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The nonconforming virtual element method for fourth-order singular perturbation problem

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Abstract
We present the nonconforming virtual element method for the fourth-order singular perturbation problem. The virtual element proposed in this paper is a variant of the C^0-continuous nonconforming virtual element presented in our previous work and allows to compute two different projection operators that are used for the construction of the discrete scheme. We show the optimal convergence in the energy norm for the nonconforming virtual element method. Further, the lowest order nonconforming method is proved to be uniformly convergent with respect to the perturbation parameter. Finally, we verify the convergence for the nonconforming virtual element method by some numerical tests.

Keywords Nonconforming virtual element · Fourth-order singular perturbation problem · Polygonal mesh

Mathematics Subject Classification (2010) 65N30 · 65N12

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with boundary $\partial\Omega$. We consider the following fourth-order singular perturbation problem of the form:

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$$\begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^2(\Omega)$, $\mathbf{n}$ is the unit outward normal vector along the boundary $\partial\Omega$ and ε is a real number such that $0 < \varepsilon \leq 1$. It is obvious that the differential equation (1.1) formally degenerates to the Poisson equation when ε tends to zero. Hence, problem (1.1) is a plate model which may degenerate toward an elastic membrane problem.

For fourth-order problems, C^1-continuity must be required in the construction of conforming finite elements, that makes the element complicated. By contrast, nonconforming finite elements are of great interest. However, some of them are not uniformly convergent for problem (1.1) with respect to the perturbation parameter ε, such as the well-known Morley element [25]. Hence, some modified elements for problem (1.1) have been proposed in order to obtain the uniform convergence. For example, Nilssen et al. [26] presented a nine-parameter C^0 triangular element. Wang et al. [30] derived a modified Morley element method by using the triangular Morley element or rectangular Morley element, where the discrete variational formulation was changed by the linear or bilinear approximation of finite element functions in the lower part of the bilinear form. Later, Wang and Meng [29] extended this method to three dimensions. In fact, the non-C^0 rectangular Morley element was shown to be uniformly convergent in the energy norm with respect to the perturbation parameter in [28], where a C^0 extended high-order rectangular Morley element was also presented. Chen et al. [17] proposed two non-C^0 nonconforming elements with double set parameters. An anisotropic nonconforming element was constructed by the double set parameter method in [16]. Following [26], Guzman et al. [20] presented a family of nonconforming elements by adding bubble functions to Lagrange elements. For further works on nonconforming plate elements with uniform convergency, see the references [12, 13, 31]. In addition, see [6, 21] for a C^0 interior penalty method or a tailored finite point method.

Recently, the virtual element method (VEM) has aroused a vast concern because of its high flexibility of the mesh handling and properties of the scheme by avoiding an explicit construction of the discrete shape function. The VEM can deal with the polygonal meshes, see [4] for the basic principle of VEM. The nonconforming VEM is also developed in [3] where the nonconforming virtual element is first constructed for Poisson problem. Especially for the fourth-order problems, it is much easier to treat the C^1-continuity and construct the virtual element with any order of convergence. For example, Brezzi and Marini proposed a conforming VEM with any order of convergence for the plate bending problem in [9]. In [33], we presented a nonconforming VEM for the plate bending problem, where a C^0-continuous nonconforming virtual element was constructed for any order of accuracy. Like the classical nonconforming finite elements, it relaxes the continuity requirement for the function space to some extent. Further, the fully nonconforming virtual element for fourth-order elliptic problems was designed in [2, 34] by different ways, which can be taken as the extension of the well-known Morley element to polygonal meshes. However, the Morley-type virtual element [2, 34] should not be uniformly convergent for problem (1.1) with respect to ε like the Morley element, since they both contain the same types of the degrees of freedom. As a generalization of Morley-type virtual element [2, 34], a uniform construction of the fully H^m-nonconforming virtual elements of any order k is developed for the m-harmonic equation in $\mathbb{R}^n$ with constraints $m \leq n$ and $k \geq m$ in [15]. Besides, we also mention that the nonconforming virtual element presented in [3] has been applied to solve other problems such as general elliptic problems[11], convection-diffusion-reaction problem [5], Stokes problem [10, 23], linear elasticity problem [32], and parabolic problem [24, 36]. For further development on nonconforming VEM, we refer to the recent paper [35] where the divergence-free nonconforming VEM for the Stokes problem is presented.

In this paper, we develop the nonconforming VEM for the fourth-order singular perturbation problem (1.1) based on the C^0-continuous nonconforming virtual element presented in [33]. Specifically, taking inspiration from the idea of [1, 18], we enlarge the original shape function space and then use a projection operator to impose a restrictive condition for the enlarged shape function space. Accordingly, the modified virtual element space has the same dimension as the original one and the original degrees of freedom are unisolvent for the modified virtual element space. Moreover, we are able to compute two different projection operators that are used for the construction of the discrete scheme. Then we follow the argument in an abstract framework from [33] and prove the optimal error estimate in the energy norm associated with the bilinear form for the proposed nonconforming VEM. It is worth mentioning that, for the lowest order case $k = 2$, the error estimate for the proposed nonconforming method is proved to be uniform with respect to parameter ε. Finally, we verify the convergence for the nonconforming VEM by two numerical tests. The first one confirms the optimal convergence for the problem without boundary layers, and the other reflects the uniform convergence for the problem with boundary layers.

Throughout the paper, let S be any given open subset of Ω. $(\cdot,\cdot)_S$ and $\|\cdot\|_S$ denote the usual integral inner product and the corresponding norm of both $L^2(S)$ and $L^2(S)^2$, respectively. For a positive integer m, we shall use the common notation for the Sobolev spaces $H^m(S)$ and $H_0^m(S)$ with the corresponding norms $\|\cdot\|_{m,S}$ and $|\cdot|_{m,S}$ (see, e.g., [19]). If $S = \Omega$, the subscript will be omitted. For a given nonnegative integer k, $\mathbb{P}_k(S)$ denotes the space of polynomials of order k or less.

For an edge or element S, we denote $|S|$ its length or area, h_S its diameter, and $\mathbf{x}_S$ its barycenter. For integer $l \geq 0$, let $\mathcal{M}_l^S$ denote the set of scaled monomials

$$\mathcal{M}_l^S = \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_S}{h_S} \right)^{\beta};\ |\beta| \leq l \right\},$$

where $\beta = (\beta_1, \ldots, \beta_d)$ is the nonnegative multi-index, $|\beta| = \beta_1 + \cdots + \beta_d$ and $\mathbf{x}^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$.

2 The continuous problem

In order to define weak solution of problem (1.1), it could be convenient to split the associated bilinear form $a(u, v)$ as

$$a(u, v) = \varepsilon^2 a_\Delta(u, v) + a_\nabla(u, v) = \varepsilon^2(D^2u, D^2v) + (\nabla u, \nabla v).$$

In addition, $a^K_{\Delta}(u, v)$, $a^K_{\nabla}(u, v)$, and $a^K(u, v)$ are the restrictions of bilinear forms $a_{\Delta}(u, v)$, $a_{\nabla}(u, v)$, and $a(u, v)$ on the subset K, respectively. The weak formulation of problem (1.1) reads: find $u \in H^2_0(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall\, v \in H^2_0(\Omega). \tag{2.1}$$

Associated with the bilinear form $a(u, v)$, we define the energy norm $\|\cdot\|$ by

$$\|v\|^2 = \varepsilon^2 |v|_2^2 + |v|_1^2.$$

Obviously for $v \in H^2_0(\Omega)$, it is indeed a norm. With this choice of norm, it is easy to see that the bilinear form $a(\cdot, \cdot)$ is bounded and coercive, i.e., there exist two constants M and α such that

$$\begin{aligned} a(u, v) &\le M \|u\| \|v\|, \quad \forall\, u, v \in H^2_0(\Omega), \\ a(v, v) &\ge \alpha \|v\|^2, \qquad \forall\, v \in H^2_0(\Omega), \end{aligned}$$

where $\alpha = M = 1$ in the present setting. Hence, Eq. (2.1) has a unique solution, see, e.g., [19].

3 The nonconforming virtual element

We present the local shape function space for the C^0-continuous nonconforming virtual element from [33]. For $k \ge 2$ and a convex polygon K with n edges, the local shape function space V_h^K is defined by

$$V_h^K = \{v \in H^2(K);\ \Delta^2 v \in \mathbb{P}_{k-4}(K), v|_e \in \mathbb{P}_k(e), \Delta v|_e \in \mathbb{P}_{k-2}(e), \forall e \subset \partial K\},$$

with the usual convention that $\mathbb{P}_{-1}(K) = \mathbb{P}_{-2}(K) = \{0\}$.

As is shown in [33, Section 4.1], a function in V_h^K can be uniquely determined by the following degrees of freedom:

$$\bullet \text{ The values of } v(a), \quad \forall \text{ vertex } a, \tag{3.1}$$

$$\bullet \text{ The moments } \frac{1}{h_e}\int_e q v \mathrm{d}s, \quad \forall q \in \mathcal{M}^e_{k-2}, \quad \forall \text{ edge } e, \tag{3.2}$$

$$\bullet \text{ The moments } \int_e q \frac{\partial v}{\partial \mathbf{n}_e} \mathrm{d}s, \quad \forall q \in \mathcal{M}^e_{k-2}, \quad \forall \text{ edge } e, \tag{3.3}$$

$$\bullet \text{ The moments } \frac{1}{h_K^2}\int_K q v \mathrm{d}x, \quad \forall q \in \mathcal{M}^K_{k-4}, \tag{3.4}$$

where $\mathbf{n}_e$ denotes a given unit normal vector of edge e. Here, the dimension of V_h^K and the total number of the above degrees of freedom are

$$N_V^K = n(2k-1) + \frac{1}{2}(k-2)(k-3).$$

Following some ideas in [1, 18], we introduce a new space W_h^K to be used in place of V_h^K. To this end, we enlarge V_h^K to

$$\widetilde{V_h^K} = \{v \in H^2(K);\ \Delta^2 v \in \mathbb{P}_{k-2}(K), v|_e \in \mathbb{P}_k(e), \Delta v|_e \in \mathbb{P}_{k-2}(e), \forall e \subset \partial K\}.$$

Then the dimension of $\widetilde{V_h^K}$ is

$$\widetilde{N_V^K} = n(2k-1) + \frac{1}{2}k(k-1)$$

and a function in $\widetilde{V_h^K}$ can be uniquely determined by the following degrees of freedom:

$$\bullet \text{ The values of } v(a), \quad \forall \text{ vertex } a, \tag{3.5}$$

$$\bullet \text{ The moments } \frac{1}{h_e}\int_e qv\mathrm{d}s, \quad \forall q \in \mathcal{M}_{k-2}^e, \quad \forall \text{ edge } e, \tag{3.6}$$

$$\bullet \text{ The moments } \int_e q\frac{\partial v}{\partial \mathbf{n}_e}\mathrm{d}s, \quad \forall q \in \mathcal{M}_{k-2}^e, \quad \forall \text{ edge } e, \tag{3.7}$$

$$\bullet \text{ The moments } \frac{1}{h_K^2}\int_K qv\mathrm{d}x, \quad \forall q \in \mathcal{M}_{k-2}^K. \tag{3.8}$$

Using the degrees of freedom (3.5)–(3.8), one can exactly compute $a_\Delta^K(\psi, q)$ when $\psi \in \widetilde{V_h^K}$ and $q \in \mathbb{P}_k(K)$. Then we can define a projection operator $\Pi_\Delta^K : \widetilde{V_h^K} \to \mathbb{P}_k(K) \subseteq \widetilde{V_h^K}$ by finding the solution $\Pi_\Delta^K\psi$ of

$$\begin{cases} a_\Delta^K(\Pi_\Delta^K\psi, q) = a_\Delta^K(\psi, q), & \forall q \in \mathbb{P}_k(K), \\ \widehat{\Pi_\Delta^K\psi} = \widehat{\psi}, & \\ \int_{\partial K} \nabla\Pi_\Delta^K\psi\mathrm{d}s = \int_{\partial K} \nabla\psi\mathrm{d}s, & \end{cases}$$

for any given $\psi \in \widetilde{V_h^K}$, such that $\Pi_\Delta^K v = v$ for $v \in \mathbb{P}_k(K)$ and Π_Δ^K is computable from the degrees of freedom (3.5)–(3.8), where the quasi-average $\widehat{\psi}$ is defined by

$$\widehat{\psi} = \frac{1}{n}\sum_{i=1}^{n}\psi(a_i),$$

and a_i $(i = 1, 2, \ldots, n)$ are the vertices of K. For the details, see [33].

Finally we restrict $\widetilde{V_h^K}$ to a subspace W_h^K having the same dimension as the original V_h^K, but where the moments of orders $k-3$ and $k-2$ of w and $\Pi_\Delta^K w$ coincide for $w \in W_h^K$. More precisely, we set

$$W_h^K = \left\{ w \in \widetilde{V_h^K};\ (w - \Pi_\Delta^K w, q)_K = 0,\ \ \forall q \in \mathbb{P}_{k-2}(K)/\mathbb{P}_{k-4}(K) \right\}, \tag{3.9}$$

where the symbol $\mathbb{P}_{k-2}(K)/\mathbb{P}_{k-4}(K)$ denotes the subspace of $\mathbb{P}_{k-2}(K)$ containing polynomials that are $L^2(K)$-orthogonal to $\mathbb{P}_{k-4}(K)$.

Following the discussion in [33, Section 4.3], it is easily verified that the projection Π_Δ^K from W_h^K to $\mathbb{P}_k(K)$ can still be exactly computed by using only the degrees of freedom (3.1)–(3.4). Then we have the unisolvence result.

Lemma 3.1 *The degrees of freedom (3.1)–(3.4) are unisolvent for* W_h^K.

Proof Without checking the independence of the additional $2k-3$ conditions in Eq. (3.9), the dimension N_W^K of W_h^K satisfies at least

$$N_W^K \geq \widetilde{N_V^K} - (2k-3) = n(2k-1) + \frac{1}{2}(k-2)(k-3), \tag{3.10}$$

where n is the total number of edges of K.

For any given function $w \in W_h^K$ with the vanishing degrees of freedom (3.1)–(3.4), it is immediate to see that $\Pi_\Delta^K w$ is zero since $\Pi_\Delta^K w$ is exactly computed by the degrees of freedom (3.1)–(3.4). Thus, from the definition (3.9) of W_h^K, it yields that the moments of degrees $k-3$ and $k-2$ of w are also zero on K. This implies w is zero as a function in $\widetilde{V_h^K}$ with the vanishing degrees of freedom (3.5)–(3.8) and, together with inequality (3.10), the dimension of W_h^K is equal to $n(2k-1)+(k-2)(k-3)/2$. Therefore, the degrees of freedom (3.1)–(3.4) are unisolvent for W_h^K. □

Remark 3.1 According to the definition of W_h^K, for $w \in W_h^K$, the L^2-projection $P_{k-2}^K w$ onto the polynomial space $\mathbb{P}_{k-2}(K)$ is computable by the degrees of freedom (3.1)–(3.4).

In Fig. 1, we show the degrees of freedom for the first two low-order elements with $k = 2$ and 3. For any $w \in W_h^K$, when $k = 2$, we have $w|_e \in \mathbb{P}_2(e)$ and $\Delta w|_e \in \mathbb{P}_0(e)$, and when $k = 3$, $w|_e \in \mathbb{P}_3(e)$ and $\Delta w|_e \in \mathbb{P}_1(e)$, on each edge $e \subset \partial K$.

Let $\{\mathcal{T}_h\}_h$ be a family of decompositions of Ω into polygonal elements and $\mathcal{E}_h$ denote the set of edges of $\mathcal{T}_h$, where h stands for the maximum of the diameters of elements in $\mathcal{T}_h$. We also assume that each polygon $K \in \mathcal{T}_h$ is convex such that the definitions of shape function spaces in Section 3 are meaningful. Following [4], we make the following assumption on the family of decompositions:

H0 There exists a positive constant r such that, for every h, and every $K \in \mathcal{T}_h$,

- the ratio between the shortest edge and the diameter h_K of K is bigger than r,
- K is star-shaped with respect to all the points of a ball of radius $\geq r h_K$.

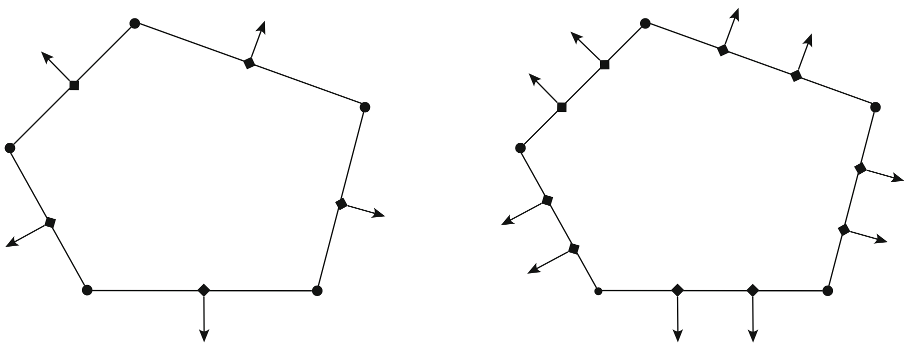

Fig. 1 Local degrees of freedom for the first two low-order elements. $k = 2$ (left) and $k = 3$ (right)

For any $K \in \mathcal{T}_h$, $\mathbf{n}_K$ ($\mathbf{t}_K$) always denotes its unit outward normal (counterclockwise tangential) vector along the boundary ∂K. We shall use the notation $\mathbf{n}_e$ ($\mathbf{t}_e$) for a unit normal (tangential) of an edge $e \in \mathcal{E}_h$, whose orientation is chosen arbitrarily but fixed for internal edges and coinciding with the outward normal (tangential) of Ω for boundary edges.

For an internal edge e shared by $K, L \in \mathcal{T}_h$ such that $\mathbf{n}_e$ points from K to L, we defined the jump of function w through the edge e by

$$[\![w]\!] = (w|_K)|_e - (w|_L)|_e.$$

For the boundary edge e, set $[\![w]\!] = w|_e$.

Observe that for $w \in W_h^K$, the restriction of w to the boundary of K is uniquely determined by the degrees of freedom (3.1)–(3.2), where K is an element in $\mathcal{T}_h$. Therefore, for every decomposition $\mathcal{T}_h$ and $k \geq 2$, we define the global virtual element space W_h as

$$W_h = \Big\{ w \in H_0^1(\Omega);\ w|_K \in W_h^K,\ \forall K \in \mathcal{T}_h,\ \int_e q[\![\frac{\partial w}{\partial \mathbf{n}_e}]\!]\mathrm{d}s = 0, \\ \forall q \in \mathbb{P}_{k-2}(e),\ \forall e \in \mathcal{E}_h \Big\}.$$

According to the previous discussion, the global degrees of freedom for W_h can then be taken as

- The values of $w(a)$, $\forall$ internal vertex a, (3.11)
- The moments $\frac{1}{h_e}\int_e q w \mathrm{d}s$, $\forall q \in \mathcal{M}_{k-2}^e$, $\forall$ internal edge e, (3.12)
- The moments $\int_e q \frac{\partial w}{\partial \mathbf{n}_e}\mathrm{d}s$, $\forall q \in \mathcal{M}_{k-2}^e$, $\forall$ internal edge e, (3.13)
- The moments $\frac{1}{h_K^2}\int_K q w \mathrm{d}x$, $\forall q \in \mathcal{M}_{k-4}^K$, $\forall$ element K. (3.14)

Note that here $W_h \not\subseteq H_0^2(\Omega)$ but W_h is C^0-continuous. Thus, the virtual element space W_h is nonconforming. As it happens for the local space W_h^K, the dimension of W_h coincides with the total number of degrees of freedom (3.11)–(3.14), which is given by

$$N = N_V + 2(k-1)N_E + \frac{(k-2)(k-3)}{2}N_K,$$

where N_V is the number of internal vertices of $\mathcal{T}_h$, N_E is the number of internal edges, and N_K is the number of elements. The unisolvence for the local space W_h^K given in Lemma 3.1 implies the unisolvence for the global space W_h.

Next we define an interpolation operator in W_h having optimal approximation properties. To this end, for each element $K \in \mathcal{T}_h$, we denote by χ_i the operator associated with the ith local degree of freedom, $i = 1, 2, \ldots, N_W^K$. It follows easily from the above construction that for every smooth enough function w, there exists a unique element $w_I \in W_h^K$ such that

$$\chi_i(w - w_I) = 0, \quad i = 1, 2, \ldots, N_W^K.$$

For the interpolation error, we have the following estimate, of which the proof is given in the Appendix.

Lemma 3.2 *For every* $K \in \mathcal{T}_h$ *and every* $w \in H^s(K)$ *with* $2 \le s \le k+1$*, it holds that*

$$\|w - w_I\|_{m,K} \le C h_K^{s-m} |w|_{s,K}, \quad m = 0, 1, 2.$$

Remark 3.2 We mention the works from [7, 14] where the interpolation error for the H^1-conforming virtual element [4] is rigorously proved by different ways. However, it is not clear whether those techniques used in [7, 14] can be applied for the interpolation error analysis for the C^0-continuous H^2-nonconforming virtual element. Thus, in the Appendix, we display another way to estimate the interpolation error for the virtual element presented here.

In what follows, we need a lower order estimate for the interpolation error, but the dependence on the function is weaker. As is shown in [26], following the standard trace inequality

$$\|w\|_{\partial \widehat{K}} \le C \|w\|_{\widehat{K}}^{\frac{1}{2}} \|w\|_{1,\widehat{K}}^{\frac{1}{2}}, \quad \forall K \in \mathcal{T}_h \tag{3.15}$$

and the Bramble-Hilbert argument, we obtain

$$\|w - w_I\|_1 \le C h^{\frac{1}{2}} |w|_1^{\frac{1}{2}} |w|_2^{\frac{1}{2}}, \quad \text{for } w \in H_0^2(\Omega), \tag{3.16}$$

where $\widehat{K}$ is the reference polygon defined by the transformation $\mathbf{x} = h_K \hat{\mathbf{x}}$.

Remark 3.3 The proof of trace inequality (3.15) is immediate. To this end, we establish a virtual triangulation $\mathcal{T}_{\widehat{K}}$ of $\widehat{K}$ such that $\mathcal{T}_{\widehat{K}}$ is regular and quasi-uniform, and the size of each triangle is comparable with that of $\widehat{K}$ with $h_{\widehat{K}} = 1$, which is due to the mesh assumption **H0**. Note that each edge of $\widehat{K}$ is a side of a certain triangle in $\mathcal{T}_{\widehat{K}}$. Thus, for each edge $\hat{e}$ of $\widehat{K}$, we have the following trace inequality:

$$\|w\|_{\hat{e}}^2 \le C \|w\|_{K_{\hat{e}}} \|w\|_{1,K_{\hat{e}}}, \quad \hat{e} \subset \partial K_{\hat{e}}, \quad K_{\hat{e}} \in \mathcal{T}_{\widehat{K}},$$

where the constant C depends only on the mesh regularity parameter r and the quasi-uniformity, but is independent of h. Therefore, we have

$$\|w\|_{\partial \widehat{K}}^2 = \sum_{\hat{e} \subset \partial \widehat{K}} \|w\|_{\hat{e}}^2 \le C \sum_{\hat{e} \subset \partial \widehat{K}} \|w\|_{K_{\hat{e}}} \|w\|_{1,K_{\hat{e}}} \le C \|w\|_{\widehat{K}} \|w\|_{1,\widehat{K}}.$$

In addition, we need to introduce a local approximation with optimal approximation properties. In view of the mesh regularity assumption **H0**, we have the following result. For the details, see [8].

Lemma 3.3 *For every* $K \in \mathcal{T}_h$ *and every* $w \in H^s(K)$ *with* $2 \le s \le k+1$*, there exists a polynomial* $w_\pi \in \mathbb{P}_k(K)$ *such that*

$$\|w - w_\pi\|_{m,K} \le C h_K^{s-m} |w|_{s,K}, \quad m = 0, 1, 2.$$

4 The discretization and error analysis

We are now in a position to construct a symmetric and computable discrete bilinear form $a_h(\cdot,\cdot)$. To start with, we define $\Pi_\nabla^K : W_h^K \to \mathbb{P}_k(K) \subset W_h^K$ as the solution of

$$\begin{cases} a_\nabla^K(\Pi_\nabla^K \psi, q) = a_\nabla^K(\psi, q), \quad \forall q \in \mathbb{P}_k(K), \\ \widehat{\Pi_\nabla^K \psi} = \widehat{\psi}, \end{cases} \tag{4.1}$$

where $\psi \in W_h^K$. The projection Π_∇^K is computable by the degrees of freedom (3.1)–(3.4). Indeed, we have

$$a_\nabla^K(\psi, q) = (\nabla\psi, \nabla q)_K = -\int_K \Delta q \psi \mathrm{d}x + \int_{\partial K} \frac{\partial q}{\partial \mathbf{n}_K} \psi \mathrm{d}s. \tag{4.2}$$

Note that $\Delta q \in \mathbb{P}_{k-2}(K)$, then the first term in the right hand side of Eq. (4.2) can be obtained from the degrees of freedom (3.4) and

$$\int_K q\psi \mathrm{d}x = \int_K q \Pi_\Delta^K \psi \mathrm{d}x, \quad \forall q \in \mathbb{P}_{k-2}(K)/\mathbb{P}_{k-4}(K),$$

once the projection Π_Δ^K has been computed. The second term in the right hand side of Eq. (4.2) is computable as we can easily compute ψ on each edge e by the degrees of freedom (3.1)–(3.2). Hence, all terms in the right hand side of Eq. (4.2) are computable using only the degrees of freedom of ψ.

For each polygon $K \in \mathcal{T}_h$, we define

$$\begin{aligned} a_h^K(v_h, w_h) =& \varepsilon^2 a_\Delta^K(\Pi_\Delta^K v_h, \Pi_\Delta^K w_h) + \varepsilon^2 h_K^{-2} S^K(v_h - \Pi_\Delta^K v_h, w_h - \Pi_\Delta^K w_h) \\ &+ a_\nabla^K(\Pi_\nabla^K v_h, \Pi_\nabla^K w_h) + S^K(v_h - \Pi_\nabla^K v_h, w_h - \Pi_\nabla^K w_h), \quad \forall v_h, w_h \in W_h^K, \end{aligned}$$

where the term $S^K(\cdot,\cdot)$ will be chosen to be a symmetric and positive definite bilinear form satisfying

$$C_0 a_\Delta^K(v_h, v_h) \le h_K^{-2} S^K(v_h, v_h) \le C_1 a_\Delta^K(v_h, v_h), \quad \forall v_h \in \ker(\Pi_\Delta^K), \tag{4.3}$$

$$C_0 a_\nabla^K(v_h, v_h) \le S^K(v_h, v_h) \le C_1 a_\nabla^K(v_h, v_h), \quad \forall v_h \in \ker(\Pi_\nabla^K). \tag{4.4}$$

By using the properties of Π_Δ^K and Π_∇^K, the standard arguments [4, 9] show that the bilinear form $a_h^K(\cdot,\cdot)$ satisfies the k-consistency and stability, i.e.,

- k-consistency: for all $p \in \mathbb{P}_k(K)$ and $v_h \in W_h^K$, it holds that

$$a_h^K(p, v_h) = a^K(p, v_h).$$

- stability: there exist two positive constants α_* and α^*, independent of h and ε, such that

$$\alpha_* a^K(v_h, v_h) \le a_h^K(v_h, v_h) \le \alpha^* a^K(v_h, v_h), \quad \forall v_h \in W_h^K.$$

The remaining thing is the choice of the bilinear form $S^K(\cdot,\cdot)$ based on the degrees of freedom such that (4.3–4.4) is satisfied. Observing that the degrees of freedom are of the same dimension, we set

$$S^K(v_h, w_h) = \sum_{i=1}^{N_W^K} \chi_i(v_h)\chi_i(w_h).$$

In the usual way, the global bilinear form $a_h(\cdot,\cdot)$ is given by

$$a_h(v_h, w_h) = \sum_{K\in\mathcal{T}_h} a_h^K(v_h, w_h), \quad \forall v_h, w_h \in W_h.$$

For the convenience of discussion, we reformulate the bilinear form $a(\cdot,\cdot)$ as

$$a(v_h, w_h) = \sum_{K\in\mathcal{T}_h} a^K(v_h, w_h)$$

such that it makes sense for $v_h, w_h \in W_h$.

The definition of the right hand side is simpler than that in [33]. For $k \geq 2$, we can simply take f_h on each element K as the L^2-projection of the load f onto the space $\mathbb{P}_{k-2}(K)$, that is,

$$f_h|_K = P_{k-2}^K f, \quad K \in \mathcal{T}_h.$$

Now we introduce the discrete problem: find $u_h \in W_h$ such that

$$a_h(u_h, v_h) = (f_h, v_h), \quad \forall\, v_h \in W_h. \tag{4.5}$$

Since $W_h \not\subseteq H_0^2(\Omega)$, the VEM (4.5) is nonconforming. Note that the right hand side is computable by the degrees of freedom (3.1)–(3.4), since

$$(f_h, v_h) = \sum_{K\in\mathcal{T}_h} (P_{k-2}^K f, v_h)_K = \sum_{K\in\mathcal{T}_h} (f, P_{k-2}^K v_h)_K$$

and $P_{k-2}^K v_h$ is computable according to Remark 3.1.

In order to measure the error, for any $s > 0$, we introduce the broken H^s-seminorm

$$|v_h|_{s,h}^2 = \sum_{K\in\mathcal{T}_h} |v_h|_{s,K}^2, \quad v_h \in W_h,$$

and then define the discrete energy norm $|\!|\!|v_h|\!|\!|_h = (\varepsilon^2|v_h|_{2,h}^2 + |v_h|_{1,h}^2)^{\frac{1}{2}}$, the restriction of which on an element K is denoted by $|\!|\!|v_h|\!|\!|_K$. It can be shown that for the space W_h, $|\!|\!|\cdot|\!|\!|_h$ is indeed a norm, see [33] for the details.

Following the arguments in [33], we have the following convergence theorem.

Theorem 4.1 *The discrete problem (4.5) has a unique solution $u_h \in W_h$. Moreover, for every approximation u_I of u in W_h and for every approximation u_π of u in discontinuous piecewise k-order polynomial space, we have*

$$|\!|\!|u - u_h|\!|\!|_h \leq C(|\!|\!|u - u_I|\!|\!|_h + |\!|\!|u - u_\pi|\!|\!|_h + \|f - f_h\|_{W_h'} + E_h), \tag{4.6}$$

where C is a positive constant depending only on α_*, α^* and

$$\|f - f_h\|_{W_h'} = \sup_{v_h \in W_h} \frac{(f - f_h, v_h)}{|\!|\!| v_h |\!|\!|_h}, \quad E_h = \sup_{v_h \in W_h} \frac{a(u, v_h) - (f, v_h)}{|\!|\!| v_h |\!|\!|_h}.$$

For $\|f - f_h\|_{W_h'}$, we have the following estimates:

$$\begin{aligned}(f_h, v_h)_K - (f, v_h)_K &= (P_{k-2}^K f, v_h)_K - (f, v_h)_K \\ &= (P_{k-2}^K f - f, v_h - P_0^K v_h)_K \\ &\le Ch_K^k |f|_{k-1,K} |\!|\!| v_h |\!|\!|_K, \quad \forall v_h \in W_h. \end{aligned} \tag{4.7}$$

Hence, for $k \ge 2$, this ensures the optimal $O(h^k)$ error bound for data approximation.

In view of the following analysis, for $k = 2$, it is useful to obtain a variant of this estimate (4.7), which is lower order with respect to h but requires a weaker dependence on the function f. In fact,

$$\begin{aligned}(f_h, v_h)_K - (f, v_h)_K &= (P_0^K f, v_h)_K - (f, v_h)_K \\ &= (P_0^K f - f, v_h - P_0^K v_h)_K \\ &\le Ch_K \|f\|_K |\!|\!| v_h |\!|\!|_K, \quad \forall v_h \in W_h. \end{aligned} \tag{4.8}$$

The optimal interpolation error and approximation error estimates have been presented in Lemmas 3.2–3.3. From Theorem 4.1 and (4.7), we know that what remains to do is to estimate the consistency error for the nonconforming VEM, which will be presented in the following lemma.

Lemma 4.1 *Assume that $u \in H^{k+1}(\Omega)$. Then we have the estimate*

$$E_h \le C\varepsilon h^{k-1} \|u\|_{k+1},$$

where E_h is the consistency error from inequality (4.6).

Proof The proof is the same as in [33, Lemma 5.2]. In order to be convenient for the following discussion on the uniform convergence, we still give the proof.

For any $v_h \in W_h$, from Green's formula [19, 22], it holds that

$$\begin{aligned}\int_K \Delta u \Delta v_h \mathrm{d}x &= \int_{\partial K} \Delta u \frac{\partial v_h}{\partial \mathbf{n}_K} \mathrm{d}s - \int_K \nabla \Delta u \cdot \nabla v_h \mathrm{d}x, \\ &\int_K \left(2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v_h}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v_h}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v_h}{\partial x_1^2} \right) \mathrm{d}x \\ &= \int_{\partial K} \left(\frac{\partial^2 u}{\partial \mathbf{n}_K \partial \mathbf{t}_K} \frac{\partial v_h}{\partial \mathbf{t}_K} - \frac{\partial^2 u}{\partial \mathbf{t}_\mathbf{K}^2} \frac{\partial v_h}{\partial \mathbf{n}_K} \right) \mathrm{d}s. \end{aligned}$$

Noting that $W_h \subset H_0^1(\Omega)$, then it implies

$$a(u, v_h) - (f, v_h) = \varepsilon^2 \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\Delta u - \frac{\partial^2 u}{\partial \mathbf{t}_K^2}\right) \frac{\partial v_h}{\partial \mathbf{n}_K} \mathrm{d}s$$
$$= \varepsilon^2 \sum_{e\in\mathcal{E}_h} \int_e \left(\Delta u - \frac{\partial^2 u}{\partial \mathbf{t}_e^2}\right) [\![\frac{\partial v_h}{\partial \mathbf{n}_e}]\!] \mathrm{d}s. \tag{4.9}$$

Since

$$\int_e [\![\frac{\partial v_h}{\partial \mathbf{n}_e}]\!] p \mathrm{d}s = 0, \quad \forall p \in \mathbb{P}_{k-2}(e), \quad \forall e \in \mathcal{E}_h,$$

setting $w|_e = \Delta u - \frac{\partial^2 u}{\partial \mathbf{t}_e^2}$, then we have

$$\varepsilon^2 \int_e w [\![\frac{\partial v_h}{\partial \mathbf{n}_e}]\!] \mathrm{d}s = \varepsilon^2 \int_e (w - P_{k-2}^e w) [\![\frac{\partial v_h}{\partial \mathbf{n}_e} - P_0^e\left(\frac{\partial v_h}{\partial \mathbf{n}_e}\right)]\!] \mathrm{d}s$$
$$\leq \varepsilon^2 \|w - P_{k-2}^e w\|_e \| [\![\frac{\partial v_h}{\partial \mathbf{n}_e} - P_0^e\left(\frac{\partial v_h}{\partial \mathbf{n}_e}\right)]\!] \|_e, \tag{4.10}$$

where P_m^e denotes the L^2-projection onto the space $\mathbb{P}_m(e)$ on edge e.

Using standard approximation estimates [19], it implies that, for each internal edge $e = \partial K^+ \cap \partial K^-$,

$$\|w - P_{k-2}^e w\|_e \leq Ch^{k-\frac{3}{2}} \|u\|_{k+1, K^+\cup K^-}, \tag{4.11}$$

$$\| [\![\frac{\partial v_h}{\partial \mathbf{n}_e} - P_0^e\left(\frac{\partial v_h}{\partial \mathbf{n}_e}\right)]\!] \|_e \leq Ch^{\frac{1}{2}} (\|v_h\|_{2,K^+}^2 + \|v_h\|_{2,K^-}^2)^{\frac{1}{2}}. \tag{4.12}$$

For boundary edges, the adjustment is obvious.

Hence, combining (4.9)–(4.12), the proof is concluded. □

Combining Lemmas 3.2–3.3, (4.7), Lemma 4.1, and Theorem 4.1, we obtain the convergence for the nonconforming VEM (4.5) as follows.

Theorem 4.2 *Assume $f \in L^2(\Omega) \cap H^{k-1}(\Omega)$ and $u \in H_0^2(\Omega) \cap H^{k+1}(\Omega)$ with $k \geq 2$. For the nonconforming VEM (4.5), we have the following error estimate*

$$|\!|\!| u - u_h |\!|\!|_h \leq \begin{cases} C(\varepsilon h^{k-1} + h^k)|u|_{k+1} + h^k \|f\|_{k-1}, \\ Ch^{k-1}(\varepsilon|u|_{k+1} + |u|_k) + h^k\|f\|_{k-1}. \end{cases} \tag{4.13}$$

From Theorem 4.2, we see that for problems without boundary layers, the estimate (4.13) ensures the first-order convergence of the lowest order VEM. Further, it arrives at the second-order convergence as ε tends to 0. However, the estimate (4.13) is not uniform in ε for problems with boundary layers. Hence, we shall treat the dependence of $|u|_2$ and $|u|_3$ on ε in the remaining.

Let u^0 be the solution of the following boundary value problem:

$$\begin{cases} -\Delta u^0 = f, & \text{in } \Omega, \\ u^0 = 0, & \text{on } \partial\Omega. \end{cases}$$

The following a priori regular estimate was presented in [26].

Lemma 4.2 *If Ω is convex, then*

$$|u|_2 + \varepsilon|u|_3 \le C\varepsilon^{-1/2}\|f\| \quad \text{and} \quad |u-u^0|_1 \le C\varepsilon^{1/2}\|f\|$$

for all $f \in L^2(\Omega)$.

Theorem 4.3 *Assume $f \in L^2(\Omega)$ and $u \in H_0^2(\Omega)\cap H^3(\Omega)$ is the corresponding weak solution of Eq. (2.1). For the nonconforming VEM (4.5) with $k=2$, we have the following error estimate*

$$|||u-u_h|||_h \le Ch^{1/2}\|f\|.$$

Proof We first show that

$$|||u-u_I|||_h \le Ch^{1/2}\|f\|. \tag{4.14}$$

From the triangle inequality, Lemma 4.2, and the regularity $\|u^0\|_2 \le C\|f\|$, we obtain

$$\begin{aligned} |u-u_I|_1 &\le |u-u^0-(u-u^0)_I|_1 + |u^0-(u^0)_I|_1 \\ &\le Ch^{\frac{1}{2}}|u-u^0|_1^{\frac{1}{2}}|u-u^0|_2^{\frac{1}{2}} + Ch|u^0|_2 \\ &\le Ch^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\|f\|^{\frac{1}{2}}(|u|_2^{\frac{1}{2}} + |u^0|_2^{\frac{1}{2}}) + Ch\|f\| \\ &\le Ch^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\|f\|^{\frac{1}{2}}(\varepsilon^{-\frac{1}{4}}\|f\|^{\frac{1}{2}} + \|f\|^{\frac{1}{2}}) + Ch\|f\| \\ &\le Ch^{\frac{1}{2}}\|f\| + Ch\|f\| \\ &\le Ch^{\frac{1}{2}}\|f\|, \end{aligned}$$

where we have also used the fact that $\varepsilon \le 1$, $h \le 1$ and inequality (3.16).

For $\varepsilon|u-u_I|_{2,h}$, by applying Lemmas 3.2 and 4.2, it is not difficult to see that

$$\begin{aligned} \varepsilon|u-u_I|_{2,h} &= \varepsilon|u-u_I|_{2,h}^{\frac{1}{2}}|u-u_I|_{2,h}^{\frac{1}{2}} \\ &\le C\varepsilon|u|_2^{\frac{1}{2}}h^{\frac{1}{2}}|u|_3^{\frac{1}{2}} \\ &\le C\varepsilon h^{\frac{1}{2}}\varepsilon^{-\frac{1}{4}}\|f\|^{\frac{1}{2}}\varepsilon^{-\frac{3}{4}}\|f\|^{\frac{1}{2}} \\ &\le Ch^{\frac{1}{2}}\|f\|. \end{aligned}$$

Similarly, we can obtain

$$|||u-u_\pi|||_h \le Ch^{1/2}\|f\|. \tag{4.15}$$

As in the proof of Lemma 4.1, we show that

$$\begin{aligned} a(u,v_h)-(f,v_h) &= \varepsilon^2\sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\Delta u - \frac{\partial^2 u}{\partial \mathbf{t}_K^2}\right)\frac{\partial v_h}{\partial \mathbf{n}_K}\mathrm{d}s \\ &= \varepsilon^2\sum_{e\in\mathcal{E}_h}\int_e\left(\Delta u - \frac{\partial^2 u}{\partial \mathbf{t}_e^2}\right)[\![\frac{\partial v_h}{\partial \mathbf{n}_e}]\!]\mathrm{d}s. \end{aligned}$$

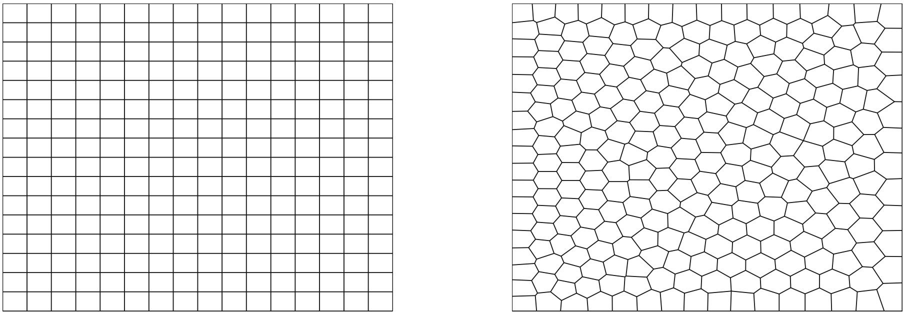

Fig. 2 The uniform rectangular (left) and unstructured polygonal (right) meshes

Setting $w|_e = \Delta u - \frac{\partial^2 u}{\partial \mathbf{t}_e^2}$, then for the internal edge $e = \partial K^+ \cap \partial K^-$, we use the trace inequality (3.15) and the estimates (4.12) to obtain

$$\begin{aligned}\varepsilon^2 \int_e w[\![\frac{\partial v_h}{\partial \mathbf{n}_e}]\!]\mathrm{d}s &= \varepsilon^2 \int_e (w - P_0^K w)[\![\frac{\partial v_h}{\partial \mathbf{n}_e} - P_0^e \frac{\partial v_h}{\partial \mathbf{n}_e}]\!]\mathrm{d}s \\ &\leq \varepsilon^2 \|w - P_0^K w\|_e \|[\![\frac{\partial v_h}{\partial \mathbf{n}_e} - P_0^e \frac{\partial v_h}{\partial \mathbf{n}_e}]\!]\|_e \\ &\leq C\varepsilon^2 \|w\|_K^{\frac{1}{2}} |w|_{1,K}^{\frac{1}{2}} h^{\frac{1}{2}} (\|v_h\|_{2,K^+}^2 + \|v_h\|_{2,K^-}^2)^{\frac{1}{2}} \\ &\leq C\varepsilon^2 |u|_{2,K}^{\frac{1}{2}} |u|_{3,K}^{\frac{1}{2}} h^{\frac{1}{2}} (\|v_h\|_{2,K^+}^2 + \|v_h\|_{2,K^-}^2)^{\frac{1}{2}} \\ &\leq C h^{\frac{1}{2}} \varepsilon |u|_{2,K}^{\frac{1}{2}} |u|_{3,K}^{\frac{1}{2}} |\!|\!| v_h |\!|\!|_K,\end{aligned}$$

where K is K^+ or K^-. For boundary edges, the adjustment is obvious. Then we have the estimate

$$E_h \leq C h^{\frac{1}{2}} \|f\|,$$

which, together with (4.14–4.15) and (4.8), concludes the proof. □

Table 1 The relative error on the rectangular meshes for Example 5.1

ε \ h	0.353553	0.176777	0.088388	0.044194	0.022097	0.011049	Rate
2^0	0.823790	0.764662	0.461869	0.243241	0.123256	0.061835	0.9952
2^{-2}	0.331237	0.460695	0.353670	0.206225	0.107778	0.054513	0.9835
2^{-4}	0.301400	0.064592	0.024436	0.014416	0.007656	0.003889	0.9773
2^{-6}	0.350617	0.096449	0.024485	0.006796	0.002405	0.001054	1.1902
2^{-8}	0.352104	0.097687	0.024683	0.005979	0.001302	0.000303	2.1035
2^{-10}	0.352371	0.097978	0.024970	0.006259	0.001553	0.000378	2.0387

Table 2 The relative error on the polygonal meshes for Example 5.1

$\varepsilon \backslash h$	0.355304	0.197791	0.102512	0.049521	0.023756	0.012230	Rate
2^{0}	1.134582	0.915272	0.516979	0.268102	0.138576	0.069649	1.0362
2^{-2}	0.322225	0.319815	0.196424	0.103555	0.053761	0.026952	1.0400
2^{-4}	0.320965	0.073840	0.024606	0.011891	0.006268	0.003062	1.0790
2^{-6}	0.361906	0.100991	0.029972	0.010308	0.004418	0.002124	1.1031
2^{-8}	0.361873	0.097818	0.025555	0.005947	0.001276	0.000513	1.3724
2^{-10}	0.362132	0.098193	0.025956	0.006340	0.001570	0.000381	2.1328

5 Numerical tests

In order to confirm the theoretical results developed in this paper, we carry out some numerical tests for two different kinds of meshes, which are the uniform rectangular and unstructured polygonal meshes, see Fig. 2. For the generation of the polygonal meshes, we use the code PolyMesher [27]. For simplicity, we use the lowest order element ($k = 2$) to solve problem (1.1) on the domain $\Omega = (0, 1) \times (0, 1)$ in all tests. The relative error is computed in the discrete energy norm

$$\left(\frac{a_h(u_h - u_I, u_h - u_I)}{a_h(u_I, u_I)}\right)^{\frac{1}{2}}.$$

For each fixed ε, the convergence rate with respect to h is computed by using the numerical results over the last two meshes.

Example 5.1 [26] Consider problem (1.1) with $f = \varepsilon^2\Delta^2 u - \Delta u$ and $u = (\sin(\pi x)\sin(\pi y))^2$. We compute the relative error for different values of ε and mesh size h. The numerical results for the rectangular and polygonal meshes are listed in Tables 1 and 2, respectively.

As the exact solution u of Example 5.1 has no boundary layers, from Tables 1 and 2 we see that the nonconforming VEM (4.5) ensures the first-order convergence

Table 3 The relative error on the rectangular meshes for Example 5.2

$\varepsilon \backslash h$	0.353553	0.176777	0.088388	0.044194	0.022097	0.011049	Rate
10^{0}	0.077916	0.055992	0.032847	0.017397	0.008867	0.004509	0.9757
10^{-1}	0.084437	0.054721	0.048862	0.029083	0.015255	0.007726	0.9816
10^{-2}	0.526614	0.367667	0.220245	0.126494	0.071078	0.037377	0.9273
10^{-3}	0.593512	0.481776	0.360651	0.247767	0.152596	0.077915	0.9698
10^{-4}	0.600000	0.494843	0.383800	0.284407	0.203865	0.141750	0.5243
10^{-5}	0.600642	0.496133	0.386085	0.288087	0.209448	0.149928	0.4824

Table 4 The relative error on the polygonal meshes for Example 5.2

$\varepsilon \backslash h$	0.355304	0.197791	0.102512	0.049521	0.023756	0.012230	Rate
10^{0}	0.217230	0.162498	0.094796	0.050149	0.026121	0.013336	1.0125
10^{-1}	0.058715	0.063935	0.045202	0.027532	0.013086	0.006316	1.0972
10^{-2}	0.442064	0.303586	0.189184	0.113090	0.067755	0.036239	0.9425
10^{-3}	0.504951	0.393573	0.307208	0.201594	0.120931	0.060807	1.0355
10^{-4}	0.511565	0.404339	0.330023	0.231290	0.163008	0.112622	0.5569
10^{-5}	0.512223	0.405411	0.332291	0.234325	0.167683	0.119197	0.5140

rate for all $\varepsilon \in (0, 1)$. More precisely, as ε becomes small, it yields the second-order convergence rate. Hence, these numerical results are in fact consistent with the theoretical results presented in Theorem 4.2.

Example 5.2 [28] Consider problem (1.1) with $f = 2x_2$ and $u = \varepsilon(\mathrm{e}^{-x_1/\varepsilon} + \mathrm{e}^{-x_2/\varepsilon}) - x_1^2 x_2$. The corresponding Dirichlet boundary condition holds. We note that the exact solution u has boundary layers for sufficiently small ε. We compute the relative error for different values of ε and mesh size h. The numerical results for the rectangular and polygonal meshes are listed in Tables 3 and 4. From Tables 3 and 4, we can see that the nonconforming VEM (4.5) ensures the 1/2-order convergence as $\varepsilon \to 0$. This is consistent with the theoretical result presented in Theorem 4.3.

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Appendix

First by using the bubble functions, we show some inverse inequalities for the local virtual space $\widetilde{V_h^K}$ on every $K \in \mathcal{T}_h$, of which the definition can be found in Section 3. Similar ideas can be found in [14, 15]. These inverse inequalities are used to prove the interpolation error estimates for the C^0-continuous H^2-nonconforming virtual element.

Lemma A.1 *For every given $K \in \mathcal{T}_h$, it holds that*

$$\|\Delta^2 v\|_K \leq C h_K^{-2} \|\Delta v\|_K, \qquad \forall v \in \widetilde{V_h^K}. \tag{A.1}$$

Proof For a given $K \in \mathcal{T}_h$, let $\lambda_e \in \mathbb{P}_1(K)$ be the function associated with the edge e of K defined by setting $\lambda_e = -\alpha(\mathbf{x}-\mathbf{x}_e)\cdot\mathbf{n}_K/|e|$ such that $\lambda_e = 0$ on e, where the constant $\alpha > 0$ is chosen to make sure $\|\lambda_e\|_{\infty,K} = 1$. b_K is the bubble function on K obtained by multiplying all the edge functions λ_e, so b_K vanishes on ∂K and $b_K \geq 0$ in K where we have also used the fact that K is convex. With the help of b_K, for any given $v \in \widetilde{V_h^K}$, we have

$$C\|\Delta^2 v\|_K^2 \leq (b_K^2\Delta^2 v, \Delta^2 v)_K = (\Delta(b_K^2\Delta^2 v), \Delta v)_K \leq |b_K^2\Delta^2 v|_{2,K}\|\Delta v\|_K.$$

Observing the fact that $\Delta^2 v \in \mathbb{P}_{k-2}(K)$, we use the inverse inequality on polynomial space [7] to obtain

$$\|\Delta^2 v\|_K^2 \leq C h_K^{-2}\|\Delta^2 v\|_K\|\Delta v\|_K,$$

which leads to the inverse inequality (A.1). □

Further, we use the edge bubble function to prove the so-called trace inverse inequality for the local virtual space $\widetilde{V_h^K}$ on every $K \in \mathcal{T}_h$.

Lemma A.2 *For every given $K \in \mathcal{T}_h$ and $e \subset \partial K$, it holds that*

$$\|\Delta v\|_e \leq C h_K^{-\frac{1}{2}}\|\Delta v\|_K, \qquad \forall v \in \widetilde{V_h^K}. \tag{A.2}$$

Proof For any given edge e of K, let $b_e = b_K/\lambda_e$ where b_K is the bubble function on K and λ_e the edge function defined in the proof of Lemma A.1. Obviously it holds that $b_e = 0$ on $\partial K\backslash e$ and $b_e \in \mathbb{P}_{n-1}(K)$ in K, where n is the number of edges of K. For any given $v \in \widetilde{V_h^K}$, the norm equivalence on polynomial space on edge yields

$$\|\Delta v\|_e \leq C\|b_e\Delta v\|_e, \tag{A.3}$$

since $\Delta v \in \mathbb{P}_{k-2}(e)$ on e. Moreover, we observe that

$$\Delta(b_e\Delta v) \in H^{-1}(K) \text{ in } K,\quad b_e\Delta v \in \mathbb{P}_{k+n-3}(e) \text{ on } e,\quad b_e\Delta v = 0 \text{ on } \partial K\backslash e,$$
$$b_e\Delta v \in C^0(\partial K) \text{ on } \partial K.$$

Thus, we have $b_e\Delta v \in H^1(K)$. Further, the trace inequality and the Poincaré-Friedrichs inequality [7] imply

$$\|b_e\Delta v\|_e \leq C h_K^{\frac{1}{2}}|b_e\Delta v|_{1,K},$$

which, together with inequality (A.3), leads to

$$\|\Delta v\|_e \leq C h_K^{\frac{1}{2}}|b_e\Delta v|_{1,K}. \tag{A.4}$$

Let $A = b_e\Delta\widetilde{V_h^K}$ and $w = b_e\Delta v \in A$. Because A is a finite dimensional subspace of $L^2(K)$, it holds

$$C|w|_{1,K}^2 \leq (b_K\nabla w, \nabla w)_K,$$

where the constant C is independent of h_K. Generally, for a finite dimensional subspace $V^K \subset L^2(K)$, it holds the norm equivalence

$$c\|b_K^{\frac{1}{2}}\phi\|_K \leq \|\phi\|_K \leq C\|b_K^{\frac{1}{2}}\phi\|_K, \quad \forall\phi \in V^K.$$

Using the generalized scaling argument, it is easy to show the constants c and C are independent of h_K.

By using integration by parts, we have

$$C|w|_{1,K}^2 \leq (b_K \nabla w, \nabla w)_K = -(\nabla b_K \cdot \nabla w, \nabla w)_K - (b_K \Delta w, w)_K.$$

For Δw, it holds the following identity:

$$\begin{aligned}\Delta w &= \Delta b_e \Delta v + b_e \Delta^2 v + 2\nabla b_e \cdot \nabla(\Delta v)\\ &= \Delta b_e \Delta v + b_e \Delta^2 v + 2b_e^{-1} \nabla b_e \cdot (b_e \nabla(\Delta v))\\ &= \Delta b_e \Delta v + b_e \Delta^2 v + 2b_e^{-1} \nabla b_e \cdot (\nabla w - \Delta v \nabla b_e).\end{aligned}$$

Thus, we have

$$\begin{aligned}C|w|_{1,K}^2 \leq& -(\nabla b_K \cdot \nabla w, w)_K - (b_K \Delta b_e \Delta v, w)_K - (b_K b_e \Delta^2 v, w)_K\\ &-2(\lambda_e \nabla b_e \cdot \nabla w, w)_K + 2(\lambda_e \nabla b_e \cdot \nabla b_e \Delta v, w)_K.\end{aligned}$$

For all the terms on the right hand side of the above inequality, the inverse inequality on polynomial space and Lemma A.1 imply

$$\begin{aligned}&-(\nabla b_K \cdot \nabla w, w)_K \leq C_1 h_K^{-1} |w|_{1,K} \|w\|_K \leq \frac{1}{4} C|w|_{1,K}^2 + C^{-1} C_1^2 h_K^{-2} \|w\|_K^2,\\ &-(b_K \Delta b_e \Delta v, w)_K \leq C_2 h_K^{-2} \|\Delta v\|_K \|w\|_K,\\ &-(b_K b_e \Delta^2 v, w) \leq \|b_K b_e\|_{\infty,K} \|\Delta^2 v\|_K \|w\|_K \leq C_3 h_K^{-2} \|\Delta v\|_K \|w\|_K,\\ &-2(\lambda_e \nabla b_e \cdot \nabla w, w)_K \leq C_4 h_K^{-1} |w|_{1,K} \|w\|_K \leq \frac{1}{4} C|w|_{1,K}^2 + C^{-1} C_4^2 h_K^{-2} \|w\|_K^2,\\ &2(\lambda_e \nabla b_e \cdot \nabla b_e \Delta v, w)_K \leq C_5 h_K^{-2} \|\Delta v\|_K \|w\|_K.\end{aligned}$$

Collecting up the above inequalities, we obtain

$$\begin{aligned}\frac{1}{2} C|w|_{1,K}^2 \leq& C^{-1} C_1^2 h_K^{-2} \|w\|_K^2 + C_2 h_K^{-2} \|\Delta v\|_K \|w\|_K + C_3 h_K^{-2} \|\Delta v\|_K \|w\|_K\\ &+C^{-1} C_4^2 h_K^{-2} \|w\|_K^2 + C_5 h_K^{-2} \|\Delta v\|_K \|w\|_K.\end{aligned}$$

Therefore, observing $w = b_e \Delta v$, we obtain

$$|b_e \Delta v|_{1,K} \leq C h_K^{-1} \|\Delta v\|_K,$$

which, together with inequality (A.4), yields (A.2). □

We introduce the global space for the H^1-conforming virtual element [4] defined by

$$U_h = \{w \in H_0^1(\Omega);\ w|_K \in U_h^K,\ \forall K \in \mathcal{T}_h\},$$

where

$$U_h^K = \{w \in H^2(K);\ \Delta w \in \mathbb{P}_{k-2}(K),\ w|_e \in \mathbb{P}_k(e),\ \forall e \subset \partial K\}.$$

In [4], it has been shown that the following interpolation error estimates holds.

Lemma A.3 *For every $w \in H_0^1(\Omega) \cap H^s(\Omega)$ with $2 \le s \le k+1$, there exists a interpolation function $w_I^c \in U_h$ satisfying*

$$\begin{aligned} w_I^c(a) &= w(a), && \forall \text{ vertex } a, \\ \int_e w_I^c q \mathrm{d}s &= \int_e wq \mathrm{d}s, && \forall q \in \mathcal{M}_{k-2}^e, \ \forall \text{ edge } e, \\ \int_K w_I^c q \mathrm{d}x &= \int_K wq \mathrm{d}x, && \forall q \in \mathcal{M}_{k-2}^K. \end{aligned}$$

Further, it holds

$$\|w - w_I^c\|_{m,K} \le Ch^{s-m}|w|_{s,K}, \quad m = 0, 1, \quad \forall K \in \mathcal{T}_h.$$

Next, we show an inverse inequality for the local virtual space U_h^K on every $K \in \mathcal{T}_h$ by using the bubble function defined in the proof of Lemma A.1.

Lemma A.4 *For every given $K \in \mathcal{T}_h$, it holds that*

$$|v|_{2,K} \le Ch_K^{-1}|v|_{1,K}, \quad \forall v \in U_h^K. \tag{A.5}$$

Proof For any given $K \in \mathcal{T}_h$ and $v \in U_h^K$, let $\phi = \nabla v$, then $|\phi|_{1,K} = |v|_{2,K}$ and $\Delta\phi = \nabla(\Delta v)$. Further, we have

$$C|\phi|_{1,K}^2 \le (b_K \nabla\phi, \nabla\phi)_K = -(\nabla b_K \cdot \nabla\phi, \phi)_K - (b_K \Delta\phi, \phi)_K.$$

Here the constant C in the first inequality can be also shown to be independent of h_K by the similar argument in the proof of Lemma A.2. Thus, the inverse inequality on polynomial space and the fact that $\Delta v \in \mathbb{P}_{k-2}(K)$ imply

$$\begin{aligned} |v|_{2,K}^2 &= |\phi|_{1,K}^2 \\ &\le C(\|\nabla b_K\|_{\infty,K}|\phi|_{1,K} + \|\Delta\phi\|_K)\|\phi\|_K \\ &\le C(h_K^{-1}|\phi|_{1,K} + \|\Delta\phi\|_K)\|\phi\|_K \\ &= C(h_K^{-1}|v|_{2,K} + |\Delta v|_{1,K})|v|_{1,K} \\ &\le Ch_K^{-1}(|v|_{2,K} + \|\Delta v\|_K)|v|_{1,K} \\ &\le Ch_K^{-1}|v|_{2,K}|v|_{1,K}, \end{aligned}$$

which leads to the inverse inequality (A.5). □

With the above preparations, we show an approximation result for the global virtual space $\widetilde{V}_h$ which is defined by the local space $\widetilde{V_h^K}$ as follows:

$$\widetilde{V}_h = \left\{ w \in H_0^1(\Omega);\ w|_K \in \widetilde{V_h^K},\ \forall K \in \mathcal{T}_h,\ \int_e q [\![\frac{\partial w}{\partial \mathbf{n}_e}]\!] \mathrm{d}s = 0,\ \forall q \in \mathbb{P}_{k-2}(e),\ \forall e \in \mathcal{E}_h \right\}.$$

To this end, we define the interpolation $\tilde{w}_I \in \widetilde{V}_h$ for any $w \in H_0^2(\Omega)$ by requiring that the values of the degrees of freedom (3.5)–(3.8) of $\tilde{w}_I$ are equal to the corresponding ones of w. Then we have the following interpolation error estimates.

Lemma A.5 *For every $w \in H_0^2(\Omega) \cap H^s(\Omega)$ with $2 \le s \le k+1$, it holds that*

$$\|w - \tilde{w}_I\|_{m,K} \le Ch^{s-m}|w|_{s,K}, \quad m = 0, 1, 2, \quad \forall K \in \mathcal{T}_h.$$

Proof From Green's formula [19, 22] it holds that

$$\begin{aligned}
|w|_{2,K}^2 &= (\Delta w, \Delta w)_K + \int_K \left(2\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 w}{\partial x_1 \partial x_2} - \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 w}{\partial x_1^2} \right) \mathrm{d}x \\
&= \int_K w \Delta^2 w \mathrm{d}x - \int_{\partial K} w \frac{\partial \Delta w}{\partial \mathbf{n}_K} \mathrm{d}s + \int_{\partial K} \frac{\partial w}{\partial \mathbf{n}_K} \left(\Delta w - \frac{\partial^2 w}{\partial \mathbf{t}_\mathbf{K}^2} \right) \mathrm{d}s \\
&\quad + \int_{\partial K} \frac{\partial w}{\partial \mathbf{t}_K} \frac{\partial^2 w}{\partial \mathbf{t}_K \partial \mathbf{n}_K} \mathrm{d}s,
\end{aligned}$$

where $K \in \mathcal{T}_h$. For the details, see [33, Remark 4.2]. Thus, for any given $K \in \mathcal{T}_h$, we have

$$\begin{aligned}
|w_I^c - \tilde{w}_I|_{2,K}^2 &= \int_K (w_I^c - \tilde{w}_I)\Delta^2(w_I^c - \tilde{w}_I)\mathrm{d}x - \int_{\partial K} (w_I^c - \tilde{w}_I) \frac{\partial \Delta(w_I^c - \tilde{w}_I)}{\partial \mathbf{n}_K} \mathrm{d}s \\
&\quad + \int_{\partial K} \frac{\partial (w_I^c - \tilde{w}_I)}{\partial \mathbf{n}_K} \left(\Delta(w_I^c - \tilde{w}_I) - \frac{\partial^2 (w_I^c - \tilde{w}_I)}{\partial \mathbf{t}_\mathbf{K}^2} \right) \mathrm{d}s \\
&\quad + \int_{\partial K} \frac{\partial (w_I^c - \tilde{w}_I)}{\partial \mathbf{t}_K} \frac{\partial^2 (w_I^c - \tilde{w}_I)}{\partial \mathbf{t}_K \partial \mathbf{n}_K} \mathrm{d}s.
\end{aligned}$$

Observing the fact that w_I^c and $\tilde{w}_I$ belongs to $\mathbb{P}_k(e)$ on each edge e of K and are uniquely determined by the same degrees of freedom (3.5)–(3.6) of w, we have

$$w_I^c - \tilde{w}_I = 0, \quad \frac{\partial (w_I^c - \tilde{w}_I)}{\partial \mathbf{t}_K} = 0, \quad \frac{\partial^2 (w_I^c - \tilde{w}_I)}{\partial \mathbf{t}_\mathbf{K}^2} = 0, \quad \text{on } \partial K.$$

Then we obtain

$$|w_I^c - \tilde{w}_I|_{2,K}^2 = \int_K (w_I^c - \tilde{w}_I)\Delta^2(w_I^c - \tilde{w}_I)\mathrm{d}x + \int_{\partial K} \frac{\partial (w_I^c - \tilde{w}_I)}{\partial \mathbf{n}_K} \Delta(w_I^c - \tilde{w}_I)\mathrm{d}s,$$

which, together with the interpolation properties of $\tilde{w}_I$, implies

$$|w_I^c - \tilde{w}_I|_{2,K}^2 = \int_K (w_I^c - w)\Delta^2(w_I^c - \tilde{w}_I)\mathrm{d}x + \int_{\partial K} \frac{\partial (w_I^c - w)}{\partial \mathbf{n}_K} \Delta(w_I^c - \tilde{w}_I)\mathrm{d}s. \quad \text{(A.6)}$$

For the first term in Eq. (A.6), we use the inverse inequality (A.1) and Lemma A.3 to obtain

$$\begin{aligned}
\int_K (w_I^c - w)\Delta^2(w_I^c - \tilde{w}_I)\mathrm{d}x &\le \|w - w_I^c\|_K \|\Delta^2(w_I^c - \tilde{w}_I)\|_K \\
&\le Ch_K^{-2}\|w - w_I^c\|_K \|\Delta(w_I^c - \tilde{w}_I)\|_K \\
&\le Ch_K^{s-2}|w|_{s,K}|w_I^c - \tilde{w}_I|_{2,K}. \quad \text{(A.7)}
\end{aligned}$$

For the second term in Eq. (A.6), we use the trace inequality, Lemmas A.2–A.4 and Lemma 3.3 to obtain

$$\begin{aligned}\int_e \frac{\partial(w_I^c - w)}{\partial \mathbf{n}_K}\Delta(w_I^c - \tilde{w}_I)\mathrm{d}s &\le \|\frac{\partial(w_I^c - w)}{\partial \mathbf{n}_K}\|_e\|\Delta(w_I^c - \tilde{w}_I)\|_e \\ &\le Ch_K^{-\frac{1}{2}}(h_K^{-\frac{1}{2}}|w_I^c - w|_{1,K} + h_K^{\frac{1}{2}}|w_I^c - w|_{2,K})\|\Delta(w_I^c - \tilde{w}_I)\|_K \\ &\le C(h_K^{-1}|w_I^c - w|_{1,K} + |w_I^c - w_\pi|_{2,K} + |w_\pi - w|_{2,K})|w_I^c - \tilde{w}_I|_{2,K} \\ &\le C(h_K^{-1}|w_I^c - w|_{1,K} + h_K^{-1}|w_I^c - w_\pi|_{1,K} + |w_\pi - w|_{2,K})|w_I^c - \tilde{w}_I|_{2,K} \\ &\le C(h_K^{-1}|w - w_I^c|_{1,K} + h_K^{-1}|w - w_\pi|_{1,K} + |w - w_\pi|_{2,K})|w_I^c - \tilde{w}_I|_{2,K} \\ &\le Ch_K^{s-2}|w|_{s,K}|w_I^c - \tilde{w}_I|_{2,K}, \quad e \subset \partial K. \end{aligned} \tag{A.8}$$

Substituting (A.7)–(A.8) into (A.6), we obtain

$$|w_I^c - \tilde{w}_I|_{2,K} \le Ch_K^{s-2}|w|_{s,K},$$

which, together with the triangle inequality, inverse inequality (A.5), Lemma 3.3, and Lemma A.3, yields

$$\begin{aligned}|w - \tilde{w}_I|_{2,K} &\le |w - w_\pi|_{2,K} + |w_\pi - \tilde{w_I^c}|_{2,K} + |w_I^c - \tilde{w}_I|_{2,K} \\ &\le |w - w_\pi|_{2,K} + Ch_K^{-1}|w_\pi - \tilde{w_I^c}|_{1,K} + |w_I^c - \tilde{w}_I|_{2,K} \\ &\le |w - w_\pi|_{2,K} + Ch_K^{-1}(|w_\pi - w|_{1,K} + |w - w_I^c|_{1,K}) + |w_I^c - \tilde{w}_I|_{2,K} \\ &\le Ch_K^{s-2}|w|_{s,K}.\end{aligned}$$

By using the Poincaré-Friedrichs inequality [7] and the interpolation properties of $\tilde{w}_I$, we further obtain

$$\|w - \tilde{w}_I\|_K \le Ch_K|w - \tilde{w}_I|_{1,K} \le Ch_K^2|w - \tilde{w}_I|_{2,K} \le Ch_K^s|w|_{s,K}.$$

The proof is complete. □

Finally, we show the proof of Lemma 3.2 as follows.

The proof of Lemma 3.2 Observing the fact that the projection operator Π_Δ^K is uniquely determined by the degrees of freedom (3.11)–(3.14) of w_I which are also the degrees of freedom of $\tilde{w}_I$, we have

$$\Pi_\Delta^K \tilde{w}_I = \Pi_\Delta^K w_I = \Pi_\Delta^K w.$$

By similar arguments in the proof of Eq. (A.6), the interpolation properties of $\tilde{w}_I$ and w_I imply

$$|\tilde{w}_I - w_I|_{2,K}^2 = (\tilde{w}_I - w_I, \Delta^2(\tilde{w}_I - w_I))_K.$$

For convenience, let $\phi = \Delta^2(\tilde{w}_I - w_I) \in \mathbb{P}_{k-2}(K)$. Recalling the definition (3.9) of W_h^K, the property of Π_Δ^K, the Poincaré-Friedrichs inequality, and inverse inequality (A.1), we obtain

$$\begin{aligned}
|\tilde{w}_I - w_I|_{2,K}^2 &= (\tilde{w}_I - w_I, \phi - P_{k-4}^K\phi)_K \\
&= (\tilde{w}_I - \Pi_\Delta^K w_I, \phi - P_{k-4}^K\phi)_K \\
&= (\tilde{w}_I - \Pi_\Delta^K \tilde{w}_I, \phi - P_{k-4}^K\phi)_K \\
&\le \|\tilde{w}_I - \Pi_\Delta^K \tilde{w}_I\|_K \|\phi - P_{k-4}^K\phi\|_K \\
&\le Ch_K^2 |\tilde{w}_I - \Pi_\Delta^K \tilde{w}_I|_{2,K} \|\phi\|_K \\
&= Ch_K^2 |(\tilde{w}_I - w_\pi) - \Pi_\Delta^K(\tilde{w}_I - w_\pi|_{2,K} \|\Delta^2(\tilde{w}_I - w_I)\|_K \\
&\le C|\tilde{w}_I - w_\pi|_{2,K} \|\Delta(\tilde{w}_I - w_I)\|_K \\
&\le C(|w - \tilde{w}_I|_{2,K} + |w - w_\pi|_{2,K})|\tilde{w}_I - w_I|_{2,K},
\end{aligned}$$

which, together with Lemma 3.3 and Lemma A.5, leads to

$$|\tilde{w}_I - w_I|_{2,K} \le Ch_K^{s-2}|w|_{s,K}.$$

By using the triangle inequality and Lemma A.5, we obtain

$$|w - w_I|_{2,K} \le Ch_K^{s-2}|w|_{s,K}.$$

By using the Poincaré-Friedrichs inequality [7] and the interpolation property of w_I, we further obtain

$$\|w - w_I\|_K \le Ch_K|w - w_I|_{1,K} \le Ch_K^2|w - w_I|_{2,K} \le Ch_K^s|w|_{s,K}.$$

The proof of Lemma 3.2 is complete.

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