Convergence rates for boundedly regular systems

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Abstract

In this work, we consider a continuous dynamical system associated with the fixed point set of a nonexpansive operator which was originally studied by Boţ and Csetnek (J. Dyn. Diff. Equat. **29**(1), pp. 155–168, 2017). Our main results establish convergence rates for the system's trajectories when the nonexpansive operator satisfies an additional regularity property. This setting is the natural continuous-time analogue to discrete-time results obtained in Bauschke, Noll and Phan (J. Math. Anal. Appl. **421**(1), pp. 1–20, 2015) and Borwein, Li and Tam (SIAM J. Optim. **27**(1), pp. 1–33, 2017) by using the same regularity properties. Closure properties of the class of Hölder regular operators under taking convex combinations and compositions are also derived.

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1 Introduction

Let $\mathcal{H}$ denote a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. In this work, we consider the continuous-time dynamical system with initial point $x_0 \in \mathcal{H}$ given by

$$\dot{x}(t) = \lambda(t) \left(T(x(t)) - x(t) \right), \quad x(0) = x_0, \tag{1}$$

where $T : \mathcal{H} \to \mathcal{H}$ is nonexpansive and $\lambda : [0, +\infty) \to [0, 1]$ is Lebesgue measurable. We remark that the parameter function λ has an interpretation as a time-scaling factor, through which Eq. 1 can be shown equivalent to the case with $\lambda(t) = 1$ for all $t \geq 0$. For details, see [12, Section 4].

We shall investigate the behaviour of trajectories of Eq. 1 which are understood in the sense of *strong global solutions*.

Definition 1 (Strong global solution) A trajectory $x : [0, +\infty) \to \mathcal{H}$ is a *strong global solution* of Eq. 1 if the following properties are satisfied:

(i) x is absolutely continuous on each interval $[0, b]$ for $0 < b < +\infty$.
(ii) $\dot{x}(t) = \lambda(t) \left(T(x(t)) - x(t) \right)$ for almost all $t \in [0, +\infty)$.
(iii) $x(0) = x_0$.

Here, absolute continuity of the trajectory x on $[0, b]$ is understood in the vector-valued sense (see, for instance, [4, Definition 2.1]) which implies

$$x(t) = x(0) + \int_0^t \dot{x}(s) \, ds \quad \forall t \in [0, b].$$

The existence and uniqueness of a strong global solution for each $x_0 \in \mathcal{H}$ follows as a consequence of the *Cauchy–Lipschitz theorem*. The detailed argument can be found in [12, Section 2].

Convergence of these trajectories (without rates) was established by Boț and Csetnek [12].

Theorem 1 [12, Theorem 6] *Suppose* $T : \mathcal{H} \to \mathcal{H}$ *is nonexpansive with* Fix $T \neq \emptyset$ *and* $\lambda : [0, +\infty) \to [0, 1]$ *be Lebesgue measurable with either*

$$\int_0^{+\infty} \lambda(t) \left(1 - \lambda(t) \right) dt = +\infty \quad or \quad \inf_{t \geq 0} \lambda(t) > 0.$$

Let x *denote the unique strong global solution of* Eq. 1. *Then, the following assertions hold.*

(i)   *The trajectory x is bounded and* $\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty$.
(ii) $\lim_{t \to +\infty} \left(T(x(t)) - x(t) \right) = 0$.
(iii) $\lim_{t \to +\infty} \dot{x}(t) = 0$.
(iv) $x(t)$ *converges weakly to a point* $\bar{x} \in$ Fix T *as* $t \to +\infty$.

The dynamical system (1) can be viewed as a continuous-time analogue to the discrete-time system given by

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k T(x_k). \tag{2}$$

More precisely, the sequence (x_k) in Eq. 2 can be viewed as a discretisation of the trajectory $x(t)$ in Eq. 1 along unit stepsizes. In other words, for $k \in \mathbb{N}$, we take $\lambda_k \approx \lambda(k)$ and $x_k \approx x(k)$ together with the forward discretisation $\dot{x}(k) \approx x_{k+1} - x_k$. In the literature, the discrete system (2) is well-known as the *Krasnoselskii–Mann* iteration [15] corresponding to T. By choosing the operator T appropriately, many iterative algorithms can be understood within this framework (see, for instance, [8, Section 26]).

In analogue with Theorem 1, it can be shown that the sequence $(x_k)_{k \in \mathbb{N}}$ generated by Eq. 2 converges weakly to a point in Fix T provided that (λ_k) satisfies $\sum_{k=1}^{\infty} \lambda_k(1 - \lambda_k) = +\infty$ [8, Theorem 5.15]. Furthermore, when T satisfies appropriate regularity conditions, information about the rate of convergence of (x_k) can also be provided—it converges R-linearly when T is *boundedly linearly regular*, and sublinearly when T is *boundedly Hölder regular*. Although we defer formally defining these regularity notions until Section 2, we will nevertheless state the following result for completeness.

Theorem 2 *Let* $T : \mathcal{H} \to \mathcal{H}$ *be an nonexpansive operator with* Fix $T \neq \emptyset$. *Let* $x_0 \in \mathcal{H}$ *and consider the sequence* (x_k) *given by* Eq. 2 *with* $(\lambda_k) \subseteq [0, 1]$ *such that* $\inf_{k \in \mathbb{N}} \lambda_k(1 - \lambda_k) > 0$. *Then there exists a point* $\bar{x} \in$ Fix T *such that the following assertions hold.*

(i)   *If* T *is boundedly linearly regular, then* $x_k \to \bar{x}$ *with at least R-linear rate, that is, with at least rate* $O(r^k)$ *for some* $r \in [0, 1)$.

(ii) *If* T *is boundedly Hölder regular, then* $x_k \to \bar{x}$ *with at least rate* $O(k^{-\rho})$ *for some* $\rho > 0$.

Proof (i): See [9, Theorem 6.1]. (ii): See [11, Corollary 3.9]. For generalisations, see [22]. □

In this work, we show that the analogue statements about convergence rates given in Theorem 2 also hold in the continuous-time setting. From the perspective of iterative algorithms in optimisation, understanding the interplay between the corresponding discrete and continuous-time systems provides insight into the conditions required convergence as well as a technology for deriving new schemes. For specific examples, see [19, 26]. For other recent works which study the interplay between discrete and continuous-time systems, the reader is referred to [1, 3, 6, 25, 29].

The remainder of this work is structured as follows. In Section 2, we review notions of bounded regularity for operators. These notions are then used in Section 3 to prove convergence rates for the strong global trajectories of Eq. 1. Closure properties of the classes of boundedly regular operators are studied in Section 4. These properties are of interest in their own right and complement the results in [17].

Finally, Section 5 uses these closure properties to deduce several extensions of the results from Section 3.

2 Boundedly regular operators

In this section, we recall two notions of boundedly regular operators as well as providing examples of each. These notions are a kind of *error bound* in that, when satisfied, they bound the distance to the fixed point set of an operator in terms of its residual.

The first notion, based on linear regularity, was proposed for projection operators by Bauschke and Borwein [7] and for the general case by Bauschke, Noll, and Phan [9].

Definition 2 (Linearly regular operators) An operator $T : \mathcal{H} \to \mathcal{H}$ is *linearly regular* on $U \subseteq \mathcal{H}$ if there exists a constant $\kappa > 0$ such that

$$d(y, \text{Fix } T) \leq \kappa \|y - T(y)\| \quad \forall y \in U.$$

If T is linearly regular on every bounded subset of $\mathcal{H}$, it is said to be *boundedly linearly regular*.

Recall that a set is *polyhedral* if it can be expressed as the intersection of finitely many closed half-spaces and/or hyperplanes, and that an operator is *polyhedral* if its graph is the union of finitely many polyhedral sets. For remarks on this terminology, see [27, p. 76].

Proposition 1 *Let $\mathcal{H} = \mathbb{R}^n$. If $T : \mathcal{H} \to \mathcal{H}$ is polyhedral with $\text{Fix } T \neq \emptyset$, then T is boundedly linearly regular.*

Proof Since Id and T are polyhedral and the class of polyhedral operators is closed under addition [28, p. 206], the operator $F := \text{Id} - T$ is also polyhedral. By [28, Corollary] applied to F, there exist $\kappa_1 > 0$ and $\epsilon > 0$ such that

$$d(x, \text{Fix } T) = d(x, F^{-1}(0)) \leq \kappa_1 d(0, F(x)) = \kappa_1 \|x - T(x)\| \qquad (3)$$

for all $x \in \mathcal{H}$ with $\|x - T(x)\| < \epsilon$. Let $U \subseteq \mathcal{H}$ be a nonempty bounded set. Then $\kappa_2 := \sup_{x \in U} d(x, \text{Fix } T) < +\infty$. Thus, for all $x \in U$ with $\|x - T(x)\| \geq \epsilon$, we have

$$\frac{d(x, \text{Fix } T)}{\|x - T(x)\|} \leq \frac{d(x, \text{Fix } T)}{\epsilon} \leq \frac{\kappa_2}{\epsilon}. \qquad (4)$$

By combining (3) and (4), we deduce

$$d(x, \text{Fix } T) \leq \max\left\{\kappa_1, \frac{\kappa_2}{\epsilon}\right\} \|x - T(x)\| \quad \forall x \in U,$$

which establishes the claimed result. □

One drawback of linear regularity is that is often too restrictive to hold or too difficult to verify in practice (i.e., beyond polyhedral settings such as Example 1).

For further examples, see [9, Section 2]. To overcome this shortcoming, the following Hölder counterpart of Definition 2 was introduced in [11, Definition 2.7].

Definition 3 (Hölder regular operators) An operator $T: \mathcal{H} \to \mathcal{H}$ is *Hölder regular* on $U \subseteq \mathcal{H}$ if there exists a constant $\kappa > 0$ and $\gamma \in (0, 1)$ such that

$$\mathrm{d}(y, \mathrm{Fix}\, T) \leq \kappa \|y - T(y)\|^{\gamma} \quad \forall y \in U.$$

If T is Hölder regular on every bounded subset of $\mathcal{H}$, it is said to be *boundedly Hölder regular*.

Recall that a set is *semi-algebraic* if it can be expressed as the union of finitely many sets, each of which can be defined by finitely many polynomial equalities and inequalities. An operator is *semi-algebraic* if its graph is a semi-algebraic set.

Proposition 2 *Let $\mathcal{H} = \mathbb{R}^n$. If $T: \mathcal{H} \to \mathcal{H}$ is continuous and semi-algebraic with $\mathrm{Fix}\, T \neq \emptyset$, then T is boundedly Hölder regular.*

Proof Let U be a nonempty bounded set. Then there exists an $R > 0$ such that

$$U \subseteq \mathbb{B}(0, R) := \{x \in \mathcal{H} : \|x\| \leq R\}$$

where we note that $\mathbb{B}(0, R)$ is semi-algebraic. Consider the continuous functions

$$\phi(y) := \|y - T(y)\| \quad \text{and} \quad \psi(y) := \mathrm{d}(y, \mathrm{Fix}\, T).$$

Since $\| \cdot \|$ and $\mathrm{Id} - T$ are semi-algebraic as, their composition, the function ϕ is also semi-algebraic [10, Proposition 2.2.6(i)]. Since $\mathrm{Fix}\, T = \phi^{-1}(0)$ and ϕ is semi-algebraic, the set $\mathrm{Fix}\, T$ is also semi-algebraic by [10, Proposition 2.2.7]. By [10, Proposition 2.2.8(i)], it then follows that $\mathrm{d}(\cdot, \mathrm{Fix}\, T)$ is semi-algebraic. Thus, since ϕ and ψ are continuous semi-algebraic functions with $\psi^{-1}(0) = \phi^{-1}(0) = \mathrm{Fix}\, T \neq \emptyset$, Łojasiewicz's inequality [10, Corollary 2.6.7] implies that there exist constants $\kappa > 0$ and $\gamma \in (0, 1)$ such that

$$\mathrm{d}(x, \mathrm{Fix}\, T) = |\psi(x)| \leq \kappa |\phi(x)|^{\gamma} = \kappa \|x - T(x)\|^{\gamma} \quad \forall x \in \mathbb{B}(0, R),$$

and the claimed result follows. $\square$

Example 1 (Forward-backward operator) Let $\mathcal{H} = \mathbb{R}^n$ and consider the *monotone inclusion*

$$0 \in (A + B)(x), \tag{5}$$

where $A: \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone and $B: \mathcal{H} \to \mathcal{H}$ is monotone and continuous. This problem arises, for instance, as the optimality conditions of the minimisation problem

$$\min_{x \in \mathcal{H}} g(x) + f(x), \tag{6}$$

where $g: \mathcal{H} \to (-\infty, +\infty]$ is proper, lsc, convex, and $f: \mathcal{H} \to \mathbb{R}$ is convex and differentiable. More precisely, by setting $A = \partial g$ (i.e. the convex subdifferential of g) and $B = \nabla f$.

The *forward-backward operator* $T \colon \mathcal{H} \to \mathcal{H}$ for Eq. 5 with stepsize $\lambda > 0$ is given by

$$T := (\mathrm{Id} + \lambda A)^{-1} \circ (\mathrm{Id} - \lambda B),$$

where the *resolvent operator* $(\mathrm{Id} + \lambda A)^{-1}$ is single-valued and continuous with full domain [8, Proposition 23.10]. Then, T is continuous and $\mathrm{Fix}\, T = (A + B)^{-1}(0)$. Moreover, T is semi-algebraic, and hence boundedly Hölder regular by Proposition 2, whenever A and B are semi-algebraic. Indeed, if A and B are semi-algebraic, then so are $\mathrm{Id} + \lambda A$ and $\mathrm{Id} - \lambda B$. And, since $(u, v) \in \mathrm{gra}(\mathrm{Id} + \lambda A)$ if and only if $(v, u) \in \mathrm{gra}(\mathrm{Id} + \lambda A)^{-1}$, the resolvent operator is also semi-algebraic. As the composition of two semi-algebraic operators, T is therefore also semi-algebraic.

Since the subdifferential of a convex semi-algebraic function is again semi-algebraic (see, for instance, [20, 21]), we also note that, in particular, the forward-backward operator applied to Eq. 6 is boundedly Hölder regular when f and g are semi-algebraic.

Remark 1 Let $T \colon \mathcal{H} \to \mathcal{H}$ and let $z \in \mathcal{H}$. Then, it is immediate from the respective definitions, that T is boundedly linearly (resp. Hölder) regular if and only if T is boundedly linearly (resp. Hölder) regular on $\mathbb{B}(z, R)$ for all $R > 0$.

3 Convergence of trajectories with regularity

In this section, we show a refinement of Theorem 1. Namely, that the convergence rate of the trajectories in Eq. 1 can be given when the operator T is boundedly regular. Although it will not always be explicitly stated within this section's proofs to avoid repetition, identities will sometimes be understood to hold for almost all $t \in [0, +\infty)$ due the identity for $\dot{x}$ in Definition 1(ii).

We shall require the following lemmata as well as the well-known identity:

$$\|(1 - \alpha)u + \alpha v\|^2 + \alpha(1 - \alpha)\|u - v\|^2 = (1 - \alpha)\|u\|^2 + \alpha\|v\|^2$$
$$\forall \alpha \in \mathbb{R}, \ \forall u, v \in \mathcal{H}. \tag{7}$$

Lemma 1 *Let x be the unique strong global solution of Eq. 1, let $x^* \in \mathrm{Fix}\, T$ and suppose $\inf_{t \geq 0} \lambda(t) > 0$. For almost all $t \in [0, +\infty)$, we have*

$$\|\dot{x}(t) + x(t) - x^*\|^2 + \frac{1 - \lambda(t)}{\lambda(t)}\|\dot{x}(t)\|^2 \leq \|x(t) - x^*\|^2.$$

Proof By applying (7) followed by nonexpansivity of T, we obtain

$$
\begin{aligned}
\|\dot{x}(t) + x(t) - x^*\|^2 &= \|(1 - \lambda(t))(x(t) - x^*) + \lambda(t)(T(x(t)) - x^*)\|^2 \\
&= (1 - \lambda(t))\|x(t) - x^*\|^2 + \lambda(t)\|T(x(t)) - x^*\|^2 \\
&\quad - \lambda(t)(1 - \lambda(t))\|x(t) - T(x(t))\|^2 \\
&\leq \|x(t) - x^*\|^2 - \frac{1 - \lambda(t)}{\lambda(t)}\|\dot{x}(t)\|^2,
\end{aligned}
$$

which completes the proof of the result. $\qquad\square$

Proposition 3 [8, Corollary 12.31] *Let $C \subseteq \mathcal{H}$ be a nonempty closed convex set. Then $x \mapsto d^2(x, C)$ is Fréchet differentiable on $\mathcal{H}$ with $\nabla d^2(\cdot, C) = 2(\mathrm{Id} - P_C)$.*

Lemma 2 *Let x be the unique strong global solution of* Eq. 1. *Suppose $\mathrm{Fix}\, T \neq \emptyset$ and $\inf_{t \geq 0} \lambda(t) > 0$. Then, for almost all $t \in [0, +\infty)$, we have*

(i) $\dfrac{d}{dt} d^2(x(t), \mathrm{Fix}\, T) \leq -\lambda(t) \|x(t) - T(x(t))\|^2$, *and*

(ii) $\dfrac{d}{dt} \|x(t) - x^*\|^2 \leq -\lambda(t)(1 - \lambda(t)) \|x(t) - T(x(t))\|^2 - \|\dot{x}(t)\|^2$ *for all $x^* \in \mathrm{Fix}\, T$.*

Proof (i): Since T is nonexpansive, $F := \mathrm{Fix}\, T$ is nonempty, closed, and convex [8, Proposition 4.13]. The chain-rule together with Proposition 3 therefore implies

$$
\begin{aligned}
\frac{d}{dt} d^2(x(t), F) &= \langle \dot{x}(t), \nabla d^2(\cdot, F)(x(t)) \rangle \\
&= 2 \langle \dot{x}(t), x(t) - P_F(x(t)) \rangle \\
&= \|\dot{x}(t) + x(t) - P_F(x(t))\|^2 - \|\dot{x}(t)\|^2 - \|x(t) - P_F(x(t))\|^2 \\
&= \|\dot{x}(t) + x(t) - P_F(x(t))\|^2 - \lambda(t)^2 \|x(t) - T(x(t))\|^2 \\
&\quad - \|x(t) - P_F(x(t))\|^2.
\end{aligned}
$$

Since $P_F(x(t)) \in F = \mathrm{Fix}\, T$, Lemma 1 then gives

$$
\|\dot{x}(t) + x(t) - P_F(x(t))\|^2 \leq \|x(t) - P_F(x(t))\|^2 - \lambda(t)\,(1 - \lambda(t))\,\|x(t) - T(x(t))\|^2.
$$

The claimed inequality follows by combining the previous two equations.

(ii): For any $\bar{x} \in \mathrm{Fix}\, T$, we have

$$
\begin{aligned}
\frac{d}{dt} \|x(t) - \bar{x}\|^2 &= 2 \langle \dot{x}(t), x(t) - \bar{x} \rangle \\
&= \|\dot{x}(t) + x(t) - \bar{x}\|^2 - \|\dot{x}(t)\|^2 - \|x(t) - \bar{x}\|^2.
\end{aligned}
$$

The result then follows by combining this equality with Lemma 1. $\qquad\square$

We shall also require the following well-known, classical result.

Lemma 3 (Grönwall's inequality) *Let $u \colon [0, +\infty) \to [0, +\infty)$ be absolutely continuous. Suppose there exists $\alpha > 0$ such that, for almost all $t \in [0, +\infty)$, we have*

$$
\frac{d}{dt} u(t) \leq -\alpha u(t).
$$

Then $u(t) \leq \exp(-\alpha t) u(0)$ for all $t \in [0, +\infty)$.

The following theorem is our first main result. It shows that the dynamical system Eq. (1) is *exponentially stable* when T is boundedly linearly regular.

Theorem 3 *Suppose $T \colon \mathcal{H} \to \mathcal{H}$ is nonexpansive with $\mathrm{Fix}\, T \neq \emptyset$ and $\lambda \colon [0, +\infty) \to [0, 1]$ is Lebesgue measurable with $\lambda^* := \inf_{t \geq 0} \lambda(t) > 0$. Let x be*

the unique strong global solution of Eq. 1. *If T is boundedly linearly regular, then there exists $\bar{x} \in \mathrm{Fix}\, T$ and $\kappa > 0$ such that, for almost all $t \in [0, +\infty)$, we have*

$$\|x(t) - \bar{x}\| \leq 2\exp\left(-\frac{\lambda^*}{2\kappa^2}t\right)\mathrm{d}(x_0, \mathrm{Fix}\, T).$$

That is, the trajectory $x(t)$ converges exponentially to $\bar{x}$ as $t \to +\infty$.

Proof By Theorem 1, the trajectory x is bounded and $x(t)$ converges weakly to a point $\bar{x} \in \mathrm{Fix}\, T$ as $t \to +\infty$. Thus, since T is boundedly linearly regular, there exists $\kappa > 0$ such that

$$\mathrm{d}(x(t), \mathrm{Fix}\, T) \leq \kappa\|x(t) - T(x(t))\|.$$

Combining this with Lemma 2(i) yields

$$\frac{d}{dt}\mathrm{d}^2(x(t), \mathrm{Fix}\, T) \leq -\lambda(t)\|x(t) - T(x(t))\|^2 \leq -\frac{\lambda^*}{\kappa^2}\mathrm{d}^2(x(t), \mathrm{Fix}\, T).$$

By applying Grönwall's inequality (Lemma 3) to the function $t \mapsto \mathrm{d}^2(x(t), \mathrm{Fix}\, T)$, we obtain

$$\mathrm{d}^2(x(t), \mathrm{Fix}\, T) \leq \exp\left(-\frac{\lambda^*}{\kappa^2}t\right)\mathrm{d}^2(x_0, \mathrm{Fix}\, T). \tag{8}$$

Let $x^* \in \mathrm{Fix}\, T$ be arbitrary. By Lemma 2(ii), we have $\frac{d}{dt}\|x(t) - x^*\|^2 \leq 0$ and hence the function $t \mapsto \|x(t) - x^*\|^2$ is nonincreasing. Assuming that $s > t$, we deduce

$$\|x(t) - x(s)\| \leq \|x(t) - x^*\| + \|x(s) - x^*\| \leq 2\|x(t) - x^*\|.$$

Using weak lower semicontinuity of the norm and setting $x^* = P_{\mathrm{Fix}\, T}(x(t))$ then gives

$$\|x(t) - \bar{x}\| \leq \liminf_{s \to +\infty}\|x(t) - x(s)\| \leq 2\,\mathrm{d}(x(t), \mathrm{Fix}\, T). \tag{9}$$

The result then follows by combining (8) and (9). □

In the following theorem, we make use of the following generalisation of Grönwall's inequality.

Lemma 4 (Bihari–LaSalle inequality) *Let $u\colon [0, +\infty) \to [0, +\infty)$ be absolutely continuous. Suppose there exists $\alpha > 0$ and $\gamma \in (0, 1)$ such that, for almost all $t \in [0, +\infty)$, we have*

$$\frac{d}{dt}u(t) \leq -\alpha u(t)^{\frac{1}{\gamma}}. \tag{10}$$

Then there exists a constant $M > 0$ such that $u(t) \leq Mt^{-\frac{\gamma}{1-\gamma}}$ for all $t \in [0, +\infty)$.

Proof If there exists $t_0 \geq 0$ such that $u(t_0) = 0$, then (10) implies that $u(t) = 0$ for all $t \geq t_0$ and the result trivially holds. Thus, we suppose that $u > 0$. In this case, since $1 - 1/\gamma < 0$, we have

$$\frac{d}{dt}\left(u(t)^{1-\frac{1}{\gamma}} + \left(1 - \frac{1}{\gamma}\right)\alpha t\right) = \left(1 - \frac{1}{\gamma}\right)u(t)^{-\frac{1}{\gamma}}\left(\frac{d}{dt}u(t) + \alpha u(t)^{\frac{1}{\gamma}}\right) \geq 0.$$

Thus, since $t \mapsto u(t)^{1-\frac{1}{\gamma}} + \left(1 - \frac{1}{\gamma}\right) \alpha t$ is non-decreasing and absolutely continuous, we have

$$u(t)^{1-\frac{1}{\gamma}} + \left(1 - \frac{1}{\gamma}\right) \alpha t \geq u(0)^{1-\frac{1}{\gamma}} \geq 0,$$

which implies

$$u(t) \leq \left(\frac{\gamma}{\alpha(1-\gamma)}\right)^{\frac{\gamma}{1-\gamma}} t^{-\frac{\gamma}{1-\gamma}}.$$

This establishes the result and completes the proof. $\qquad\square$

The following theorem is the Hölder regular analogue of Theorem 3. It is our second main result.

Theorem 4 *Suppose* $T : \mathcal{H} \to \mathcal{H}$ *is nonexpansive with* $\mathrm{Fix}\, T \neq \emptyset$ *and* $\lambda : [0, +\infty) \to [0, 1]$ *is Lebesgue measurable with* $\lambda^* := \inf_{t \geq 0} \lambda(t) > 0$. *Let* x *be the unique strong global solution of Eq. 1. If* T *is boundedly Hölder regular, then there exists* $\bar{x} \in \mathrm{Fix}\, T$, $M > 0$ *and* $\gamma \in (0, 1)$ *such that, for almost all* $t \in [0, +\infty)$, *we have*

$$\|x(t) - \bar{x}\| \leq M\, t^{-\frac{\gamma}{2(1-\gamma)}}.$$

That is, the trajectory $x(t)$ *converges with order* $\rho := \frac{\gamma}{2(1-\gamma)} > 0$ *to* $\bar{x}$ *as* $t \to +\infty$.

Proof By Theorem 1, $x(t)$ converges weakly to a point $\bar{x} \in \mathrm{Fix}\, T$. In particular, the trajectory x is bounded and hence, as T is boundedly Hölder regular, there exists $\kappa > 0$ and $\gamma \in (0, 1)$ such that

$$\mathrm{d}(x(t), \mathrm{Fix}\, T) \leq \kappa \|x(t) - T(x(t))\|^{\gamma}.$$

Combining this with Lemma 2(i) yields

$$\frac{d}{dt}\, \mathrm{d}^2(x(t), \mathrm{Fix}\, T) \leq -\lambda(t)\|x(t) - T(x(t))\|^2 \leq -\frac{\lambda^*}{\kappa^{2/\gamma}}\, \mathrm{d}^{2/\gamma}(x(t), \mathrm{Fix}\, T).$$

By applying the Bihari–LaSalle inequality (Lemma 4) to the function $t \mapsto \mathrm{d}^2(x(t), \mathrm{Fix}\, T)$, we deduce the existence of a constant $M_0 > 0$ such that

$$\mathrm{d}(x(t), \mathrm{Fix}\, T) \leq M_0\, t^{-\frac{\gamma}{2(1-\gamma)}}.$$

Let $x^* \in \mathrm{Fix}\, T$ be arbitrary. By using the same argument as used in Theorem 3 to obtain (9), we deduce

$$\|x(t) - \bar{x}\| \leq 2\, \mathrm{d}(x(t), \mathrm{Fix}\, T).$$

Combining the previous two inequalities gives

$$\|x(t) - \bar{x}\| \leq M\, t^{-\frac{\gamma}{2(1-\gamma)}} \quad \text{where } M := 2M_0,$$

which completes the proof. $\qquad\square$

An interesting direction for further investigation would be to study convergence rates under regularity properties for T for second-order dynamical systems. Specially, given initial points $u_0, v_0 \in \mathcal{H}$, it is natural to consider the system

$$\begin{cases} \ddot{x}(t) + \gamma \dot{x}(t) + \lambda(t)\,(x(t) - T(x(t))) = 0 \\ x(0) = u_0, \quad \dot{x}(0) = v_0 \end{cases} \tag{11}$$

where $\gamma > 0$ and $\lambda > 0$ is as considered above.

The motivation for studying (11) is that its time discretisation leads to iterative schemes involving inertial effects, which have had a great impact in the research community due to the works of Polyak [14], Nesterov [23, 24], etc. In the particular case when $\lambda(t) = 1$ for all $t \in [0, +\infty)$, the dynamical system (11) has been investigated in [2, Theorem 3.2] (see also [13]).

4 Further properties of regular operators

In this section, we study closure properties of the classes of boundedly linearly/Hölder regular operators under convex combinations and compositions. In order to establish these properties, we shall work with the following class which includes averaged nonexpansive operators as a special cases.

Definition 4 (Strongly quasinonexpansive operators) An operator $T : \mathcal{H} \to \mathcal{H}$ is *ρ-strongly quasinonexpansive (ρ-SQNE)* if $\rho > 0$ and

$$\|T(x) - x^*\|^2 + \rho\|x - T(x)\|^2 \le \|x - x^*\|^2 \quad \forall x \in \mathcal{H}, \ \forall x^* \in \operatorname{Fix} T.$$

Remark 2 Although this paper will only use the results from this section applied to averaged-nonexpansive operators, our motivation for studying SQNE operators is twofold. Firstly, as every averaged-nonexpansive operator is also SNQE, there is no loss of generality in considering SNQE operators. Secondly, many of the results in this section can be seen as extensions of those in [17] which considered SNQE operators. Thus, it is natural for us to consider the same setting.

The fixed points of SQNE operators satisfy the following properties.

Proposition 4 [16, Theorem 2.1.26] *Let $T_i : \mathcal{H} \to \mathcal{H}$ be SQNE with a common fixed point. Then, the identity*

$$\operatorname{Fix} T = \cap_{i=1}^n \operatorname{Fix} T_i$$

holds provided that T has one of the following forms:

(i) $T = \sum_{i=1}^n \omega_i T_i$ with $\sum_{i=1}^n \omega_i = 1$ and $\omega_i > 0$ for all $i \in \{1, \dots, n\}$.
(ii) $T = T_n \dots T_2 T_1$

We also require the following regularity notion for collections of sets.

Definition 5 (Linearly regular collections of sets) A collection of sets $\{C_1, \ldots, C_n\}$ is *linearly regular* on U if there exists $\tau > 0$ such that

$$d(x, \cap_{i=1}^n C_i) \leq \tau \max_{i=1,\ldots,n} d(x, C_i) \quad \forall x \in U.$$

If the collection $\{C_1, \ldots, C_n\}$ is linearly regular on every bounded subset of $\mathcal{H}$, it is said to be *boundedly linearly regular*.

Theorem 5 [17, Corollary 5.3] *Let* $T_i \colon \mathcal{H} \to \mathcal{H}$ *be* ρ_i-*SQNE for all* $i \in \{1, \ldots, n\}$. *Assume* $\min_{i=1,\ldots,n} \omega_i > 0$, $\sum_{i=1}^n \omega_i = 1$ *and* $\cap_{i=1}^n \text{Fix}\, T_i \neq \emptyset$. *Suppose the following assertions hold.*

  (i)   *The operator* T_i *is boundedly linearly regular for all* $i \in \{1, \ldots, n\}$.
  (ii) *The collection* $\{\text{Fix}\, T_i\}_{i=1}^n$ *is boundedly linearly regular.*

Then $T := \sum_{i=1}^n \omega_i T_i$ *is boundedly linearly regular.*

Theorem 6 [17, Corollary 5.6] *Let* $T_i \colon \mathcal{H} \to \mathcal{H}$ *be* ρ_i-*SQNE for all* $i \in \{1, \ldots, n\}$. *Assume* $\cap_{i=1}^n \text{Fix}\, T_i \neq \emptyset$ *and denote* $U := \mathbb{B}(z, R)$ *where* $z \in \cap_{i=1}^n \text{Fix}\, T_i$ *and* $R > 0$. *Suppose the following assertions hold.*

  (i)   *The operator* T_i *is linearly regular on* U *for all* $i \in \{1, \ldots, n\}$.
  (ii) *The collection* $\{\text{Fix}\, T_i\}_{i=1}^n$ *is linearly regular on* U.

Then $T := T_n \ldots T_2 T_1$ *is linearly regular on* U.

The following is immediate from the definitions.

Corollary 1 *Let* $T_i \colon \mathcal{H} \to \mathcal{H}$ *be* ρ_i-*SQNE for all* $i \in \{1, \ldots, n\}$. *Assume* $\cap_{i=1}^n \text{Fix}\, T_i \neq \emptyset$. *Suppose the following assertions hold.*

  (i)   *The operator* T_i *is boundedly linearly regular for all* $i \in \{1, \ldots, n\}$.
  (ii) *The collection* $\{\text{Fix}\, T_i\}_{i=1}^n$ *is boundedly linearly regular.*

Then $T := T_n \ldots T_2 T_1$ *is boundedly linearly regular.*

Proof Follows by combining Theorem 6 with Remark 1. □

The following regularity notion is the Hölder analogue of Definition 5.

Definition 6 (Hölder regular collections of sets) A collection of sets $\{C_1, \ldots, C_n\}$ is *Hölder regular* on U if there exists $\tau > 0$ and $\theta \in (0, 1)$ such that

$$d(x, \cap_{i=1}^n C_i) \leq \tau \max_{i=1,\ldots,n} d(x, C_i)^\theta \quad \forall x \in U.$$

If the collection $\{C_1, \ldots, C_n\}$ is Hölder regular on every bounded subset of $\mathcal{H}$, it is said to be *boundedly Hölder regular*.

Lemma 5 *Let* $0 < \gamma \leq \theta$ *and* $b > 0$. *There exists* $M > 0$ *such that* $\alpha^\theta \leq M \alpha^\gamma$ *for all* $\alpha \in [0, b]$.

Proof Since $\theta - \gamma > 0$ by assumption, we have $\alpha^{\theta-\gamma} \leq b^{\theta-\gamma}$. Thus, for all $\alpha \in [0, b]$, we have $\alpha^\theta = \alpha^{\theta-\gamma} \alpha^\gamma \leq M\alpha^\gamma$ with $M = b^{\theta-\gamma}$. $\square$

The following lemma is due to Cegielski and Zalas [18]. Since we need a slightly different version result to one which appears in [18, Proposition 4.5], we include its proof.

Lemma 6 *Suppose $T_i \colon \mathcal{H} \to \mathcal{H}$ is ρ_i-SQNE for all $i \in \{1, \ldots, n\}$ and denote $T := \sum_{i=1}^n \omega_i T_i$ where $\sum_{i=1}^n \omega_i = 1$ and $\omega_i > 0$ for all $i \in \{1, \ldots, n\}$. Assume that $\mathrm{Fix}\, T = \cap_{i=1}^n \mathrm{Fix}\, T_i \neq \emptyset$. Then*

$$\sum_{i=1}^n \omega_i \rho_i \|x - T_i(x)\|^2 \leq 2\, \mathrm{d}(x, \mathrm{Fix}\, T)\|x - T(x)\| \quad \forall x \in \mathcal{H}.$$

Proof Let $z = P_{\mathrm{Fix}\, T}(x)$. Since T_i is ρ_i-SQNE, we have

$$\|T(x) - z\|^2 \leq \sum_{i=1}^n \omega_i \|T_i(x) - z\|^2 \leq \|x - z\|^2 - \sum_{i=1}^n \omega_i \rho_i \|x - T_i(x)\|^2.$$

Using the Cauchy–Schwarz inequality, we deduce

$$\begin{aligned} \|T(x) - z\|^2 &= \|T(x) - x\|^2 + \|x - z\|^2 + 2\langle T(x) - x, x - z\rangle \\ &\geq \|T(x) - x\|^2 + \|x - z\|^2 - 2\, \mathrm{d}(x, \mathrm{Fix}\, T)\|T(x) - x\|. \end{aligned}$$

The claimed result follows by combining the previous two inequalities. $\square$

Theorem 7 *Let $T_i \colon \mathcal{H} \to \mathcal{H}$ be ρ_i-SQNE for all $i \in \{1, \ldots, n\}$ and assume $\mathrm{Fix}\, T = \cap_{i=1}^n \mathrm{Fix}\, T_i \neq \emptyset$. Suppose that the following assertions hold.*

(i) *The operator T_i is boundedly Hölder regular.*
(ii) *The collection $\{\mathrm{Fix}\, T_i\}_{i=1}^n$ is boundedly Hölder regular.*

Then $T := \sum_{i=1}^n \omega_i T_i$ is boundedly Hölder regular whenever $\sum_{i=1}^n \omega_i = 1$ and $\omega_i > 0$ for all $i \in \{1, \ldots, n\}$.

Proof Let U be a nonempty bounded set. Since T_i is boundedly Hölder regular, there exist constants $\kappa_i > 0$ and $\gamma_i \in (0, 1)$ such that

$$\mathrm{d}(x, \mathrm{Fix}\, T_i) \leq \kappa_i \|x - T_i(x)\|^{\gamma_i} \quad \forall x \in U.$$

Denote $\gamma = \min_{i=1,\ldots,n} \gamma_i \in (0, 1)$. Since U is bounded and $\gamma \leq \gamma_i$, Lemma 5 implies the existence of constants $M_i > 0$ such that

$$\|x - T_i(x)\|^{\gamma_i} \leq M_i \|x - T_i(x)\|^\gamma \quad \forall x \in U.$$

Denote $\kappa = \max_{i=1,\ldots,n} \kappa_i M_i$. Then, combining the previous two inequalities gives

$$\mathrm{d}(x, \mathrm{Fix}\, T_i) \leq \kappa_i M_i \|x - T_i(x)\|^\gamma \leq \kappa \|x - T_i(x)\|^\gamma \quad \forall x \in U.$$

Let $x \in U$ and $z \in \operatorname{Fix} T$. Set $\omega = \min_{j=1,\ldots,n} \omega_j$ and set $\rho = \min_{j=1,\ldots,n} \rho_j$. Then

$$\omega_i \operatorname{d}(x, \operatorname{Fix} T_i) \leq \sum_{j=1}^{n} \omega_j \operatorname{d}(x, \operatorname{Fix} T_j) \leq \kappa \sum_{j=1}^{n} \omega_j \|x - T_j x\|^{\gamma},$$

and so convexity of $t \mapsto t^{2/\gamma}$ together with Lemma 6 implies

$$\omega^{2/\gamma} \operatorname{d}^{2/\gamma}(x, \operatorname{Fix} T_i) \leq \kappa^{2/\gamma} \sum_{j=1}^{n} \omega_j \|x - T_j x\|^2 \leq \frac{2\kappa^{2/\gamma}}{\rho} \operatorname{d}(x, \operatorname{Fix} T)\|x - T(x)\|. \quad (12)$$

Thus, using the fact that the collection $\{\operatorname{Fix} T_i\}_{i=1}^{n}$ is Hölder regular on U together with Eq. 12, we deduce the existence of a $\tau > 0$ and a $\theta \in (0, 1)$ such that

$$\operatorname{d}^{\frac{2}{\gamma\theta}}(x, \operatorname{Fix} T) \leq \tau^{\frac{2}{\gamma\theta}} \max_{i=1,\ldots,n} \operatorname{d}^{2/\gamma}(x, \operatorname{Fix} T_i) \leq \tau^{\frac{2}{\gamma\theta}} \frac{2\kappa^{2/\gamma}}{\rho\omega^{2/\gamma}} \operatorname{d}(x, \operatorname{Fix} T)\|x - T(x)\|,$$

from which the result follows. $\qquad \square$

The following lemma is due to Cegielski and Zalas [18]. Since we need a slightly different version result to one which appears in [18, Proposition 4.6], we include its proof.

Lemma 7 *Suppose $T_i \colon \mathcal{H} \to \mathcal{H}$ is ρ_i-SQNE for all $i \in \{1, \ldots, n\}$ and denote $T := T_n \ldots T_2 T_1$. Assume that $\operatorname{Fix} T = \cap_{i=1}^{n} \operatorname{Fix} T_i \neq \emptyset$. Then*

$$\sum_{i=1}^{n} \rho_i \|Q_{i-1}(x) - Q_i(x)\|^2 \leq 2 \operatorname{d}(x, \operatorname{Fix} T)\|x - T(x)\| \quad \forall x \in \mathcal{H},$$

where we denote $Q_0 := \operatorname{Id}$ and $Q_i := T_i \ldots T_1$ for all $i \in \{1, \ldots, n\}$.

Proof Let $z = P_{\operatorname{Fix} T}(x)$. Since T_i is ρ_i-SQNE, we have

$$\|T(x) - z\|^2 \leq \|x - z\|^2 - \sum_{i=1}^{n} \rho_i \|Q_{i-1}(x) - Q_i(x)\|^2.$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|T(x) - z\|^2 &= \|T(x) - x\|^2 + \|x - z\|^2 + 2\langle T(x) - x, x - z\rangle \\ &\geq \|T(x) - x\|^2 + \|x - z\|^2 - 2\operatorname{d}(x, \operatorname{Fix} T)\|T(x) - x\|.\end{aligned}$$

The claimed result follows by combining the previous two inequalities. $\qquad \square$

Theorem 8 *Let $T_i \colon \mathcal{H} \to \mathcal{H}$ be ρ_i-SQNE for $i \in \{1, \ldots, n\}$ and let $T := T_n \ldots T_2 T_1$. Assume that $\operatorname{Fix} T = \cap_{i=1}^{n} \operatorname{Fix} T_i \neq \emptyset$ and denote $U = \mathbb{B}(z, R)$ for some $z \in \operatorname{Fix} T$ and $R > 0$. Suppose that the following assertions hold.*

(i)   *The operator T_i is Hölder regular on U for all $i \in \{1, \ldots, n\}$.*
(ii) *The collection $\{\operatorname{Fix} T_i\}_{i=1}^{n}$ is Hölder regular on U.*

Then T is Hölder regular on U.

Proof Denote $Q_0 = \text{Id}$ and $Q_i = T_i \ldots T_2 T_1$ for all $i \in \{1, \ldots, n\}$. Since T_i is Hölder regular on U, there exist constants $\kappa_i > 0$ and $\gamma_i \in (0, 1)$ such that

$$d(x, \text{Fix } T_i) \le \kappa_i \|x - T_i(x)\|^{\gamma_i} \quad \forall x \in U.$$

Denote $\gamma = \min_{i=1,\ldots,n} \gamma_i \in (0, 1)$. By using the same argument as in Theorem 7, we deduce the existence of $\kappa > 0$ such that, for all $i \in \{1, \ldots, n\}$, we have

$$d(x, \text{Fix } T_i) \le \kappa \|x - T_i(x)\|^{\gamma} \quad \forall x \in U. \tag{13}$$

Let $x \in U$. Then, since T_i is ρ_i-SQNE for all $i \in \{1, \ldots, n\}$, we have that

$$
\begin{aligned}
R^2 \ge \|x - z\|^2 &\ge \|Q_1(x) - z\|^2 + \rho_1 \|Q_0(x) - Q_1(x)\|^2 \\
&\ge \|Q_2(x) - z\|^2 + \rho_2 \|Q_1(x) - Q_2(x)\|^2 + \rho_1 \|Q_0(x) - Q_1(x)\|^2 \\
&\vdots \\
&\ge \|T(x) - z\|^2 + \sum_{i=1}^{n} \rho_i \|Q_{i-1}(x) - Q_i(x)\|^2.
\end{aligned}
$$

From this, it follows that $Q_i(x) \in U$ for all $i \in \{1, \ldots, n\}$ and that

$$\max_{i=1,\ldots,n} \|Q_{i-1}(x) - Q_i(x)\| \in \left[0, \frac{R}{\sqrt{\rho}}\right] \quad \text{where } \rho = \min_{i=1,\ldots,n} \rho_i.$$

By Lemma 5, there exists a constant $\mu > 0$ such that, for all $i \in \{1, \ldots, n\}$, we have

$$\|Q_{i-1}(x) - Q_i(x)\| \le \mu \|Q_{i-1}(x) - Q_i(x)\|^{\gamma} \quad \text{for all } x \in U. \tag{14}$$

Set $M := \max\{\mu, \kappa\}$ and $j \in \{1, \ldots, n\}$. Applying the triangle inequality, followed by Eqs. 13 and 14, gives

$$
\begin{aligned}
d(x, \text{Fix } T_j) &\le \|x - P_{\text{Fix } T_j}(Q_{j-1}(x))\| \\
&\le \|x - Q_1(x)\| + \|Q_1(x) - Q_2(x)\| + \ldots \\
&\quad + \|Q_{j-1}(x) - P_{\text{Fix } T_j}(Q_{j-1}(x))\| \\
&\le \|x - Q_1(x)\| + \|Q_1(x) - Q_2(x)\| + \cdots + \kappa \|Q_{j-1}(x) - Q_j(x)\|^{\gamma_j} \\
&\le M \sum_{i=1}^{n} \|Q_{i-1}(x) - Q_i(x)\|^{\gamma}.
\end{aligned}
$$

Set $\rho := \min_{i=1,\ldots,n} \rho_i$. Using convexity of $t \mapsto t^{2/\gamma}$ followed by Lemma 7, we deduce

$$
\begin{aligned}
d^{2/\gamma}(x, \text{Fix } T_j) &\le n^{(\gamma/2-1)} M^{2/\gamma} \sum_{i=1}^{n} \|Q_{i-1}(x) - Q_i(x)\|^2 \\
&\le \frac{2n^{(\gamma/2-1)} M^{2/\gamma}}{\rho} d(x, \text{Fix } T) \|x - T(x)\|. \tag{15}
\end{aligned}
$$

Thus, using the fact that the collection $\{\text{Fix } T_j\}_{j=1}^{n}$ is Hölder regular on U together with (15), we deduce the existence of a $\tau > 0$ and a $\theta \in (0, 1)$ such that

$$d^{\frac{2}{\gamma\theta}}(x, \text{Fix } T) \leq \tau^{\frac{2}{\gamma\theta}} \max_{j=1,\ldots,n} d^{2/\gamma}(x, \text{Fix } T_j)$$

$$\leq \frac{2n^{(\gamma/2-1)} M^{2/\gamma} \tau^{\frac{2}{\gamma\theta}}}{\rho} d(x, \text{Fix } T) \|x - T(x)\|.$$

The result then follows on observing that $\frac{\gamma\theta}{2-\gamma\theta} < 1$ as $\gamma, \theta \in (0, 1)$. $\qquad\square$

Corollary 2 *Let* $T_i \colon \mathcal{H} \to \mathcal{H}$ *be* ρ_i-*SQNE for all* $i \in \{1, \ldots, n\}$. *Assume* $\cap_{i=1}^{n} \text{Fix } T_i \neq \emptyset$. *Suppose the following assertions hold.*

(i)   *The operator* T_i *is boundedly Hölder regular for all* $i \in \{1, \ldots, n\}$.
(ii) *The collection* $\{\text{Fix } T_i\}_{i=1}^{n}$ *is boundedly Hölder regular.*

Then $T := T_n \ldots T_2 T_1$ *is boundedly Hölder regular.*

Proof Follows by combining Theorem 8 with Remark 1. $\qquad\square$

5 Convergence rates for combinations and compositions

In this section, we further refine the results from Section 3. More precisely, we consider the dynamical system (1) in the setting when the operator T can be expressed in terms of a convex combination or a composition of operators $T_i \colon \mathcal{H} \to \mathcal{H}$ for $i \in \{1, \ldots, n\}$ with $\cap_{i=1}^{n} \text{Fix } T_i \neq \emptyset$. In other words, we consider the system

$$\dot{x}(t) = \lambda(t) \left(T(x(t)) - x(t) \right), \tag{16}$$

where T is given by ether:

(i) $T = \sum_{i=1}^{n} \omega_i T_i$ with $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i > 0$ for all $i \in \{1, \ldots, n\}$, or
(ii) $T = T_n \ldots T_2 T_1$.

Situations of this kind naturally arise in the study of continuous-time *projection algorithms* for solving the *feasibility problem*. This problem asks for a point in the intersection of closed, convex constraints $C_1, \ldots, C_n$. In the simplest such algorithm, the *method of cyclic projections*, $T_i = P_{C_i}$ where P_C denotes the *nearest point projector* onto a set C given by

$$P_C(x) = \{c \in C : \|x - c\| \leq \|x - z\| \, \forall z \in C\},$$

and $T = P_{C_n} \ldots P_{C_2} P_{C_1}$ is the *cyclic projections* operator. Another example is provided by *Douglas–Rachford* methods in which each operator T_i is a *Douglas–Rachford* operator of the form

$$\frac{\text{Id} + (2P_{C_j} - \text{Id})(2P_{C_l} - \text{Id})}{2} = \text{Id} + P_{C_j}(2P_{C_l} - \text{Id}) - P_{C_l}$$

for pair indices $j, l \in \{1, \ldots, n\}$. For further details on projection algorithms (with $\mathcal{H}$ potentially infinite dimensional) in linearly regular settings, see [9, 17], and in Hölder regular settings, see [11].

We obtain the results in this section by combining the results from the previous two sections. To do so, we require the following class of operators which are both nonexpansive and strongly quasinonexpansive.

Definition 7 (Averaged nonexpansive [5]) An operator $T : \mathcal{H} \to \mathcal{H}$ is α-*averaged nonexpansive* if $\alpha \in (0, 1)$ such that one of the following two equivalent properties holds.

(i)   There exists a nonexpansive operator $R : \mathcal{H} \to \mathcal{H}$ such that $T = (1-\alpha)\,\mathrm{Id} + \alpha R$.
(ii) For all $x, y \in \mathcal{H}$, we have

$$\|T(x) - T(y)\|^2 + \frac{1 - \alpha}{\alpha} \|(\mathrm{Id} - T)(x) - (\mathrm{Id} - T)(y)\|^2 \leq \|x - y\|^2.$$

Note that it is immediate from the respective definitions that an α-averaged operator is ρ-SQNE with $\rho = (1 - \alpha)/\alpha$.

Corollary 3 *Let $T_i : \mathcal{H} \to \mathcal{H}$ be α_i-averaged nonexpansive with $\cap_{i=1}^n \mathrm{Fix}\, T_i \neq \emptyset$. Suppose $\lambda : [0, +\infty) \to [0, 1]$ is Lebesgue measurable with $\inf_{t \geq 0} \lambda(t)\,(1 - \lambda(t)) > 0$. Let x be the unique strong global solution of* Eq. 16. *Furthermore, suppose that the following assertions hold.*

(i)   *The operator T_i is boundedly linearly regular for $i \in \{1, \ldots, n\}$.*
(ii) *The collection $\{\mathrm{Fix}\, T_i\}_{i=1}^n$ is boundedly linearly regular.*

Then there exist $\bar{x} \in \cap_{i=1}^n \mathrm{Fix}\, T_i$ and constants $M, r > 0$ such that, for almost all $t \in [0, +\infty)$, we have

$$\|x(t) - \bar{x}\| \leq M \exp(-rt).$$

In particular, the trajectory $x(t)$ converges strongly to $\bar{x}$ as $t \to +\infty$.

Proof By either Theorem 5 or Corollary 1, the operator T is boundedly Hölder regular. The result then follows by Theorem 3. ☐

Corollary 4 *Let $T_i : \mathcal{H} \to \mathcal{H}$ be α_i-averaged nonexpansive with $\cap_{i=1}^n \mathrm{Fix}\, T_i \neq \emptyset$. Suppose $\lambda : [0, +\infty) \to [0, 1]$ is Lebesgue measurable with $\inf_{t \geq 0} \lambda(t)\,(1 - \lambda(t)) > 0$. Let x be the unique strong global solution of* Eq. 16. *Furthermore, suppose that the following assertions hold.*

(i)   *The operator T_i is boundedly Hölder regular for all $i \in \{1, \ldots, n\}$.*
(ii) *The collection $\{\mathrm{Fix}\, T_i\}_{i=1}^n$ is boundedly Hölder regular.*

Then there exists $\bar{x} \in \cap_{i=1}^n \mathrm{Fix}\, T_i$, $M > 0$ and $\gamma \in (0, 1)$ such that, for almost all $t \in [0, +\infty)$, we have

$$\|x(t) - \bar{x}\| \leq M\, t^{-\frac{\gamma}{2(1-\gamma)}}.$$

In particular, the trajectory $x(t)$ converges strongly to $\bar{x}$ as $t \to +\infty$.

Proof By either Theorem 7 or Corollary 2, the operator T is boundedly Hölder regular. The result then follows by Theorem 4.   □

References

1. Abbas, B., Attouch, H., Svaiter, B.F.: Newton-like dynamics and forward-backward methods for structured monotone inclusions in Hilbert spaces. J. Optim. Theory Appl. **161**(2), 331–360 (2014)
2. Attouch, H., Alvarez, F.: The heavy ball with friction dynamical system for convex constrained minimization problems. In: Optimization (Namur, 1998), Lecture Notes in Economics and Mathematical Systems 481, pp. 25–35. Springer, Berlin (2000)
3. Attouch, H., Cabot, A.: Convergence of a relaxed inertial forward-backward algorithm for structured monotone inclusions. Appl. Math. Optim. **80**(3), 547–598 (2019)
4. Attouch, H., Svaiter, B.F.: A continuous dynamical Newton-like approach to solving monotone inclusions. SIAM J. Control. Optim. **49**(2), 574–598 (2011)
5. Baillon, J.-B., Bruck, R.E., Reich, S.: On the asymptotic behavior of nonexpansive mappings and semigroups. Houst. J. Math. **4**, 1–9 (1978)
6. Banert, S., Boţ, R.I.: A forward-backward-forward differential equation and its asymptotic properties. J. Convex Anal. **25**(2), 371–388 (2018)
7. Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**(3), 367–426 (1996)
8. Bauschke, H.H., Combettes, P.L. Convex Analysis and Monotone Operator Theory in Hilbert Spaces vol. 408 of CMS Books in Mathematics, 2nd edn. Springer, New York (2017)
9. Bauschke, H.H., Noll, D., Phan, H.M.: Linear and strong convergence of algorithms involving averaged nonexpansive operators. J. Math. Anal. Appl. **421**(1), 1–20 (2015)
10. Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry. Springer-Verlag, New York (1998)
11. Borwein, J.M., Li, G., Tam, M.K.: Convergence rate analysis for averaged fixed point iterations in common fixed point problems. SIAM J. Optim. **27**(1), 1–33 (2017)
12. Boţ, R.I., Csetnek, E.R.: A dynamical system associated with the fixed points set of a nonexpansive operator. J. Dyn. Diff. Equat. **29**(1), 155–168 (2015)
13. Boţ, R.I., Csetnek, E.R.: Second order forward-backward dynamical systems for monotone inclusion problems. SIAM J. Control. Optim. **54**(3), 1423–1443 (2016)
14. Polyak, B.T.: Introduction to Optimization (Translated from the Russian), Translations Series in Mathematics and Engineering, Optimization Software, Inc. Publications Division, New York (1987)
15. Bravo, M., Cominetti, R., Pavez-Signé, M.: Rates of convergence for inexact Krasnosel'skii–Mann iterations in Banach spaces. Math. Program. **175**, 241–262 (2019)
16. Cegielski, A.: Iterative Methods for Fixed Point Problems in Hilbert Spaces. Springer-Verlag, Berlin Heidelberg (2012)
17. Cegielski, A., Reich, S., Zalas, R.: Regular sequences of quasi-nonexpansive operators and their applications. SIAM J. Optim. **28**(2), 1508–1532 (2018)
18. Cegielski, A., Zalas, R.: Properties of a class of approximately shrinking operators and their applications. Fixed Point Theory **15**, 399–426 (2014)
19. Csetnek, E.R., Malitsky, Y., Tam, M.K.: Shadow Douglas–Rachford splitting for monotone inclusions. Appl. Math. Optim. **80**(3), 665–678 (2019)
20. Drusvyatskiy, D., Lewis, A.S.: Semi-algebraic functions have small subdifferentials. Math. Program. **140**(1), 5–29 (2013)
21. Ioffe, A.D.: An invitation to tame optimization. SIAM J. Optim. **19**(4), 1894–1917 (2009)

22. Luke, D.R., Nguyen, H.T., Tam, M.K.: Implicit error bounds for Picard iterations on Hilbert spaces. Vietnam J. Math. **46**(2), 243–258 (2018)
23. Nesterov, Y.: A method for unconstrained convex minimization problem with the rate of convergence $O(1/k^2)$. Doklady AN SSSR (translated as Soviet Math. Docl.) **269**, 543–547 (1983)
24. Nesterov, Y.: Introductory Lectures on Convex Optimization: A Basic Course. Kluwer Academic Publishers, Dordrecht (2004)
25. Peypouquet, J., Sorin, S.: Evolution equations for maximal monotone operators: asymptotic analysis in continuous and discrete time. J. Convex Anal. **17**, 1113–1163 (2010)
26. Rieger, J., Tam, M.K.: Backward-forward-reflected-backward splitting for three operator monotone inclusions. Appl. Math. Comput. **381**, 125248 (2020)
27. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer Science & Business Media, New York (2009)
28. Robinson, S.M.: Some continuity properties of polyhedral multifunctions. Math. Program. Study **14**, 206–214 (1981)
29. Ryu, E.K., Kun, Y., Wotao, Y.: ODE analysis of stochastic gradient methods with optimism and anchoring for minimax problems and GANs. arXiv:1905.10899 (2019)