J}(v_j^1+v_j^0)+\frac{h\tau}{3}\sum_{j=1}^{J}u_j^0(u_j^1)_{\hat{x}}-\frac{h^3\tau}{6}\sum_{j=1}^{J}(u_j^0)_{x\bar{x}}(u_j^1)_{\hat{x}}\\
&-\frac{h^5\tau}{72}\sum_{j=1}^{J}(v_j^0)_{\hat{x}}v_j^1-\frac{h^7\tau}{864}\sum_{j=1}^{J}(v_j^0)_{\hat{x}}(v_j^1)_{x\bar{x}}\\
=\;&2h\sum_{j=1}^{J}(u_j^0-v_j^0)+\frac{h\tau}{3}\sum_{j=1}^{J}u_j^0(u_j^0)_{\hat{x}}-\frac{h^3\tau}{6}\sum_{j=1}^{J}(u_j^0)_{x\bar{x}}(u_j^0)_{\hat{x}}-\frac{h^5\tau}{72}\sum_{j=1}^{J}(v_j^0)_{\hat{x}}v_j^0\\
&-\frac{h^7\tau}{864}\sum_{j=1}^{J}(v_j^0)_{\hat{x}}(v_j^0)_{x\bar{x}}\\
&h\sum_{j=1}^{J}(\rho_j^1+\rho_j^0)=2h\sum_{j=1}^{J}\rho_j^0,
\end{aligned}$$

then we have $M_1^1=M_1^0$, $M_2^1=M_2^0$. Hence, we have Eqs. (4.1) and (4.2).

Taking the inner product of Eq. (3.19) with $2u^{\bar{n}}$, of Eq. (3.20) with $2\rho^{\bar{n}}$ and applying Lemma 2.1, we have

$$\|u^n\|_{\hat{t}}^2-\langle v_{\hat{t}}^n,2u^{\bar{n}}\rangle+\langle\rho_{\hat{x}}^{\bar{n}},2u^{\bar{n}}\rangle-\frac{h^2}{6}\langle w_{\hat{x}}^{\bar{n}},2u^{\bar{n}}\rangle=0, \tag{4.5}$$

$$\|\rho^n\|_{\hat{t}}^2+\langle u_{\hat{x}}^{\bar{n}},2\rho^{\bar{n}}\rangle-\frac{h^2}{6}\langle v_{\hat{x}}^{\bar{n}},2\rho^{\bar{n}}\rangle=0. \tag{4.6}$$

From Lemmas 2.4, 2.5 and taking $R=0$, $S=0$, we have

$$\langle v_{\hat{t}}^n,2u^{\bar{n}}\rangle=-\|u_x^n\|_{\hat{t}}^2-\frac{h^2}{12}\|v^n\|_{\hat{t}}^2+\frac{h^4}{144}\|v_x^n\|_{\hat{t}}^2,\quad \langle\rho_{\hat{x}}^{\bar{n}},2u^{\bar{n}}\rangle+\langle u_{\hat{x}}^{\bar{n}},2\rho^{\bar{n}}\rangle=0,$$
$$\langle w_{\hat{x}}^{\bar{n}},2u^{\bar{n}}\rangle+\langle v_{\hat{x}}^{\bar{n}},2\rho^{\bar{n}}\rangle=0.$$

Adding Eq. (4.5) to (4.6), we have

$$\|u^n\|_{\hat{t}}^2+\|u_x^n\|_{\hat{t}}^2+\frac{h^2}{12}\|v^n\|_{\hat{t}}^2-\frac{h^4}{144}\|v_x^n\|_{\hat{t}}^2+\|\rho^n\|_{\hat{t}}^2=0,$$

then we obtain

$$E^{n+1}=E^n,\quad 1\le n\le N-1. \tag{4.7}$$

Taking the inner product of Eq. (3.17) with $2u^{\frac{1}{2}}$ and of Eq. (3.18) with $2\rho^{\frac{1}{2}}$, similar to Eq. (4.7), we obtain

$$\|u^0\|_t^2 + \|u_x^0\|_t^2 + \frac{h^2}{12}\|v^0\|_t^2 - \frac{h^4}{144}\|v_x^0\|_t^2 + \|\rho^0\|_t^2 = 0,$$

which implies

$$\begin{aligned} &\|u^1\|^2 + \|u^0\|^2 + \|u_x^1\|^2 + \|u_x^0\|^2 + \frac{h^2}{12}(\|v^1\|^2 + \|v^0\|^2) \\ &-\frac{h^4}{144}(\|v_x^1\|^2 + \|v_x^0\|^2) + \|\rho^1\|^2 + \|\rho^0\|^2 \\ = \ & 2(\|u^0\|^2 + \|u_x^0\|^2) + \frac{h^2}{6}\|v^0\|^2 - \frac{h^4}{72}\|v_x^0\|^2 + 2\|\rho^0\|^2, \end{aligned}$$

then we have $E^1 = E^0$. Hence, we obtain Eq. (4.3). This completes the proof. □

5 Boundedness

Lemma 5.1 [7] *Suppose that $u_0(x) \in H^1(\Omega)$, $\rho_0(x) \in L^2(\Omega)$, then the solutions of the problem (1.1)–(1.4) satisfy*

$$\|u\|_{L^2} \le C, \quad \|u_x\|_{L^2} \le C, \quad \|\rho\|_{L^2} \le C, \quad \|u\|_{L^\infty} \le C, \quad t \in (0, T],$$

for a constant C.

Lemma 5.2 [9] *Suppose that $u_0(x) \in H^2(\Omega)$, $\rho_0(x) \in H^1(\Omega)$, then there exists a positive constant $C = C(\|u_0\|_2, \|\rho\|_1)$ such that*

$$\|u_x\|_{L^2} \le C, \quad \|u_{xx}\|_{L^2} \le C, \quad \|\rho_x\|_{L^2} \le C, \quad t \in (0, T].$$

Furthermore,

$$\|u_x\|_{L^\infty} \le C, \quad \|\rho\|_{L^\infty} \le C, \quad t \in (0, T].$$

Theorem 5.1 *Suppose that $u_0(x) \in H^2(\Omega)$, $\rho_0(x) \in H^1(\Omega)$ and $u(x,t), \rho(x,t) \in C_{x,t}^{5,3}(\Omega \times (0,T])$, there exist sufficiently small positive constants τ_0, h_0 such that $\tau \le \tau_0$, $h \le h_0$, then the solution (u^n, ρ^n) of the compact scheme (3.17)–(3.25) satisfies*

$$\|u^n\| \le C, \quad \|u_x^n\| \le C, \quad \|\rho^n\| \le C, \quad \|u^n\|_\infty \le C, \quad 0 \le n \le N.$$

Furthermore, we have

$$\|u_{x\bar{x}}^n\| \le C, \quad \|\rho_x^n\| \le C, \quad \|u_x^n\|_\infty \le C, \quad \|\rho^n\|_\infty \le C, \quad 0 \le n \le N.$$

Proof From Eqs. (3.12)–(3.14), there exists a positive constant $\widehat{c}$ satisfies

$$\max_{(x,t)\in\Omega\times(0,T]}\{|U|, |U_x|, |U_{xx}|, |\phi|, |\phi_x|, |\phi_{xx}|, |V|, |V_x|, |V_{xx}|, |W|, |W_x|, |W_{xx}|\} \le \widehat{c}.$$

It follows from Eq. (4.3) and Lemma 2.2 that

$$E^0 = E^{n+1} \geq \frac{1}{2}(\|u^{n+1}\|^2 + \|u_x^{n+1}\|^2 + \|\rho^{n+1}\|^2) + \frac{h^2}{36}\|v^{n+1}\|^2, \quad 0 \leq n \leq N-1.$$

There exists a positive constant h_0 such that $h \leq h_0$ and $E^0 \leq (3 + h_0^2/12)\widehat{c}^2$, we have

$$\|u^n\| \leq \sqrt{2E^0}, \quad \|u_x^n\| \leq \sqrt{2E^0}, \quad \|\rho^n\| \leq \sqrt{2E^0}, \quad \|v^n\| \leq \frac{6\sqrt{E^0}}{h}, \quad 1 \leq n \leq N.$$

Using Lemma 2.2, we obtain

$$\|u^n\|_\infty \leq \frac{\sqrt{L}}{2}\|u_x^n\| \leq \sqrt{\frac{LE^0}{2}}, \quad 1 \leq n \leq N.$$

Taking the inner product of Eq. (3.17) with $-2u_{x\bar{x}}^{\frac{1}{2}}$, of Eq. (3.18) with $-2\rho_{x\bar{x}}^{\frac{1}{2}}$ and applying Lemma 2.1, we have

$$\|u_x^0\|_t^2 + \langle v_t^0, 2u_{x\bar{x}}^{\frac{1}{2}}\rangle - \langle \rho_{\hat{x}}^{\frac{1}{2}}, 2u_{x\bar{x}}^{\frac{1}{2}}\rangle + \frac{h^2}{6}\langle w_{\hat{x}}^{\frac{1}{2}}, 2u_{x\bar{x}}^{\frac{1}{2}}\rangle - \langle \psi(u^0, u^{\frac{1}{2}}), 2u_{x\bar{x}}^{\frac{1}{2}}\rangle$$
$$+\frac{h^2}{2}\langle \psi(v^0, u^{\frac{1}{2}}), 2u_{x\bar{x}}^{\frac{1}{2}}\rangle = 0, \tag{5.1}$$

$$\|\rho_x^0\|_t^2 - \langle u_{\hat{x}}^{\frac{1}{2}}, 2\rho_{x\bar{x}}^{\frac{1}{2}}\rangle + \frac{h^2}{6}\langle v_{\hat{x}}^{\frac{1}{2}}, 2\rho_{x\bar{x}}^{\frac{1}{2}}\rangle = 0. \tag{5.2}$$

Using Lemmas 2.4, 2.5 and letting $R = S = 0$, we have

$$\langle v_t^0, 2u_{x\bar{x}}^{\frac{1}{2}}\rangle = \|u_{x\bar{x}}^0\|_t^2 + \frac{h^2}{12}\|v_x^0\|_t^2 - \frac{h^4}{144}\|v_{x\bar{x}}^0\|_t^2, \tag{5.3}$$

$$\langle \rho_{\hat{x}}^{\frac{1}{2}}, u_{x\bar{x}}^{\frac{1}{2}}\rangle + \langle u_{\hat{x}}^{\frac{1}{2}}, \rho_{x\bar{x}}^{\frac{1}{2}}\rangle = 0, \quad \langle w_{\hat{x}}^{\frac{1}{2}}, u_{x\bar{x}}^{\frac{1}{2}}\rangle + \langle v_{\hat{x}}^{\frac{1}{2}}, \rho_{x\bar{x}}^{\frac{1}{2}}\rangle = 0, \tag{5.4}$$

Noticing

$$\psi(v_j, u_j) = \frac{1}{3}[v_j(u_j)_{\hat{x}} + (v_j u_j)_{\hat{x}}] = \frac{1}{3}[(v_j + v_{j+1})(u_j)_{\hat{x}} + (v_j)_{\hat{x}} u_{j-1}], \quad 1 \leq j \leq J.$$

Applying Cauchy-Schwarz inequality and Lemmas 2.1, 2.2, we have

$$\begin{aligned}\langle \psi(u^0, u^{\frac{1}{2}}), 2u_{x\bar{x}}^{\frac{1}{2}}\rangle &\leq \frac{2}{3}\left(2\|u^0\|_\infty \|u_x^{\frac{1}{2}}\|\|u_{x\bar{x}}^{\frac{1}{2}}\| + \|u_{\hat{x}}^0\|_\infty \|u^{\frac{1}{2}}\|\|u_{x\bar{x}}^{\frac{1}{2}}\|\right)\\ &\leq \frac{\widehat{c}}{3}\left(2 + \frac{L}{\sqrt{6}}\right)(\|u_x^{\frac{1}{2}}\|^2 + \|u_{x\bar{x}}^{\frac{1}{2}}\|^2),\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\frac{h^2}{2}\langle \psi(v^0, u^{\frac{1}{2}}), 2u_{x\bar{x}}^{\frac{1}{2}}\rangle &\leq \frac{h^2}{3}\left(2\|v^0\|_\infty \|u_x^{\frac{1}{2}}\|\|u_{x\bar{x}}^{\frac{1}{2}}\| + \|v_{\hat{x}}^0\|_\infty \|u^{\frac{1}{2}}\|\|u_{x\bar{x}}^{\frac{1}{2}}\|\right)\\ &\leq \frac{\widehat{c}h^2}{6}\left(2 + \frac{L}{\sqrt{6}}\right)(\|u_x^{\frac{1}{2}}\|^2 + \|u_{x\bar{x}}^{\frac{1}{2}}\|^2).\end{aligned} \tag{5.6}$$

Letting

$$\widetilde{\mathscr{F}}^1 = \|u_x^1\|^2 + \|u_{x\bar{x}}^1\|^2 + \frac{h^2}{12}\|v_x^1\|^2 - \frac{h^4}{144}\|v_{x\bar{x}}^1\|^2 + \|\rho_x^1\|^2,$$

and using Lemma 2.2, we have

$$\widetilde{\mathscr{F}}^1 \geq \|u_x^1\|^2 + \|u_{x\bar{x}}^1\|^2 + \frac{h^2}{18}\|v_x^1\|^2 + \|\rho_x^1\|^2.$$

Adding Eqs. (5.1)–(5.2), substituting Eqs. (5.3)–(5.6) to Eqs. (5.1)–(5.2) and using Cauchy-Schwarz inequality and Lemma 2.2, there exist sufficiently small positive constants τ_0, h_0 such that $\tau \leq \tau_0, h \leq h_0$, we have

$$\begin{aligned}\widetilde{\mathscr{F}}^1 - \widetilde{\mathscr{F}}^0 &\leq \left(2 + \frac{L}{\sqrt{6}}\right)\left(\frac{1}{6} + \frac{h^2}{12}\right)\widehat{c}\tau\left(\|u_x^1\|^2 + \|u_x^0\|^2 + \|u_{x\bar{x}}^1\|^2 + \|u_{x\bar{x}}^0\|^2\right)\\ &\leq c_1\tau(\widetilde{\mathscr{F}}^1 + \widetilde{\mathscr{F}}^0),\end{aligned}$$

where $c_1 = (2+L/\sqrt{6})(2+h_0^2)\widehat{c}/12$. When $c_1\tau \leq 1/3$, we have $\widetilde{\mathscr{F}}^0 \leq (3+h_0^2/12)\widehat{c}^2$ and

$$\widetilde{\mathscr{F}}^1 \leq (1 + 3c_1\tau_0)\widetilde{\mathscr{F}}^0 \leq C.$$

Using Lemma 2.2, we obtain $\|u_{x\bar{x}}^1\| \leq C$, $\|\rho_x^1\| \leq C$, $\|u_x^1\|_\infty \leq C$, $\|\rho^1\|_\infty \leq C$.

Taking the inner product of Eq. (3.19) with $-2u_{x\bar{x}}^{\bar{n}}$ and of Eq. (3.20) with $-2\rho_{x\bar{x}}^{\bar{n}}$, we have

$$\begin{aligned}&\|u_x^n\|_{\hat{t}}^2 + \langle v_{\hat{t}}^n, 2u_{x\bar{x}}^{\bar{n}}\rangle - \langle \rho_{\hat{x}}^{\bar{n}}, 2u_{x\bar{x}}^{\bar{n}}\rangle + \frac{h^2}{6}\langle w_{\hat{x}}^{\bar{n}}, 2u_{x\bar{x}}^{\bar{n}}\rangle - \langle \psi(u^n, u^{\bar{n}}), 2u_{x\bar{x}}^{\bar{n}}\rangle\\ &+\frac{h^2}{2}\langle \psi(v^n, u^{\bar{n}}), 2u_{x\bar{x}}^{\bar{n}}\rangle = 0, \end{aligned} \tag{5.7}$$

$$\|\rho_x^n\|_{\hat{t}}^2 - \langle u_{\hat{x}}^{\bar{n}}, 2\rho_{x\bar{x}}^{\bar{n}}\rangle + \frac{h^2}{6}\langle v_{\hat{x}}^{\bar{n}}, 2\rho_{x\bar{x}}^{\bar{n}}\rangle = 0, \tag{5.8}$$

for $1 \leq n \leq N-1$. Similar to Eqs. (5.3)–(5.6), we have

$$\langle v_{\hat{t}}^n, 2u_{x\bar{x}}^{\bar{n}}\rangle = \|u_{x\bar{x}}^n\|_{\hat{t}}^2 + \frac{h^2}{12}\|v_x^n\|_{\hat{t}}^2 - \frac{h^4}{144}\|v_{x\bar{x}}^n\|_{\hat{t}}^2, \quad 1 \leq n \leq N-1, \tag{5.9}$$

$$\langle \rho_{\hat{x}}^{\bar{n}}, u_{x\bar{x}}^{\bar{n}}\rangle + \langle u_{\hat{x}}^{\bar{n}}, \rho_{x\bar{x}}^{\bar{n}}\rangle = 0, \quad \langle w_{\hat{x}}^{\bar{n}}, u_{x\bar{x}}^{\bar{n}}\rangle + \langle v_{\hat{x}}^{\bar{n}}, \rho_{x\bar{x}}^{\bar{n}}\rangle = 0, \quad 1 \leq n \leq N-1, \tag{5.10}$$

and

$$\begin{aligned}\langle \psi(u^n, u^{\bar{n}}), 2u_{x\bar{x}}^{\bar{n}}\rangle &\leq \frac{2}{3}(2\|u^n\|_\infty\|u_x^{\bar{n}}\|\|u_{x\bar{x}}^{\bar{n}}\| + \|u^{\bar{n}}\|_\infty\|u_x^n\|\|u_{x\bar{x}}^{\bar{n}}\|)\\ &\leq \frac{\sqrt{2LE^0}}{3}(2\|u_x^{\bar{n}}\|\|u_{x\bar{x}}^{\bar{n}}\| + \|u_x^n\|\|u_{x\bar{x}}^{\bar{n}}\|)\\ &\leq \frac{\sqrt{2LE^0}}{3}\left(\|u_x^{\bar{n}}\|^2 + \frac{1}{2}\|u_x^n\|^2 + \frac{3}{2}\|u_{x\bar{x}}^{\bar{n}}\|^2\right),\end{aligned} \tag{5.11}$$

$$\begin{aligned}\frac{h^2}{2}\langle\psi(v^n,u^{\bar n}),2u^{\bar n}_{x\bar x}\rangle &\le \frac{h^2}{3}(2\|v^n\|_\infty\|u^{\bar n}_x\|\|u^{\bar n}_{x\bar x}\|+\|u^{\bar n}\|_\infty\|v^n_x\|\|u^{\bar n}_{x\bar x}\|)\\ &\le \frac{\sqrt{L}h^2}{2}\|v^n_x\|\|u^{\bar n}_x\|\|u^{\bar n}_{x\bar x}\|\\ &\le 3\sqrt{LE^0}(\|u^{\bar n}_x\|^2+\|u^{\bar n}_{x\bar x}\|^2). \end{aligned} \tag{5.12}$$

Letting

$$\begin{aligned}\mathscr{F}^{n+1} &= \|u^{n+1}_x\|^2+\|u^n_x\|^2+\|u^{n+1}_{x\bar x}\|^2+\|u^n_{x\bar x}\|^2+\|\rho^{n+1}_x\|^2+\|\rho^n_x\|^2\\ &\quad+\frac{h^2}{12}(\|v^{n+1}_x\|^2+\|v^n_x\|^2)-\frac{h^4}{144}(\|v^{n+1}_{x\bar x}\|^2+\|v^n_{x\bar x}\|^2),\quad 0\le n\le N-1,\end{aligned}$$

and using Lemma 2.2, we have

$$\mathscr{F}^{n+1}\ge\|u^{n+1}_x\|^2+\|u^{n+1}_{x\bar x}\|^2+\|\rho^{n+1}_x\|^2+\frac{h^2}{18}\|v^{n+1}_x\|^2,\quad 0\le n\le N-1.$$

Adding Eq. (5.7) to Eq. (5.8), substituting Eqs. (5.9)–(5.12) to Eqs. (5.7)–(5.8) and using Cauchy-Schwarz inequality and Lemma 2.2, there exist sufficiently small positive constants τ_0, h_0 such that $\tau\le\tau_0$, $h\le h_0$, we have

$$\begin{aligned}\mathscr{F}^{n+1}-\mathscr{F}^n &\le 2\sqrt{2LE^0}\tau\left[\left(\frac{\sqrt{2}}{3}+3\right)\|u^{\bar n}_x\|^2+\left(\frac{\sqrt{2}}{2}+3\right)\|u^{\bar n}_{x\bar x}\|^2+\frac{1}{2}\|u^n_x\|^2\right]\\ &\le 4\sqrt{LE^0}\tau\left(\|u^{n+1}_x\|^2+\|u^n_x\|^2+\|u^{n-1}_x\|^2+\|u^{n+1}_{x\bar x}\|^2+\|u^{n-1}_{x\bar x}\|^2\right)\\ &\le c_2\tau(\mathscr{F}^{n+1}+\mathscr{F}^n),\quad 1\le n\le N-1,\end{aligned}$$

where $c_2=4\sqrt{L(3+h_0^2/12)\widehat{c}}$. When $c_2\tau\le 1/3$, applying discrete Gronwall's inequality, we have

$$\mathscr{F}^{n+1}\le\exp(3c_2T)\mathscr{F}^1=\exp(3c_2T)(\widetilde{\mathscr{F}}^1+\widetilde{\mathscr{F}}^0)\le C,\quad 1\le n\le N-1,$$

which implies $\|u^{n+1}_{x\bar x}\|\le C$, $\|\rho^{n+1}_x\|\le C$ for $1\le n\le N-1$. Hence, we have $\|u^{n+1}_x\|_\infty\le C$, $\|\rho^{n+1}\|_\infty\le C$ for $1\le n\le N-1$ by Lemma 2.2. This completes the proof. □

6 Uniqueness

Theorem 6.1 *The compact difference scheme (3.17)–(3.25) is uniquely solvable.*

Proof Obviously, it is easy to know that u^0, ρ^0, v^0, w^0 have been determined from Eqs. (3.21)–(3.23). The first level u^1, ρ^1, v^1, w^1 are computed by Eqs. (3.17), (3.18), (3.21) and (3.22). Now, we consider its homogenous system

$$\begin{aligned}&\frac{1}{\tau}(u^1_j-v^1_j)+\frac{1}{2}(\rho^1_j)_{\hat x}-\frac{h^2}{12}(w^1_j)_{\hat x}+\frac{1}{2}\psi(u^0_j,u^1_j)-\frac{h^2}{4}\psi(v^0_j,u^1_j)=0,\\ &\quad 1\le j\le J,\end{aligned} \tag{6.1}$$

$$\frac{1}{\tau}\rho_j^1 + \frac{1}{2}(u_j^1)_{\hat{x}} - \frac{h^2}{12}(v_j^1)_{\hat{x}} = 0, \quad 1 \le j \le J, \tag{6.2}$$

$$v_j^1 = (u_j^1)_{x\bar{x}} - \frac{h^2}{12}(v_j^1)_{x\bar{x}}, \quad 1 \le j \le J, \tag{6.3}$$

$$w_j^1 = (\rho_j^1)_{x\bar{x}} - \frac{h^2}{12}(w_j^1)_{x\bar{x}}, \quad 1 \le j \le J. \tag{6.4}$$

Taking the inner product of Eq. (6.1) with u^1, of Eq. (6.2) with ρ^1 and applying Lemma 2.1, we have

$$\frac{1}{\tau}\|u^1\|^2 - \frac{1}{\tau}\langle v^1, u^1\rangle + \frac{1}{2}\langle \rho_{\hat{x}}^1, u^1\rangle - \frac{h^2}{12}\langle w_{\hat{x}}^1, u^1\rangle = 0, \tag{6.5}$$

$$\frac{1}{\tau}\|\rho^1\|^2 + \frac{1}{2}\langle u_{\hat{x}}^1, \rho^1\rangle - \frac{h^2}{12}\langle v_{\hat{x}}^1, \rho^1\rangle = 0. \tag{6.6}$$

Adding Eq. (6.5) to Eq. (6.6), using Lemmas 2.2, 2.4, 2.5 and taking $R = S = 0$, we have

$$0 = \frac{1}{\tau}(\|u^1\|^2 + \|\rho^1\| - \langle v^1, u^1\rangle) \ge \frac{1}{\tau}\left(\|u^1\|^2 + \|\rho^1\| + \|u_x^1\|^2 + \frac{h^2}{18}\|v^1\|^2\right). \tag{6.7}$$

Thus, it holds that $\|u^1\| = 0$, $\|\rho^1\| = 0$, $\|v^1\| = 0$ and $\|w^1\| = 0$ by Eq. (6.4), which implies Eqs. (3.17), (3.18), (3.21) and (3.22) determined u^1, ρ^1, v^1, w^1 uniquely.

Now, we suppose u^k, ρ^k, v^k, w^k for $0 \le k \le n$ have been determined. u^{n+1}, ρ^{n+1}, v^{n+1}, w^{n+1} are computed by Eqs. (3.19)–(3.22), we consider its homogenous system

$$\frac{1}{2\tau}(u_j^{n+1} - v_j^{n+1}) + \frac{1}{2}(\rho_j^{n+1})_{\hat{x}} - \frac{h^2}{12}(w_j^{n+1})_{\hat{x}} + \frac{1}{2}\psi(u_j^n, u_j^{n+1}) - \frac{h^2}{4}\psi(v_j^n, u_j^{n+1}) = 0, \quad 1 \le j \le J, \tag{6.8}$$

$$\frac{1}{2\tau}\rho_j^{n+1} + \frac{1}{2}(u_j^{n+1})_{\hat{x}} - \frac{h^2}{12}(v_j^{n+1})_{\hat{x}} = 0, \quad 1 \le j \le J, \tag{6.9}$$

$$v_j^{n+1} = (u_j^{n+1})_{x\bar{x}} - \frac{h^2}{12}(v_j^{n+1})_{x\bar{x}}, \quad 1 \le j \le J, \tag{6.10}$$

$$w_j^{n+1} = (\rho_j^{n+1})_{x\bar{x}} - \frac{h^2}{12}(w_j^{n+1})_{x\bar{x}}, \quad 1 \le j \le J. \tag{6.11}$$

Taking the inner product of Eq. (6.8) with u^{n+1}, of Eq. (6.9) with ρ^{n+1} and applying Lemma 2.1, we have

$$\frac{1}{2\tau}\|u^{n+1}\|^2 - \frac{1}{2\tau}\langle v^{n+1}, u^{n+1}\rangle + \frac{1}{2}\langle \rho_{\hat{x}}^{n+1}, u^{n+1}\rangle - \frac{h^2}{12}\langle w_{\hat{x}}^{n+1}, u^{n+1}\rangle = 0, \tag{6.12}$$

$$\frac{1}{2\tau}\|\rho^{n+1}\|^2 + \frac{1}{2}\langle u_{\hat{x}}^{n+1}, \rho^{n+1}\rangle - \frac{h^2}{12}\langle v_{\hat{x}}^{n+1}, \rho^{n+1}\rangle = 0. \tag{6.13}$$

Similar to Eq. (6.7), we have

$$
\begin{aligned}
0 &= \frac{1}{2\tau}(\|u^{n+1}\|^2 + \|\rho^{n+1}\| - \langle v^{n+1}, u^{n+1}\rangle) \\
&\geq \frac{1}{2\tau}(\|u^{n+1}\|^2 + \|\rho^{n+1}\| + \|u_x^{n+1}\|^2 + \frac{h^2}{18}\|v^{n+1}\|^2),
\end{aligned}
$$

then it holds that $\|u^{n+1}\| = 0$, $\|\rho^{n+1}\| = 0$, $\|v^{n+1}\| = 0$ and $\|w^{n+1}\| = 0$ by Eq. (6.11), which implies Eqs. (3.17)–(3.19) determined u^{n+1}, ρ^{n+1}, v^{n+1}, w^{n+1} uniquely. This completes the proof. □

7 Convergence and stability

Theorem 7.1 *Suppose that $u_0(x) \in H^2(\Omega)$, $\rho_0(x) \in H^1(\Omega)$ and $u(x,t), \rho(x,t) \in C_{x,t}^{5,3}(\Omega \times (0,T])$. Let (U^n, ϕ^n, V^n, W^n) be the exact solution of the* Eqs. (3.1)–(3.7) *and (u^n, ρ^n, v^n, w^n) be the numerical solution of the compact scheme (3.17)–(3.25). Denote*

$$
e^n = U^n - u^n,\ \eta^n = \phi^n - \rho^n,\ f^n = V^n - v^n,\ \xi^n = W^n - w^n.
$$

There exist sufficiently small positive constants τ_0, h_0 such that $\tau \leq \tau_0$, $h \leq h_0$, then we have

$$
\begin{aligned}
&\|e^n\| + \|e_x^n\| + \|e_{x\bar{x}}^n\| \leq C(\tau^2 + h^4), \quad \|\eta^n\| + \|\eta_x^n\| \leq C(\tau^2 + h^4), \\
&\|e^n\|_\infty + \|e_x^n\|_\infty + \|\eta^n\|_\infty \leq C(\tau^2 + h^4).
\end{aligned}
$$

Proof Substituting Eqs. (3.8)–(3.16) from Eqs. (3.17)–(3.25), we obtain the following error system:

$$
\begin{aligned}
&(e_j^0)_t - (f_j^0)_t + (\eta_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(\xi_j^{\frac{1}{2}})_{\hat{x}} + \psi(U_j^0, U_j^{\frac{1}{2}}) - \psi(u_j^0, u_j^{\frac{1}{2}}) \\
&- \frac{h^2}{2}\left[\psi(V_j^0, U_j^{\frac{1}{2}}) - \psi(v_j^0, u_j^{\frac{1}{2}})\right] = P_j^0, \quad 1 \leq j \leq J, \qquad (7.1)
\end{aligned}
$$

$$
(\eta_j^0)_t + (e_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(f_j^{\frac{1}{2}})_{\hat{x}} = Q_j^0, \quad 1 \leq j \leq J, \qquad (7.2)
$$

$$
\begin{aligned}
&(e_j^n)_{\hat{t}} - (f_j^n)_{\hat{t}} + (\eta_j^{\bar{n}})_{\hat{x}} - \frac{h^2}{6}(\xi_j^{\bar{n}})_{\hat{x}} + \psi(U_j^n, U_j^{\bar{n}}) - \psi(u_j^n, u_j^{\bar{n}}) \\
&- \frac{h^2}{2}\left[\psi(V_j^n, U_j^{\bar{n}}) - \psi(v_j^n, u_j^{\bar{n}})\right] = P_j^n, \quad 1 \leq j \leq J, \quad 1 \leq n \leq N-1, \qquad (7.3)
\end{aligned}
$$

$$
(\eta_j^n)_{\hat{t}} + (e_j^{\bar{n}})_{\hat{x}} - \frac{h^2}{6}(f_j^{\bar{n}})_{\hat{x}} = Q_j^n, \quad 1 \leq j \leq J, \quad 1 \leq n \leq N-1, \qquad (7.4)
$$

$$
f_j^n = (e_j^n)_{x\bar{x}} - \frac{h^2}{12}(f_j^n)_{x\bar{x}} + R_j^n, \quad 1 \leq j \leq J, \quad 0 \leq n \leq N, \qquad (7.5)
$$

$$\xi_j^n = (\eta_j^n)_{x\bar{x}} - \frac{h^2}{12}(\xi_j^n)_{x\bar{x}} + S_j^n, \quad 1 \le j \le J, \quad 0 \le n \le N, \tag{7.6}$$

$$e_j^0 = 0, \quad \eta_j^0 = 0, \quad 0 \le j \le J, \tag{7.7}$$

$$e_j^n = e_{j+J}^n, \quad \eta_j^n = \eta_{j+J}^n, \quad 0 \le j \le J, \quad 0 \le n \le N, \tag{7.8}$$

$$f_j^n = f_{j+J}^n, \quad \xi_j^n = \xi_{j+J}^n, \quad 0 \le j \le J, \quad 0 \le n \le N. \tag{7.9}$$

From Eq. (7.7), considering Eqs. (7.1), (7.2), (7.5) and (7.6) when $n = 0$, we have

$$\frac{1}{\tau}(e_j^1 - f_j^1) + \frac{1}{2}(\eta_j^1)_{\hat{x}} - \frac{h^2}{12}(\xi_j^1)_{\hat{x}} + \psi(U_j^0, e_j^1) - \frac{h^2}{2}\psi(V_j^0, e_j^1) = P_j^0, \quad 1 \le j \le J, \tag{7.10}$$

$$\frac{1}{\tau}\eta_j^1 + \frac{1}{2}(e_j^1)_{\hat{x}} - \frac{h^2}{12}(f_j^1)_{\hat{x}} = Q_j^0, \quad 1 \le j \le J, \tag{7.11}$$

$$f_j^1 = (e_j^1)_{x\bar{x}} - \frac{h^2}{12}(f_j^1)_{x\bar{x}} + R_j^1, \quad 1 \le j \le J, \tag{7.12}$$

$$\xi_j^1 = (\eta_j^1)_{x\bar{x}} - \frac{h^2}{12}(\xi_j^1)_{x\bar{x}} + S_j^1, \quad 1 \le j \le J. \tag{7.13}$$

Taking the inner product of Eq. (7.10) with e^1, of Eq. (7.11) with η^1 and applying Lemma 2.1, we have

$$\frac{1}{\tau}\|e^1\|^2 - \frac{1}{\tau}\langle f^1, e^1\rangle + \frac{1}{2}\langle \eta_{\hat{x}}^1, e^1\rangle - \frac{h^2}{12}\langle \xi_{\hat{x}}^1, e^1\rangle = \langle P^0, e^1\rangle, \tag{7.14}$$

$$\frac{1}{\tau}\|\eta^1\|^2 + \frac{1}{2}\langle e_{\hat{x}}^1, \eta^1\rangle - \frac{h^2}{12}\langle f_{\hat{x}}^1, \eta^1\rangle = \langle Q^0, \eta^1\rangle. \tag{7.15}$$

Combining Lemmas 2.4, 2.5, we have

$$\langle f^1, e^1\rangle = -\|e_x^1\|^2 - \frac{h^2}{12}\|f^1\|^2 + \frac{h^4}{144}\|f_x^1\|^2 + \frac{h^2}{12}\langle R^1, f^1\rangle + \langle R^1, e^1\rangle,$$

$$\langle \xi_{\hat{x}}^1, e^1\rangle + \langle f_{\hat{x}}^1, \eta^1\rangle = \frac{h^2}{12}(\langle \xi_{\hat{x}}^1, R^1\rangle + \langle f_{\hat{x}}^1, S^1\rangle) - \langle S^1, e_{\hat{x}}^1\rangle - \langle R^1, \eta_{\hat{x}}^1\rangle,$$

$$\langle \eta_{\hat{x}}^1, e^1\rangle + \langle e_{\hat{x}}^1, \eta^1\rangle = 0.$$

Adding Eq. (7.14) to Eq. (7.15), we have

$$\begin{aligned}
&\|e^1\|^2 + \|e_x^1\|^2 + \frac{h^2}{12}\|f^1\|^2 - \frac{h^4}{144}\|f_x^1\|^2 + \|\eta^1\|^2 \\
= &\frac{h^2}{12}\langle R^1, f^1\rangle + \langle R^1, e^1\rangle + \frac{h^4\tau}{144}(\langle \xi_{\hat{x}}^1, R^1\rangle + \langle f_{\hat{x}}^1, S^1\rangle) \\
&-\frac{h^2\tau}{12}(\langle S^1, e_{\hat{x}}^1\rangle + \langle R^1, \eta_{\hat{x}}^1\rangle) \\
&+\tau(\langle P^0, e^1\rangle + \langle Q^0, \eta^1\rangle).
\end{aligned} \tag{7.16}$$

Taking the inner product of Eq. (7.10) with $-e^1_{x\bar{x}}$ and of Eq. (7.11) with $-\eta^1_{x\bar{x}}$, we have

$$\frac{1}{\tau}\|e^1_x\|^2+\frac{1}{\tau}\langle f^1,e^1_{x\bar{x}}\rangle-\frac{1}{2}\langle\eta^1_{\hat{x}},e^1_{x\bar{x}}\rangle+\frac{h^2}{12}\langle\xi^1_{\hat{x}},e^1_{x\bar{x}}\rangle-\langle\psi(U^0,e^1),e^1_{x\bar{x}}\rangle$$
$$+\frac{h^2}{2}\langle\psi(V^0,e^1),e^1_{x\bar{x}}\rangle=-\langle P^0,e^1_{x\bar{x}}\rangle \tag{7.17}$$

and

$$\frac{1}{\tau}\|\eta^1_x\|^2-\frac{1}{2}\langle e^1_{\hat{x}},\eta^1_{x\bar{x}}\rangle+\frac{h^2}{12}\langle f^1_{\hat{x}},\eta^1_{x\bar{x}}\rangle=-\langle Q^0,\eta^1_{x\bar{x}}\rangle. \tag{7.18}$$

Applying Lemmas 2.4, 2.5, we have

$$\langle\xi^1_{\hat{x}},e^1_{x\bar{x}}\rangle+\langle f^1_{\hat{x}},\eta^1_{x\bar{x}}\rangle=-\langle\xi^1_{\hat{x}},R^1\rangle-\langle f^1_{\hat{x}},S^1\rangle,\quad \langle\eta^1_{\hat{x}},e^1_{x\bar{x}}\rangle+\langle e^1_{\hat{x}},\eta^1_{x\bar{x}}\rangle=0.$$

It follows from Eq. (7.12) that

$$\begin{aligned}\langle f^1,e^1_{x\bar{x}}\rangle&=\langle e^1_{x\bar{x}}-\frac{h^2}{12}f^1_{x\bar{x}}+R^1,e^1_{x\bar{x}}\rangle\\&=\|e^1_{x\bar{x}}\|^2-\frac{h^2}{12}\langle f^1_{x\bar{x}},f^1+\frac{h^2}{12}f^1_{x\bar{x}}-R^1\rangle+\langle R^1,e^1_{x\bar{x}}\rangle\\&=\|e^1_{x\bar{x}}\|^2+\frac{h^2}{12}\|f^1_x\|^2-\frac{h^4}{144}\|f^1_{x\bar{x}}\|^2+\frac{h^2}{12}\langle f^1_{x\bar{x}},R^1\rangle+\langle R^1,e^1_{x\bar{x}}\rangle.\end{aligned}$$

With the help of the above expression, summing Eqs. (7.17), (7.18), we have

$$\begin{aligned}&\|e^1_x\|^2+\|e^1_{x\bar{x}}\|^2+\frac{h^2}{12}\|f^1_x\|^2-\frac{h^4}{144}\|f^1_{x\bar{x}}\|^2+\|\eta^1_x\|^2\\&=-\frac{h^2}{12}\langle f^1_{x\bar{x}},R^1\rangle-\langle R^1,e^1_{x\bar{x}}\rangle+\frac{h^2\tau}{12}(\langle\xi^1_{\hat{x}},R^1\rangle+\langle f^1_{\hat{x}},S^1\rangle)\\&\quad-\tau(\langle P^0,e^1_{x\bar{x}}\rangle+\langle Q^0,\eta^1_{x\bar{x}}\rangle)\\&\quad+\tau\left(\langle\psi(U^0,e^1),e^1_{x\bar{x}}\rangle-\frac{h^2}{2}\langle\psi(V^0,e^1),e^1_{x\bar{x}}\rangle\right).\end{aligned} \tag{7.19}$$

Summing Eqs. (7.16), (7.19), we have

$$\begin{aligned}&\|e^1\|^2+2\|e^1_x\|^2+\|e^1_{x\bar{x}}\|^2+\frac{h^2}{12}\|f^1\|^2+\left(\frac{h^2}{12}-\frac{h^4}{144}\right)\|f^1_x\|^2\\&-\frac{h^4}{144}\|f^1_{x\bar{x}}\|^2+\|\eta^1\|^2+\|\eta^1_x\|^2\\&=\frac{h^2}{12}(\langle R^1,f^1\rangle-\langle f^1_{x\bar{x}},R^1\rangle)+\langle R^1,e^1\rangle-\langle R^1,e^1_{x\bar{x}}\rangle-\frac{h^2\tau}{12}(\langle S^1,e^1_{\hat{x}}\rangle+\langle R^1,\eta^1_{\hat{x}}\rangle)\\&\quad+\left(\frac{h^4\tau}{144}+\frac{h^2\tau}{12}\right)(\langle\xi^1_{\hat{x}},R^1\rangle+\langle f^1_{\hat{x}},S^1\rangle)+\tau(\langle P^0,e^1\rangle+\langle Q^0,\eta^1\rangle\\&\quad-\langle P^0,e^1_{x\bar{x}}\rangle-\langle Q^0,\eta^1_{x\bar{x}}\rangle)\\&\quad+\tau\left(\langle\psi(U^0,e^1),e^1_{x\bar{x}}\rangle-\frac{h^2}{2}\langle\psi(V^0,e^1),e^1_{x\bar{x}}\rangle\right).\end{aligned} \tag{7.20}$$

Using Lemma 2.2, we have

$$\begin{aligned}
&\|e^1\|^2 + 2\|e_x^1\|^2 + \|e_{x\bar{x}}^1\|^2 + \frac{h^2}{12}\|f^1\|^2 + \left(\frac{h^2}{12} - \frac{h^4}{144}\right)\|f_x^1\|^2 - \frac{h^4}{144}\|f_{x\bar{x}}^1\|^2 \\
&+\|\eta^1\|^2 + \|\eta_x^1\|^2 \\
\geq\ &\|e^1\|^2 + \|e_{x\bar{x}}^1\|^2 + \frac{1}{2}(\|e_x^1\|^2 + \|\eta^1\|^2 + \|\eta_x^1\|^2) + \frac{h^2}{18}(\|f^1\|^2 + \|f_x^1\|^2). \quad (7.21)
\end{aligned}$$

It follows from Lemma 2.2 and Eq. (7.13) that

$$\begin{aligned}
\|\xi^n\|^2 &= \langle \eta_{x\bar{x}}^n, \xi^n\rangle - \frac{h^2}{12}\langle \xi_{x\bar{x}}^n, \xi^n\rangle + \langle S^n, \xi^n\rangle \leq \|\eta_{x\bar{x}}^n\|\|\xi^n\| + \frac{h^2}{12}\|\xi_x^n\|^2 + \|S^n\|\|\xi^n\| \\
&\leq \frac{2}{3}\|\xi^n\|^2 + \frac{6}{h^2}\|\eta_x^n\|^2 + \frac{3}{2}\|S^n\|^2, \quad 0 \leq n \leq N,
\end{aligned}$$

we have

$$\|\xi^n\|^2 \leq \frac{18}{h^2}\|\eta_x^n\|^2 + \frac{9}{2}\|S^n\|^2, \quad 0 \leq n \leq N. \quad (7.22)$$

Similar to Eqs. (5.5), (5.6), we have

$$\langle \psi(U^0, e^1), e_{x\bar{x}}^1\rangle \leq \frac{\widehat{c}}{3}\left(2 + \frac{L}{\sqrt{6}}\right)\|e_x^1\|\|e_{x\bar{x}}^1\|, \quad (7.23)$$

$$\frac{h^2}{2}\langle \psi(V^0, e^1), e_{x\bar{x}}^1\rangle \leq \frac{\widehat{c}h^2}{6}\left(2 + \frac{L}{\sqrt{6}}\right)\|e_x^1\|\|e_{x\bar{x}}^1\|. \quad (7.24)$$

Substituting Eqs. (7.21)–(7.24) to Eq. (7.20) and applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\|e^1\|^2 + \|e_{x\bar{x}}^1\|^2 + \frac{1}{2}(\|e_x^1\|^2 + \|\eta^1\|^2 + \|\eta_x^1\|^2) + \frac{h^2}{18}(\|f^1\|^2 + \|f_x^1\|^2) \\
\leq\ &\frac{h^2}{12}(\|R^1\|\|f^1\| + \|f_x^1\|\|R_x^1\|) + \|R^1\|\|e^1\| + \|R^1\|\|e_{x\bar{x}}^1\| \\
&+\frac{h^2\tau}{12}(\|S^1\|\|e_x^1\| + \|R^1\|\|\eta_x^1\|) \\
&+\frac{h^4\tau}{144}\|\xi_x^1\|\|R^1\| + \frac{h^2\tau}{12}\|\xi^1\|\|R_x^1\| + \left(\frac{h^4\tau}{144} + \frac{h^2\tau}{12}\right)\|f_x^1\|\|S^1\| \\
&+\tau(\|P^0\|\|e^1\| + \|Q^0\|\|\eta^1\| + \|P^0\|\|e_{x\bar{x}}^1\| + \|Q_x^0\|\|\eta_x^1\|) \\
&+\frac{\widehat{c}\tau}{6}\left(2 + \frac{L}{\sqrt{6}}\right)(2 + h^2)\|e_x^1\|\|e_{x\bar{x}}^1\| \\
\leq\ &\frac{h^2}{36}(\|f^1\|^2 + \|f_x^1\|^2) + \frac{h^2}{16}(\|R^1\|^2 + \|R_x^1\|^2) + \|R^1\|^2 \\
&+\frac{1}{2}(\|e_{x\bar{x}}^1\|^2 + \|e^1\|^2) + \frac{h^4\tau}{288}\|R^1\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{h^2\tau}{24}\|R_x^1\|^2+\frac{h^2\tau}{18}\|\xi^1\|^2+\frac{h^2\tau}{24}(\|S^1\|^2+\|e_x^1\|^2+\|R^1\|^2+\|\eta_x^1\|^2)\\
&+\frac{\tau}{2}(2\|P^0\|^2+\|e^1\|^2+\|Q^0\|^2+\|\eta^1\|^2+\|e_{x\bar{x}}^1\|^2+\|Q_x^0\|^2+\|\eta^1\|^2)\\
&+\frac{\widehat{c}\tau}{12}\left(2+\frac{L}{\sqrt{6}}\right)(2+h^2)(\|e_x^1\|^2+\|e_{x\bar{x}}^1\|^2)+\left(\frac{h^4\tau}{288}+\frac{h^2\tau}{24}\right)(\|f_x^1\|^2+\|S^1\|^2)\\
\le\ &\frac{1}{2}(\|e^1\|^2+\|e_{x\bar{x}}^1\|^2)+\frac{h^2}{36}(\|f^1\|^2+\|f_x^1\|^2)+\frac{\tau}{2}(\|e^1\|^2+\|\eta^1\|^2)\\
&+\left(\frac{h^4}{288}+\frac{h^2}{24}\right)\tau\|f_x^1\|^2\\
&+\left(\frac{h^2}{24}+\frac{1}{2}\right)\tau\|\eta_x^1\|^2+\left[\frac{1}{2}+\frac{h^2}{24}+\frac{\widehat{c}}{12}\left(2+\frac{L}{\sqrt{6}}\right)(2+h^2)\right]\tau(\|e_x^1\|^2+\|e_{x\bar{x}}^1\|^2)\\
&+\tau\|P^0\|^2\\
&+\frac{\tau}{2}(\|Q^0\|^2+\|Q_x^0\|^2)+\left[\frac{5}{4}+\frac{h^2}{16}+\left(\frac{h^2}{24}+\frac{h^4}{288}+\frac{1}{6}\right)\tau\right]\|R^1\|^2\\
&+\left(\frac{h^2}{12}+\frac{h^4}{288}\right)\tau\|S^1\|^2.
\end{aligned}
$$

Letting

$$
\widetilde{\mathscr{G}}^1=\frac{1}{2}(\|e^1\|^2+\|e_x^1\|^2+\|e_{x\bar{x}}^1\|^2+\|\eta^1\|^2+\|\eta_x^1\|^2)+\frac{h^2}{36}(\|f^1\|^2+\|f_x^1\|^2),
$$

there exist sufficiently small positive constants τ_0, h_0 such that $\tau \le \tau_0$, $h \le h_0$, we have

$$
\widetilde{\mathscr{G}}^1 \le c_3\tau\widetilde{\mathscr{G}}^1+c_4(\|R^1\|^2+\|S^1\|^2)+2\tau\|P^0\|^2+\tau(\|Q^0\|^2+\|Q_x^0\|^2),
$$

where

$$
c_3=1+\frac{h_0^2}{12}+\frac{\widehat{c}}{6}\left(2+\frac{L}{\sqrt{6}}\right)(2+h_0^2),\quad c_4=\frac{5}{2}+\frac{h_0^2}{8}+\frac{h_0^2\tau_0}{12}+\frac{h_0^4\tau_0}{288}+\frac{\tau_0}{6}.
$$

When $c_3\tau \le 1/2$, we have

$$
\widetilde{\mathscr{G}}^1 \le 2c_4(\|R^1\|^2+\|S^1\|^2)+4\tau\|P^0\|^2+2\tau(\|Q^0\|^2+\|Q_x^0\|^2)\le C(\tau^2+h^4)^2,
$$

which implies

$$
\begin{aligned}
&\|e^1\|+\|e_x^1\|+\|e_{x\bar{x}}^1\|\le C(\tau^2+h^4),\quad \|\eta^1\|+\|\eta_x^1\|\le C(\tau^2+h^4),\\
&\|f^1\|+\|f_x^1\|\le\frac{C}{h}(\tau^2+h^4).
\end{aligned}
$$

Thus, we have

$$
\|e^1\|_\infty+\|e_x^1\|_\infty+\|\eta^1\|_\infty\le C(\tau^2+h^4),
$$

by Lemma 2.2.

Now, we suppose that

$$\|e^k\| + \|e_x^k\| + \|e_{x\bar{x}}^k\| \le C(\tau^2 + h^4), \quad \|\eta^k\| + \|\eta_x^k\| \le C(\tau^2 + h^4),$$
$$\|e^k\|_\infty + \|e_x^k\|_\infty + \|\eta^k\|_\infty \le C(\tau^2 + h^4),$$

for $0 \le k \le n$. Taking the inner product of Eq. (7.3) with $2e^{\bar{n}}$ and the inner product of Eq. (7.4) with $2\eta^{\bar{n}}$, we have

$$\|e^n\|_{\hat{t}}^2 - \langle f_{\hat{t}}^n, 2e^{\bar{n}}\rangle + \langle \eta_{\hat{x}}^{\bar{n}}, 2e^{\bar{n}}\rangle - \frac{h^2}{6}\langle \xi_{\hat{x}}^{\bar{n}}, 2e^{\bar{n}}\rangle + \langle \psi(U^n, U^{\bar{n}}) - \psi(u^n, u^{\bar{n}}), 2e^{\bar{n}}\rangle$$
$$-\frac{h^2}{2}\langle \psi(V^n, U^{\bar{n}}) - \psi(v^n, u^{\bar{n}}), 2e^{\bar{n}}\rangle = \langle P^n, 2e^{\bar{n}}\rangle, \quad 1 \le n \le N-1, \tag{7.25}$$
$$\|\eta^n\|_{\hat{t}}^2 + \langle e_{\hat{x}}^{\bar{n}}, 2\eta^{\bar{n}}\rangle - \frac{h^2}{6}\langle f_{\hat{x}}^{\bar{n}}, 2\eta^{\bar{n}}\rangle = \langle Q^n, 2\eta^{\bar{n}}\rangle, \quad 1 \le n \le N-1. \tag{7.26}$$

Applying Lemmas 2.4, 2.5, we have

$$\langle \eta_{\hat{x}}^{\bar{n}}, e^{\bar{n}}\rangle + \langle e_{\hat{x}}^{\bar{n}}, \eta^{\bar{n}}\rangle = 0, \tag{7.27}$$
$$\langle f_{\hat{t}}^n, 2e^{\bar{n}}\rangle = -\|e_x^n\|_{\hat{t}}^2 - \frac{h^2}{12}\|f^n\|_{\hat{t}}^2 + \frac{h^4}{144}\|f_x^n\|_{\hat{t}}^2 + \frac{h^2}{12}\langle f_{\hat{t}}^n, 2R^{\bar{n}}\rangle + \langle R_{\hat{t}}^n, 2e^{\bar{n}}\rangle, \tag{7.28}$$
$$\langle \xi_{\hat{x}}^{\bar{n}}, e^{\bar{n}}\rangle + \langle f_{\hat{x}}^{\bar{n}}, \eta^{\bar{n}}\rangle = \frac{h^2}{12}(\langle \xi_{\hat{x}}^{\bar{n}}, R^n\rangle + \langle f_{\hat{x}}^{\bar{n}}, S^n\rangle) - \langle S^n, e_{\hat{x}}^{\bar{n}}\rangle - \langle R^n, \eta_{\hat{x}}^{\bar{n}}\rangle. \tag{7.29}$$

Substituting Eqs. (7.27)–(7.29) to Eqs. (7.25), (7.26) and adding Eq. (7.25) to Eq. (7.26), we have

$$\begin{aligned}
&\|e^n\|_{\hat{t}}^2 + \|e_x^n\|_{\hat{t}}^2 + \frac{h^2}{12}\|f^n\|_{\hat{t}}^2 - \frac{h^4}{144}\|f_x^n\|_{\hat{t}}^2 + \|\eta^n\|_{\hat{t}}^2 \\
=\ & \frac{h^2}{12}\langle f_{\hat{t}}^n, 2R^{\bar{n}}\rangle + \langle R_{\hat{t}}^n, 2e^{\bar{n}}\rangle + \frac{h^4}{36}(\langle \xi_{\hat{x}}^{\bar{n}}, R^n\rangle + \langle f_{\hat{x}}^{\bar{n}}, S^n\rangle) \\
&-\frac{h^2}{3}(\langle S^n, e_{\hat{x}}^{\bar{n}}\rangle + \langle R^n, \eta_{\hat{x}}^{\bar{n}}\rangle) + \langle P^n, 2e^{\bar{n}}\rangle + \langle Q^n, 2\eta^{\bar{n}}\rangle \\
&-\langle \psi(U^n, U^{\bar{n}}) - \psi(u^n, u^{\bar{n}}), 2e^{\bar{n}}\rangle + \frac{h^2}{2}\langle \psi(V^n, U^{\bar{n}}) - \psi(v^n, u^{\bar{n}}), 2e^{\bar{n}}\rangle, \\
&\quad 1 \le n \le N-1.
\end{aligned} \tag{7.30}$$

Again taking the inner product of Eq. (7.3) with $-2e_{x\bar{x}}^{\bar{n}}$ and the inner product of Eq. (7.4) with $-2\eta_{x\bar{x}}^{\bar{n}}$, we have

$$\|e_x^n\|_{\hat{t}}^2 + \langle f_{\hat{t}}^n, 2e_{x\bar{x}}^{\bar{n}}\rangle - \langle \eta_{\hat{x}}^{\bar{n}}, 2e_{x\bar{x}}^{\bar{n}}\rangle + \frac{h^2}{6}\langle \xi_{\hat{x}}^{\bar{n}}, 2e_{x\bar{x}}^{\bar{n}}\rangle - \langle \psi(U^n, U^{\bar{n}}) - \psi(u^n, u^{\bar{n}}), 2e_{x\bar{x}}^{\bar{n}}\rangle$$
$$+\frac{h^2}{2}\langle \psi(V^n, U^{\bar{n}}) - \psi(v^n, u^{\bar{n}}), 2e_{x\bar{x}}^{\bar{n}}\rangle = -\langle P^n, 2e_{x\bar{x}}^{\bar{n}}\rangle, \quad 1 \le n \le N-1, \tag{7.31}$$
$$\|\eta_x^n\|_{\hat{t}}^2 - \langle e_{\hat{x}}^{\bar{n}}, 2\eta_{x\bar{x}}^{\bar{n}}\rangle + \frac{h^2}{6}\langle f_{\hat{x}}^{\bar{n}}, 2\eta_{x\bar{x}}^{\bar{n}}\rangle = -\langle Q^n, 2\eta_{x\bar{x}}^{\bar{n}}\rangle, \quad 1 \le n \le N-1. \tag{7.32}$$

Applying Lemmas 2.4, 2.5, we have

$$\langle \eta_{\hat{x}}^{\bar{n}}, e_{x\bar{x}}^{\bar{n}}\rangle + \langle e_{\hat{x}}^{\bar{n}}, \eta_{x\bar{x}}^{\bar{n}}\rangle = 0, \tag{7.33}$$

$$\begin{aligned}\langle f_{\hat{t}}^{n}, 2e_{x\bar{x}}^{\bar{n}}\rangle &= \|e_{x\bar{x}}^{n}\|_{\hat{t}}^{2} + \frac{h^2}{12}\|f_x^n\|_{\hat{t}}^{2} - \frac{h^4}{144}\|f_{x\bar{x}}^{n}\|_{\hat{t}}^{2}\\ &+\frac{h^2}{12}\langle f_{x\bar{x}\hat{t}}^{n}, 2R^{\bar{n}}\rangle + \langle R_{\hat{t}}^{n}, 2e_{x\bar{x}}^{\bar{n}}\rangle,\end{aligned} \tag{7.34}$$

$$\langle \xi_{\hat{x}}^{\bar{n}}, e_{x\bar{x}}^{\bar{n}}\rangle + \langle f_{\hat{x}}^{\bar{n}}, \eta_{x\bar{x}}^{\bar{n}}\rangle = -\langle \xi_{\hat{x}}^{\bar{n}}, R^{n}\rangle - \langle f_{\hat{x}}^{\bar{n}}, S^{n}\rangle. \tag{7.35}$$

Substituting Eqs. (7.33)–(7.35) to Eqs. (7.31), (7.32) and adding Eq. (7.31) to Eq. (7.32), we have

$$\begin{aligned}&\|e_x^n\|_{\hat{t}}^2 + \|e_{x\bar{x}}^n\|_{\hat{t}}^2 + \frac{h^2}{12}\|f_x^n\|_{\hat{t}}^2 - \frac{h^4}{144}\|f_{x\bar{x}}^n\|_{\hat{t}}^2 + \|\eta_x^n\|_{\hat{t}}^2\\ =& -\frac{h^2}{12}\langle f_{x\bar{x}\hat{t}}^{n}, 2R^{\bar{n}}\rangle - \langle R_{\hat{t}}^{n}, 2e_{x\bar{x}}^{\bar{n}}\rangle + \frac{h^2}{3}(\langle \xi_{\hat{x}}^{\bar{n}}, R^n\rangle + \langle f_{\hat{x}}^{\bar{n}}, S^n\rangle)\\ &+\langle \psi(U^n, U^{\bar{n}}) - \psi(u^n, u^{\bar{n}}), 2e_{x\bar{x}}^{\bar{n}}\rangle\\ &-\frac{h^2}{2}\langle \psi(V^n, U^{\bar{n}}) - \psi(v^n, u^{\bar{n}}), 2e_{x\bar{x}}^{\bar{n}}\rangle - \langle P^n, 2e_{x\bar{x}}^{\bar{n}}\rangle - \langle Q^n, 2\eta_{x\bar{x}}^{\bar{n}}\rangle,\\ &1 \le n \le N-1.\end{aligned} \tag{7.36}$$

Summing Eqs. (7.30), (7.36), we have

$$\begin{aligned}&\|e^n\|_{\hat{t}}^2 + 2\|e_x^n\|_{\hat{t}}^2 + \|e_{x\bar{x}}^n\|_{\hat{t}}^2 + \frac{h^2}{12}\|f^n\|_{\hat{t}}^2 + \left(\frac{h^2}{12} - \frac{h^4}{144}\right)\|f_x^n\|_{\hat{t}}^2\\ &-\frac{h^4}{144}\|f_{x\bar{x}}^n\|_{\hat{t}}^2 + \|\eta^n\|_{\hat{t}}^2 + \|\eta_x^n\|_{\hat{t}}^2\\ =& \frac{h^2}{12}\langle f_{\hat{t}}^n - f_{x\bar{x}\hat{t}}^n, 2R^{\bar{n}}\rangle + \left(\frac{h^2}{3} + \frac{h^4}{36}\right)(\langle \xi_{\hat{x}}^{\bar{n}}, R^n\rangle + \langle f_{\hat{x}}^{\bar{n}}, S^n\rangle)\\ &-\frac{h^2}{3}(\langle S^n, e_{\hat{x}}^{\bar{n}}\rangle + \langle R^n, \eta_{\hat{x}}^{\bar{n}}\rangle) + \langle Q^n, 2(\eta^{\bar{n}} - \eta_{x\bar{x}}^{\bar{n}})\rangle\\ &\quad+\Big\langle R_{\hat{t}}^n - (\psi(U^n, U^{\bar{n}}) - \psi(u^n, u^{\bar{n}})) + \frac{h^2}{2}(\psi(V^n, U^{\bar{n}})\\ &-\psi(v^n, u^{\bar{n}})) + P^n, 2(e^{\bar{n}} - e_{x\bar{x}}^{\bar{n}})\Big\rangle,\end{aligned} \tag{7.37}$$

for $1 \le n \le N-1$. Noticing

$$\begin{aligned}&\langle f_{\hat{t}}^n - f_{x\bar{x}\hat{t}}^n, 2R^{\bar{n}}\rangle\\ =& \left\langle f^{n+1} - f^{n-1}, 2R_{\hat{t}}^n + \frac{2}{\tau}R^{n-1}\right\rangle + \left\langle f_x^{n+1} - f_x^{n-1}, 2R_{x\hat{t}}^n + \frac{2}{\tau}R_x^{n-1}\right\rangle,\\ \le& 2\|f^{n+1} - f^{n-1}\|\left\|R_{\hat{t}}^n + \frac{2}{\tau}R^{\bar{n}}\right\| + 2\|f_x^{n+1} - f_x^{n-1}\|\left\|R_{x\hat{t}}^n + \frac{2}{\tau}R_x^{\bar{n}}\right\|\end{aligned} \tag{7.38}$$

and

$$
\begin{aligned}
&\psi(V_j, U_j) - \psi(v_j, u_j)\\
&= \frac{1}{3}[V_j(U_j)_{\hat{x}} + (V_jU_j)_{\hat{x}} - v_j(u_j)_{\hat{x}} - (v_ju_j)_{\hat{x}}]\\
&= \frac{1}{3}[(v_j + f_j)(U_j)_{\hat{x}} + ((v_j + f_j)U_j)_{\hat{x}} - v_j(u_j)_{\hat{x}} - (v_ju_j)_{\hat{x}}]\\
&= \frac{1}{3}[v_j(e_j)_{\hat{x}} + f_j(U_j)_{\hat{x}} + (v_je_j + f_jU_j)_{\hat{x}}]\\
&= \frac{1}{3}[(v_j + v_{j+1})(e_j)_{\hat{x}} + (v_j)_{\hat{x}}e_{j+1} + (f_j + f_{j+1})(U_j)_{\hat{x}} + (f_j)_{\hat{x}}U_{j+1}].
\end{aligned}
$$

Using Lemmas 2.1, 2.2, Theorem 5.1 and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\frac{h^2}{2}\langle\psi(V^n, U^{\bar{n}}) - \psi(v^n, u^{\bar{n}}), 2e^{\bar{n}}\rangle &= \frac{h^2}{3}\langle f^nU^{\bar{n}}_{\hat{x}} + (f^nU^{\bar{n}})_{\hat{x}}, e^{\bar{n}}\rangle\\
&\le \frac{h^2}{3}(2\|U^{\bar{n}}_{\hat{x}}\|_\infty\|f^n\| + \|U^{\bar{n}}\|_\infty\|f^n_x\|)\|e^{\bar{n}}\|\\
&\le \frac{\widehat{c}h^2}{3}(2\|f^n\| + \|f^n_x\|)\|e^{\bar{n}}\|, \qquad (7.39)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{h^2}{2}\langle\psi(V^n, U^{\bar{n}}) - \psi(v^n, u^{\bar{n}}), 2e^{\bar{n}}_{x\bar{x}}\rangle\\
&= \frac{h^2}{3}\langle v^ne^{\bar{n}}_{\hat{x}} + f^nU^{\bar{n}}_{\hat{x}} + (v^ne^{\bar{n}} + f^nU^{\bar{n}})_{\hat{x}}, e^{\bar{n}}_{x\bar{x}}\rangle\\
&\le \frac{h^2}{3}(2\|v^n\|_\infty\|e^{\bar{n}}_{\hat{x}}\| + \|e^{\bar{n}}\|_\infty\|v^n_x\| + 2\|U^{\bar{n}}_{\hat{x}}\|_\infty\|f^n\| + \|U^{\bar{n}}\|_\infty\|f^n_x\|)\|e^{\bar{n}}_{x\bar{x}}\|\\
&\le \frac{\sqrt{L}h^2}{2}\|v^n_x\|\|e^{\bar{n}}\|\|e^{\bar{n}}_{x\bar{x}}\| + \frac{\widehat{c}h^2}{3}(2\|f^n\| + \|f^n_x\|)\|e^{\bar{n}}_{x\bar{x}}\|\\
&\le 6\sqrt{LE^0}\|e^{\bar{n}}\|\|e^{\bar{n}}_{x\bar{x}}\| + \frac{\widehat{c}h^2}{3}(2\|f^n\| + \|f^n_x\|)\|e^{\bar{n}}_{x\bar{x}}\|. \qquad (7.40)
\end{aligned}
$$

Similarly,

$$
\langle\psi(U^n, U^{\bar{n}}) - \psi(u^n, u^{\bar{n}}), 2e^{\bar{n}}\rangle \le \frac{2\widehat{c}}{3}(2\|e^n\| + \|e^n_x\|)\|e^{\bar{n}}\|, \qquad (7.41)
$$

$$
\begin{aligned}
&\langle\psi(U^n, U^{\bar{n}}) - \psi(u^n, u^{\bar{n}}), 2e^{\bar{n}}_{x\bar{x}}\rangle \le \sqrt{2LE^0}\|e^{\bar{n}}\|\|e^{\bar{n}}_{x\bar{x}}\|\\
&+\frac{2\widehat{c}}{3}(2\|e^n\| + \|e^n_x\|)\|e^{\bar{n}}_{x\bar{x}}\|. \qquad (7.42)
\end{aligned}
$$

Letting

$$
\begin{aligned}
\mathscr{G}^{n+1} = &\|e^{n+1}\|^2 + \|e^n\|^2 + 2(\|e^{n+1}_x\|^2 + \|e^n_x\|^2) + \|e^{n+1}_{x\bar{x}}\|^2 + \|e^n_{x\bar{x}}\|^2\\
&+\|\eta^{n+1}\|^2 + \|\eta^n\|^2 + \|\eta^{n+1}_x\|^2 + \|\eta^n_x\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{h^2}{12}(\|f^{n+1}\|^2+\|f^n\|^2)+\left(\frac{h^2}{12}-\frac{h^4}{144}\right)(\|f_x^{n+1}\|^2+\|f_x^n\|^2)\\
&-\frac{h^4}{144}(\|f_{x\bar{x}}^{n+1}\|^2+\|f_{x\bar{x}}^n\|^2),
\end{aligned}
$$

for $1 \le n \le N-1$. Using Lemma 2.2, we have

$$
\begin{aligned}
\mathscr{G}^{n+1} \ge\ & \|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2+\|e_{x\bar{x}}^{n+1}\|^2+\|e_{x\bar{x}}^n\|^2+\|\eta^{n+1}\|^2\\
&+\|\eta^n\|^2+\|\eta_x^{n+1}\|^2+\|\eta_x^n\|^2\\
&+\frac{h^2}{18}(\|f^{n+1}\|^2+\|f^n\|^2+\|f_x^{n+1}\|^2+\|f_x^n\|^2).
\end{aligned}
$$

Substituting Eqs. (7.22), (7.38)–(7.42) to Eq. (7.37), we have

$$
\begin{aligned}
&\frac{1}{2\tau}(\mathscr{G}^{n+1}-\mathscr{G}^n)\\
\le\ & \frac{h^2}{6}\left(\|f^{n+1}-f^{n-1}\|\left\|R_{\hat{t}}^n+\frac{2}{\tau}R^{\bar{n}}\right\|+\|f_x^{n+1}-f_x^{n-1}\|\left\|R_{x\hat{t}}^n+\frac{2}{\tau}R_x^{\bar{n}}\right\|\right)\\
&+\left(\frac{h^2}{3}+\frac{h^4}{36}\right)\|f_x^{\bar{n}}\|\|S^n\|\\
&+2(\|e^{\bar{n}}\|+\|e_{x\bar{x}}^{\bar{n}}\|)(\|R_{\hat{t}}^n\|+\|P^n\|)+\frac{h^2}{3}\|\xi^{\bar{n}}\|\|R_x^n\|+\frac{h^4}{36}\|\xi_x^{\bar{n}}\|\|R^n\|\\
&+(6+\sqrt{2})\sqrt{LE^0}\|e^{\bar{n}}\|\|e_{x\bar{x}}^{\bar{n}}\|\\
&+\frac{\widehat{c}h^2}{3}(2\|f^n\|+\|f_x^n\|)(\|e^{\bar{n}}\|+\|e_{x\bar{x}}^{\bar{n}}\|)+\frac{\widehat{c}}{3}(2\|e^n\|+\|e_x^n\|)(\|e^{\bar{n}}\|+\|e_{x\bar{x}}^{\bar{n}}\|)\\
&+2\|Q^n\|\|\eta^{\bar{n}}\|\\
&+\frac{h^2}{3}(\|S^n\|\|e_x^{\bar{n}}\|+\|R^n\|\|\eta_x^{\bar{n}}\|)+2\|Q_x^n\|\|\eta_x^{\bar{n}}\|\\
\le\ & \frac{h^2}{6}\left(\|f^{n+1}\|^2+\|f^{n-1}\|^2+\|R_{\hat{t}}^n\|^2+\frac{4}{\tau^2}\|R^{\bar{n}}\|^2+\|f_x^{n+1}\|^2+\|f_x^{n-1}\|^2\right)\\
&+\frac{2}{3}\|R_{\hat{t}}^n\|^2+\frac{8}{3\tau^2}\|R^{\bar{n}}\|^2\\
&+\left(\frac{h^2}{6}+\frac{h^4}{72}\right)(\|f_x^{\bar{n}}\|^2+\|S^n\|^2)+2(\|e^{\bar{n}}\|^2+\|e_{x\bar{x}}^{\bar{n}}\|^2+\|R_{\hat{t}}^n\|^2+\|P^n\|^2)\\
&+\|\eta_x^{\bar{n}}\|^2+h^2\|S^{\bar{n}}\|^2\\
&+\left(\frac{2}{3}+\frac{h^4}{72}\right)\|R^n\|^2+(3+\sqrt{2})\sqrt{LE^0}(\|e^{\bar{n}}\|^2+\|e_{x\bar{x}}^{\bar{n}}\|^2)+\|Q^n\|^2+\|Q_x^n\|^2\\
&+\|\eta^{\bar{n}}\|^2+\|\eta_x^{\bar{n}}\|^2\\
&+\frac{\widehat{c}h^2}{3}(4\|f^n\|^2+\|f_x^n\|^2+\|e^{\bar{n}}\|^2+\|e_{x\bar{x}}^{\bar{n}}\|^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\widehat{c}}{3}(4\|e^n\|^2+\|e_x^{n}\|^2+\|e^{\bar{n}}\|^2+\|e_{x\bar{x}}^{\bar{n}}\|^2)\\
&+\frac{h^2}{6}(\|S^n\|^2+\|e_x^{\bar{n}}\|^2+\|R^n\|^2+\|\eta_x^{\bar{n}}\|^2)\\
\le&\left[1+\frac{3+\sqrt{2}}{2}\sqrt{LE^0}+\frac{\widehat{c}h^2}{6}+\frac{4\widehat{c}}{3}\right](\|e^{n+1}\|^2+\|e^{n-1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2\\
&++\|e_x^{n-1}\|^2+\|e_x^n\|^2\\
&+\|e_{x\bar{x}}^{n+1}\|^2+\|e_{x\bar{x}}^{n-1}\|^2)+\frac{h^2}{18}\left(24\widehat{c}+\frac{9}{2}\right)(\|f^{n+1}\|^2+\|f^{n-1}\|^2+\|f^n\|^2\\
&++\|f_x^{n+1}\|^2+\|f_x^{n-1}\|^2\\
&+\|f_x^n\|^2)+\frac{5}{2}(\|\eta^{n+1}\|^2+\|\eta^{n-1}\|^2+\|\eta_x^{n+1}\|^2+\|\eta_x^{n-1}\|^2)\\
&++\left(\frac{2}{3}+\frac{h^2}{6}+\frac{h^4}{72}\right)\|R^n\|^2+2\|P^n\|^2\\
&+\|Q^n\|^2+\|Q_x^n\|^2+\frac{2(h^2+4)}{3\tau^2}\|R^{\bar{n}}\|^2+\left(\frac{8}{3}+\frac{h^2}{6}\right)\|R_{\hat{t}}^n\|^2\\
&++\left(\frac{h^2}{3}+\frac{h^4}{72}\right)\|S^n\|^2+h^2\|S^{\bar{n}}\|^2.
\end{aligned}
$$

When sufficiently small positive constants τ_0, h_0 satisfy $\tau\le\tau_0$, $h\le h_0$, we have

$$
\mathscr{G}^{n+1}-\mathscr{G}^n\le 2c_5\tau(\mathscr{G}^{n+1}+\mathscr{G}^n)+2Cc_6\tau(\tau^2+h^4)^2,\quad 1\le n\le N-1,
$$

where

$$
\begin{aligned}
c_5&=\max\left\{1+\frac{(3+\sqrt{2})\widehat{c}}{2}\sqrt{\frac{(36+h_0^2)L}{12}}+\frac{\widehat{c}h_0^2}{6}+\frac{4\widehat{c}}{3},\ \frac{\widehat{c}}{3}+\frac{h_0^2}{12},\ \frac{9}{2}+24\widehat{c}\right\},\\
c_6&=\frac{17}{3}+\frac{7h_0^2}{6}+\frac{h_0^4}{36}.
\end{aligned}
$$

According to the Gronwall inequality, when $2c_5\tau\le 1/3$, we have

$$
\mathscr{G}^{n+1}\le\exp(6Tc_5)\cdot\left[\mathscr{G}^1+\frac{Cc_6}{c_5}(\tau^2+h^4)^2\right]\le C(\tau^2+h^4)^2,\quad 1\le n\le N-1,
$$

where

$$
\begin{aligned}
\mathscr{G}^1&\le\|e^1\|^2+2\|e_x^1\|^2+\|e_{x\bar{x}}^1\|^2+\frac{h^2}{12}(\|f^1\|^2+\|f_x^1\|^2)+\|\eta^1\|^2+\|\eta_x^1\|^2\\
&\le C(\tau^2+h^4)^2,
\end{aligned}
$$

by Eq. (7.7), then we obtain

$$
\|e^{n+1}\|+\|e_x^{n+1}\|+\|e_{x\bar{x}}^{n+1}\|\le C(\tau^2+h^4),\quad \|\eta^{n+1}\|+\|\eta_x^{n+1}\|\le C(\tau^2+h^4),
$$
$$
1\le n\le N-1.
$$

Hence,

$$\|e_x^{n+1}\|_\infty + \|e^{n+1}\|_\infty + \|\eta^{n+1}\|_\infty \le C(\tau^2 + h^4),$$

for $1 \le n \le N-1$ by Lemma 2.2. This completes the proof. □

Theorem 7.2 *Suppose that $u_0(x) \in H^2(\Omega)$, $\rho_0(x) \in H^1(\Omega)$ and $u(x,t), \rho(x,t) \in C_{x,t}^{5,3}(\Omega \times (0,T])$, then the solution (u^n, ρ^n) of the compact scheme (3.17)–(3.25) is stable with respect to the initial conditions in the L^∞-norm.*

Proof Assume that $(\tilde{u}^n, \tilde{\rho}^n, \tilde{v}^n, \tilde{w}^n)$ is the solution of the compact scheme (3.17)–(3.22) with the initial conditions:

$$\tilde{u}_j^0 = u_0(x_j) + \varepsilon_0(x_j), \quad \tilde{\rho}_j^0 = \rho_0(x_j) + \varepsilon_1(x_j), \quad 1 \le j \le J,$$

where $\varepsilon_0(x)$, $\varepsilon_1(x)$ are perturbation functions. Setting

$$\tilde{e}^n = u^n - \tilde{u}^n,\ \tilde{\eta}^n = \rho^n - \tilde{\rho}^n,\ \tilde{f}^n = v^n - \tilde{v}^n,\ \tilde{\xi}^n = w^n - \tilde{w}^n.$$

Using a similar proof for Theorem 7.1, we can conclude that

$$\|\tilde{e}^n\|_\infty^2 + \|\tilde{\eta}^n\|_\infty^2 \le C(\|\varepsilon_0\|_\infty^2 + \|\varepsilon_1\|_\infty^2),$$

which indicates that $(\tilde{e}^n, \tilde{\eta}^n)$ is controlled by the initial conditions $\varepsilon_0(x)$, $\varepsilon_1(x)$, implying that the compact scheme (3.17)–(3.25) is stable in the L^∞-norm. This completes the proof. □

8 Numerical examples

In this section, numerical examples are presented to test the conservation and convergence order of the compact difference scheme (3.17)–(3.25). Before presenting experiments, we give an algorithm to solving the present compact scheme. Denote

$$\mathbf{u}^n = (u_1^n, u_2^n, \ldots, u_J^n)^\top, \quad \rho^n = (\rho_1^n, \rho_2^n, \ldots, \rho_J^n)^\top, \quad \mathbf{v}^n = (v_1^n, v_2^n, \ldots, v_J^n)^\top,$$
$$\mathbf{w}^n = (w_1^n, w_2^n, \ldots, w_J^n)^\top,$$

for $0 \le n \le N$. The algorithm of the compact difference scheme (3.17)–(3.25) can be described as follows:

Step 1. Solve $\mathbf{v}^0$ and $\mathbf{w}^0$ based on Eqs. (3.3)–(3.5).
Step 2. Define $k_1 = \frac{1}{2\tau}$, $k_2 = \frac{1}{12h}$ and $k_3 = \frac{h}{24}$.
For $n = 0, 1, \ldots, N-1$,

- If $n = 0$, let $a = 2k_1$ and $\mathbf{z} = (\mathbf{u}^n, \rho^n, \mathbf{v}^n, \mathbf{w}^n)^\top$.
- If $n \ge 1$, let $a = k_1$ and $\mathbf{z} = (\mathbf{u}^{n-1}, \rho^{n-1}, \mathbf{v}^{n-1}, \mathbf{w}^{n-1})^\top$.

Solve $\mathbf{u}^{n+1}, \rho^{n+1}, \mathbf{v}^{n+1}, \mathbf{w}^{n+1}$ based on the following matrix-vector form

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 & -\mathbf{C}_1 & \mathbf{D} \\ \mathbf{B}_1 & \mathbf{C}_1 & \mathbf{D} & \mathbf{O} \\ -\mathbf{B}_2 & \mathbf{O} & \mathbf{C}_2 & \mathbf{O} \\ \mathbf{O} & -\mathbf{B}_2 & \mathbf{O} & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \rho^{n+1} \\ \mathbf{v}^{n+1} \\ \mathbf{w}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2 & -\mathbf{B}_1 & -\mathbf{C}_1 & -\mathbf{D} \\ -\mathbf{B}_1 & \mathbf{C}_1 & -\mathbf{D} & \mathbf{O} \\ \mathbf{B}_2 & \mathbf{O} & -\mathbf{C}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_2 & \mathbf{O} & -\mathbf{C}_2 \end{bmatrix} \mathbf{z}.$$

End For.

Here, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{D}$ are $J \times J$ circulate matrices defined by

$$\mathbf{A}_1 = \begin{bmatrix} a & a_{2,1} & 0 & \cdots & a_{3,1} \\ a_{3,2} & a & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{3,J-1} & a & a_{2,J-1} \\ a_{2,J} & \cdots & 0 & a_{3,J} & a \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} a & -a_{2,1} & 0 & \cdots & -a_{3,1} \\ -a_{3,2} & a & -a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -a_{3,J-1} & a & -a_{2,J-1} \\ -a_{2,J} & \cdots & 0 & -a_{3,J} & a \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 & 3k_2 & 0 & \cdots & -3k_2 \\ -3k_2 & 0 & 3k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -3k_2 & 0 & 3k_2 \\ 3k_2 & \cdots & 0 & -3k_2 & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 1 & \cdots & 0 & 1 & -2 \end{bmatrix},$$

$$\mathbf{C}_1 = \begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a & 0 \\ 0 & \cdots & 0 & 0 & a \end{bmatrix}, \quad \mathbf{C}_2 = \frac{1}{12}\begin{bmatrix} 10 & 1 & 0 & \cdots & 1 \\ 1 & 10 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 10 & 1 \\ 1 & \cdots & 0 & 1 & 10 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 0 & -k_3 & 0 & \cdots & k_3 \\ k_3 & 0 & -k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & k_3 & 0 & -k_3 \\ -k_3 & \cdots & 0 & k_3 & 0 \end{bmatrix},$$

where

$$a_{2,j} = k_2(u_j^n + u_{j+1}^n) - k_3(v_j^n + v_{j+1}^n), \quad a_{3,j} = -k_2(u_j^n + u_{j-1}^n) + k_3(v_j^n + v_{j-1}^n),$$

for $1 \leq j \leq J$, $0 \leq n \leq N-1$.

For convenience, we denote the maximum errors and convergence orders as

$$E_{\infty,u}^n(h,\tau) = \|U^n(h,\tau) - u^n(h,\tau)\|_\infty, \quad E_{\infty,\rho}^n(h,\tau) = \|\phi^n(h,\tau) - \rho^n(h,\tau)\|_\infty,$$
$$Order\ 1 = \log_2\left(\frac{E_\infty^n(h,\tau)}{E_\infty^n(\frac{h}{2},\frac{\tau}{2})}\right), \quad Order\ 2 = \log_2\left(\frac{E_\infty^n(h,\tau)}{E_\infty^n(\frac{h}{2},\frac{\tau}{4})}\right).$$

Example 1 We consider the following initial conditions [23]:

$$u_0(x) = \frac{3(\upsilon^2 - 1)}{\upsilon}\,\mathrm{sech}^2\left(\sqrt{\frac{\upsilon^2 - 1}{4\upsilon^2}}(x + x_0)\right),$$

$$\rho_0(x) = \frac{3(\upsilon^2 - 1)}{\upsilon^2}\,\mathrm{sech}^2\left(\sqrt{\frac{\upsilon^2 - 1}{4\upsilon^2}}(x + x_0)\right).$$

Then the initial-boundary problem (1.1)–(1.4) has the exact solutions as

$$u(x,t) = \frac{3(\upsilon^2 - 1)}{\upsilon}\,\mathrm{sech}^2\left(\sqrt{\frac{\upsilon^2 - 1}{4\upsilon^2}}(x - \upsilon t + x_0)\right),$$

$$\rho(x,t) = \frac{3(\upsilon^2 - 1)}{\upsilon^2}\,\mathrm{sech}^2\left(\sqrt{\frac{\upsilon^2 - 1}{4\upsilon^2}}(x - \upsilon t + x_0)\right),$$

where $\upsilon^2 > 1$ is a variable parameter that allows the existence of bidirectional propagation and x_0 is a dislocation parameter.

We chose $\upsilon = 1.5, \sqrt{2}$ and $x_0 = 0$. The comparisons of maximum error and convergence order were reported in Table 1 and Fig. 1. It can be seen that the present scheme has much higher convergence order and small error than the schemes [20, 23, 26]. Moreover, the results of convergence order computed by the present scheme in space at $T = 10$ with $\upsilon = \sqrt{2}$, $h = 0.4$, $\tau = h^2$ were listed in Table 2. The present scheme is second-order accurate in time and fourth-order accurate in space as seen in Tables 1 and 2. Furthermore, the results of M_1^n, M_2^n and E^n computed by the present scheme at different times with $\upsilon = \sqrt{2}$, $h = 0.2$, $\tau = h^2$ were displayed in Table 3. To test the conservation of long-time discrete energy and mass, we calculated the absolute errors of M_1^n, M_2^n and E^n at different times with $\upsilon = 1.5$, $T = 500$, $h = 0.2$, $\tau = h^2$ in Fig. 2. From Table 3 and Fig. 2, we see that the present scheme preserves the discrete conservative laws very well, even for long-time simulations. The exact traveling solitons and numerical solitons at different times with $\upsilon = \sqrt{2}$, $h = 0.125$, $\tau = h^2$ were showed in Fig. 3. It can be seen that numerical solitons agree well with the exact solitons.

Table 1 The comparison results of maximum error and convergence order at $T = 1$ with $\upsilon = 1.5$, $\Omega = [-20, 20]$ for Example 1

Scheme		$E^n_{\infty,u}$	*Order* 1	$E^n_{\infty,\rho}$	*Order* 1
Present Scheme	$h = 5\tau = 0.2$	6.4639e-04	–	6.0505e-04	–
	$h = 5\tau = 0.1$	1.2273e-04	2.3969	1.3652e-04	2.1479
	$h = 5\tau = 0.05$	3.0636e-05	2.0022	3.0935e-05	2.1418
Zhao et al. [26]	$h = 5\tau = 0.2$	8.6006e-04	–	0.0011	–
	$h = 5\tau = 0.1$	2.2651e-04	1.9249	2.7354e-04	2.0077
	$h = 5\tau = 0.05$	5.8060e-05	1.9640	6.8300e-05	2.0018

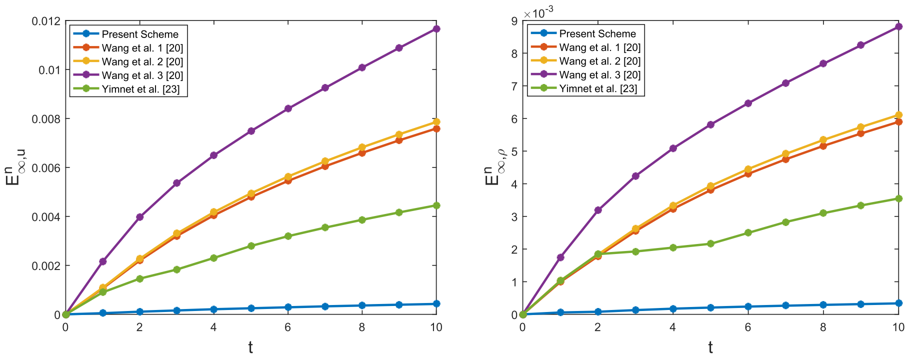

Fig. 1 The comparison results of maximum error at different times with $\upsilon = \sqrt{2}$, $\Omega = [-20, 80]$, $h = 0.125$, $\tau = h^2$ for Example 1

Table 2 The results of maximum error and convergence order at $T = 10$ with $\upsilon = \sqrt{2}$, $\Omega = [-20, 80]$, $h = 0.4$, $\tau = h^2$ for Example 1

	$E^n_{\infty,u}$	$Order\ 2$	$E^n_{\infty,\rho}$	$Order\ 2$
$h = 0.4$	4.2584e-02	–	3.3637e-02	–
$h = 0.2$	2.7761e-03	3.9392	2.1967e-03	3.9366
$h = 0.1$	1.7413e-04	3.9948	1.3779e-04	3.9948

Table 3 M_1^n, M_2^n and E^n computed by the present scheme at different times with $\upsilon = \sqrt{2}$, $\Omega = [-20, 80]$, $h = 0.2$, $\tau = h^2$ for Example 1

T	M_1^n	M_2^n	E^n
0	12.0003199286120	8.48527451034010	27.1529032353602
10	12.0003199187332	8.48527450284519	27.1529032341956
20	12.0003199248562	8.48527454360423	27.1529032334682
30	12.0003199152076	8.48527447685053	27.1529032319831
40	12.0003199317686	8.48527448569040	27.1529032311352

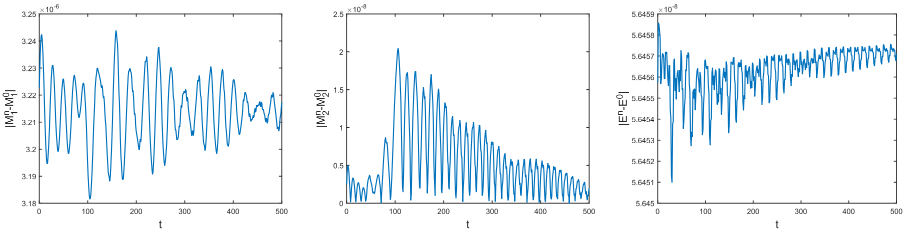

Fig. 2 The absolute errors of long-time discrete conservation of M_1^n, M_2^n and E^n with $\upsilon = 1.5$, $t \in (0, 500]$, $\Omega = [-20, 1000]$, $h = 0.2$, $\tau = h^2$ for Example 1

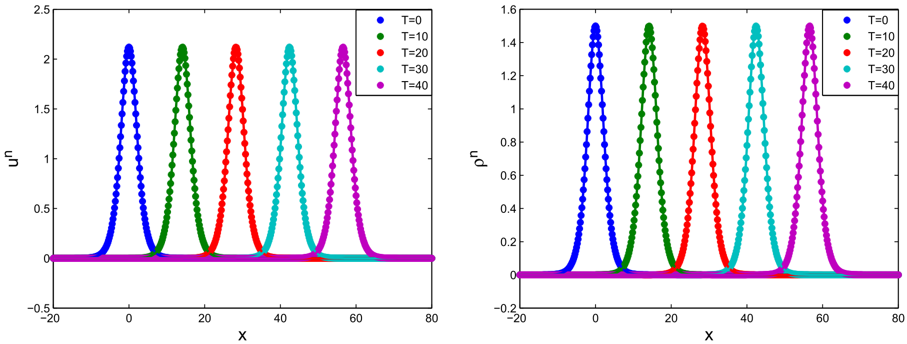

Fig. 3 Travelling solitons at different times with $\upsilon = \sqrt{2}$, $h = 0.125$, $\tau = h^2$ for Example 1

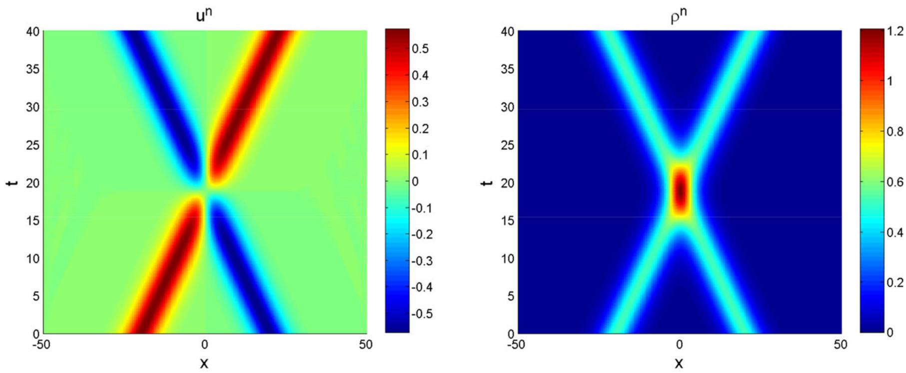

Fig. 4 Profile of head-on collision of two solitons for Example 2

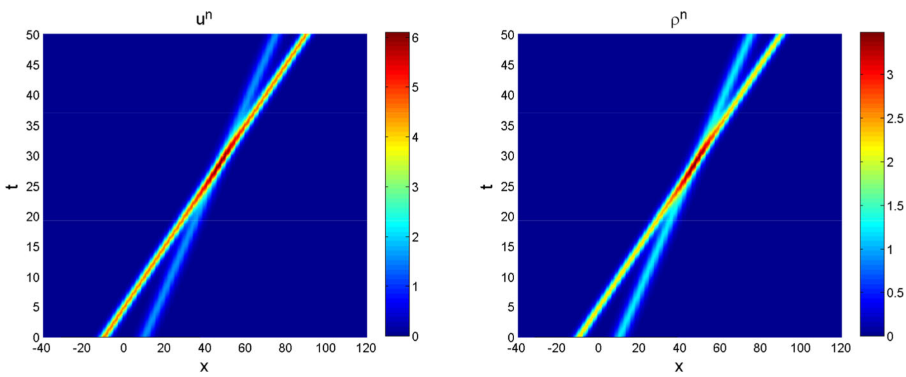

Fig. 5 Profile of catch-up collision of two solitons for Example 2

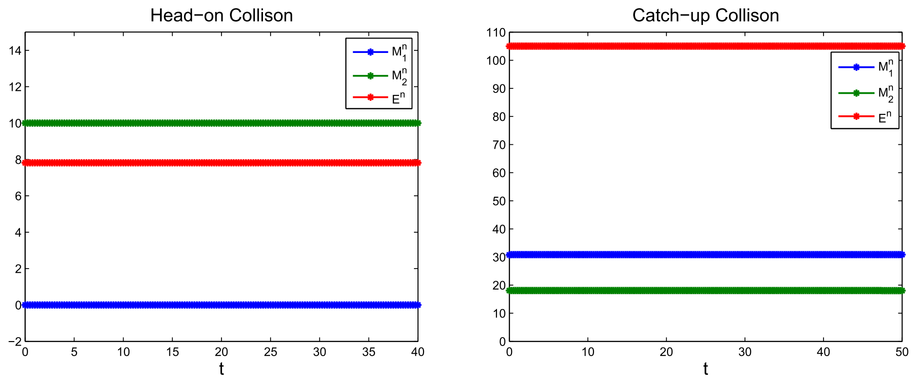

Fig. 6 M_1^n, M_2^n and E^n of two solitary collision at different times for Example 2

Example 2 We investigate the soliton-soliton collision with the initial conditions [9]:

$$u_0(x) = \sum_{i=1}^{2}\left[\frac{3(\upsilon_i^2-1)}{\upsilon_i}\,\mathrm{sech}^2\left(\sqrt{\frac{\upsilon_i^2-1}{4\upsilon_i^2}}(x+x_i)\right)\right],$$

$$\rho_0(x) = \sum_{i=1}^{2}\left[\frac{3(\upsilon_i^2-1)}{\upsilon_i^2}\,\mathrm{sech}^2\left(\sqrt{\frac{\upsilon_i^2-1}{4\upsilon_i^2}}(x+x_i)\right)\right].$$

We analyzed the collision of two solitons by considering two cases:

Head-on collision: We take the parameters $\upsilon_1 = -\upsilon_2 = -1.1$, $x_1 = -x_2 = -20$, $\Omega = [-50, 50]$, $T = 40$ and $h = 0.125$, $\tau = h^2$.

Catch-up collision: We take the parameters $\upsilon_1 = 1.3$, $\upsilon_2 = 2$, $x_1 = -x_2 = -10$, $\Omega = [-40, 120]$, $T = 50$ and $h = 0.125$, $\tau = h^2$.

Numerical results computed by the present scheme were given in Figs. 4, 5, and 6. In Fig. 4, the two solitary waves for u disappear and the peak of the interaction for ρ occurs at about $T = 20$. In Fig. 5, the higher wave catch up with the lower wave as time goes on. Moreover, two solitary waves back to its original shape and velocity and move forward without any changes after collision as seen in Figs. 4 and 5. M_1^n, M_2^n and E^n of head-on collision and catch-up collision solitons at different times were shown in Fig. 6. It is easy to see the present scheme keep the discrete conservative law, which supports Theorem 4.1.

9 Conclusions

A new linearized fourth-order conservative compact difference scheme for the SRLW equations is constructed based on the reduction order method. The discrete conservative laws, boundedness and unique solvability of the present compact scheme are proved in detail. The convergence order $\mathscr{O}(\tau^2+h^4)$ in the L^∞-norm and stability are strictly proved by the discrete energy method. Numerical experiments show that the present scheme supports the theoretical analysis.

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Declarations

Conflict of interest The authors declare no competing interests.

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