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Generalized Born-Jordan distributions and applications

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Abstract

One of the most popular time-frequency representations is certainly the Wigner distribution. Its quadratic nature is, however, at the origin of unwanted interferences or artefacts. The desire to suppress these artefacts is the reason why engineers, mathematicians and physicists have been looking for related time-frequency distributions, many of them being members of the Cohen class. Among these, the Born-Jordan distribution has recently attracted the attention of many authors, since the so-called ghost frequencies are grandly damped, and the noise is, in general, reduced; it also seems to play a key role in quantum mechanics. The central insight relies on the kernel of such a distribution, which contains the *sinus cardinalis* sinc, the Fourier transform of the first B-spline B_1. The idea is to replace the function B_1 with the spline or order n, denoted by B_n, yielding the function $(\mathrm{sinc})^n$ when Fourier transformed, whose speed of decay at infinity increases with n. The related Cohen kernel is given by $\Theta^n(z_1, z_2) = \mathrm{sinc}^n(z_1 \cdot z_2)$, $n \in \mathbb{N}$, and the corresponding time-frequency distribution is called *generalized Born-Jordan distribution of order* n. We show that this new representation has a great potential to damp unwanted interference effects and this damping effect increases with n. Our proofs of these properties require an interdisciplinary approach, using tools from both microlocal and time-frequency analysis. As a by-product, a new quantization rule and a related pseudo-differential calculus are investigated.

Keywords Time-frequency analysis · Wigner distribution · Born-Jordan distribution · B-splines · Interferences · Wave-front set · Fourier-Lebesgue spaces

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1 Introduction

The time-frequency analysis of real-world signals is an intrinsically interdisciplinary topic, involving engineering, physics and mathematics. It is an essential topic in various applications (see for instance the papers [5–7, 24, 33, 34, 39, 40, 47]). In the present paper, we introduce a new family of time-frequency representations defined by exponentiating the *sinus cardinalis* kernel; we call the members of this family *generalized Born-Jordan distributions*. These new distributions form a subclass of the Cohen class containing several important and well-known distributions (Wigner and Born–Jordan). The interest of this new class of time-frequency distributions comes from the fact that its members efficiently damp the artefacts stemming from the interaction between distinct time-frequency components in a given signal, which are due to the bilinear nature of Cohen class distributions. These damping properties will be made explicit by a precise study of the smoothing effects induced by our generalized Born–Jordan distributions.

Now, one of the most popular time-frequency representations of a signal f is the Wigner distribution

$$Wf(x,\omega) = \int_{\mathbb{R}^d} f\big(x+\frac{y}{2}\big)\overline{f\big(x-\frac{y}{2}\big)}e^{-2\pi i y\omega}\,dy, \qquad x,\omega\in\mathbb{R}^d, \tag{1}$$

where the signal f can be thought of as a function in $L^2(\mathbb{R}^d)$ (or more generally as a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$). It is however well-known that the quadratic nature of the Wigner distribution generates undesired (usually oscillatory) interferences between signal components separated in time-frequency. To overcome this issue, the so-called Cohen class of time-frequency distributions was introduced in [6] and widely studied by many authors (see [1–3, 7, 34] and references therein). The Cohen class members Qf are generated by convolving the Wigner distribution of a signal f with a smoothing distribution $\theta \in \mathcal{S}'(\mathbb{R}^{2n})$ (Cohen kernel) in order to try to suppress the oscillatory artefacts:

$$Qf = Wf * \theta. \tag{2}$$

Choosing $\theta = \mathcal{F}_\sigma\Theta^1$, where $\mathcal{F}_\sigma\Theta^1$ is the symplectic Fourier transform of

$$\Theta^1(x,\omega) = \operatorname{sinc}(x\omega) = \begin{cases} \dfrac{\sin(\pi x\omega)}{\pi x\omega} & \text{for } x\omega \neq 0 \\ 1 & \text{for } x\omega = 0 \end{cases} \tag{3}$$

leads to the Born-Jordan distribution:

$$Q^1 f = Wf * \mathcal{F}_\sigma(\Theta^1), \quad f\in L^2(\mathbb{R}^d), \tag{4}$$

see [2, 6–8, 11, 26, 29, 34] and references therein.

In the present paper, we introduce new Cohen kernels and related distributions using the B-spline functions B_n. Recall that the sequence of B-splines $\{B_n\}_{n\in\mathbb{N}_+}$ is defined inductively as follows: the first B-spline is

$$B_1(t) = \chi_{\left[-\frac{1}{2},\frac{1}{2}\right]}(t).$$

Assuming that we have defined B_n, for some $n \in \mathbb{N}_+$, the spline B_{n+1} is then defined by

$$B_{n+1}(t) = (B_n * B_1)(t) = \int_{\mathbb{R}} B_n(t-y)B_1(y)dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} B_n(t-y)dy. \quad (5)$$

B_n is a piecewise polynomial of degree at most $n-1$, $n \in \mathbb{N}_+$, and satisfying $B_n \in \mathcal{C}^{n-2}(\mathbb{R})$, $n \geq 2$. For the main properties of B_n, we refer e.g. to [4].

Observe that $\operatorname{sinc}(\xi) = \mathcal{F}B_1(\xi)$ hence by induction on n

$$\operatorname{sinc}^n(\xi) = \mathcal{F}B_n(\xi), \quad n \in \mathbb{N}_+. \quad (6)$$

Definition 1 For $n \in \mathbb{N}$, the n^{th} Born-Jordan kernel is the function Θ^n on $\mathbb{R}^{2d}$ defined by

$$\Theta^n(x,\omega) = \operatorname{sinc}^n(x\omega), \quad (x,\omega) \in \mathbb{R}^{2d}. \quad (7)$$

The Born-Jordan distribution of order n (BJDn) is given by

$$Q^n f = Wf * \mathcal{F}_\sigma(\Theta^n), \quad f \in L^2(\mathbb{R}^d). \quad (8)$$

The cross-BJDn is given by

$$Q^n(f,g) = W(f,g) * \mathcal{F}_\sigma(\Theta^n), \quad f,g \in L^2(\mathbb{R}^d). \quad (9)$$

We write $Q^n(f,f) = Q^n f$ for every $f \in L^2(\mathbb{R}^d)$.

Remark 1 Note that $\Theta^0 \equiv 1$, hence $\mathcal{F}_\sigma(\Theta^0) = \delta$ and $Q^0 f = Wf$, the Wigner distribution of f.

In the sequel, we study central properties of the newly introduced distributions and thereby address the following issues:

(i) *Regularity and Smoothness Properties of* Q^n;
(ii) *Damping of interferences in comparison with the Wigner distribution*;
(iii) *Visual comparison in dimension* $d = 1$ *between* Q^n *and the Wigner Distribution*;
(iv) *Born–Jordan quantization of order n and related pseudo-differential calculus.*

The most suitable framework to handle these aspects is provided by *modulation spaces* (see [19] and also the textbook [31]), recalled in Section 2.3. Their definition is based on the *the short-time Fourier transform (STFT)* $V_g f$, defined, for a fixed Schwartz function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, by

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(y)\, \overline{g(y-x)}\, e^{-2\pi i y\omega}\, dy, \quad (x,\omega) \in \mathbb{R}^{2d}. \quad (10)$$

For $1 \leq p,q \leq \infty$, the (unweighted) modulation space $M^{p,q}(\mathbb{R}^d)$ is then the subspace of tempered distributions f such that

$$\|f\|_{M^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \, dx \right)^{q/p} d\omega \right)^{1/q} < \infty$$

(with standard modifications for $p = \infty$ or $q = \infty$).

Regularity of Q^n While it seems intuitively clear that a signal's Born-Jordan distribution of order n cannot be rougher than the corresponding Wigner distribution, we will prove several related precise statements. In Proposition 6, we will show that the n-th Born-Jordan kernel belongs to the Wiener amalgam space $W(\mathcal{F}L^1, L^\infty)$, defined in Section 2.3 below, for every $n \in \mathbb{N}_+$. This observation is the key tool for proving the following result:

Theorem 1 *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{p,q}(\mathbb{R}^{2d})$ for some $1 \leq p, q \leq \infty$. Then $Q^n f \in M^{p,q}(\mathbb{R}^{2d})$, for every $n \in \mathbb{N}_+$.*

The previous statement holds in more generality and can be rephrased for members in the Cohen class as follows.

Theorem 2 *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{p,q}(\mathbb{R}^{2d})$ for some $1 \leq p, q \leq \infty$ and the Cohen kernel θ defined in (2) belonging to the modulation space $M^{1,\infty}(\mathbb{R}^{2d})$. Then, the corresponding Cohen member Qf belongs to $M^{p,q}(\mathbb{R}^{2d})$.*

Our central concern is the discussion of the new distributions' capacity for the damping of interferences in comparison with the Wigner distribution, a topic connected with the smoothness of Q^n and measured using the Fourier-Lebesgue wave-front set.

The notion of wave-front set of a distribution is nowadays a standard technique in the study of singularities for solutions to partial differential (or pseudo-differential) equations. The basic idea is to detect the location and orientation of the singularities of a distribution f by looking at which directions the Fourier transform of φf fails to decay rapidly, where φ is a cut-off function supported in a neighbourhood of any given point x_0. This test is performed in the framework of edge detection, where often the Fourier transform is replaced by other transforms, see e.g. [38] and the references therein.

We shall use the Fourier-Lebesgue wave-front set, introduced in [41–43], and related to the Fourier-Lebesgue spaces $\mathcal{F}L^q_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 \leq q \leq \infty$. Recall that the norm in the space $\mathcal{F}L^q_s(\mathbb{R}^d)$, $1 \leq q \leq \infty$, is given by

$$\|f\|_{\mathcal{F}L^q_s(\mathbb{R}^d)} = \|\widehat{f}(\omega)\langle\omega\rangle^s\|_{L^q(\mathbb{R}^d)}, \tag{11}$$

with $\langle\omega\rangle = (1+|\omega|^2)^{1/2}$. Inspired by this definition, given a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, its wave-front set $WF_{\mathcal{F}L^q_s}(f) \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ is the set of points $(x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d$, $\omega_0 \neq 0$, where the following condition *is not satisfied*: for some cut-off function φ (i.e. φ is smooth and compactly supported on $\mathbb{R}^d$), with $\varphi(x_0) \neq 0$, and some open conic neighbourhood $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of ω_0 it holds

$$\|\mathcal{F}[\varphi f](\omega)\langle\omega\rangle^s\|_{L^q(\Gamma)} < \infty. \tag{12}$$

Observe that $WF_{\mathcal{F}L^2_s}(f) = WF_{H^s}(f)$ is the standard H^s wave-front set (see [35, Chapter XIII] and Section 2 below). Roughly speaking, $(x_0, \omega_0) \notin WF_{\mathcal{F}L^q_s}(f)$ means that f has regularity $\mathcal{F}L^q_s$ at x_0 and in the direction ω_0. We are interested in the $\mathcal{F}L^q_s$ wave-front set of the Born-Jordan distribution of order n of a given signal $f \in L^2(\mathbb{R}^d)$.

Here is the mathematical explanation of the Q^n's smoothing effects:

Theorem 3 *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{\infty,q}(\mathbb{R}^{2d})$ for some $1 \le q \le \infty$. Let $(z,\zeta) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$, with $\zeta = (\zeta_1,\zeta_2)$ satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. Then*

$$(z,\zeta) \notin WF_{\mathcal{F}L^q_{2n}}(Q^n f).$$

This means that if the Wigner distribution Wf has $\mathcal{F}L^q$ local regularity and is somewhat controlled at infinity, then $Q^n f$ is smoother, having $s = 2n$ additional derivatives, at least in the directions $\zeta = (\zeta_1,\zeta_2)$ satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. In dimension $d = 1$, this condition reduces to $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$. Hence, this result explains the smoothing property of such distributions, which involves all the possible directions except those of the coordinates axes. That is why the interferences of two components which do not share the same time or frequency localization come out substantially reduced. Observe that for $n = 1$, we recapture the damping phenomenon of the classical Born–Jordan distribution (cf. [13, Theorem 1.2]).

For signals in $L^2(\mathbb{R}^d)$, the previous result can be rephrased in terms of the Hörmander's wave-front set as follows:

Corollary 1 *Let $f \in L^2(\mathbb{R}^d)$, so that $Wf \in L^2(\mathbb{R}^{2d})$. Let (z,ζ) be as in the statement of Theorem 3. Then, $(z,\zeta) \notin WF_{H^{2n}}(Q^n f)$, i.e. $Q^n f$ has regularity H^{2n} at z and in the direction ζ.*

The pictorial examples below suggest that the smoothing effects of the BJDn do not occur in the directions $\zeta_1 \cdot \zeta_2 = 0$. From a mathematical point of view, this is explained by the following theorem.

Theorem 4 *Suppose that for some $1 \le p, q_1, q_2 \le \infty$, $n \in \mathbb{N}_+$ and $C > 0$, we have*

$$\|Q^n f\|_{M^{p,q_1}} \le C\|Wf\|_{M^{p,q_2}}, \tag{13}$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$. Then, $q_1 \ge q_2$.

In other words, for a general signal, the BJDn is not everywhere smoother than the Wigner distribution. As expected, the problems arise in the directions $\zeta = (\zeta_1,\zeta_2)$ such that $\zeta_1 \cdot \zeta_2 = 0$.

Visual comparison in dimension $d = 1$ between Q^n and the Wigner distribution We now illustrate the effect of using higher order cross-term suppression by means of the generalized BJDn. We display the time-frequency distributions of both synthetic and real signals. More precisely, Fig. 1 shows a comparison of the Wigner transform, the Born-Jordan transform and generalized Born-Jordan transform of the sum of four rotated Gaussian windows. It is clearly visible that the amount of cross-term suppression increases for higher order smoothing.

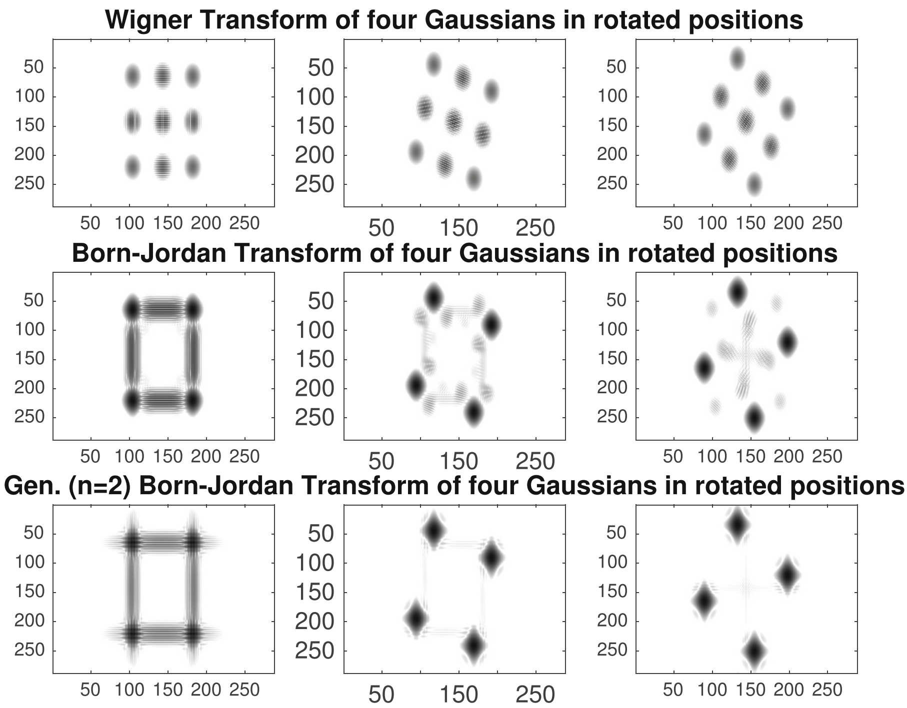

Fig. 1 Four Gaussian Windows in rotated positions: Comparison of Wigner distribution, Born-Jordan and generalized Born-Jordan distribution

The second example, shown in Fig. 2, depicts the Wigner transform, the Born–Jordan transform and two versions of generalized Born-Jordan transform ($n = 10$ and $n = 100$) of another synthetic signal consisting of two linear chirps. Note that the geometry of this example is different from the previous one in the sense of that it lacks symmetry around zero. As a final example, shown in Fig. 3, we applied the Wigner transform, the Born–Jordan transform and two versions of generalized Born–Jordan transform to a classical real signal, namely a bat call. As in the first example, the cross-term suppression increases for exponent $n = 2$, while, when applying even higher order smoothing, we observe a loss of concentration in time-frequency. As in the case of the two chirps, the geometry of this example lacks central symmetry.

The Born-Jordan quantization of order *n* This procedure arises as the natural extension of the $n = 1$ case (that is, the usual Born–Jordan quantization). Observe that choosing $n = 0$, it reduces to the Weyl quantization. We denote by $\hbar$ a positive parameter; in physics it is viewed as the reduced Planck constant.

Definition 2 For $n \in \mathbb{N}$, the Born–Jordan quantization of order n is the mapping

$$a \in \mathcal{S}'(\mathbb{R}^{2d}) \mapsto \widehat{A}_{\mathrm{BJ},n} = \mathrm{Op}_{BJ,n}(a) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbb{R}^{2d}} (\mathcal{F}_\sigma a)(z)\Theta^n(z)\widehat{T}(z)dz, \quad (14)$$

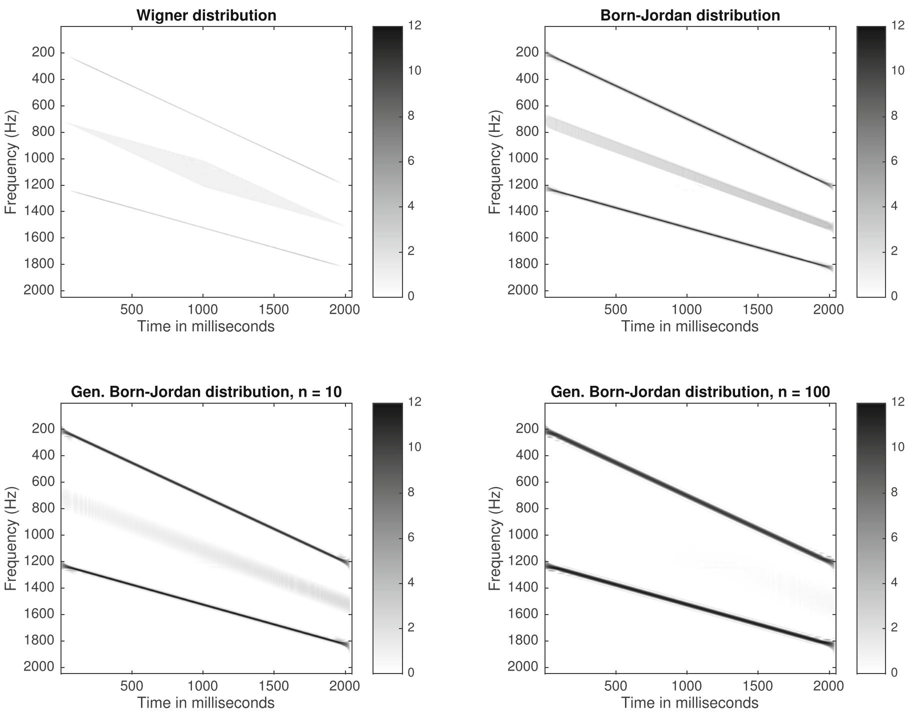

Fig. 2 Two linear chirps: Comparison of Wigner distribution, Born-Jordan and generalized Born-Jordan distribution

where $\widehat{T}(z) = e^{-i\sigma(\widehat{z},z)/\hbar}$ is the Heisenberg operator and σ the standard symplectic form (see the notation below).

The case $n = 0$ ($\Theta^0 \equiv 1$) is the well-known Weyl quantization.

In the sequel, we shall set $\hbar = 1/2\pi$, as is customary in time-frequency analysis. Hence, the constant in front of the integrals in (14) disappears.

2 Preliminaries

2.1 Notation

We use the notation $x\omega = x \cdot \omega = x_1\omega_1 + \ldots + x_d\omega_d$ for the scalar product in $\mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(\mathbb{R}^d)$ and for the duality pairing between Schwartz functions and temperate distributions (it is antilinear in the second argument by convention). Given functions f, g, we write $f \lesssim g$ if $f(x) \leq Cg(x)$ for every x and some constant $C > 0$, and similarly for $\gtrsim$. The notation $f \asymp g$ means $f \lesssim g$ and $f \gtrsim g$.

We write $\mathcal{C}_c^\infty(\mathbb{R}^d)$ for the class of smooth functions on $\mathbb{R}^d$ with compact support.

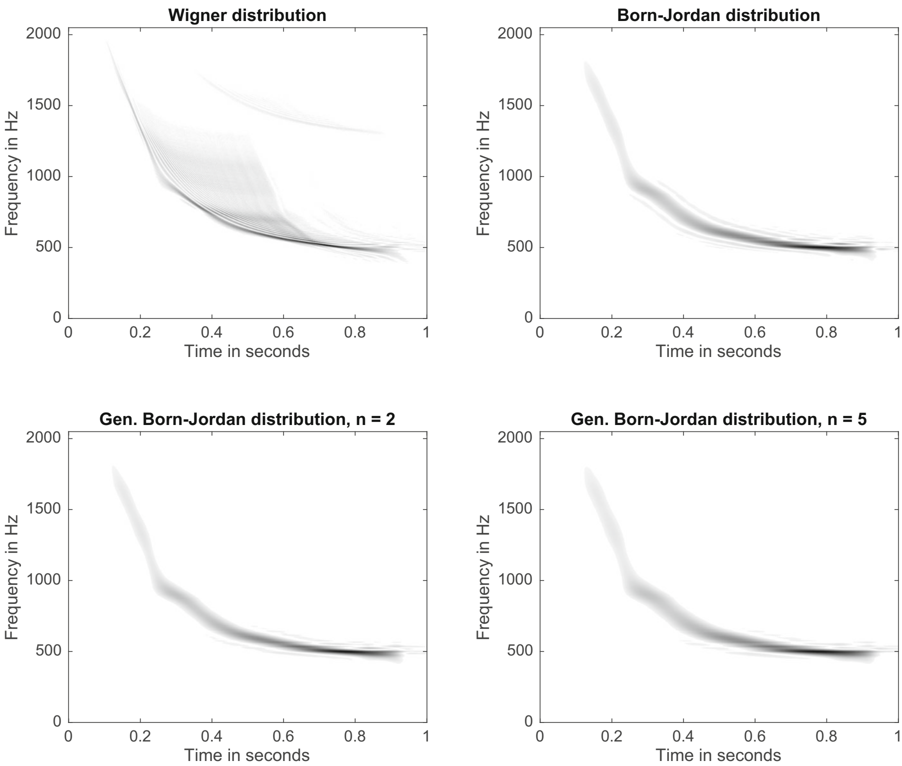

Fig. 3 Bat call signal: Comparison of Wigner distribution, Born-Jordan and generalized Born-Jordan distribution

We denote by σ the standard symplectic form on the phase space $\mathbb{R}^{2d} \equiv \mathbb{R}^d \times \mathbb{R}^d$; the phase space variable is denoted $z = (x, \omega)$ and the dual variable by $\zeta = (\zeta_1, \zeta_2)$. By definition $\sigma(z, \zeta) = Jz \cdot \zeta = \omega \cdot \zeta_1 - x \cdot \zeta_2$, where

$$J = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & 0_{d\times d} \end{pmatrix}.$$

The Fourier transform of a function $f(x)$ in $\mathbb{R}^d$ is

$$\mathcal{F} f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x\omega} f(x)\, dx,$$

and the symplectic Fourier transform of a function $F(z)$ in the phase space $\mathbb{R}^{2d}$ is defined by

$$\mathcal{F}_\sigma F(\zeta) = \int_{\mathbb{R}^{2d}} e^{-2\pi i \sigma(\zeta, z)} F(z)\, dz.$$

The symplectic Fourier transform is an involution, i.e. $\mathcal{F}_\sigma(\mathcal{F}_\sigma F) = F$. Moreover, $\mathcal{F}_\sigma F(\zeta) = \mathcal{F} F(J\zeta)$.

Observe that $\Theta^n(J(\zeta_1, \zeta_2)) = \Theta^n(\zeta_1, \zeta_2)$ so that

$$\mathcal{F}_\sigma(\Theta^n) = \mathcal{F}(\Theta^n), \quad \forall n \in \mathbb{N}_+. \tag{15}$$

For $s \in \mathbb{R}$, the L^2-based Sobolev space $H^s(\mathbb{R}^d)$ is constituted by the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{H^s} := \|\widehat{f}(\omega)\langle\omega\rangle^s\|_{L^2} < \infty. \tag{16}$$

2.2 Time-frequency representations and main properties

2.2.1 Wigner distribution and ambiguity function [25, 31]

We already defined in the "Introduction", see (1), the Wigner distribution Wf of a signal $f \in \mathcal{S}'(\mathbb{R}^d)$. In general, we have $Wf \in \mathcal{S}'(\mathbb{R}^{2d})$. When $f \in L^2(\mathbb{R}^d)$, we have $Wf \in L^2(\mathbb{R}^{2d})$ and in fact it turns out

$$\|Wf\|_{L^2(\mathbb{R}^{2d})} = \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{17}$$

In the sequel, we will encounter several times the symplectic Fourier transform of Wf, which is known as Woodward's (radar) *ambiguity function* Af. We have the formula

$$Af(\zeta_1, \zeta_2) = \mathcal{F}_\sigma Wf(\zeta_1, \zeta_2) = \int_{\mathbb{R}^d} f\big(y + \frac{1}{2}\zeta_1\big)\overline{f\big(y - \frac{1}{2}\zeta_1\big)} e^{-2\pi i \zeta_2 y}\, dy. \tag{18}$$

We refer to [25, Chapter 9] and [28] for more details.

2.2.2 Marginal properties of Q^n

The members of the Cohen class are also called *pseudo-density functions* since they are supposed to indicate how the signal density is distributed over time and frequency. The terminology *pseudo-density* comes from the fact that such distributions in general are not positive functions and can take not only negative but even complex values. In order for Q^n to be a pseudo-density function, it must satisfy certain requirements. In particular, the marginal densities

$$\int_{\mathbb{R}^d} Q^n f(x,\omega) d\omega = |f(x)|^2, \quad \int_{\mathbb{R}^d} Q^n f(x,\omega) dx = |\hat{f}(\omega)|^2, \tag{19}$$

for every f in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. It can be shown (see [36] or [28, Proposition 97]) that those conditions are equivalent to the requirements

$$\mathcal{F}(\Theta^n)(x, 0) = 1, \ \forall x \in \mathbb{R}^d, \quad \mathcal{F}(\Theta^n)(0, \omega) = 1, \ \forall \omega \in \mathbb{R}^d. \tag{20}$$

In this case, using (15), (8) and (7), one sees that they are trivially satisfied, since $\mathrm{sinc}^n(0) = 1$, for every $n \in \mathbb{N}$.

2.2.3 The Moyal identity is not satisfied

A quite convenient property of Cohen's kernel (2) is Moyal's identity [15, Theorem 14.2 and 27.15]

$$\langle Q(f_1, g_1), Q(f_2, g_2)\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R}^d)}, \quad f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d). \tag{21}$$

It plays an essential role in quantum mechanics (but perhaps not in signal analysis, as already observed by Janssen in [36]). While the Wigner distribution, the STFT and the ambiguity function satisfy (21), *the BJDn* Q^n *does not for* $n \in \mathbb{N}_+$. To prove this, we will use the following characterization (cf. [36, Section 3] and [27]):

Proposition 1 *A member of the Cohen class, cf.* (2)*, satisfies Moyal's identity* (21) *if and only if*

$$|\theta(x,\omega)| = 1, \quad \textit{for all} \quad (x,\omega) \in \mathbb{R}^{2d}. \tag{22}$$

Choosing $Q = Q^n$, $n \in \mathbb{N}_+$, we have $\theta(x,\omega) = \mathrm{sinc}^n(x\omega)$, so that condition (22) is not satisfied for any $n \in \mathbb{N}_+$. Observe that for $n = 0$ (the Wigner distribution), the previous conditions holds, as expected.

2.3 Modulation spaces [20–22, 25, 31]

Modulation spaces are used in time-frequency analysis to measure the time-frequency concentration of a signal. As already observed in the "Introduction", the construction of these functional spaces relies on the notion of short-time (or windowed) Fourier transform defined in (10).

Let now $s \in \mathbb{R}$, $1 \le p, q \le \infty$. The *modulation space* $M^{p,q}_s(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_s} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \langle\omega\rangle^{sp}\, dx\right)^{q/p} d\omega\right)^{1/q} < \infty \tag{23}$$

(with obvious changes for $p = \infty$ or $q = \infty$). When $s = 0$, we write $M^{p,q}(\mathbb{R}^d)$ instead of $M^{p,q}_0(\mathbb{R}^d)$. We will also use the shorthand notation $M^p_s(\mathbb{R}^d)$ for $M^{p,p}_s(\mathbb{R}^d)$. The spaces $M^{p,q}_s(\mathbb{R}^d)$ are Banach spaces for any $1 \le p, q \le \infty$, and every non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ yields an equivalent norm in (23).

Modulation spaces generalize and include as special cases several function spaces arising in Harmonic Analysis. In particular for $p = q = 2$, we have

$$M^2_s(\mathbb{R}^d) = H^s(\mathbb{R}^d),$$

whereas $M^1(\mathbb{R}^d)$ coincides with the Segal algebra $S_0(\mathbb{R}^d)$ (cf. [18]), and $M^{\infty,1}(\mathbb{R}^d)$ is the so-called Sjöstrand class [32].

For members of $M^{p,q}_s$, the exponent p is a measure of average decay at infinity in the scale of spaces ℓ^p, whereas the exponent q is a measure of smoothness in the scale $\mathcal{F}L^q$. The number s is a further regularity index, completely analogous to that appearing in the Sobolev spaces $H^s(\mathbb{R}^d)$.

Other modulation spaces, also known as *Wiener amalgam spaces*, are obtained by exchanging the order of integration in (23). Precisely, the modulation spaces $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$, for $p, q \in [1, +\infty)$, are given by the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p\, d\omega\right)^{q/p} dx\right)^{1/q} < \infty$$

(with obvious changes for $p = \infty$ or $q = \infty$). Using Parseval's identity in (10), we can write the so-called fundamental identity of time-frequency analysis

$$V_g f(x, \omega) = e^{-2\pi i x\omega} V_{\hat{g}} \hat{f}(\omega, -x),$$

hence,

$$|V_g f(x, \omega)| = |V_{\hat{g}} \hat{f}(\omega, -x)| = |\mathcal{F}(\hat{f}\, T_\omega \overline{\hat{g}})(-x)|$$

so that

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d} \|\hat{f}\, T_\omega \overline{\hat{g}}\|_{\mathcal{F}L^p}^q \, d\omega\right)^{1/q} = \|\hat{f}\|_{W(\mathcal{F}L^p, L^q)}.$$

This means that Wiener amalgam spaces can be viewed as the images by a Fourier transform of modulation spaces: $\mathcal{F}(M^{p,q}) = W(\mathcal{F}L^p, L^q)$.

We will frequently use the following product property of Wiener amalgam spaces [20, Theorem 1 (v)]: for $1 \leq p, q \leq \infty$,

$$\textit{if } f \in W(\mathcal{F}L^1, L^\infty) \textit{ and } g \in W(\mathcal{F}L^p, L^q) \textit{ then } fg \in W(\mathcal{F}L^p, L^q). \tag{24}$$

Taking $p = 1, q = \infty$, we see that $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ is an algebra under pointwise multiplication.

Proposition 2 *Let* $1 \leq p, q \leq \infty$ *and* $A \in GL(d, \mathbb{R})$. *Then, for every* $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$,

$$\|f(A\,\cdot)\|_{W(\mathcal{F}L^p, L^q)} \leq C|\det A|^{(1/p-1/q-1)}(\det(I + A^*A))^{1/2}\|f\|_{W(\mathcal{F}L^p, L^q)}. \tag{25}$$

In particular, for $A = \lambda I$, $\lambda > 0$,

$$\|f(A\,\cdot)\|_{W(\mathcal{F}L^p, L^q)} \leq C\lambda^{d\left(\frac{1}{p}-\frac{1}{q}-1\right)}(\lambda^2 + 1)^{d/2}\|f\|_{W(\mathcal{F}L^p, L^q)}. \tag{26}$$

In the proof of Theorem 4, we will use the following dilation properties of Gaussians (first proved in [46, Lemma 1.8], see also [10, Lemma 3.2]):

Lemma 1 *Let* $\varphi(x) = e^{-\pi|x|^2}$ *and* $\lambda > 0$. *For* $1 \leq p, q \leq \infty$,

$$\|\varphi(\lambda\,\cdot)\|_{M^{p,q}} \asymp \lambda^{-d/q'} \quad \text{as } \lambda \to +\infty,$$

where q' *is the conjugate exponent of* q, *that is* $1/q + 1/q' = 1$.

2.4 Wave-front set for Fourier-Lebesgue spaces [35, 41]

The notion of H^s wave-front set allows to quantify the regularity of a function or distribution in the Sobolev scale at any given point and direction. This is done by microlocalizing the definition of the H^s norm in (16) as follows (cf. [35, Chapter XIII]).

Given a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, we define its wave-front set $WF_{H^s}(f) \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, as the set of points $(x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d$, $\omega_0 \neq 0$, for which the following condition is *not* satisfied: for some cut-off function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(x_0) \neq 0$ and some open conic neighbourhood of $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of ω_0. we have

$$\|\mathcal{F}[\varphi f](\omega)\langle\omega\rangle^s\|_{L^2(\Gamma)} < \infty.$$

More generally one can start from the Fourier-Lebesgue spaces $\mathcal{F}L^q_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 \le q \le \infty$, which is the space of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm in (11) is finite. Arguing exactly as above (with the space L^2 replaced by L^q), one then arrives in a natural way to a corresponding notion of wave-front set $WF_{\mathcal{F}L^q_s}(f)$ as we anticipated in the "Introduction" (see (12)).

We need to recall some basic results about the action of constant coefficient linear partial differential operators on such wave-front set (cf. [41]). Given the operator

$$P = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha, \quad c_\alpha \in \mathbb{C};$$

it is straightforward to see that, for $1 \le q \le \infty$, $s \in \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$WF_{\mathcal{F}L^q_s}(Pf) \subset WF_{\mathcal{F}L^q_{s+m}}(f).$$

Consider now the inverse inclusion. We say that $\zeta \in \mathbb{R}^d$, $\zeta \ne 0$, is non-characteristic for the operator P if

$$\sum_{|\alpha|=m} c_\alpha \zeta^\alpha \ne 0$$

i.e. the operator P is elliptic in the direction ζ. The following result is a microlocal version of the classical regularity result of elliptic operators (see [41, Corollary 1 (2)]):

Proposition 3 *Let $1 \le q \le \infty$, $s \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Let $z \in \mathbb{R}^d$ and suppose that $\zeta \in \mathbb{R}^d \setminus \{0\}$ is non-characteristic for P. Then, if $(z, \zeta) \notin WF_{\mathcal{F}L^q_s}(Pf)$, we have $(z, \zeta) \notin WF_{\mathcal{F}L^q_{s+m}}(f)$.*

3 Generalized Born–Jordan kernels for monomials

Let $\mathbb{C}[x, \omega]$ be the commutative ring of polynomials generated by x and ω; it consists of all finite sums $a(x, \omega) = \sum \lambda_{m\ell} a_{m\ell}(x, \omega)$ ($\lambda_{m\ell} \in \mathbb{C}$) where $a_{m\ell}(x, \omega) = \omega^m x^\ell$ with $(m, \ell) \in \mathbb{N}^2$. We identify $\mathbb{C}[x, \omega]$ with the ring of polynomial functions in the variables $(x, \omega) \in \mathbb{R}^2$. We denote by $\mathbb{C}[\widehat{x}, \widehat{\omega}]$ the corresponding Weyl algebra; it is realized as the non-commutative unital algebra generated by the two operators $\widehat{x}$ and $\widehat{\omega}$ satisfying $[\widehat{x}, \widehat{\omega}] = (i/2\pi) I_{\mathrm{d}}$. These operators are realized as the unbounded operators defined on $L^2(\mathbb{R})$ by $\widehat{x} f = xf$ and $\widehat{\omega} f = -(i/2\pi)\partial_x f$. We will call *quantization of* $\mathbb{C}[x, \omega]$ any continuous linear mapping $\mathrm{Op} : \mathbb{C}[x, \omega] \longrightarrow \mathbb{C}[\widehat{x}, \widehat{\omega}]$ having the following properties:

(Q1) Triviality: $\mathrm{Op}(1) = I_{\mathrm{d}}$, $\mathrm{Op}(x) = \widehat{x}$, and $\mathrm{Op}(\omega) = \widehat{\omega}$;

(Q2) Dirac's restricted rule:

$$[x, \mathrm{Op}(a_{m\ell})] = (i/2\pi)\,\mathrm{Op}(\{x, a_{m\ell}\}) \ , \ [\omega, \mathrm{Op}(a_{m\ell})] = (i/2\pi)\,\mathrm{Op}(\{\omega, a_{m\ell}\});$$

(Q3) Self-adjointness: If $a \in \mathbb{C}[x, \omega]$ then $\mathrm{Op}(a)$ is self-adjoint on its domain.

One shows [16] (also see [8]) that for every quantization of $\mathbb{C}[x,\omega]$, there exists [8, 16] a function f with $f(0)=1$ and $e^{-it/2}f$ real such that

$$\operatorname{Op}(a_{m\ell}) = \sum_{j=0}^{\min(m,\ell)} j!\binom{m}{j}\binom{\ell}{j} f^{(j)}(0)(2\pi)^{-j}\widehat{\omega}^{m-j}\widehat{x}^{\ell-j}. \tag{27}$$

Let $(a_{m\ell})_\sigma = \mathcal{F}_\sigma a_{m\ell}$ be the symplectic Fourier transform of $a_{m\ell}$ and $\widehat{T}(z) = e^{-2\pi i\sigma(\widehat{z},z)}$ the Heisenberg operator.

Proposition 4 *Let* $\operatorname{Op}: \mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ *be a quantization having the properties (Q1), (Q2), (Q3). (i) The restriction of* Op *to* $\mathbb{C}[x,\omega]$ *is then given by*

$$\operatorname{Op}(a_{m\ell}) = \int (a_{m\ell})_\sigma(x,\omega)\Phi(2\pi x\omega)\widehat{T}(x,\omega)d\omega dx \tag{28}$$

where $\Phi(t) = e^{-it/2}f(t)$. *(ii) The Cohen kernel* θ *of* Op *thus has symplectic Fourier transform* $\mathcal{F}_\sigma\theta$ *given by*

$$\mathcal{F}_\sigma\theta(x,\omega) = \Phi(2\pi x\omega). \tag{29}$$

Proof A detailed proof is given in Domingo and Galapon [16] (formulas (10) and (14)). Notice that formula (28) readily follows from (27) using the elementary formula

$$\mathcal{F}(\omega^m \otimes x^\ell) = (i/2\pi)^{m+\ell}\delta^{(m)}(\omega)\otimes\delta^{(\ell)}(x).$$

Formula (29) follows since (28) is the Weyl representation of the operator with twisted symbol $(a_{m\ell})_\sigma\Phi$ [the twisted symbol is the symplectic Fourier transform of the usual symbol]. □

Remark 2 This result shows that if one limits oneself to pseudo-differential calculi satisfying the Dirac conditions (Q2) then the Cohen kernel is of a very particular type: its Fourier transform only depends on the product ωx. In particular, the associated quasidistribution $Q\psi = W\psi * \theta$ satisfies the marginal conditions since $\mathcal{F}_\sigma\theta(0) = \Phi(0) = 1$ (see [27], formula (7.29), p. 107).

We now focus on the case where the symplectic Fourier transform of the Cohen kernel is given by

$$\mathcal{F}_\sigma\theta(x,\omega) = \operatorname{sinc}^n(\pi x\omega)\ , n\in\mathbb{N} = \{0,1,2,...\}.$$

With the notation above, we thus have $\Phi(\pi x\omega) = \operatorname{sinc}^n(\pi x\omega)$ so that $\Phi(t) = \operatorname{sinc}^n(t/2)$ and hence $f(t) = e^{it/2}\operatorname{sinc}^n(t/2)$. Suppose first $n=0$; then $f^{(j)}(0) = (i/2)^j$ hence formula (27) yields

$$\operatorname{Op}(a_{m\ell}) = \sum_{j=0}^{\min(m,\ell)} \binom{m}{j}\binom{\ell}{j} j!\left(\frac{i}{4\pi}\right)^j \widehat{\omega}^{m-j}\widehat{x}^{\ell-j}$$

so that $\mathrm{Op}(a_{m\ell}) = \mathrm{Op}^{\mathrm{W}}(a_{m\ell}) = \mathrm{Op}_{BJ,0}(a_{m\ell})$ (see (14)) is just the Weyl ordering of the monomial $a_{m\ell}$ ([16] and [27], p.34). Suppose next $n = 1$. Then, $f^{(j)}(0) = i^j/(j+1)$ and

$$\mathrm{Op}(a_{m\ell}) = \sum_{j=0}^{\min(m,\ell)} \binom{m}{j}\binom{\ell}{j}\frac{j!}{j+1}\left(\frac{i}{2\pi}\right)^j \widehat{\omega}^{m-j}\widehat{x}^{\ell-j};$$

here $\mathrm{Op}(a_{m\ell}) = \mathrm{Op}_{BJ,1}(a_{m\ell})$ is the Born-Jordan ordering [16] and [27, page 34].

In the case of a general n, we have, by Leibniz's formula,

$$f^{(j)}(0) = \sum_{k=0}^{j}\binom{j}{k}\left(\frac{i}{2}\right)^{j-k}\left(\frac{1}{2}\right)^k\left(\frac{d^k}{dt^k}\operatorname{sinc}^n\right)(0). \tag{30}$$

The derivatives of sinc^n at $t = 0$ can be calculated using Faà di Bruno's formula [17] for the derivatives of the composition of two functions

$$(g\circ h)^{(k)}(t) = \sum_{\kappa\cdot\alpha=k}\binom{k}{\alpha} g^{(|\alpha|)}(h(t))\Pi_\alpha(t) \tag{31}$$

where $\kappa = (1, 2, ..., k)$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in \mathbb{N}^k$ and

$$\Pi_\alpha(t) = \left(\frac{1}{1!}h^{(1)}(t)\right)^{\alpha_1}\left(\frac{1}{2!}h^{(2)}(t)\right)^{\alpha_2}\cdots\left(\frac{1}{k!}h^{(k)}(t)\right)^{\alpha_k}.$$

Choosing $g(t) = x^n$ and $h(t) = \operatorname{sinc}(t/2)$ this formula yields

$$\frac{d^k}{dt^k}\operatorname{sinc}^n(0) = \sum_{\substack{\kappa\cdot\alpha=k\\|\alpha|\le n}}\binom{k}{\alpha}\binom{n}{|\alpha|}|\alpha|!\Pi_\alpha(0);$$

since $\operatorname{sinc}^{(2m+1)}(0) = 0$ and $\operatorname{sinc}^{(2m)}(0) = (-1)^m/(2m+1)$ we have

$$\Pi_\alpha(0) = \frac{1}{1!(\alpha_1+1)^{\alpha_1}2!(\alpha_2+1)^{\alpha_2}\cdots k!(\alpha_k+1)^{\alpha_k}}.$$

4 Time-frequency analysis of the n^{th} Born-Jordan kernel

The Born–Jordan kernel Θ^1 in (3) belongs to the space $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$, as proved in [13]:

Proposition 5 *The function Θ^1 in (3) belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

The previous property is true for any Θ^n, $n \in \mathbb{N}_+$, as shown below.

Proposition 6 *For $n \in \mathbb{N}_+$, the function Θ^n defined in (7) belongs to the Wiener algebra $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

Proof The result is attained by induction on n. We know that $\Theta^1 \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ by Proposition 5. If we assume $\Theta^n \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$, for a certain integer $n > 1$, we obtain

$$\Theta^{n+1} = \Theta^n \cdot \Theta^1 \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}) \cdot W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}) \hookrightarrow W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}),$$

since the Banach space $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ is an algebra by pointwise product. This gives the claim. □

We now have the tools we need to prove Theorem 1 stated in the "Introduction".

Proof of Theorem 1 We need to show that $Q^n f \in M^{p,q}(\mathbb{R}^{2d})$. Taking the symplectic Fourier transform in (4), we are reduced to prove that

$$\Theta^n \mathcal{F}_\sigma(Wf) = \Theta^n Af \in W(\mathcal{F}L^p, L^q)$$

where $\mathcal{F}_\sigma(Wf) = Af$ is the ambiguity function of f in (18). The claim is proven using the product property (24): by Proposition 6, the function Θ^n is in $W(\mathcal{F}L^1, L^\infty)$ and in view of the assumption $Wf \in M^{p,q}(\mathbb{R}^{2d})$ so that $\mathcal{F}(Wf) \in W(\mathcal{F}L^p, L^q)$. Therefore, $\mathcal{F}_\sigma(Wf)(\zeta) = \mathcal{F}(Wf)(J\zeta) \in W(\mathcal{F}L^p, L^q)$ by Proposition 2 and we are done. □

An alternative proof relies on the continuity of the mapping

$$A : a \longmapsto a * \Theta^1_\sigma, \tag{32}$$

which was shown to be bounded on $M^{p,q}(\mathbb{R}^{2d})$ in [12, Proposition 5.1], see also the subsequent work [30]. By induction, it then follows that the same continuity property holds for Q^n in (8), with $a = Wf$, and Theorem 1.2 is thus proved.

Actually, the previous issue is a special case of the general result for members of the Cohen class stated in Theorem 2 (recall that, if $\Theta^n \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$, then $\mathcal{F}_\sigma \Theta^n \in M^{1,\infty}(\mathbb{R}^{2d})$), which can be proved as follows.

Proof of Theorem 2 It is a consequence of the convolution relations for modulation spaces (cf. [9]):

$$M^{p,q}(\mathbb{R}^{2d}) * M^{1,\infty}(\mathbb{R}^{2d}) \hookrightarrow M^{p,q}(\mathbb{R}^{2d}),$$

for any $1 \le p, q \le \infty$. □

In [13], the following property for the chirp function was proven:

Proposition 7 *The function $F(\zeta_1, \zeta_2) = e^{2\pi i \zeta_1 \zeta_2}$ belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

Since $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ can be characterized as the space of pointwise multipliers on the Feichtinger algebra $W(\mathcal{F}L^1, L^1)(\mathbb{R}^{2d})$ [23, Corollary 3.2.10], the result in Proposition 7 could also be deduced from general results about the action of second order characters on the Feichtinger algebra, cf. [18, 45].

By Proposition 7 and by the dilation properties for Wiener amalgam spaces (25), we can state:

Corollary 2 *For $\zeta = (\zeta_1, \zeta_2)$, consider the function $F_J(\zeta) = F(J\zeta) = e^{-2\pi i \zeta_1 \zeta_2}$. Then, $F_J \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

5 Smoothness of the Born-Jordan distribution of order *n*

In the present section, we compare the smoothness of the Born–Jordan distribution of order n with that of the Wigner distribution. In particular, we will prove Theorem 3.

We begin by stating and proving the following global result.

Theorem 5 *Let* $f \in \mathcal{S}'(\mathbb{R}^d)$ *be a signal such that* $Wf \in M^{p,q}(\mathbb{R}^{2d})$ *for some* $1 \le p, q \le \infty$. *Then,*

$$Q^n f \in M^{p,q}(\mathbb{R}^{2d})$$

and moreover

$$(\nabla_x \cdot \nabla_\omega)^n Q^n f \in M^{p,q}(\mathbb{R}^{2d}). \tag{33}$$

Here we used the notation

$$\nabla_x \cdot \nabla_\omega = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j \partial \omega_j}.$$

Proof The property $Q^n f \in M^{p,q}(\mathbb{R}^{2d})$ is proven in Theorem 1.

Let us now prove (33). Taking the symplectic Fourier transform, we see that it is sufficient to prove that

$$(\zeta_1\zeta_2)^n \operatorname{sinc}^n(\zeta_1\zeta_2)\mathcal{F}_\sigma Wf = \frac{1}{\pi^n}\sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q).$$

We have

$$\sin(\pi\zeta_1\zeta_2) = \frac{e^{\pi i\zeta_1\zeta_2} - e^{-\pi i\zeta_1\zeta_2}}{2i} \in W(\mathcal{F}L^1, L^\infty), \tag{34}$$

by Proposition 7, Corollary 2 and Proposition 2, with the scaling $\lambda = 1/\sqrt{2}$.

Hence, for $n = 1$,

$$\frac{1}{\pi}\sin(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q)$$

by the product property (24). Assume now that, for a certain $n \in \mathbb{N}_+$,

$$\frac{1}{\pi^n}\sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q).$$

Then,

$$\frac{1}{\pi^{n+1}}\sin^{n+1}(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf = \underbrace{\frac{1}{\pi}\sin(\pi\zeta_1\zeta_2)}_{\in W(\mathcal{F}L^1, L^\infty)} \cdot \underbrace{\frac{1}{\pi^n}\sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf}_{\in W(\mathcal{F}L^p, L^q)} \in W(\mathcal{F}L^p, L^q),$$

by (34) and the product property (24) again. By induction, we attain the result. □

We are now ready to prove Theorem 3.

Proof of Theorem 3 Consider $n \in \mathbb{N}_+$. We will apply Proposition 3 to the $2n$-th order operator P^n, where $P = \nabla_x \cdot \nabla_\omega$ in $\mathbb{R}^{2d}$. The non-characteristic directions for P^n are given by the vectors $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^d \times \mathbb{R}^d$, satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. By (33) (with $p = \infty$), we have

$$WF_{\mathcal{F}L^q}(P^n Q^n f) = \emptyset,$$

because $\varphi F \in \mathcal{F}L^q$ if $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and $F \in M^{\infty,q}(\mathbb{R}^{2d})$ (here $F = P^n Q^n f$). Hence, we obtain

$$(z, \zeta) \notin WF_{\mathcal{F}L^q}(P^n Q^n f), \quad \forall (z, \zeta) \text{ such that } \zeta = (\zeta_1, \zeta_2),\ \zeta_1 \cdot \zeta_2 \neq 0.$$

Since ζ is non-characteristic for the operator P^n, by Proposition 3 we infer

$$(z, \zeta) \notin WF_{\mathcal{F}L^q_{2n}}(Q^n f)$$

for every $z \in \mathbb{R}^{2d}$. □

Proof of Corollary 1 Apply Theorem 3 with $q = 2$. Indeed, for $f \in L^2(\mathbb{R}^d)$, Moyal's identity gives $Wf \in L^2(\mathbb{R}^{2d}) = M^{2,2}(\mathbb{R}^d) \subset M^{\infty,2}(\mathbb{R}^{2d})$ (*cf.* (17)). Observe that the $\mathcal{F}L^2_{2n}$ wave-front set coincides with the H^{2n} wave-front set. □

The proof of Theorem 4 requires Lemma 5.1 in [13]:

Lemma 2 *Let $\chi \in C_c^\infty(\mathbb{R})$. Then, the function $\chi(\zeta_1\zeta_2)$ belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

As announced in the "Introduction", the smoothing properties of the Q^n distributions do not hold in the whole phase space. We do not have any gain in the directions $\zeta_1 \cdot \zeta_2 = 0$ as it comes up clearly from the proof of the following issue.

Proof of Theorem 4 The pattern is similar to that of Theorem 1.4 in [13]. We detail the main steps for sake of clarity. The idea is to test the estimate (13) using rescaled Gaussian functions $f(x) = \varphi(\lambda x)$, with $\lambda > 0$ large parameter. We shall prove that, restricting to a neighbourhood of $\zeta_1 \cdot \zeta_2 = 0$, the constrain $q_1 \geq q_2$ must be satisfied.

An easy computation (see e.g. [31, Formula (4.20)]) yields

$$W(\varphi(\lambda\, \cdot))(x, \omega) = 2^{d/2}\lambda^{-d}\varphi(\sqrt{2}\lambda\, x)\varphi(\sqrt{2}\lambda^{-1}\, \omega). \tag{35}$$

For every $1 \leq p, q \leq \infty$, the above formula gives

$$\|W(\varphi(\lambda\, \cdot))\|_{M^{p,q}} = 2^{d/2}\lambda^{-d}\|\varphi(\sqrt{2}\lambda\, \cdot)\|_{M^{p,q}}\|\varphi(\sqrt{2}\lambda^{-1}\, \cdot)\|_{M^{p,q}}.$$

By the dilation properties of Gaussians in Lemma 1

$$\|W(\varphi(\lambda\, \cdot))\|_{M^{p,q}} \asymp \lambda^{-2d+d/q+d/p} \quad \text{as } \lambda \to +\infty. \tag{36}$$

We now study the $M^{p,q}$-norm of the BJDn $Q^n(\varphi(\lambda\, \cdot))$. The idea is to estimate this norm from below obtaining the same expansion as in (36).

$$\|Q^n(\varphi(\lambda\, \cdot))\|_{M^{p,q}} = \|\mathcal{F}_\sigma(\Theta^n) * W(\varphi(\lambda\, \cdot))\|_{M^{p,q}}.$$

By taking the symplectic Fourier transform and using Lemma 2 and the product property (24), we have

$$\begin{aligned}\|\mathcal{F}_\sigma(\Theta^n) * W(\varphi(\lambda\, \cdot))\|_{M^{p,q}} &\asymp \|\Theta^n \mathcal{F}_\sigma[W(\varphi(\lambda\, \cdot))]\|_{W(\mathcal{F}L^p, L^q)} \\ &\gtrsim \|\Theta^n(\zeta_1, \zeta_2)\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda\, \cdot))]\|_{W(\mathcal{F}L^p, L^q)}\end{aligned}$$

for any $\chi \in C_c^\infty(\mathbb{R})$ and $n \in \mathbb{N}_+$. Choosing χ supported in the interval $[-1/4, 1/4]$ and $\chi \equiv 1$ in the interval $[-1/8, 1/8]$ (the latter condition will be used later), we write

$$\chi(\zeta_1\zeta_2) = \chi(\zeta_1\zeta_2)\Theta^n(\zeta_1, \zeta_2)\Theta^{-n}(\zeta_1, \zeta_2)\tilde{\chi}(\zeta_1\zeta_2),$$

with $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ supported in $[-1/2, 1/2]$ and $\tilde{\chi} = 1$ on $[-1/4, 1/4]$, therefore on the support of χ. Since by Lemma 2 the function $\Theta^{-n}(\zeta_1, \zeta_2)\tilde{\chi}(\zeta_1\zeta_2)$ belongs to $W(\mathcal{F}L^1, L^\infty)$, by the product property, the last expression can be estimated from below as

$$\gtrsim \|\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda\,\cdot))]\|_{W(\mathcal{F}L^p,L^q)}.$$

We end up with the same object that was already estimated in the proof of Theorem 1.4 in [13], where it was shown that

$$\|\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda\,\cdot))]\|_{W(\mathcal{F}L^p,L^q)} \gtrsim \lambda^{-2d+d/p+d/q} \quad \text{as } \lambda \to +\infty. \tag{37}$$

Comparing (37) with (36), we obtain the desired conclusion. □

6 Pseudo-differential calculus

The Weyl quantization was introduced by Weyl in [47] and is the $n = 0$ case of the Born–Jordan quantization of order n in (14):

$$a \in \mathcal{S}'(\mathbb{R}^{2d}) \mapsto \widehat{A}_{\mathrm{W}} = \mathrm{Op}_{\mathrm{W}}(a) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbb{R}^{2d}} \mathcal{F}_\sigma a(z)\widehat{T}(z)dz.$$

Comparing with (14), we infer the symbol relation

$$\mathcal{F}_\sigma a_{BJ,n}\Theta^n = \mathcal{F}_\sigma a_W$$

(observe that $a_{BJ,n}$ denotes the symbol of $\widehat{A}_{\mathrm{BJ},n}$ whereas a_W is the Weyl symbol) that is

$$a_{BJ,n} * \mathcal{F}_\sigma(\Theta^n) = a_W. \tag{38}$$

Using the weak definition for Weyl operators via the Wigner distribution

$$\langle Op_W(a)f, g\rangle = \langle a, W(g, f)\rangle, \quad a \in \mathcal{S}'(\mathbb{R}^{2d}),\ f, g \in \mathcal{S}(\mathbb{R}^d)$$

and the convolution property (whenever is well-defined)

$$\langle F * G, H\rangle = \langle F, H * G\rangle$$

we can also define, for $n \in \mathbb{N}$, the n-th Born-Jordan pseudo-differential operator with symbol $a \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle Op_{BJ,n}(a)f, g\rangle = \langle a, Q^n(g, f)\rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{39}$$

(Observe that $n = 1$ is the standard Born–Jordan operator, whereas $n = 0$ gives the Weyl operator).

We aim at studying continuity properties of such operators and of the related distributions on modulation spaces.

First, we analyze the quadratic representations Q^n.

Theorem 6 *Assume $s \geq 0$, $p_1, q_1, p, q \in [1, \infty]$ such that*

$$2\min\{\frac{1}{p_1}, \frac{1}{q_1}\} \geq \frac{1}{p} + \frac{1}{q}. \tag{40}$$

If $f \in M^{p_1,q_1}_{v_s}(\mathbb{R}^d)$, the Cohen distribution $Q^n f$, $n \in \mathbb{N}_+$, is in $M^{p,q}_{1\otimes v_s}(\mathbb{R}^{2d})$, with

$$\|Q^n f\|_{M^{p,q}_{1\otimes v_s}(\mathbb{R}^{2d})} \lesssim \|\Theta^n\|_{W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^{2d})} \|f\|^2_{M^{p_1,q_1}_{v_s}(\mathbb{R}^d)}. \tag{41}$$

Proof In [14, Theorem 1.2], two of us proved that if the Cohen kernel θ, defined in (2), is in $M^{1,\infty}(\mathbb{R}^{2d})$, then the related Cohen distribution Qf satisfies

$$\|Q^n f\|_{M^{p,q}_{1\otimes v_s}(\mathbb{R}^{2d})} \lesssim \|\theta\|_{M^{1,\infty}(\mathbb{R}^{2d})} \|f\|^2_{M^{p_1,q_1}_{v_s}(\mathbb{R}^d)}$$

where the indices $p_1, q_1, p, q \in [1, \infty]$ are related by condition (40).

By Proposition 6, the function Θ^n is in $W(\mathcal{F}L^1, L^\infty)$, so that the BJ kernel $\mathcal{F}_\sigma(\Theta^n)$ is in $M^{1,\infty}(\mathbb{R}^{2d})$ with $\|\mathcal{F}_\sigma(\Theta^n)\|_{M^{1,\infty}} \asymp \|\Theta^n\|_{W(\mathcal{F}L^1,L^\infty)}$ and the thesis follows. □

We write q' for the conjugate exponent of $q \in [1, \infty]$ (it is defined by $1/q + 1/q' =$). The n-th Born-Jordan operator enjoys the same continuity properties as for the $n = 1$ case, proved in [12, Theorem 1.1]. Indeed, we can state:

Theorem 7 *Consider $1 \leq p, q, r_1, r_2 \leq \infty$, such that*

$$p \leq q' \tag{42}$$

and

$$q \leq \min\{r_1, r_2, r_1', r_2'\}. \tag{43}$$

Then, the Born-Jordan operator $Op_{BJ,n}(a)$, from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, having symbol $a \in M^{p,q}(\mathbb{R}^{2d})$, extends uniquely to a bounded operator on $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$, with the estimate

$$\|Op_{BJ,n}(a)f\|_{\mathcal{M}^{r_1,r_2}} \lesssim \|a\|_{M^{p,q}} \|f\|_{\mathcal{M}^{r_1,r_2}}, \quad f \in \mathcal{M}^{r_1,r_2}. \tag{44}$$

Conversely, if this conclusion holds true, the constraints (42) are satisfied and it must hold

$$\max\left\{\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_1'}, \frac{1}{r_2'}\right\} \leq \frac{1}{q} + \frac{1}{p}, \tag{45}$$

that is (43) for $p = \infty$.

Proof The sufficient conditions are proved by induction on n. The result holds true for $n = 1$ by Theorem [12, Theorem 1.1]. Assume now that the result is true for a certain $n \in \mathbb{N}_+$ and observe, by definition (14), that

$$Op_{BJ,n+1}(a) = Op_{BJ,n}(b), \quad \text{with} \quad a = b * \mathcal{F}_\sigma\Theta.$$

The claim follows from the convolution relation $M^{p,q}(\mathbb{R}^{2d}) * M^{1,\infty}(\mathbb{R}^{2d}) \hookrightarrow M^{p,q}(\mathbb{R}^{2d})$.

The necessary conditions are obtained arguing exactly as in the case $n = 1$; for details, we refer to the proof of Theorem 1.1 given in [12]. □

7 Technical notes

The figures in the "Introduction" were produced using LTFAT (The Large Time-Frequency Analysis Toolbox), cf. [44] as well as the Time-Frequency Toolbox (TFTB), distributed under the terms of the GNU Public Licence: http://tftb.nongnu.org/.

The bat sonar signal in Fig. 3 was recorded as a .mat file in the latter toolbox.

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References

1. Boggiatto, P., Carypis, E., Oliaro, A.: Windowed-Wigner representations in the Cohen class and uncertainty principles. J. Geom. Anal. **23**(4), 1753–1779 (2013)
2. Boggiatto, P., De Donno, G., Oliaro, A.: Time-frequency representations of Wigner type and pseudo-differential operators. Trans. Amer. Math. Soc. **362**(9), 4955–4981 (2010)
3. Boggiatto, P., Oliaro, A., Wahlberg, P.: The wave front set of the Wigner distribution and instantaneous frequency. J. Fourier Anal. Appl. **18**(2), 410–438 (2012)
4. Christensen, O.: An Introduction to Frames and Riesz Bases. Basel, Birkhäuser (2016)
5. Choi, H., Williams, W.J.: Improved time-frequency representation of multicomponent signals using exponential kernels. IEEE Trans. Acoust. Speech Signal Process. **37**(6), 862–871 (1989)
6. Cohen, L.: Generalized phase-space distribution functions. J. Math. Phys. **7**, 781–786 (1966)
7. Cohen, L.: Time Frequency Analysis: Theory and Applications. Prentice Hall (1995)
8. Cohen, L.: The Weyl Operator and its Generalization. Springer Science & Business Media (2012)
9. Cordero, E., Gröchenig, K.: Time-frequency analysis of Localization operators. J. Funct. Anal. **205**(1), 107–131 (2003)
10. Cordero, E., Nicola, F.: Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation. J. Funct. Anal. **254**, 506–534 (2008)
11. Cordero, E., de Gosson, M., Nicola, F.: On the invertibility of Born-Jordan quantization. J. Math. Pures Appl. **105**, 537–557 (2016)
12. Cordero, E., de Gosson, M., Nicola, F.: Time-frequency analysis of Born-Jordan pseudodifferential operators. J. Funct. Anal. **272**(2), 577–598 (2017). https://doi.org/10.1016/j.jfa.2016.10.004
13. Cordero, E., de Gosson, M., Nicola, F.: On the reduction of the interferences in the Born-Jordan distribution. Appl. Comput. Harmon. Anal. **44**(2), 230–245 (2018). https://doi.org/10.1016/j.acha.2016.04.007
14. Cordero, E., Nicola, F.: Sharp integral bounds for Wigner distribution. Int. Math. Res. Notic. **2016**, 1–29 (2016)
15. de Bruijn, N.G.: A theory of generalized functions, with applications to Wigner distribution and Weyl correspondence. Njeuw Archief voor Wiskunde **21**, 205–280 (1973)
16. Domingo, H.B., Galapon, E.A.: Generalized Weyl transform for operator ordering: polynomial functions in phase space. J. Math. Phys. **56**(2), 022104 (2015)
17. Faà di Bruno, F.: Sullo Sviluppo delle Funzioni. Annali di Scienze Matematiche e Fisiche **6**, 479–480 (1855)
18. Feichtinger, H.G.: On a new Segal algebra. Monatshefte Math. **92**(4), 269–289 (1981)
19. Feichtinger, H.G.: Modulation Spaces on Locally Compact Abelian Groups, Technical Report, University Vienna, 1983, and also in "Wavelets and Their Applications", M. Krishna, R. Radha, S. Thangavelu, editors. Allied Publishers, pp. 99–140 (2003)

20. Feichtinger, H.G.: Banach convolution algebras of Wiener's type. In: Proc. Conf. "Function, Series, Operators", Budapest August 198, Colloq. Math. Soc. János Bolyai, vol. 35, pp. 509–524, North-Holland (1983)
21. Feichtinger, H.G.: Banach spaces of distributions of Wiener's type and interpolation. In: Proc. Conf. Functional Analysis and Approximation, Oberwolfach August 1980, Internat. Ser. Numer. Math., vol. 69, pp. 153–165. Birkhäuser, Boston (1981)
22. Feichtinger, H.G.: Generalized amalgams, with applications to Fourier transform. Canad. J. Math. **42**(3), 395–409 (1990)
23. Feichtinger, H.G., Zimmermann, G.: A Banach space of test functions for Gabor analysis. In: Feichtinger, H.G., Strohmer, T. (eds.) Applied and Numerical Harmonic Analysis, pp. 123–170. Birkhäuser, Basel (1998)
24. Galleani, L., Lo Presti, L.: Application of the Wigner distribution to nonlinear systems. J. Mod. Opt. **49**(3/4), 571–579 (2002)
25. de Gosson, M.: Symplectic Methods in Harmonic Analysis and in Mathematical Physics. Birkhäuser (2011)
26. de Gosson, M.: Symplectic covariance properties for Shubin and Born–Jordan pseudo-differential operators. Trans. Amer. Math. Soc. **365**(6), 3287–3307 (2013)
27. de Gosson, M.: Born–Jordan Quantization: Theory and Applications, vol. 182. Springer (2016)
28. de Gosson, M.: The Wigner Transform. World Scientific Pub Co (2017)
29. de Gosson, M., Luef, F.: Preferred quantization rules: Born–Jordan vs. Weyl; applications to phase space quantization. J. Pseudo-Differ. Oper. Appl. **2**(1), 115–139 (2011)
30. de Gosson, M., Toft, J.: Continuity properties for Born-Jordan operators with symbols in Hörmander classes and modulation spaces, arXiv:1702.05714
31. Gröchenig, K.: Foundation of Time-Frequency Analysis. Birkhäuser, Boston (2001)
32. Gröchenig, K.: Time-frequency analysis of Sjö strand's class. Rev. Mat. Iberoamericana **22**(2), 703–724 (2006)
33. Hlawatsch, F., Flandrin, P.: The interference structure of the Wigner distribution and related time-frequency signal representations. In: Mecklenbräuker, W., Hlawatsch, F. (eds.) The Wigner Distribution – Theory and Applications in Signal Processing, pp. 59–133. Elsevier, Amsterdam (1997)
34. Hlawatsch, F., Auger, F.: Time-Frequency Analysis. Wiley (2008)
35. Hörmander, L.: Lectures on Nonlinear Hyperbolic Differential Equations. Springer (1997)
36. Janssen, A.J.E.M.: On the locus and spread of pseudodensity functions in the time-frequency plane. Philips J. Res. **37**(3), 79–110 (1982)
37. Jeong, J., Williams, W.J.: Kernel design for reduced interference distributions. IEEE Trans. Signal Process. **40**(2), 402–412 (1992)
38. Kutyniok, G., Labate, D.: Resolution of the wavefront set using continuous shearlets. Trans. Amer. Math. Soc. **361**, 2719–2754 (2009)
39. Loughlin, P., Pitton, J., Atlas, L.: New properties to alleviate interference in time-frequency representations. In: Proc. IEEE Intl. Conf. Acous., Speech and Sig. Process.'91, pp. 3205–3208 (1991)
40. Loughlin, P., Pitton, J., Atlas, L.: Bilinear time-frequency representations: new insights and properties. IEEE Trans. Sig. Process. **41**(2), 750–767 (1993)
41. Pilipović, S., Teofanov, N., Toft, J.: Wave front sets in Fourier Lebesgue spaces. Rend. Sem. Mat. Univ. Pol. Torino **66**(4), 299–319 (2008)
42. Pilipovic, S., Teofanov, N., Toft, J.: Micro-local analysis in Fourier-Lebesgue and modulation spaces. Part II. J. Pseudo-Differ. Oper. Appl. **1**, 341–376 (2010)
43. Pilipovic, S., Teofanov, N., Toft, J.: Micro-local analysis in Fourier-Lebesgue and modulation spaces. Part I. J. Fourier Anal. Appl. **17**, 374–407 (2011)
44. Prusa, Z., Sondergaard, P.L., Holighaus, N., Wiesmeyr, C., Balazs, P.: The large time-frequency analysis toolbox 2.0. Sound, music, and motion. Lect. Notes Comput. Sci., 419–442 (2014)
45. Reiter, H.: Metaplectic Groups and Segal Algebras, Lecture Notes in Mathematics, vol. 1382. Springer, Berlin (1989)
46. Toft, J.: Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I. J. Funct. Anal. **207**(2), 399–429 (2004)
47. Weyl, H.: Quantenmechanik und Gruppentheories. Zeitschrift fur Physik **46**, 1–46 (1927)

48. Zhao, Y., Atlas, L.E., Marks, R.J.: The use of cone-shaped kernels for generalized time-frequency representations of nonstationary signals. IEEE Trans. Acoust. Speech. Signal Process. **38**(7), 1084–1091 (1990)

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