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$\mathcal{H}$-matrix approximability of inverses of discretizations of the fractional Laplacian

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Abstract
The integral version of the fractional Laplacian on a bounded domain is discretized by a Galerkin approximation based on piecewise linear functions on a quasiuniform mesh. We show that the inverse of the associated stiffness matrix can be approximated by blockwise low-rank matrices at an exponential rate in the block rank.

Keywords Hierarchical matrices · Fractional Laplacian

Mathematics Subject Classification (2010) 65N30 · 65F05 · 65F30 · 65F50

1 Introduction

Fractional differential operators are non-local operators with many applications in science and technology and interesting mathematical properties; a discussion of some of their features can be found, e.g., in [51]. The non-local nature of such operators implies for numerical discretizations that the resulting system matrices are fully populated. Efficient matrix compression techniques are therefore necessary. Various data-sparse representations of discretizations of classical integral operators have been proposed in the past. We mention techniques based on multipole expansions,

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panel clustering, wavelet compression techniques, the mosaic-skeleton method, the adaptive cross approximation (ACA) method, and the hybrid cross approximation (HCA); we refer to [23] for a more detailed literature review in the context of classical boundary element methods (BEM). In fact, many of these data-sparse methods may be understood as specific incarnations of $\mathcal{H}$-matrices, which were introduced in [29, 31, 35, 36] as blockwise low-rank matrices. Although many of the abovementioned techniques were originally developed for applications in BEM, the underlying reason for their success is the so-called "asymptotic smoothness" of the kernel function, which is given for a much broader class of problems. We refer to [20] and references therein, where the question of approximability is discussed for pseudodifferential operators. Discretizations of integral versions of the fractional Laplacian such as the one considered in the present paper, (1.6), are therefore amenable to data-sparse representations with $O(N \log^\beta N)$ complexity, where N is the matrix size and $\beta \geq 0$. This compressibility has recently been observed in [61] and in [4], where an analysis and implementation of a panel clustering type matrix-vector multiplication for the stiffness matrix is presented.

The above discussion argued the compressibility of (discretized) fractional differential operators as a result of "asymptotic smoothness" of the kernel of the associated integral operator. Fractional differential operators admit several other representations (see, e.g., [47, Thm. 1.1]), for example, semigroup theoretical characterizations. Then, the Riesz-Dunford calculus may be brought to bear, which allows one to express fractional operators as suitable contour integrals. We note that also inverses of fractional operators can be obtained in this way. Compressed operators are then obtained by approximating the contour integral by a quadrature, i.e., as a sum of operators. For large classes of operators, these quadrature errors are $O(e^{-bM^\beta})$ for some $b, \beta > 0$, where M is the number of quadrature points [26, 40]. In turn, fully discrete numerical schemes can be derived from this by discretizing the M operator appearing in the quadrature. We refer to [8, 9, 21, 41, 42] for more details.

The purpose of the present paper is to show that also the inverse of the stiffness matrix of a discretization of the integral version of the fractional Laplacian can be represented in the $\mathcal{H}$-matrix format, using the same underlying block structure as employed to compress the stiffness matrix. One reason for studying the compressibility of the inverses (or the closely related question of compressibility of the LU-factors) are recent developments in fast (approximate) arithmetic for data-sparse matrix formats. For example, $\mathcal{H}$-matrices come with an (approximate) arithmetic with log-linear complexity, which includes, in particular, the (approximate) inversion and factorization of matrices. These (approximate) inverses/factors could be used either as direct solvers or as preconditioners, as advocated, for example, in a BEM context in [6, 30, 32, 33, 48] and in [49] in the context of fractional differential equations. Although our compressibility result for the inverse of the fractional Laplacian, Theorem 2.5 does not prove that the use of an $\mathcal{H}$-matrix arithmetic will be successful; it indicates that the $\mathcal{H}$-matrix format based on the standard admissibility criterion considered here is a good choice to base the arithmetic on.

We point out that the class of $\mathcal{H}$-matrices is not the only one for which inversion and factorization algorithms have been devised. Related to $\mathcal{H}$-matrices and its arithmetic are "hierarchically semiseparable matrices" [50, 59, 60] and the idea of “recursive skeletonization” [34, 43, 44]; for discretizations of partial differential equations (PDEs), we mention [27, 44, 52, 56], and particular applications to boundary integral equations are [18, 45, 53].

The underlying structure of our proof is similar to that in [23, 24] for the classical single layer and hypersingular operators of BEM. There it is exploited that these operators are traces of potentials; i.e., they are related to functions that solve an elliptic PDE. The connection of [23, 24] with the present article is given by the works of [15, 16, 58], which show that fractional powers of certain elliptic operators posed in $\mathbb{R}^d$ can be realized as the Dirichlet-to-Neumann maps for (degenerate) PDEs posed in $\mathbb{R}^{d+1}$.

1.1 The fractional Laplacian and the Caffarelli-Silvestre extension

In this section, we briefly introduce the fractional Laplacian; the discussion will remain somewhat formal as the pertinent function spaces (e.g., $\widetilde{H}^s(\Omega)$) and lifting operators (e.g., $\mathcal{L}$) will be defined in subsequent sections.

For $s \in (0,1)$, the fractional Laplacian in full space $\mathbb{R}^d$ is classically defined through the Fourier transform, $(-\Delta u)^s := \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)\right)$. As discussed in the survey [47], several equivalent definitions are available. For example, for suitable u, a pointwise characterization is given in terms of a principal value integral:

$$(-\Delta u)^s(x) = C(d,s)\,\text{P.V.}\int_{\mathbb{R}^d}\frac{u(x)-u(y)}{|x-y|^{d+2s}}\,dy, \qquad C(d,s) := -2^{2s}\frac{\Gamma(s+d/2)}{\pi^{d/2}\Gamma(-s)}.$$

The constant $C(d,s)$ and d_s in (1.1) below are given in [13, Thm. 3.1] and [47, Thm. 1.1]. Caffarelli and Silvestre [15] characterized this operator as the Dirichlet-to-Neumann operator of a (degenerate) elliptic PDE. That is, they proved with $d_s := 2^{1-2s}|\Gamma(s)|/\Gamma(1-s)$

$$(-\Delta u)^s(x) = -d_s \lim_{x_{d+1}\to 0+} x_{d+1}^{1-2s}\partial_{x_{d+1}}(\mathcal{L}u)(x, x_{d+1}), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where the *extension* $\mathcal{L}u$ is a function that solves

$$\operatorname{div}(x_{d+1}^{1-2s}\nabla\mathcal{L}u) = 0 \qquad \text{in } \mathbb{R}^{d+1}_+, \qquad \operatorname{tr}\mathcal{L}u = u. \tag{1.2}$$

In (1.1) and (1.2), the half-space is defined as $\mathbb{R}^{d+1}_+ := \{(x, x_{d+1}) \mid x \in \mathbb{R}^d, x_{d+1} > 0\}$, its boundary $\mathbb{R}^d \times \{0\}$ is identified with $\mathbb{R}^d$, and tr denotes the trace operator. In weak form, the combination of (1.1) and (1.2) therefore yields

$$d_s \int_{\mathbb{R}^{d+1}_+} x_{d+1}^{1-2s}\nabla\mathcal{L}u\cdot\nabla\mathcal{L}v = \int_{\mathbb{R}^d}(-\Delta u)^s v \qquad \forall v \in C_0^\infty(\mathbb{R}^{d+1}). \tag{1.3}$$

For suitable u and v belonging to Sobolev spaces as dictated by the double integral (1.5), we also have (cf. [47, Thm. 1.1, (e), (g)])

$$d_s \int_{\mathbb{R}^{d+1}_+} x_{d+1}^{1-2s}\nabla\mathcal{L}u\cdot\nabla\mathcal{L}v = \int_{\mathbb{R}^d}(-\Delta u)^s v \tag{1.4}$$

$$= \frac{C(d,s)}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,dx\,dy, \tag{1.5}$$

which is a form that is amenable to Galerkin discretizations.

The fractional Laplacian on a *bounded* domain $\Omega \subset \mathbb{R}^d$ can be defined in one of several *non-equivalent* ways. We consider the *integral fractional Laplacian* with the exterior "boundary" condition $u \equiv 0$ in Ω^c, which reads, cf., e.g., the discussions in [2, 51]

$$(-\Delta u)^s_I(x) = C(d,s)\text{P.V.} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,dy, \qquad x \in \Omega \tag{1.6}$$

and the understanding that $u = 0$ on Ω^c. Important for the further developments is that this version of the fractional Laplacian still admits the interpretation (1.1) as a Dirichlet-to-Neumann map for arguments $u \in \widetilde{H}^s(\Omega)$, where

$$\widetilde{H}^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^d) \mid u \equiv 0 \text{ on } \mathbb{R}^d \setminus \overline{\Omega} \right\}. \tag{1.7}$$

For a measurable subset M of $\mathbb{R}^d$, we will use standard Lebesgue and Sobolev spaces $L^2(M)$ and $H^1(M)$. Sobolev spaces of non-integer order $s \in (0,1)$ are defined via the Sobolev-Slobodecki norms

$$\|u\|^2_{H^s(M)} = \|u\|^2_{L^2(M)} + |u|^2_{H^s(M)} = \|u\|^2_{L^2(M)} + \int_M \int_M \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dxdy.$$

More generally, if necessary, we will identify subsets $\omega \subset \mathbb{R}^d$ with $\omega \times \{0\} \subset \mathbb{R}^{d+1}$. In particular, for $u, v \in \widetilde{H}^s(\Omega)$, the representations (1.4) and (1.5) are both valid.

2 Main results

2.1 Model problem and discretization

For a polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $s \in (0,1)$, we are interested in calculating the trace u on $\Omega \subset \mathbb{R}^d$ of a function $\mathbf{u}$ defined on $\mathbb{R}^{d+1}_+$, where $\mathbf{u}$ solves

$$\begin{aligned} -\text{div}\left(x_{d+1}^{1-2s}\nabla \mathbf{u}\right) &= 0 \quad \text{in } \mathbb{R}^{d+1}_+, \\ -\lim_{x_{d+1}\to 0+} x_{d+1}^{1-2s}\partial_{x_{d+1}}\mathbf{u} &= f \quad \text{on } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \mathbb{R}^d \setminus \Omega. \end{aligned} \tag{2.8}$$

Our variational formulation of (2.8) is based on the spaces $\widetilde{H}^s(\Omega)$: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$\int_{\mathbb{R}^{d+1}_+} x_{d+1}^{1-2s} \nabla \mathcal{L}u \cdot \nabla \mathcal{L}v\,dx = \int_\Omega f v\,dx \quad \text{for all } v \in \widetilde{H}^s(\Omega). \tag{2.9}$$

Here, $\mathcal{L}$ is the *harmonic extension* operator associated with the PDE given in (2.8). It has already appeared in (1.2) and is formally defined in (3.18). We will show in Section 3 ahead that the left-hand side of the above equation introduces a bounded and elliptic bilinear form. Hence, the Lax-Milgram Lemma proves that the variational formulation (2.9) is well posed. Given a quasiuniform mesh $\mathcal{T}_h$ on Ω with mesh width h, we discretize problem (2.9) using the conforming finite element space

$$\mathcal{S}^1_0(\mathcal{T}_h) := \left\{ u \in C(\mathbb{R}^d) \mid \text{supp}\, u \subset \overline{\Omega} \text{ and } u|_K \in \mathcal{P}_1 \forall K \in \mathcal{T}_h \right\} \subset \widetilde{H}^s(\Omega),$$

where $\mathcal{P}_1$ denotes the space of polynomials of degree 1. We emphasize that $\mathcal{S}_0^1(\mathcal{T}_h)$ is the “standard” space of piecewise linear functions on Ω that are extended by zero outside Ω. Obviously, there is a unique solution $u_h \in \mathcal{S}_0^1(\mathcal{T}_h)$ of the linear system

$$\int_{\mathbb{R}_+^{d+1}} x_{d+1}^{1-2s} \nabla \mathcal{L} u_h \cdot \nabla \mathcal{L} v_h \, dx = \int_\Omega f v_h \, dx \quad \text{for all } v_h \in \mathcal{S}_0^1(\mathcal{T}_h). \tag{2.10}$$

If we consider the nodal basis $(\psi_j)_{j=1}^N$ of $\mathcal{S}_0^1(\mathcal{T}_h)$, we can write Eq. (2.10) as

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Our goal is to derive an $\mathcal{H}$-matrix approximation of the inverse $\mathbf{A}^{-1}$.

Remark 2.1 Computationally, the bilinear form (2.10) is not easily accessible. In the present paper, we use it only as a theoretical tool. For computational purposes, one possibility is to employ (1.5). For this representation of the bilinear form, the entries of the stiffness matrix $\mathbf{A}$ can be computed [1, 4].

2.2 Blockwise low-rank approximation

Let us introduce the necessary notation. Let $\mathcal{I} = \{1, \ldots, N\}$ be the set of indices of the nodal basis $(\psi_j)_{j=1}^N$ of $\mathcal{S}_0^1(\mathcal{T}_h)$. A *cluster* τ is a subset of $\mathcal{I}$. For a cluster τ, we say that $B_{R_\tau}^0 \subset \mathbb{R}^d$ is a *bounding box* if

(i) $B_{R_\tau}^0$ is a hypercube with side length R_τ,
(ii) $\operatorname{supp}(\psi_j) \subset B_{R_\tau}^0$ for all $j \in \tau$.

For an *admissibility parameter* $\eta > 0$, a pair of clusters (τ, σ) is called η-admissible, if there exist bounding boxes $B_{R_\tau}^0$ of τ and $B_{R_\sigma}^0$ of σ such that

$$\max\left\{\operatorname{diam}(B_{R_\tau}^0), \operatorname{diam}(B_{R_\sigma}^0)\right\} \leq \eta \operatorname{dist}\left(B_{R_\tau}^0, B_{R_\sigma}^0\right). \tag{2.11}$$

For a partition P of $\mathcal{I} \times \mathcal{I}$ and an admissibility parameter $\eta > 0$, a matrix $\mathbf{B}_\mathcal{H} \in \mathbb{R}^{N \times N}$ is said to be a *blockwise rank-r* matrix if for every η-admissible cluster pair (τ, σ), the block $\mathbf{B}_\mathcal{H}|_{\tau \times \sigma}$ is of rank r. The next theorem is the first main result of this work. For two admissible clusters, the associated matrix block of the inverse $\mathbf{A}^{-1}$ of the matrix associated to the linear system of problem (2.8) can be approximated by low-rank matrices with an error that is exponentially small in the rank.

Theorem 2.2 *Let $\eta > 0$ be a fixed admissibility parameter and $q \in (0, 1)$. Let (τ, σ) be a cluster pair with η-admissible bounding boxes. Then, for each $k \in \mathbb{N}$, there exist matrices $\mathbf{X}_{\tau\sigma} \in \mathbb{R}^{|\tau| \times r}$ and $\mathbf{Y}_{\tau\sigma} \in \mathbb{R}^{|\sigma| \times r}$ with rank $r \leq C_{\dim}(2+\eta)^{d+1} q^{-(d+1)} k^{d+2}$ such that*

$$\|\mathbf{A}^{-1}|_{\tau \times \sigma} - \mathbf{X}_{\tau\sigma} \mathbf{Y}_{\tau\sigma}^\top\|_2 \leq C_{\text{apx}} N^{\frac{1+d}{d}} q^k. \tag{2.12}$$

The constants $C_{\dim}$ and C_{apx} depend only on d, Ω, the shape regularity of $\mathcal{T}_h$, and on s.

Theorem 2.2 shows that individual blocks of $\mathbf{A}^{-1}$ can be approximated by low-rank matrices. $\mathcal{H}$-matrices are blockwise low-rank matrices where the blocks are organized in a tree structure, which affords the fast arithmetic of $\mathcal{H}$-matrices. The block cluster tree is based on a tree structure for the index set $\mathcal{I}$, which we described next.

Definition 2.3 (**cluster tree**) A *cluster tree* with *leaf size* $n_{\text{leaf}} \in \mathbb{N}$ is a binary tree $\mathbb{T}_{\mathcal{I}}$ with root $\mathcal{I}$ such that for each cluster $\tau \in \mathbb{T}_{\mathcal{I}}$ the following dichotomy holds: either τ is a leaf of the tree and $|\tau| \leq n_{\text{leaf}}$ or there exist two so-called sons $\tau', \tau'' \in \mathbb{T}_{\mathcal{I}}$ that are disjoint subsets of τ with $\tau = \tau' \dot{\cup} \tau''$. The *level function* level : $\mathbb{T}_{\mathcal{I}} \to \mathbb{N}_0$ is inductively defined by $\text{level}(\mathcal{I}) = 0$ and $\text{level}(\tau') := \text{level}(\tau) + 1$ for τ' a son of τ. The *depth* of a cluster tree is $\text{depth}(\mathbb{T}_{\mathcal{I}}) := \max_{\tau \in \mathbb{T}_{\mathcal{I}}} \text{level}(\tau)$.

Definition 2.4 (**far field, near field, and sparsity constant**) A partition P of $\mathcal{I} \times \mathcal{I}$ is said to be based on the cluster tree $\mathbb{T}_{\mathcal{I}}$, if $P \subset \mathbb{T}_{\mathcal{I}} \times \mathbb{T}_{\mathcal{I}}$. For such a partition P and fixed admissibility parameter $\eta > 0$, we define the *far field* and the *near field* as

$$P_{\text{far}} := \{(\tau, \sigma) \in P \ : \ (\tau, \sigma) \text{ is } \eta\text{-admissible}\}, \quad P_{\text{near}} := P \backslash P_{\text{far}}. \tag{2.13}$$

The *sparsity constant* C_{sp}, introduced in [29, 38, 39], of such a partition is defined by

$$C_{\text{sp}} := \max \left\{ \max_{\tau \in \mathbb{T}_{\mathcal{I}}} |\{\sigma \in \mathbb{T}_{\mathcal{I}} \ : \ \tau \times \sigma \in P_{\text{far}}\}|, \ \max_{\sigma \in \mathbb{T}_{\mathcal{I}}} |\{\tau \in \mathbb{T}_{\mathcal{I}} \ : \ \tau \times \sigma \in P_{\text{far}}\}| \right\}. \tag{2.14}$$

The following Theorem 2.5 shows that the matrix $\mathbf{A}^{-1}$ can be approximated by blockwise rank-r matrices at an exponential rate in the block rank r:

Theorem 2.5 *Fix the admissibility parameter $\eta > 0$. Let a partition P of $\mathcal{I} \times \mathcal{I}$ be based on a cluster tree $\mathbb{T}_{\mathcal{I}}$. Then, there is a blockwise rank-r matrix $\mathbf{B}_{\mathcal{H}}$ such that*

$$\|\mathbf{A}^{-1} - \mathbf{B}_{\mathcal{H}}\|_2 \leq C_{\text{apx}} C_{\text{sp}} N^{(d+1)/d} \text{depth}(\mathbb{T}_{\mathcal{I}}) e^{-br^{1/(d+2)}}. \tag{2.15}$$

The constant C_{apx} depends only on Ω, d, the shape regularity of the quasiuniform triangulation $\mathcal{T}_h$, and on s, while the constant $b > 0$ additionally depends on η.

Proof As it is shown in [29], [36, Lemma 6.32], norm bounds for a block matrix that is based on a cluster tree can be inferred from norm bounds for the blocks. This allows one to prove Theorem 2.5 based on the results of Theorem 2.2 (see, e.g., the proof of [22, Thm. 2] for details). □

Remark 2.6 For quasiuniform meshes with $\mathcal{O}(N)$ elements, typical clustering strategies such as the "geometric clustering" described in [36] lead to fairly balanced cluster trees $\mathbb{T}_{\mathcal{I}}$ with $\text{depth}\,\mathbb{T}_{\mathcal{I}} = \mathcal{O}(\log N)$ and a sparsity constant C_{sp} that is bounded uniformly in N. We refer to [29, 36, 38, 39] for the fact that the memory requirement to store $\mathbf{B}_{\mathcal{H}}$ is $\mathcal{O}\big((r + n_{\text{leaf}}) N \log N\big)$.

3 The Beppo-Levi space $\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$

In the present section, we formulate a functional framework for the lifting operator $\mathcal{L}$ of (1.2). We will work in the Beppo-Levi space

$$\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+) := \left\{ \mathbf{u} \in C_0^\infty(\mathbb{R}^{d+1}_+)' \mid \nabla \mathbf{u} \in L^2_\alpha(\mathbb{R}^{d+1}_+) \right\}$$

of all distributions $C_0^\infty(\mathbb{R}^{d+1}_+)'$ having all first-order partial derivatives in $L^2_\alpha(\mathbb{R}^{d+1}_+)$ for

$$\alpha = 1 - 2s \in (-1, 1), \tag{3.16}$$

where $L^2_\alpha(\mathbb{R}^{d+1}_+)$ is defined as the set of measurable functions $\mathbf{u}$ such that

$$\|\mathbf{u}\|^2_{L^2_\alpha(\mathbb{R}^{d+1}_+)} = \int_{\mathbb{R}^{d+1}_+} x^\alpha_{d+1} |\mathbf{u}(x)|^2 \, dx < \infty.$$

We denote by $L^2_{\alpha,\text{bdd}}(\mathbb{R}^{d+1}_+)$ the set of functions that are in L^2_α on every bounded subset of $\mathbb{R}^{d+1}_+$. By $\text{tr} : C^\infty(\overline{\mathbb{R}^{d+1}_+}) \to C^\infty(\mathbb{R}^d)$, we denote the trace operator $(\text{tr}\, \mathbf{u})(x_1, \ldots, x_d) := \mathbf{u}(x_1, \ldots, x_d, 0)$. The following result, which is an extension to weighted spaces of the well-known result [19, Cor. 2.1], shows that the distributions in $\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$ are actually functions. Its proof will be given below in Section 3.1.

Lemma 3.1 *For* $\alpha \in (-1, 1)$*, there holds* $\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+) \subset L^2_{\alpha,\text{bdd}}(\mathbb{R}^{d+1}_+)$*. Furthermore, for* $\alpha \in [0, 1)$ *one has* $\mathbf{u} \in L^2_{0,\text{bdd}}(\mathbb{R}^{d+1}_+)$.

We additionally define the space

$$\mathcal{B}^s(\mathbb{R}^d) := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^d) \mid |u|_{H^s(\mathbb{R}^d)} < \infty \right\}.$$

From now on, we fix a hypercube $K := K' \times (0, b_{d+1})$, $K' = \prod_{j=1}^d (a_j, b_j)$. Then, using Lemma 3.1, one can show that $\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$ and $\mathcal{B}^s(\mathbb{R}^d)$ are Hilbert spaces when endowed with the norms

$$\begin{aligned} \|\mathbf{u}\|^2_{\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)} &:= \|\mathbf{u}\|^2_{L^2_\alpha(K)} + \|\nabla \mathbf{u}\|^2_{L^2_\alpha(\mathbb{R}^{d+1}_+)}, \\ \|u\|^2_{\mathcal{B}^s(\mathbb{R}^d)} &:= \|u\|^2_{L^2(K')} + |u|^2_{H^s(\mathbb{R}^d)}. \end{aligned} \tag{3.17}$$

There holds the following density result, which can be found for bounded domains in [46, Thm. 11.11] even for higher Sobolev regularity. In the present case of first-order regularity and unbounded domains, we give a short proof below in Section 3.1.

Lemma 3.2 *For* $\alpha \in (-1, 1)$*, the set* $C^\infty(\overline{\mathbb{R}^{d+1}_+}) \cap \mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$ *is dense in* $\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$.

The trace operator can be extended to the spaces $\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$ as will also be shown below in Section 3.1. Analogous trace theorems in Sobolev spaces on smooth and bounded domains are given for $s = 1/2$ in [14, Prop. 1.8] and for $s \in (0, 1) \setminus \frac{1}{2}$ in [17, Prop. 2.1].

Lemma 3.3 *For $\alpha \in (-1, 1)$, the trace operator is a bounded linear operator* $\operatorname{tr} : \mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+) \to \mathcal{B}^s(\mathbb{R}^d)$, *where s is given by (3.16).*

We define the Hilbert space $\mathcal{B}^1_{\alpha,0}(\mathbb{R}^{d+1}_+) := \ker(\operatorname{tr})$. The following Poincaré inequality holds on this space. The proof will be given below in Section 3.1.

Corollary 3.4 *For all $\mathbf{u} \in \mathcal{B}^1_{\alpha,0}(\mathbb{R}^{d+1}_+)$, there holds $\|\mathbf{u}\|_{\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)} \lesssim \|\nabla \mathbf{u}\|_{L^2_\alpha(\mathbb{R}^{d+1}_+)}$.*

For a function $u \in H^s(\mathbb{R}^d)$, we define the *minimum norm extension* or *harmonic extension* $\mathcal{L}u \in \mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$ as

$$\mathcal{L}u = \operatorname*{arg\,min}_{\substack{\mathbf{u}\in\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)\\ \operatorname{tr}\mathbf{u}=u}} \|\nabla \mathbf{u}\|_{L^2_\alpha(\mathbb{R}^{d+1}_+)}. \tag{3.18}$$

We can characterize $\mathcal{L}u$ by

$$\begin{aligned} \int_{\mathbb{R}^{d+1}_+} x_{d+1}^{1-2s} \nabla \mathcal{L}u \cdot \nabla \mathbf{v}\, dx &= 0 \quad \text{for all } \mathbf{v} \in \mathcal{B}^1_{\alpha,0}(\mathbb{R}^{d+1}_+),\\ \operatorname{tr}\mathcal{L}u &= u. \end{aligned} \tag{3.19}$$

In view of the previous developments, the minimum norm extension exists uniquely and satisfies

$$\|\mathcal{L}u\|_{\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)} \lesssim \|u\|_{H^s(\mathbb{R}^d)}. \tag{3.20}$$

Indeed, using the extension operator $\mathcal{E}$ from Lemma 3.9, which is a right inverse of the trace operator tr, the minimum norm extension can be written as $\mathcal{L}u = \mathcal{E}u + \mathbf{u}$ and $\mathbf{u} \in \mathcal{B}^1_{\alpha,0}(\mathbb{R}^{d+1}_+)$ is given by

$$\int_{\mathbb{R}^{d+1}_+} x_{d+1}^{1-2s} \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx = -\int_{\mathbb{R}^{d+1}_+} x_{d+1}^{1-2s} \nabla \mathcal{E}u \cdot \nabla \mathbf{v}\, dx \quad \text{for all } \mathbf{v} \in \mathcal{B}^1_{\alpha,0}(\mathbb{R}^{d+1}_+).$$

This equation is uniquely solvable due to the Lax-Milgram theorem and Corollary 3.4, and this also implies the stability (3.20). Due to (3.19), we see that a variational form of our original problem (2.8) is actually given by (2.9). Next, we show that problem (2.9) is well posed. We mention that ellipticity has already been shown in [15, eq. (3.7)] using Fourier methods.

Lemma 3.5 *Problem (2.9) has a unique solution $u \in \widetilde{H}^s(\Omega)$, and*

$$\|u\|_{H^s(\mathbb{R}^d)} \lesssim \|f\|_{H^{-s}(\Omega)},$$

where $H^{-s}(\Omega)$ is the dual space of $\widetilde{H}^s(\Omega)$.

Proof Due to [2, Prop. 2.4], there holds the Poincaré inequality $\|u\|_{L^2(\Omega)} \lesssim |u|_{H^s(\mathbb{R}^d)}$ for all $u \in \widetilde{H}^s(\Omega)$. We conclude that $\|u\|_{H^s(\mathbb{R}^d)} \lesssim |u|_{H^s(\mathbb{R}^d)}$ for all $u \in \widetilde{H}^s(\Omega)$. Combining this Poincaré inequality with the trace estimate (3.28), we obtain the ellipticity of the bilinear form on the left-hand side of (2.9). The continuity of this bilinear form follows from (3.20). □

3.1 Technical details and proofs

Define the Sobolev space $H^1_\alpha(\mathbb{R}^{d+1}_+)$ as the space of functions $\mathbf{u}$ such that

$$\|\mathbf{u}\|^2_{L^2_\alpha(\mathbb{R}^{d+1}_+)} + \|\nabla \mathbf{u}\|^2_{L^2_\alpha(\mathbb{R}^{d+1}_+)} < \infty.$$

We start with a density result, whose proof is based on ideas from [46, Thm. 11.11].

Lemma 3.6 *For $\alpha \in (-1, 1)$, the space $C^\infty(\overline{\mathbb{R}^{d+1}_+}) \cap H^1_\alpha(\mathbb{R}^{d+1}_+)$ is dense in $H^1_\alpha(\mathbb{R}^{d+1}_+)$.*

Proof By [28, Thm. 1], the space $C^\infty(\mathbb{R}^{d+1}_+) \cap H^1_\alpha(\mathbb{R}^{d+1}_+)$ is dense in $H^1_\alpha(\mathbb{R}^{d+1}_+)$. Hence, without loss of generality, we may assume that $\mathbf{u} \in C^\infty(\mathbb{R}^{d+1}_+) \cap H^1_\alpha(\mathbb{R}^{d+1}_+)$. For $h > 0$, define the function $\mathbf{u}_h$ by

$$\mathbf{u}_h(x_1, \ldots, x_{d+1}) := \begin{cases} \mathbf{u}(x_1, \ldots, x_{d+1}) & \text{if } h < x_{d+1} \\ \mathbf{u}(x_1, \ldots, x_d, h) & \text{if } x_{d+1} \leq h. \end{cases}$$

By construction, $\mathbf{u}_h \in C(\overline{\mathbb{R}^{d+1}_+}) \cap H^1_\alpha(\mathbb{R}^{d+1}_+)$ and

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|^2_{H^1_\alpha(\mathbb{R}^{d+1}_+)} &= \|\mathbf{u} - \mathbf{u}_h\|^2_{H^1_\alpha(\mathbb{R}^d \times (0,h))} \\ &= \|\mathbf{u} - \mathbf{u}_h\|^2_{L^2_\alpha(\mathbb{R}^d \times (0,h))} + \|\nabla \mathbf{u}\|^2_{L^2_\alpha(\mathbb{R}^d \times (0,h))} \\ &\lesssim \|\mathbf{u}\|^2_{H^1_\alpha(\mathbb{R}^d \times (0,h))} + \|\mathbf{u}_h\|^2_{L^2_\alpha(\mathbb{R}^d \times (0,h))}. \end{aligned} \tag{3.21}$$

By Lebesgue dominated convergence, we have $\lim_{h\to 0} \|\mathbf{u}\|_{H^1_\alpha(\mathbb{R}^d \times (0,h))} = 0$. Hence, we focus on showing $\lim_{h\to 0} \|\mathbf{u}_h\|_{L^2_\alpha(\mathbb{R}^d \times (0,h))} = 0$. To that end, we use a 1D trace inequality: For $v \in C^1(0, \infty)$, we have $v(h) = v(y) - \int_h^y v'(t)\,dt$ so that

$$\begin{aligned} \int_{y=h}^{h+1} y^\alpha v^2(h)\,dy &\leq 2\int_{y=h}^{h+1} y^\alpha v^2(y)\,dy + 2\int_{y=h}^{h+1} y^\alpha \left|\int_{t=h}^{y} |v'(t)|\,dt\right|^2 dy \\ &\lesssim \|v\|^2_{L^2_\alpha(h,h+1)} + \int_{y=h}^{h+1} y^\alpha y^{1-\alpha} \int_{t=h}^{h+1} t^\alpha |v'(t)|^2\,dt\,dy \\ &\lesssim \|v\|^2_{L^2_\alpha(h,h+1)} + \|v'\|^2_{L^2_\alpha(h,h+1)}. \end{aligned}$$

Since there exists $C > 0$ such that for $h \in (0, 1]$, we have $C^{-1} \leq \int_h^{h+1} t^\alpha\,dt \leq C$, we can conclude

$$|v(h)|^2 \leq C^2_{\text{trace}} \left[\|v\|^2_{L^2_\alpha(h,h+1)} + \|v'\|^2_{L^2_\alpha(h,h+1)}\right]. \tag{3.22}$$

With this, we estimate

$$\begin{aligned} \|\mathbf{u}_h\|^2_{L^2_\alpha(\mathbb{R}^d \times (0,h))} &= \int_0^h x_{d+1}^\alpha \int_{y\in\mathbb{R}^d} \mathbf{u}_h(y,h)^2\,dy\,dx_{d+1} = \frac{h^{\alpha+1}}{\alpha+1} \int_{y\in\mathbb{R}^d} \mathbf{u}(y,h)^2\,dy \\ &\overset{(3.21)}{\lesssim} C^2_{\text{trace}} h^{\alpha+1} \|\mathbf{u}\|^2_{H^1_\alpha(\mathbb{R}^d \times (h,h+1))} \lesssim C^2_{\text{trace}} h^{\alpha+1} \|\mathbf{u}\|^2_{H^1_\alpha(\mathbb{R}^{d+1}_+)}. \end{aligned}$$

As $\alpha + 1 > 0$, we conclude that $\lim_{h\to} \|\mathbf{u}_h\|_{L^2_\alpha(\mathbb{R}^d\times(0,h))} = 0$. Since $\mathbf{u}_h$ is only piecewise smooth, we perform, as a last step, a mollification step. The above shows that, given $\varepsilon > 0$, we can fix h such that

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1_\alpha\left(\mathbb{R}^{d+1}_+\right)} \leq \varepsilon. \tag{3.23}$$

Let $\chi_h \in C^\infty(\mathbb{R}^{d+1})$ be of the form $\chi_h(x_1, \ldots, x_{d+1}) = \widetilde{\chi}_h(x_{d+1})$ for some $\widetilde{\chi}_h \in C^\infty(\mathbb{R})$ with supp $\widetilde{\chi}_h \subset ([-h/2, h/2])$ and $\widetilde{\chi}_h \equiv 1$ on $[0, h/4]$. For $\delta > 0$ define $\widetilde{\mathbf{u}}_\delta := \chi_h\mathbf{u}_h + [(1-\chi_h)\mathbf{u}_h] \star \rho_\delta$ for a mollifier ρ_δ. Then, $\widetilde{\mathbf{u}}_\delta \in C^\infty(\overline{\mathbb{R}^{d+1}_+}) \cap H^1_\alpha(\mathbb{R}^{d+1}_+)$, cf. [28], and

$$\|\mathbf{u}_h - \widetilde{\mathbf{u}}_\delta\|_{H^1_\alpha\left(\mathbb{R}^{d+1}_+\right)} = \|(1-\chi_h)\mathbf{u}_h - [(1-\chi_h)\mathbf{u}_h] * \rho_\delta\|_{H^1_\alpha(\mathbb{R}^{d+1})}. \tag{3.24}$$

Note that h is already fixed. Standard results about mollification, cf., e.g., [28], show that the term $\|(1-\chi_h)\mathbf{u}_h - [(1-\chi_h)\mathbf{u}_h] \star \rho_\delta\|_{H^1_\alpha(\mathbb{R}^{d+1})}$ converges to zero for $\delta \to 0$. Hence, choosing δ small enough, we obtain from (3.23) and (3.24) that $\|\mathbf{u} - \widetilde{\mathbf{u}}_\delta\|_{H^1_\alpha\left(\mathbb{R}^{d+1}_+\right)} \leq 2\varepsilon$, which proves the result.

□

Next, we show that the trace operator tr extends continuously to weighted Sobolev spaces.

Lemma 3.7 *Let $\alpha \in (-1, 1)$. The trace operator* tr *has a unique extension as a bounded linear operator $H^1_\alpha(\mathbb{R}^{d+1}_+) \to L^2(\mathbb{R}^d)$, and there holds the trace inequality*

$$\|\operatorname{tr} \mathbf{u}\|_{L^2(\Omega)} \leq C_{tr}\left(\|\mathbf{u}\|_{L^2_\alpha(\Omega_+)} + \|\mathbf{u}\|^{(1-\alpha)/2}_{L^2_\alpha(\Omega_+)} \cdot \|\partial_{d+1}\mathbf{u}\|^{(1+\alpha)/2}_{L^2_\alpha(\Omega_+)}\right) \tag{3.25}$$

for all measurable subsets $\Omega \subseteq \mathbb{R}^d$, where $\Omega_+ := \Omega \times (0, \infty)$. The constant C_{tr} does not depend on Ω. If the support of $\mathbf{u}$ *is contained in a strip $\mathbb{R}^d \times [0, b_{d+1}]$ with $b_{d+1} > 1$, then there holds the multiplicative trace inequality*

$$\|\operatorname{tr} \mathbf{u}\|_{L^2(\Omega)} \leq C_{tr}\|\mathbf{u}\|^{(1-\alpha)/2}_{L^2_\alpha(\Omega_+)} \cdot \|\partial_{d+1}\mathbf{u}\|^{(1+\alpha)/2}_{L^2_\alpha(\Omega_+)}. \tag{3.26}$$

Proof In order to prove all statements of the lemma, we note that due to Lemma 3.6, it is sufficient to show the multiplicative estimate (3.26) for smooth functions $\mathbf{u} \in C^\infty(\overline{\mathbb{R}^{d+1}_+}) \cap H^1_\alpha(\mathbb{R}^{d+1}_+)$. The estimate (3.25) then follows with the aid of (3.25) by multiplication with an appropriate cut-off function. Using the abbreviation $v(x) = \mathbf{u}(x_1, \ldots, x_d, x)$, we note that due to Hölder's inequality

$$|v(0)| \leq |v(y)| + |\textstyle\int_0^y v'(t)\,dt| \lesssim |v(y)| + y^{(1-\alpha)/2}\|v'\|_{L^2_\alpha(\mathbb{R}_+)}.$$

A one-dimensional trace inequality and a scaling argument show for $y > 0$

$$|v(y)|^2 \lesssim y^{-1}\textstyle\int_y^{2y} |v(t)|^2\,dt + y\int_y^{2y} |v'(t)|^2\,dt.$$

For $t \in (y, 2y)$, we have $1 \leq y^{-\alpha}t^\alpha$ if $\alpha \in [0, 1)$ and $1 \leq 2^{-\alpha}y^{-\alpha}t^\alpha$ if $\alpha \in (-1, 0)$, and we conclude

$$|v(y)|^2 \lesssim y^{-1-\alpha}\|v\|^2_{L^2_\alpha(\mathbb{R}_+)} + y^{1-\alpha}\|v'\|^2_{L^2_\alpha(\mathbb{R}_+)}.$$

If $\|v'\|_{L^2_\alpha(\mathbb{R}_+)} \neq 0$, we select $y = \|v\|_{L^2_\alpha(\mathbb{R}_+)} \cdot \|v'\|^{-1}_{L^2_\alpha(\mathbb{R}_+)}$ and get

$$|v(0)|^2 \lesssim \|v\|^{1-\alpha}_{L^2_\alpha(\mathbb{R}_+)} \cdot \|v'\|^{1+\alpha}_{L^2_\alpha(\mathbb{R}_+)}. \tag{3.27}$$

We note that (3.27) is also valid if $\|v'\|_{L^2_\alpha(\mathbb{R}_+)} = 0$ since our assumption $\operatorname{supp} \mathbf{u} \subseteq \mathbb{R}^d \times [0, b_{d+1}]$ implies in this degenerate case $v \equiv 0$. Integrating $\mathbf{u}(\cdot, 0)$ over Ω and using (3.27) shows (3.26). □

Lemma 3.8 *Let $\alpha \in (-1, 1)$ and s be given by (3.16). The trace operator* tr *is bounded as* $\operatorname{tr} : H^1_\alpha(\mathbb{R}^{d+1}_+) \to H^s(\mathbb{R}^d)$*, and*

$$|\operatorname{tr} \mathbf{u}|_{H^s(\mathbb{R}^d)} \lesssim \|\nabla \mathbf{u}\|_{L^2_\alpha(\mathbb{R}^{d+1}_+)}. \tag{3.28}$$

Proof Due to Lemma 3.6, it suffices to show (3.28) for $\mathbf{u} \in C^\infty(\overline{\mathbb{R}^{d+1}_+}) \cap H^1_\alpha(\mathbb{R}^{d+1}_+)$. Combining (3.28) with Lemma 3.7 then proves that $\operatorname{tr} : H^1_\alpha(\mathbb{R}^{d+1}_+) \to H^s(\mathbb{R}^d)$ is bounded. Upon writing $y = x + r\phi$ with polar coordinates $r > 0$, $\phi \in S^{d-1} := \partial B_1(0) \subset \mathbb{R}^d$, we obtain with the triangle inequality and symmetry arguments

$$\begin{aligned} |\operatorname{tr} \mathbf{u}|^2_{H^s(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\mathbf{u}(x,0) - \mathbf{u}(y,0)|^2}{|x-y|^{d+2s}} \, dy \, dx \\ &\lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\mathbf{u}(\frac{x+y}{2}, \frac{|x-y|}{2}) - \mathbf{u}(x,0)|^2}{|x-y|^{d+2s}} \, dy \, dx \\ &\sim \int_{x \in \mathbb{R}^d} \int_{\phi \in S^{d-1}} \int_{r=0}^\infty \frac{|\mathbf{u}(x + \frac{r}{2}\phi, \frac{r}{2}) - \mathbf{u}(x,0)|^2}{r^{1+2s}} \, dr \, d\phi \, dx. \end{aligned}$$

The fundamental theorem of calculus gives

$$\mathbf{u}(x + \tfrac{r}{2}\phi, \tfrac{r}{2}) - \mathbf{u}(x,0) = \int_0^r \nabla_x \mathbf{u}(x + \tfrac{y}{2}\phi, \tfrac{y}{2}) \cdot \phi + \partial_{d+1} \mathbf{u}(x + \tfrac{y}{2}\phi, \tfrac{y}{2}) \, dy,$$

and the weighted Hardy inequality from [62, I, Thm. 9.16] (cf. also [54, (1.1)]) then implies

$$\int_{r=0}^\infty \frac{|\mathbf{u}(x + \frac{r}{2}\phi, \frac{r}{2}) - \mathbf{u}(x,0)|^2}{r^{1+2s}} \lesssim \int_{r=0}^\infty r^{1-2s} |\nabla_x \mathbf{u}(x + \tfrac{r}{2}\phi, \tfrac{r}{2}) \cdot \phi \\ + \partial_{d+1} \mathbf{u}(x + \tfrac{r}{2}\phi, \tfrac{r}{2})|^2.$$

Hence,

$$|\operatorname{tr} \mathbf{u}|^2_{H^s(\mathbb{R}^d)} \lesssim \int_{x \in \mathbb{R}^d} \int_{\phi \in S^{d-1}} \int_{r=0}^\infty r^{1-2s} |\nabla \mathbf{u}(x + \tfrac{r}{2}\phi, \tfrac{r}{2})|^2 \, dr \, d\phi \, dx \lesssim \|\nabla \mathbf{u}\|^2_{L^2_\alpha(\mathbb{R}^{d+1}_+)},$$

which proves (3.28). □

Next, we will show that the trace operator $\operatorname{tr} : H^1_\alpha(\mathbb{R}^{d+1}_+) \to H^s(\mathbb{R}^d)$ is actually onto. To that end, we generalize ideas from [25].

Lemma 3.9 *Let $\alpha \in (-1,1)$ and s be given by (3.16). There exists a bounded linear operator $\mathcal{E} : H^s(\mathbb{R}^d) \to H^1_\alpha(\mathbb{R}^{d+1}_+)$ that is a right inverse of the trace operator* tr. *Furthermore, there exists a constant $C > 0$ such that for all $h > 0$*

$$\|\mathcal{E}u\|_{L^2_\alpha(\mathbb{R}^d\times(0,h))} \leq Ch^{1-s}\|u\|_{L^2(\mathbb{R}^d)}.$$

Proof Let $\rho \in C_0^\infty(\mathbb{R}^d)$ and $\eta \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\eta \subset (-1,1)$ and $\eta \equiv 1$ in $(-1/2, 1/2)$. We denote a point in $\mathbb{R}^{d+1}_+$ by (x,t) with $x \in \mathbb{R}^d$. Define the extension operator as the mollification $\mathcal{E}u(x,t) := \eta(t)(\rho_t \star u)(x)$, where $\rho_t(y) := t^{-d}\rho(y/t)$. Since $\|\rho_t \star u\|_{L^2(\mathbb{R}^d)} \lesssim \|u\|_{L^2(\mathbb{R}^d)}$ uniformly in $t > 0$ (cf., e.g., [3, Thm. 2.29]), we immediately obtain the postulated estimate

$$\|\mathcal{E}u\|^2_{L^2_\alpha(\mathbb{R}^d\times(0,h))} \leq \|\eta\|^2_{L^\infty}\int_0^h t^\alpha\|\rho_t\star u\|^2_{L^2(\mathbb{R}^d)}\,dt \lesssim h^{2(1-s)}\|u\|^2_{L^2(\mathbb{R}^d)}.$$

Since η is compactly supported, this also shows $\|\mathcal{E}u\|_{L^2_\alpha(\mathbb{R}^{d+1}_+)} \lesssim \|u\|_{L^2(\mathbb{R}^d)}$. For the desired statement that $\mathcal{E} : H^s(\mathbb{R}^d) \to H^1_\alpha(\mathbb{R}^{d+1}_+)$ is bounded, it suffices to prove

$$\|\nabla_x(\rho_t\star u)\|^2_{L^2_\alpha(\mathbb{R}^{d+1}_+)} + \|\partial_t(\rho_t\star u)\|^2_{L^2_\alpha(\mathbb{R}^{d+1}_+)} \lesssim |u|^2_{H^s(\mathbb{R}^d)}.$$

To that end, we calculate

$$\partial_t(\rho_t\star u)(x) = -\,dt^{-d-1}\int_{\mathbb{R}^d} u(y)\rho\left(\frac{x-y}{t}\right)dy - t^{-d-2}\int_{\mathbb{R}^d} u(y)\nabla\rho\left(\frac{x-y}{t}\right)\cdot(x-y)dy.$$

Integration by parts shows $\int_{\mathbb{R}^d}\nabla\rho(z)\cdot z\,dz = -\,d$, which yields

$$-\,dt^{-d-1}\int_{\mathbb{R}^d}\rho\left(\frac{x-y}{t}\right)dy = -\,dt^{-1} = t^{-d-2}\int_{\mathbb{R}^d}\nabla\rho\left(\frac{x-y}{t}\right)\cdot(x-y)\,dy.$$

Hence, we can write

$$\begin{aligned}\partial_t(\rho_t\star u)(x) \;=\; & dt^{-d-1}\textstyle\int_{\mathbb{R}^d}[u(y)-u(x)]\rho\left(\frac{x-y}{t}\right)dy\\ & -t^{-d-2}\textstyle\int_{\mathbb{R}^d}[u(y)-u(x)]\nabla\rho\left(\frac{x-y}{t}\right)\cdot(x-y)\,dy.\end{aligned}$$

Next, we calculate for $1 \leq j \leq d$

$$\partial_{x_j}(\rho_t\star u)(x) = t^{-d-1}\int_{\mathbb{R}^d} u(y)\left(\partial_{x_j}\rho\right)\left(\frac{x-y}{t}\right)dy.$$

Integration by parts also shows that $\int(\partial_{x_j}\rho)(z)\,dz = 0$, which yields

$$\partial_{x_j}(\rho_t\star u)(x) = t^{-d-1}\int_{\mathbb{R}^d}[u(y)-u(x)]\left(\partial_{x_j}\rho\right)\left(\frac{x-y}{t}\right)dy.$$

Due to the support properties of ρ, we conclude

$$|\partial_t(\rho_t\star u)(x)| + |\nabla_x(\rho_t\star u)(x)| \lesssim t^{-d-1}\int_{B_t(x)}|u(x)-u(y)|\,dy,$$

where $B_r(x) \subset \mathbb{R}^d$ denotes the ball of radius t centered at x. Using polar coordinates and Hardy's inequality gives

$$\begin{aligned}\int_0^\infty t^\alpha \left(|\partial_t(\rho_t \star u)(x)|^2 + |\nabla_x(\rho_t \star u)(x)|^2\right) dt &\lesssim \int_0^\infty t^\alpha \left(t^{-d-1}\int_{B_t(x)} |u(y)-u(x)|\,dy\right)^2 dt \\ &\leq \int_0^\infty \left(t^{-1}\int_{B_t(x)} \frac{|u(y)-u(x)|}{|x-y|^{d-\alpha/2}}\,dy\right)^2 dt \\ &= \int_0^\infty \left(t^{-1}\int_{B_t(0)} \frac{|u(x)-u(x-z)|}{|z|^{d-\alpha/2}}\,dz\right)^2 dt \\ &= \int_0^\infty \left(t^{-1}\int_{r=0}^{t}\int_{\phi\in S^{s-1}} \frac{|u(x)-u(x-r\phi))|}{r^{1-\alpha/2}}\,d\phi\,dr\right)^2 dt \\ &\lesssim \int_{t=0}^\infty \left(\int_{\phi\in S^{d-1}} \frac{|u(x)-u(x-t\phi))|}{t^{1-\alpha/2}}\,d\phi\right)^2 dt \\ &\lesssim \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dy.\end{aligned}$$

Integrating this estimate over $x \in \mathbb{R}^d$ concludes the proof. □

We are in position to prove Lemma 3.1.

Proof of Lemma 3.1 The proof follows a standard procedure. Since it involves functions in a half-space, we present some details.

Step 1: Let $\rho \in C_0^\infty(\mathbb{R}^{d+1})$ be a symmetric, non-negative function with $\operatorname{supp}\rho \subset B_1(0)$ and set $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(x/\varepsilon)$. Introduce the translation operator τ_h by $\tau_h\varphi(x) := \varphi(x - he_{d+1})$ with $e_{d+1} = (0,0,\ldots,0,1) \in \mathbb{R}^{d+1}$. Define for $\varepsilon > 0$ the smoothing operator $\mathcal{A}_\varepsilon$ by $\mathcal{A}_\varepsilon\varphi = \rho_\varepsilon \star (\tau_{2\varepsilon}\varphi)$ and the regularized distribution $\mathbf{u}_\varepsilon$ by

$$\langle \mathbf{u}_\varepsilon, \varphi\rangle := \langle \mathbf{u}, \mathcal{A}_\varepsilon\varphi\rangle = \langle \mathbf{u}, \rho_\varepsilon \star (\tau_{2\varepsilon}\varphi)\rangle,$$

where we view $\varphi \in C_0^\infty(\mathbb{R}_+^{d+1})$ as an element of $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$ in the canonical way. Note that $\mathbf{u}_\varepsilon \in C^\infty(\mathbb{R}_+^{d+1})$ by standard arguments and $\operatorname{supp}\mathbf{u}_\varepsilon \subset \mathbb{R}^d \times (\varepsilon, \infty)$. We also note that

$$\lim_{\varepsilon\to 0}\langle \mathbf{u}_\varepsilon, \varphi\rangle = \langle \mathbf{u}, \varphi\rangle \qquad \forall \varphi \in C_0^\infty(\mathbb{R}_+^{d+1}). \tag{3.29}$$

Step 2: For $\alpha \in [0, 1)$, we claim

$$\|x_{d+1}^{-\alpha/2}\mathcal{A}_\varepsilon(x_{d+1}^{\alpha/2}\varphi)\|_{L^2(\mathbb{R}_+^{d+1})} \leq \|\varphi\|_{L^2(\mathbb{R}_+^{d+1})} \qquad \forall \varphi \in C_0^\infty(\mathbb{R}_+^{d+1}). \tag{3.30}$$

To see this, we start by noting

$$\sup_{\substack{(x,z) \\ x>\varepsilon,\, -\varepsilon<z<\varepsilon,\, x-2\varepsilon-z>0}} x^{-\alpha/2}(x-2\varepsilon-z)^{\alpha/2} \leq 1. \tag{3.31}$$

We observe $x_{d+1}^{\alpha/2}\varphi \in C_0^\infty(\mathbb{R}_+^{d+1})$ and supp $\mathcal{A}_\varepsilon(x_{d+1}^{\alpha/2}\varphi) \subset \mathbb{R}^d \times (\varepsilon, \infty)$ and write

$$x_{d+1}^{-\alpha/2}\mathcal{A}_\varepsilon(x_{d+1}^{\alpha/2}\varphi)(x) = x_{d+1}^{-\alpha/2}\int_{z\in B_\varepsilon(0)} \rho_\varepsilon(z)(x_{d+1} - 2\varepsilon - z_{d+1})^{\alpha/2}\varphi(x - 2\varepsilon e_{d+1} - z)\,dz.$$

From (3.31) and $\rho_\varepsilon \geq 0$, we get

$$\|x_{d+1}^{-\alpha/2}\mathcal{A}_\varepsilon(x_{d+1}^{\alpha/2}\varphi)\|_{L^2(\mathbb{R}_+^{d+1})} \leq \|\mathcal{A}_\varepsilon(|\varphi|)\|_{L^2(\mathbb{R}_+^{d+1})} \leq \|\varphi\|_{L^2(\mathbb{R}_+^{d+1})}.$$

Step 3: For $\alpha \in [0, 1)$, we have for every $\varepsilon > 0$

$$\|\nabla \mathbf{u}_\varepsilon\|_{L^2_\alpha(\mathbb{R}_+^{d+1})} \leq C\|\nabla \mathbf{u}\|_{L^2_\alpha(\mathbb{R}_+^{d+1})}. \tag{3.32}$$

To see (3.32), fix a bounded open $\omega \subset \mathbb{R}_+^{d+1}$. We compute for $\varphi \in C_0^\infty(\omega)$ and $\varepsilon > 0$, noting that $x_{d+1}^{\alpha/2}\varphi \in C_0^\infty(\omega)$,

$$\begin{aligned}
\left|\langle x_{d+1}^{\alpha/2}\nabla \mathbf{u}_\varepsilon, \varphi\rangle\right| &= \left|\langle \nabla \mathbf{u}_\varepsilon, x_{d+1}^{\alpha/2}\varphi\rangle\right| = \left|-\langle \mathbf{u}_\varepsilon, \nabla(x_{d+1}^{\alpha/2}\varphi)\rangle\right| \\
&= \left|-\langle \mathbf{u}, \mathcal{A}_\varepsilon\nabla(x_{d+1}^{\alpha/2}\varphi)\rangle\right| = \left|-\langle \mathbf{u}, \nabla(\mathcal{A}_\varepsilon(x_{d+1}^{\alpha/2}\varphi))\rangle\right| \\
&= \left|\langle \nabla \mathbf{u}, \mathcal{A}_\varepsilon(x_{d+1}^{\alpha/2}\varphi)\rangle\right| \\
&\leq \|\nabla \mathbf{u}\|_{L^2_\alpha(\mathbb{R}_+^{d+1})}\|x_{d+1}^{-\alpha/2}\mathcal{A}_\varepsilon(x_{d+1}^{\alpha/2}\varphi)\|_{L^2(\mathbb{R}_+^{d+1})} \\
&\overset{\text{Step 2}}{\leq} \|\nabla \mathbf{u}\|_{L^2_\alpha(\mathbb{R}_+^{d+1})}\|\varphi\|_{L^2(\omega)}.
\end{aligned}$$

Combining this with the observation

$$\|x_{d+1}^{\alpha/2}\nabla \mathbf{u}_\varepsilon\|_{L^2(\omega)} = \sup_{\varphi\in(C_0^\infty(\omega))^d} \frac{\langle x_{d+1}^{\alpha/2}\nabla \mathbf{u}_\varepsilon, \varphi\rangle}{\|\varphi\|_{L^2(\omega)}} \tag{3.33}$$

gives us $\|\nabla \mathbf{u}_\varepsilon\|_{L^2_\alpha(\omega)} \leq \|\nabla \mathbf{u}\|_{L^2_\alpha(\mathbb{R}_+^{d+1})}$. The claim (3.32) now follows since ω is arbitrary.

Step 4: For $\alpha \in (-1, 0]$, we have for every bounded open $\omega \subset \mathbb{R}_+^{d+1}$ the existence of $C_\omega > 0$ such that for $\varepsilon \in (0, 1]$

$$\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\omega)} \leq C_\omega\|\nabla \mathbf{u}\|_{L^2_\alpha(\mathbb{R}_+^{d+1})}.$$

The proof follows by inspecting the procedure of step 3 and essentially using step 2 with $\alpha = 0$ there.

Step 5: Steps 3 and 4 show that $\mathbf{u} \in H^1_{\text{loc}}(\mathbb{R}_+^{d+1})$: Fix a bounded, open, and connected $\omega \subset \mathbb{R}_+^{d+1}$ with $\varpi \subset \mathbb{R}_+^{d+1}$. Fix a $\varphi \in C_0^\infty(\omega)$ with $(\varphi, 1)_{L^2(\omega)} \neq 0$. Exploiting the norm equivalence

$$\|v\|_{H^1(\omega)} \sim \|\nabla v\|_{L^2(\omega)} + |(\varphi, v)_{L^2(\omega)}| \qquad \forall v \in H^1(\omega),$$

we infer from steps 3 and 4, and the observation $\lim_{\varepsilon\to 0}(\mathbf{u}_\varepsilon, \varphi)_{L^2(\omega)} = \langle \mathbf{u}, \varphi\rangle$ that $(\mathbf{u}_\varepsilon)_{\varepsilon\in(0,1]}$ is uniformly bounded in $H^1(\omega)$. Thus, a subsequence converges weakly in

$H^1(\omega)$ and strongly in $L^2(\omega)$ to a limit, which is the representation of the distribution $\mathbf{u}$ on ω.

Step 6: Claim: For any bounded open $\omega \subset \mathbb{R}^{d+1}_+$, we have $\mathbf{u} \in L^2_\alpha(\omega)$. It suffices to show norm bounds for bounded open sets of the form $\omega = \omega^0 \times (0, 1)$ with $\omega^0 \subset \mathbb{R}^d$. For that, consider again the regularized functions $\mathbf{u}_\varepsilon$ and assume, additionally (with the aid of a cut-off function), that $\mathbf{u}_\varepsilon(x, x_{d+1}) = 0$ for $x_{d+1} \geq 1$ and $x \in \mathbb{R}^d$. Then for $x_{d+1} \in (0, 1)$, we have

$$\mathbf{u}_\varepsilon(x, x_{d+1}) = -\int_{x_{d+1}}^1 \partial_{d+1}\mathbf{u}_\varepsilon(x, t)\,dt. \tag{3.34}$$

For $\alpha \in (-1, 0]$, we square, multiply by x^α_{d+1}, and integrate to get

$$\|\mathbf{u}_\varepsilon\|^2_{L^2_\alpha(\omega)} \leq C\|\partial_{d+1}\mathbf{u}_\varepsilon\|^2_{L^2(\omega)}.$$

Since $\|\partial_{d+1}\mathbf{u}_\varepsilon\|_{L^2(\omega)}$ can be controlled uniformly in $\varepsilon \in (0, 1]$ by steps 4 and 5, the proof is complete for $\alpha \in (-1, 0]$. For $\alpha \in [0, 1)$, we square (3.34), use a Cauchy-Schwarz inequality on the right-hand side, and integrate to get

$$\|\mathbf{u}_\varepsilon\|^2_{L^2(\omega)} \leq C_\alpha\|\partial_{d+1}\mathbf{u}_\varepsilon\|^2_{L^2(\omega)}.$$

Again, steps 3 and 5 allow us to control the right-hand side uniformly in ε. □

Proof of Lemma 3.2 We choose an open cover $(U_j)_{j\in\mathbb{N}}$ of $\mathbb{R}^{d+1}_+$ by bounded sets and a partition of unity $(\psi_j)_{j\in\mathbb{N}}$ subordinate to this cover. For $\mathbf{u} \in \mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$, we have $\mathbf{u}\psi_j \in H^1_\alpha(\mathbb{R}^{d+1}_+)$, and according to Lemma 3.6, $\mathbf{u}\psi_j$ can be approximated to arbitrary accuracy by a function $\varphi_j \in C^\infty(\overline{\mathbb{R}^{d+1}_+}) \cap H^1_\alpha(\mathbb{R}^{d+1}_+)$ in the norm of $H^1_\alpha(\mathbb{R}^{d+1}_+)$ and hence also in the norm of $\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$. By construction, only a finite number of φ_j overlap, and hence, $\sum_{j=0}^\infty \varphi_j \in C^\infty(\overline{\mathbb{R}^{d+1}_+}) \cap \mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)$. □

Proof of Lemma 3.3 Using an appropriate cut-off function, this is a simple consequence of Lemmas 3.2, 3.7, and 3.8. □

Proof of Corollary 3.4 Due to the density result of Lemma 3.2 and the definition of the trace operator, it suffices to show

$$\|\mathbf{u}\|_{L^2_\alpha(K)} \lesssim \|\operatorname{tr}\mathbf{u}\|_{L^2(K')} + \|\nabla\mathbf{u}\|_{L^2_\alpha(K)}$$

for all $\mathbf{u} \in C^\infty(\overline{\mathbb{R}^{d+1}_+})$. Using the abbreviation $v(x) = \mathbf{u}(x_1, \ldots, x_d, x)$, we note that due to Hölder's inequality,

$$|v(x)|^2 \lesssim |v(0)|^2 + |\textstyle\int_0^x v'(t)\,dt|^2 \lesssim |v(0)|^2 + x^{1-\alpha}\|v'\|^2_{L^2_\alpha(0,b_{d+1})}.$$

Multiplying the last equation by x^α and integrating over K finishes the proof. □

4 $\mathcal{H}$-matrix approximability

For any subset $D \subset \mathbb{R}^{d+1}_+$, define the space

$$\mathcal{H}_h(D) := \Big\{\mathbf{u} \in H^1_\alpha(D) \mid \exists \widetilde{\mathbf{u}} \in \mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+), \text{tr } \widetilde{\mathbf{u}} \in \mathcal{S}^1_0(\mathcal{T}_h) \subset \widetilde{H}^s(\Omega) \text{ s.t. } \mathbf{u}|_D = \widetilde{\mathbf{u}}|_D, \\ a(\widetilde{\mathbf{u}}, \mathbf{v}) = 0 \text{ for all } \mathbf{v} \in \mathcal{B}^1_{\alpha,0}(\mathbb{R}^{d+1}_+)\Big\}$$

and the space with additional orthogonality

$$\mathcal{H}_{h,0}(D) := \Big\{\mathbf{u} \in H^1_\alpha(D) \mid \exists \widetilde{\mathbf{u}} \in \mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+), \text{tr } \widetilde{\mathbf{u}} \in \mathcal{S}^1_0(\mathcal{T}_h) \subset \widetilde{H}^s(\Omega) \text{ s.t. } \mathbf{u}|_D = \widetilde{\mathbf{u}}|_D, \\ a(\widetilde{\mathbf{u}}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{B}^1_{\alpha,0}(\mathbb{R}^{d+1}_+), \text{ and} \\ a(\widetilde{\mathbf{u}}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+), \text{ tr } \mathbf{v} \in \mathcal{S}^1_0(\mathcal{T}_h) \subset \widetilde{H}^s(\Omega), \text{ supp(tr } \mathbf{v}) \subset \overline{D} \cap \mathbb{R}^d\Big\}.$$

where we define the bilinear form

$$a(\mathbf{u}, \mathbf{v}) := \int_{\mathbb{R}^{d+1}_+} x^\alpha_{d+1} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx.$$

Define the cubes with side length R (henceforth called "box") by

$$B_R := B^0_R \times (0, R) \subset \mathbb{R}^{d+1}. \tag{4.35}$$

We say that two boxes B_{R_1} and B_{R_2} are concentric if their projections on $\mathbb{R}^d$, i.e., the corresponding cubes $B^0_{R_1}$ and $B^0_{R_2}$, share the same barycenter and are concentric. For $h > 0$, we define on $H^1_\alpha(B_R)$ the norm

$$|||\mathbf{u}|||^2_{h,R} := \left(\frac{h}{R}\right)^2 \|\nabla \mathbf{u}\|^2_{L^2_\alpha(B_R)} + \frac{1}{R^2} \|\mathbf{u}\|^2_{L^2_\alpha(B_R)}.$$

We have the following Caccioppoli-type inequality.

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Let $R \in (0, 8\,\mathrm{diam}(\Omega))$, $\delta \in (0, 1)$, and $h > 0$ such that $16h \le \delta R$. Let B_R and $B_{(1+\delta)R}$ be two concentric boxes. Then, there exists a constant $C > 0$ depending only on Ω, d, the γ-shape regularity of $\mathcal{T}_h$, and s (i.e., α) such that for all $\mathbf{u} \in \mathcal{H}_{h,0}(B_{(1+\delta)R})$, there holds*

$$\|\nabla \mathbf{u}\|_{L^2_\alpha(B_R)} \le C \frac{1+\delta}{\delta} |||\mathbf{u}|||_{h,(1+\delta)R}. \tag{4.36}$$

Proof In the proof, various boxes will appear. They will always be assumed to be concentric to B_R. Choose a function $\eta \in W^{1,\infty}(\mathbb{R}^{d+1})$ with $(\text{tr }\eta)|_\Omega \in \mathcal{S}^1_0(\mathcal{T}_h)$, $\eta \equiv 1$ on B_R, $\text{supp}(\eta) \subset B_{(1+\delta/4)R}$, $0 \le \eta \le 1$, and $\|\nabla \eta\|_\infty \lesssim (\delta R)^{-1}$. We calculate

$$\|\nabla \mathbf{u}\|^2_{L^2_\alpha(B_R)} \le \|\nabla(\eta \mathbf{u})\|^2_{L^2_\alpha(B_{(1+\delta)R})} = \int_{B_{(1+\delta)R}} x^\alpha_{d+1} \nabla(\eta \mathbf{u}) \cdot \nabla(\eta \mathbf{u}) \, dx \\ = \int_{B_{(1+\delta)R}} x^\alpha_{d+1} \mathbf{u}^2 \, (\nabla \eta)^2 \, dx + \int_{B_{(1+\delta)R}} x^\alpha_{d+1} \nabla \mathbf{u} \cdot \nabla(\eta^2 \mathbf{u}) \, dx.$$

We first deal with the last integral on the right-hand side. Due to the support properties of η and the orthogonality properties of space $\mathcal{H}_{h,0}(B_{(1+\delta)R})$, we see

$$\begin{aligned}\int_{B_{(1+\delta)R}} x_{d+1}^{\alpha} \nabla \mathbf{u} \cdot \nabla(\eta^2 \mathbf{u})\,dx &= \int_{(1+\delta)R} x_{d+1}^{\alpha} \nabla \mathbf{u} \cdot \nabla\left(\widetilde{\eta}\mathcal{E}(\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u}))\right) dx \\ &\leq \|\nabla \mathbf{u}\|_{L^2_\alpha(B_{(1+\delta)R})} \cdot \|\nabla\left(\widetilde{\eta}\mathcal{E}(\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u}))\right)\|_{L^2_\alpha(B_{(1+\delta)R})}\end{aligned}$$

where $\widetilde{\eta}$ is a cut-off function with support contained $B^0_{(1+3\delta/4)R} \times (0, 3\delta R/4)$ and $\widetilde{\eta} \equiv 1$ on $B^0_{(1+\delta/2)R} \times (0, \delta R/2)$ such that $\|\nabla\widetilde{\eta}\|_\infty \lesssim (\delta R)^{-1}$. Furthermore, $I_h : C(\overline{\Omega}) \cap H^1_0(\Omega) \to S^1_0(\mathcal{T}_h)$ is the usual nodal interpolation operator (extended by zero outside Ω). Then, using Lemma 3.9, we obtain

$$\begin{aligned}&\|\nabla\left(\widetilde{\eta}\mathcal{E}(\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u}))\right)\|_{L^2_\alpha(B_{(1+\delta)R})} \\ &\quad\leq \|\nabla\left(\mathcal{E}(\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u}))\right)\|_{L^2_\alpha(B_{(1+\delta)R})} \\ &\quad+ (\delta R)^{-1}\,\|\mathcal{E}(\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u}))\|_{L^2_\alpha(B^0_{(1+3\delta/4)R}\times(0,3\delta R/4))} \\ &\quad\lesssim \|\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u})\|_{H^s(\mathbb{R}^d)} + (\delta R)^{-s}\,\|\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u})\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

For $r \in [0, 1]$, it holds

$$\|\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u})\|^2_{H^r(\mathbb{R}^d)} \lesssim h^{4-2r} \sum_{K\in\mathcal{T}_h} |\mathrm{tr}(\eta^2\mathbf{u})|^2_{H^2(K)}, \tag{4.37}$$

and a short calculation, cf. [22], and an inverse estimate show that

$$\begin{aligned}|\mathrm{tr}(\eta^2\mathbf{u})|^2_{H^2(K)} &\lesssim \frac{1}{(\delta R)^2}\|\nabla\,\mathrm{tr}(\eta\mathbf{u})\|^2_{L^2(K)} + \frac{1}{(\delta R)^4}\|\mathrm{tr}\,\mathbf{u}\|^2_{L^2(K)} \\ &\lesssim \frac{h^{2s-2}}{(\delta R)^2}\|\mathrm{tr}(\eta\mathbf{u})\|^2_{H^s(K)} + \frac{1}{(\delta R)^4}\|\mathrm{tr}\,\mathbf{u}\|^2_{L^2(K)}.\end{aligned}$$

By the support properties of η, the sum in (4.37) extends over elements $K \cap B_{(1+\delta/4)R} \neq \emptyset$. As $h \leq (\delta R)/16$, it holds $\bigcup_{K\cap B^0_{(1+\delta/4)R}\neq\emptyset} K \subset B^0_{(1+\delta/2)R}$. Then, using $h/(\delta R) \leq 1$, we conclude that

$$\begin{aligned}\|\nabla\left(\widetilde{\eta}\mathcal{E}(\mathrm{tr}(\eta^2\mathbf{u}) - I_h\,\mathrm{tr}(\eta^2\mathbf{u}))\right)\|_{L^2_\alpha(B_{(1+\delta)R})} &\lesssim \frac{h}{(\delta R)}\|\mathrm{tr}(\eta\mathbf{u})\|_{H^s(\mathbb{R}^d)} \\ &\quad+\frac{h^{2-s}}{(\delta R)^2}\|\mathrm{tr}\,\mathbf{u}\|_{L^2(B^0_{(1+\delta/2)R})}.\end{aligned}$$

Choosing a cut-off function η_2 with $\eta_2 \equiv 1$ on $B_{(1+\delta/2)R}$ and support contained in $B_{(1+3\delta/4)R}$ and employing the multiplicative trace inequality from Lemma 3.7, we see

$$\begin{aligned}\|\mathrm{tr}\,\mathbf{u}\|_{L^2(B^0_{(1+\delta/2)R})} &\leq \|\mathrm{tr}(\eta_2\mathbf{u})\|_{L^2(B^0_{(1+3\delta/4)R})} \leq C_{\mathrm{tr}}\|\eta_2\mathbf{u}\|^s_{L^2_\alpha(\mathbb{R}^{d+1})}\|\nabla(\eta_2\mathbf{u})\|^{1-s}_{L^2_\alpha(\mathbb{R}^{d+1})} \\ &\lesssim \frac{1}{(\delta R)^{1-s}}\|\mathbf{u}\|_{L^2_\alpha(B_{(1+\delta)R})} + \|\mathbf{u}\|^s_{L^2_\alpha(B_{(1+\delta)R})}\|\nabla\mathbf{u}\|^{1-s}_{L^2_\alpha(B_{(1+\delta)R})}.\end{aligned}$$

Together with the boundedness of the trace operator asserted in Lemma 3.8, i.e., $\|\operatorname{tr}(\eta\mathbf{u})\|_{H^s(\mathbb{R}^d)} \lesssim \|\mathbf{u}\|_{L^2_\alpha(B_{(1+\delta)R})} + \|\nabla(\eta\mathbf{u})\|_{L^2_\alpha(B_{(1+\delta)R})}$, this implies

$$\begin{aligned}\int_{B_{(1+\delta)R}} x_{d+1}^\alpha \nabla\mathbf{u}\cdot\nabla(\eta^2\mathbf{u})\,dx \lesssim \|\nabla\mathbf{u}\|_{L^2_\alpha(B_{(1+\delta)R})}\cdot\Big(&\frac{h}{\delta R}\|\mathbf{u}\|_{L^2_\alpha(B_{(1+\delta)R})}\\ &+\frac{h}{\delta R}\|\nabla(\eta\mathbf{u})\|_{L^2_\alpha(B_{(1+\delta)R})} + \frac{h^{2-s}}{(\delta R)^{2+(1-s)}}\|\mathbf{u}\|_{L^2_\alpha(B_{(1+\delta)R})}\\ &+\frac{h^{2-s}}{(\delta R)^2}\|\mathbf{u}\|^s_{L^2_\alpha(B_{(1+\delta)R})}\|\nabla\mathbf{u}\|^{1-s}_{L^2_\alpha(B_{(1+\delta)R})}\Big).\end{aligned}$$

The four products on the right-hand side are estimated with Young's inequality: the first three ones using the form $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ and the last one with exponents $\frac{2}{2-s}$ and $\frac{2}{s}$. We conclude that there are positive constants C_1 and C_2 such that

$$\begin{aligned}\|\nabla(\eta\mathbf{u})\|^2_{L^2_\alpha(B_{(1+\delta)R})} \le\; & C_1\frac{1}{(\delta R)^2}\|\mathbf{u}\|^2_{L^2_\alpha(B_{(1+\delta)R})}\\ &+C_2\frac{h^2}{(\delta R)^2}\|\nabla\mathbf{u}\|^2_{L^2_\alpha(B_{(1+\delta)R})} + \frac{1}{2}\|\nabla(\eta\mathbf{u})\|^2_{L^2_\alpha(B_{(1+\delta)R})}.\end{aligned}$$

Subtracting the last term from the left-hand side finishes the proof. □

Denote by $\Pi_{h,R} : \big(H^1_\alpha(B_R), |||\cdot|||_{h,R}\big) \to \big(\mathcal{H}_{h,0}(B_R), |||\cdot|||_{h,R}\big)$ the orthogonal projection. For $\mathcal{K}_H$ a shape regular triangulation of $\mathbb{R}^{d+1}_+$ of mesh width H, denote by $\Pi_H : H^1_\alpha(\mathbb{R}^{d+1}_+) \to \mathcal{S}^1(\mathcal{K}_H)$ the quasi-interpolation operator from [55], defined as

$$\Pi_H v := \sum_{\text{nodes } z \text{ of } \mathcal{K}_H} Q^0_z v(z)\phi_z,$$

where ϕ_z denotes the nodal basis functions corresponding to the node z of $\mathcal{K}_H$, and $Q^0_z v$ is an averaged Taylor polynomial of order 0 of v about the node z. This operator has the local first-order approximation property

$$\|v - \Pi_H v\|_{L^2_\alpha(K)} \lesssim H\|\nabla v\|_{L^2_\alpha(\omega_K)}, \tag{4.38}$$

where ω_K is the patch of elements touching the element K, cf. [55, Thm. 5.2].

Lemma 4.2 *Let $\delta \in (0,1)$ and $R \in (0, 4\operatorname{diam}(\Omega))$ such that $16h \le \delta R$. Let B_R, $B_{(1+\delta)R}$, and $B_{(1+2\delta)R}$ be three concentric boxes. Let $\mathbf{u} \in \mathcal{H}_{h,0}(B_{(1+2\delta)R})$ and suppose that $16H \le \delta R$. Let $\eta \in C^\infty_0(\mathbb{R}^{d+1})$ with supp$(\eta) \subset B_{(1+\delta)R}$ and $\eta = 1$ on B_R. Then it holds*

(i) $(\mathbf{u} - \Pi_{h,R}\Pi_H(\eta\mathbf{u}))|_{B_R} \in \mathcal{H}_{h,0}(B_R)$;
(ii) $|||\mathbf{u} - \Pi_{h,R}\Pi_H(\eta\mathbf{u})|||_{h,R} \le C_{\text{app}}\frac{1+2\delta}{\delta}\left(\frac{h}{R} + \frac{H}{R}\right)|||u|||_{h,(1+2\delta)R}$;
(iii) $\dim W \le C_{\text{app}}\left(\frac{(1+2\delta)R}{H}\right)^{d+1}$, *where* $W := \Pi_{h,R}\Pi_H\eta\mathcal{H}_{h,0}(B_{(1+2\delta)R})$.

Proof To see (4.2), note that if $\mathbf{u} \in \mathcal{H}_{h,0}(B_{(1+2\delta)R})$, then $\mathbf{u} \in \mathcal{H}_{h,0}(B_R)$, and $\Pi_{h,R}$ maps into $\mathcal{H}_{h,0}(B_R)$. To see (4.2), first note that due to the support properties of η and the fact that $\Pi_{h,R}$ is the orthogonal projection, it holds

$$|||\mathbf{u} - \Pi_{h,R}\Pi_H(\eta\mathbf{u})|||^2_{h,R} = |||\Pi_{h,R}(\eta\mathbf{u} - \Pi_H(\eta\mathbf{u}))|||^2_{h,R} \leq |||\eta\mathbf{u} - \Pi_H(\eta\mathbf{u})|||^2_{h,R}.$$

Furthermore, due to the approximation properties (4.38) of Π_H and the assumption $16H \leq \delta R$, we obtain

$$\begin{aligned}
|||\eta\mathbf{u} - \Pi_H(\eta\mathbf{u})|||^2_{h,R} &= \frac{h^2}{R^2}\|\nabla(\eta\mathbf{u} - \Pi_H(\eta\mathbf{u}))\|^2_{L^2_\alpha(B_R)} + \frac{1}{R^2}\|\eta\mathbf{u} - \Pi_H(\eta\mathbf{u})\|^2_{L^2_\alpha(B_R)} \\
&\lesssim \left(\frac{h^2}{R^2} + \frac{H^2}{R^2}\right)\|\nabla(\eta\mathbf{u})\|^2_{L^2_\alpha(B_{(1+\delta)R})} \\
&\lesssim \left(\frac{h^2}{R^2} + \frac{H^2}{R^2}\right)\left(\frac{1}{\delta^2 R^2}\|\mathbf{u}\|^2_{L^2_\alpha(B_{(1+\delta)R})} + \|\nabla\mathbf{u}\|^2_{L^2_\alpha(B_{(1+\delta)R})}\right).
\end{aligned}$$

Applying Lemma 4.1 with $\widetilde{\delta} = \delta/(1+\delta)$ and $\widetilde{R} = (1+\delta)R$, i.e., $(1+\widetilde{\delta})\widetilde{R} = (1+2\delta)R$, shows

$$\|\nabla\mathbf{u}\|^2_{L^2_\alpha(B_{(1+\delta)R})} \lesssim \frac{(1+2\delta)^2}{\delta^2}|||\mathbf{u}|||^2_{h,(1+2\delta)R}.$$

Together with the trivial estimate $\|\mathbf{u}\|_{L^2_\alpha(B_{(1+\delta)R})} \leq (1+2\delta)R|||\mathbf{u}|||_{h,(1+2\delta)R}$ we get (4.2). Statement (4.2) follows from the local definition of the operator Π_H. □

Lemma 4.3 *Let $q, \kappa \in (0,1)$, $R \in (0, 2\,\mathrm{diam}(\Omega))$, and $k \in \mathbb{N}$. Assume*

$$h \leq \frac{\kappa q R}{64k\max\{1, C_{\mathrm{app}}\}}, \tag{4.39}$$

where C_{app} is the constant from Lemma 4.2. Then, there exists a finite dimensional subspace $\widehat{W}_k$ of $\mathcal{H}_h(B_{(1+\kappa)R})$ with dimension $\dim \widehat{W}_k \leq C_{\dim}\left(\frac{1+\kappa^{-1}}{q}\right)^{d+1} k^{d+2}$ such that for every $\mathbf{u} \in \mathcal{H}_{h,0}(B_{(1+\kappa)R})$, there holds

$$\min_{\mathbf{v}\in\widehat{W}_k} |||\mathbf{u} - \mathbf{v}|||_{h,R} \leq q^k |||\mathbf{u}|||_{h,(1+\kappa)R}.$$

Proof Define $H := \frac{\kappa q R}{64k\max\{1,C_{\mathrm{app}}\}}$. Then $h \leq H$. Define $\delta_j := \kappa\frac{k-j}{k}$ for $j = 0,\ldots,k$. This yields $\kappa = \delta_0 > \delta_1 > \cdots > \delta_k = 0$. We will apply Lemma 4.2 k times, with $\widetilde{R}_j = (1+\delta_j)R$ and $\widetilde{\delta}_j = \frac{1}{2k(1+\delta_j)}$. This may be done, as $\widetilde{R}_j \leq 4\,\mathrm{diam}(\Omega)$, $\widetilde{\delta}_j < 1/2$, and

$$16H \leq \frac{R}{4k\max\{1, C_{\mathrm{app}}\}} \leq \frac{R}{2k(1+\delta_j)} = \widetilde{\delta}_j R \leq \widetilde{\delta}_j\widetilde{R}_j.$$

Note that $(1 + 2\widetilde{\delta}_j)\widetilde{R}_j = (1 + \delta_{j-1})R$. The first application of Lemma 4.2 yields a function $\mathbf{w}_1$ in a subspace $\widehat{W}_1$ of $\mathcal{H}_h(B_{(1+\delta_1)R})$ with $\dim W_1 \leq C_{\text{app}} \left(\frac{(1+\kappa)R}{H}\right)^{d+1}$ such that

$$|||\mathbf{u} - \mathbf{w}_1|||_{h,(1+\delta_1)R} \leq 2C_{\text{app}} \frac{1 + 2\widetilde{\delta}_1}{\widetilde{\delta}_1} \frac{H}{\widetilde{R}_1} |||\mathbf{u}|||_{h,(1+2\widetilde{\delta}_1)\widetilde{R}_1}$$
$$= 8C_{\text{app}} \frac{kH}{R} |||\mathbf{u}|||_{h,(1+\delta_0)R} \leq q\,|||\mathbf{u}|||_{h,(1+\delta_0)R}.$$

As $\mathbf{u} - \mathbf{w}_1 \in \mathcal{H}_h(B_{(1+\delta_1)R})$, a second application of Lemma 4.2 yields a function $\mathbf{w}_2$ in a subspace W_2 of $\mathcal{H}_h(B_{(1+\delta_2)R})$ such that

$$|||\mathbf{u} - \mathbf{w}_1 - \mathbf{w}_2|||_{h,(1+\delta_2)R} \leq q\,|||\mathbf{u} - \mathbf{w}_1|||_{h,(1+\delta_1)R} \leq q^2 |||\mathbf{u}|||_{h,(1+\delta_0)R}.$$

Applying k times Lemma 4.2, we obtain a function $\mathbf{v} = \sum_{j=1}^k \mathbf{w}_j$ that is an element of the subspace $V_k := \sum_{j=1}^k W_j$ of $\mathcal{H}_h(B_R)$ and fulfills $|||\mathbf{u} - \mathbf{v}|||_{h,R} \leq q^k |||\mathbf{u}|||_{h,(1+\kappa)R}$. □

Proposition 4.4 *Let $\eta > 0$ be a fixed admissibility parameter and $q \in (0, 1)$. Let (τ, σ) be a cluster pair with admissible bounding boxes $B^0_{R_\tau}$ and $B^0_{R_\sigma}$, that is, $\eta\, dist\left(B^0_{R_\tau}, B^0_{R_\sigma}\right) \geq \operatorname{diam}\left(B^0_{R_\tau}\right)$. Then, for each $k \in \mathbb{N}$, there exists a space $V_k \subset \mathcal{S}^1_0(\mathcal{T}_h)$ with $\dim V_k \leq C_{\dim}(2 + \eta)^{d+1} q^{-(d+1)} k^{d+2}$ such that if $f \in L^2(\Omega)$ with $supp(f) \subset B^0_{R_\sigma} \cap \Omega$, then the solution u_h of (2.10) satisfies*

$$\min_{v \in V_k} \|u_h - v\|_{L^2(B^0_{R_\tau})} \leq C_{\text{box}} h^{-1} q^k \|f\|_{L^2(B^0_{R_\sigma})}. \tag{4.40}$$

Proof Set $\kappa := (1 + \eta)^{-1}$. We distinguish two cases.

Case 1: Condition (4.39) is satisfied with $R = R_\tau$:
As $\operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) \geq \eta^{-1}\operatorname{diam}(B_{R_\tau}) = \eta{-}1\sqrt{d}R_\tau$, we conclude

$$\operatorname{dist}(B_{(1+\kappa)R_\tau}, B_{R_\sigma}) \geq \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) - \kappa R_\tau \sqrt{d} = \sqrt{d} R_\tau \frac{1}{\eta(1 + \eta)} > 0.$$

Hence, $\mathcal{L}u_h \in \mathcal{H}_{h,0}(B_{(1+\kappa)R})$. Lemma 4.3 implies that there is a space $\widehat{W}_k$ with

$$\min_{\mathbf{v} \in \widehat{W}_k} |||\mathcal{L}u_h - \mathbf{v}|||_{h,R_\tau} \leq q^k |||\mathcal{L}u_h|||_{h,(1+\kappa)R_\tau}.$$

Now

$$|||\mathcal{L}u_h|||_{h,(1+\kappa)R_\tau} \lesssim (1 + \frac{1}{R_\tau})\|\mathcal{L}u_h\|_{\mathcal{B}^1_\alpha(\mathbb{R}^{d+1}_+)} \lesssim (1 + \frac{1}{R_\tau})\|\Pi f\|_{L_2(\Omega)},$$

where the last estimate follows from (3.20) and Lemma 3.5. On the other hand, employing an appropriate cut-off function and the multiplicative trace estimate of Lemma 3.7 shows

$$\|u_h - \operatorname{tr} \mathbf{v}\|_{L^2(B^0_{R_\tau})} \lesssim \frac{1}{R_\tau}\|\mathcal{L}u_h - \mathbf{v}\|_{L^2_\alpha(B_{R_\tau})} + \|\nabla(\mathcal{L}u_h - \mathbf{v})\|_{L^2_\alpha(B_{R_\tau})}$$
$$\lesssim \frac{R_\tau}{h}|||\mathcal{L}u_h - \mathbf{v}|||_{h,R_\tau}.$$

Combining the last three chains of estimates, we get the desired result if we set $V_k := \operatorname{tr} \widehat{W}_k$.

Case 2: Condition (4.39) is not satisfied with $R = R_\tau$:
Select $V_k := \left\{v|_{B^0_{R_\tau}} \mid v \in \mathcal{S}^1_0(\mathcal{T}_h)\right\}$. The minimum in (4.40) is then zero and

$$\dim V_k \lesssim \left(\frac{R_\tau}{h}\right)^d \leq \left(\frac{64k \max\left\{1, C_{\text{app}}\right\}}{\kappa q}\right)^d \lesssim k^d(1+\eta)^d q^{-d}.$$

□

Proof of Theorem 2.2 Suppose first that $C_{\dim}(2+\eta)^{d+1}q^{-(d+1)}k^{d+2} \geq \min\{|\tau|, |\sigma|\}$. In the case $\min\{|\tau|, |\sigma|\} = |\tau|$, we set $\mathbf{X}_{\tau\sigma} = \mathbf{I} \in \mathbb{R}^{|\tau|\times|\tau|}$ and $\mathbf{Y}_{\tau\sigma} = \mathbf{A}^{-1}|^\top_{\tau\times\sigma}$. If $\min\{|\tau|, |\sigma|\} = |\sigma|$, we set $\mathbf{X}_{\tau\sigma} = \mathbf{A}^{-1}|_{\tau\times\sigma}$ and $\mathbf{Y}_{\tau\sigma} = \mathbf{I} \in \mathbb{R}^{|\sigma|\times\sigma}$. Now suppose that $C_{\dim}(2+\eta)^{d+1}q^{-(d+1)}k^{d+2} < \min\{|\tau|, |\sigma|\}$. For a cluster $\tau \subset \mathcal{I}$, we define $\mathbb{R}^\tau := \left\{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{x}_j = 0\ \forall j \notin \tau\right\}$. According to [57], there exist linear functionals λ_i such that $\lambda_i(\psi_j) = \delta_{ij}$ and

$$\|\lambda_i(w)\psi_i\|_{L_2(\Omega)} \lesssim \|w\|_{L_2(\operatorname{supp}(\psi_i))}, \tag{4.41}$$

where the hidden constant depends only on the shape regularity of $\mathcal{T}_h$. Define

$$\Phi_\tau : \begin{cases} \mathbb{R}^\tau \to \mathcal{S}^1_0(\mathcal{T}_h) \\ \mathbf{x} \mapsto \sum_{j\in\tau} \mathbf{x}_j\psi_j \end{cases} \quad \text{and} \quad \Lambda_\tau : \begin{cases} L_2(\Omega) \to \mathbb{R}^\tau \\ w \mapsto \boldsymbol{w} \end{cases},$$

where $\boldsymbol{w}_j = \lambda_j(w)$ for $j \in \tau$ and $\boldsymbol{w}_j = 0$ else. Note that $h^{d/2}\|\mathbf{x}\|_2 \sim \|\Phi_\tau(\mathbf{x})\|_{L_2(\Omega)}$ for $\mathbf{x} \in \mathbb{R}^\tau$ and that $\Phi_\tau \circ \Lambda_\tau$ is bounded in $L_2(\Omega)$. For $\Lambda^\star_{\mathcal{I}}$, the adjoint of $\Lambda_{\mathcal{I}}$, this implies $\|\Lambda^\star_{\mathcal{I}}\|_{\mathbb{R}^N\to L_2(\Omega)} \lesssim h^{-d/2}$. Let V_k be the space of Proposition 4.4. We define the columns of $\mathbf{X}_{\tau\sigma}$ to be an orthogonal basis of the space $\{\Lambda_\tau w \mid w \in V_k\}$ and $\mathbf{Y}_{\tau\sigma} := \mathbf{A}^{-1}|^\top_{\tau\times\sigma}\mathbf{X}_{\tau\sigma}$. The ranks of $\mathbf{X}_{\tau\sigma}$ and $\mathbf{Y}_{\tau\sigma}$ are then bounded by $C_{\dim}(2+\eta)^{d+1}q^{-(d+1)}k^{d+2}$. Now, for $\mathbf{b} \in \mathbb{R}^\sigma$, set $f := \Lambda^\star_{\mathcal{I}}(\mathbf{b})$. This yields $b_i = (f, \psi_i)_\Omega$ and $\operatorname{supp}(f) \subset \overline{B_{R_\sigma} \cap \Omega}$. According to Proposition 4.4, there exists an element $v \in V_k$ such that $\|u_h - v\|_{L_2(B^0_{R_\tau}\cap\Omega)} \lesssim h^{-1}q^k\|f\|_{L_2(B^0_{R_\sigma})}$. This implies

$$\|\Lambda_\tau u_h - \Lambda_\tau v\|_2 \lesssim h^{-d/2}\|\Phi_\tau \circ \Lambda_\tau(u_h - v)\|_{L_2(\Omega)} \lesssim h^{-d/2}\|u_h - v\|_{L_2(B^0_{R_\tau}\cap\Omega)}$$
$$\lesssim h^{-1-d/2}q^k\|f\|_{L_2(B^0_{R_\sigma})} \lesssim h^{-1-d}q^k\|\mathbf{b}\|_2.$$

For $\mathbf{z} := \mathbf{X}_{\tau\sigma}\mathbf{X}^\top_{\tau\sigma}\Lambda_\tau u_h$, it holds

$$\|\Lambda_\tau u_h - \mathbf{z}\|_2 \leq \|\Lambda_\tau u_h - \Lambda_\tau v\|_2 \lesssim h^{-1-d}q^k\|\mathbf{b}\|_2.$$

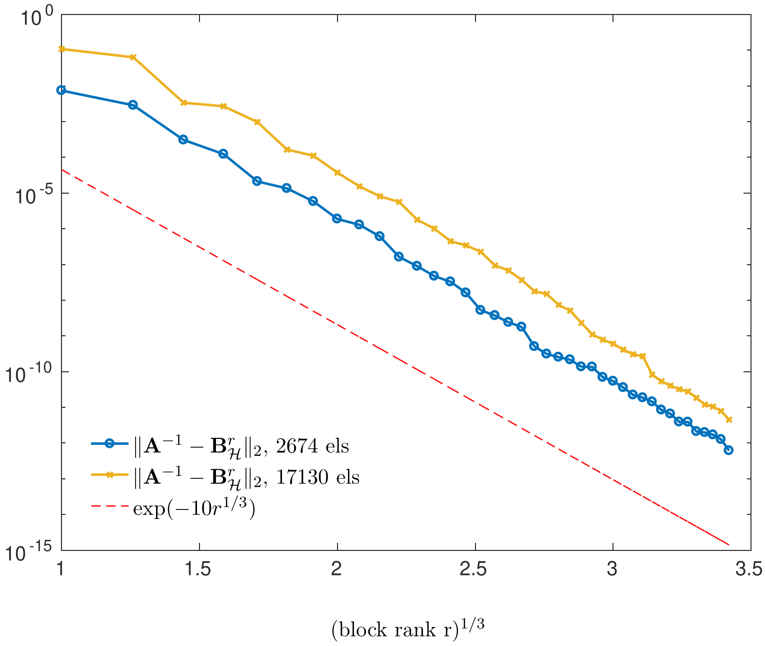

Fig. 1 Square domain. $s = 0.25$, 2674, and 17,130 elements

As $\Lambda_\tau u_h = \mathbf{A}^{-1}|_{\tau\times\sigma}\mathbf{b}|_\sigma$, we obtain

$$\|(\mathbf{A}^{-1}|_{\tau\times\sigma} - \mathbf{X}_{\tau\sigma}\mathbf{Y}^\top_{\tau\sigma})\mathbf{b}|_\sigma\|_2 \lesssim N^{\frac{1+d}{d}} q^k \|\mathbf{b}\|_2.$$

As $\mathbf{b} \in \mathbb{R}^\sigma$ was arbitrary, the result follows. □

5 Numerical experiments

We provide numerical experiments in two space dimensions, i.e., $d = 2$, that confirm our theoretical findings. The indices $\mathcal{I}$ of the standard basis of the space $S^1_0(\mathcal{T}_h)$ based on a quasiuniform triangulation of Ω are organized in a cluster tree $\mathbb{T}_{\mathcal{I}}$ that is obtained by a geometric clustering; i.e., bounding boxes are split in half perpendicular to their longest edge until the corresponding clusters are smaller than $n_{\text{leaf}} = 20$. The block cluster tree is based on that cluster tree using the admissibility parameter $\eta = 2$. In order to calculate a blockwise rank-r approximation $\mathbf{B}^r_{\mathcal{H}}$ of $\mathbf{A}^{-1}$, we compute the densely populated system matrix $\mathbf{A}$ using the MATLAB code presented in [1] and its inverse $\mathbf{A}^{-1}$. On admissible cluster pairs, we compute a rank-r approximation of the corresponding matrix block of $\mathbf{A}^{-1}$ by singular value decomposition, which produces the best possible approximation of $\mathbf{A}^{-1}$ in the Frobenius norm (in the given blockwise rank-r format). It goes without saying that this way of computing the $\mathcal{H}$-matrix approximation to $\mathbf{A}^{-1}$ has no practical relevance and is only done to illustrate Theorem 2.5 and gauge the potential of the chosen format. In practice, the matrix $\mathbf{A}$

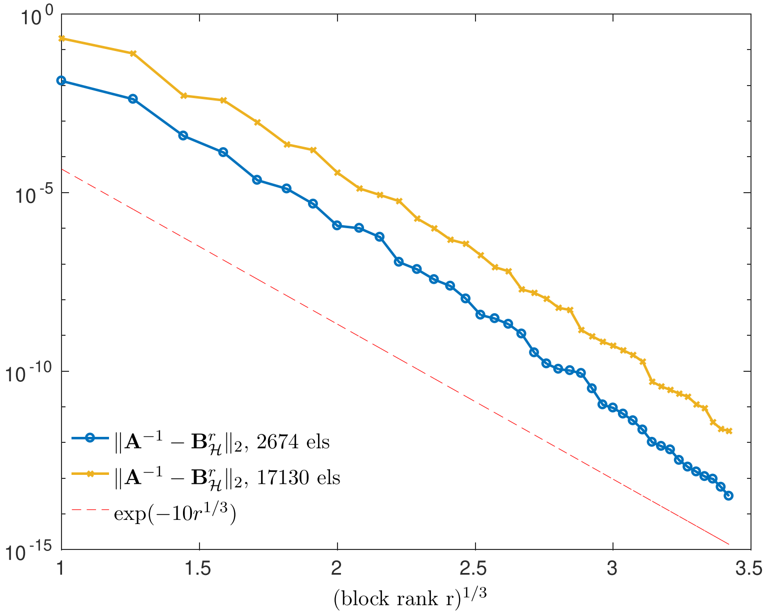

Fig. 2 Square domain. $s = 0.5$, 2674, and 17,130 elements

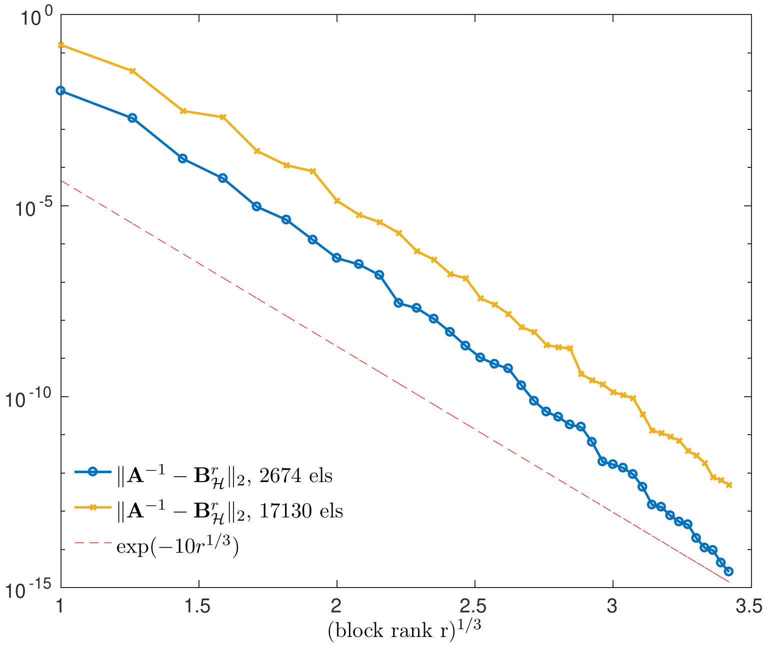

Fig. 3 Square domain. $s = 0.75$, 2674, and 17130 elements

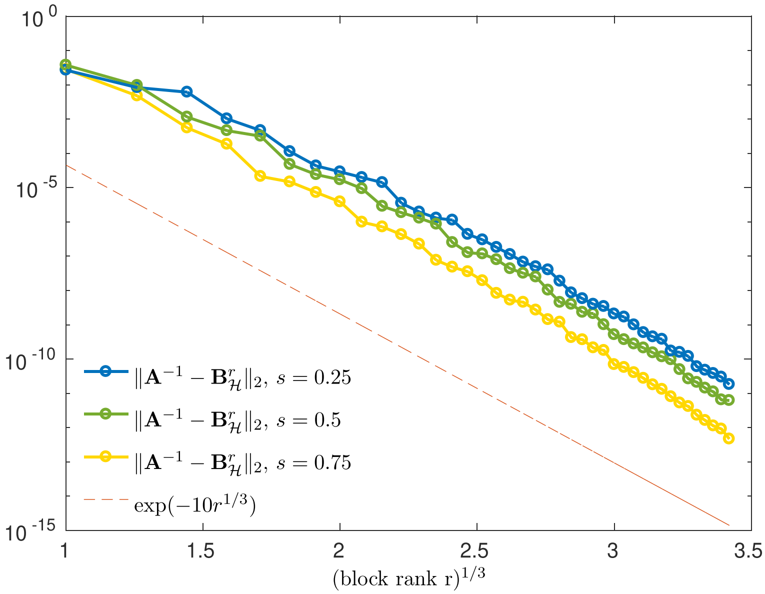

Fig. 4 L-shaped domain. $s \in \{0.25, 0.5, 0.75\}$, 6560 elements

has to be approximated during setup using kernel approximations for the far field by, e.g., Taylor expansions, Chebyshev interpolation (see [36, Sec. 4]), or with black-box techniques such as ACA [5] or HCA [12]. In practice, an $\mathcal{H}$-matrix approximation of $\mathbf{A}^{-1}$ can be obtained using (approximate) inversion or factorization techniques discussed in the introduction.

We carried out experiments for $s \in \{0.25, 0.5, 0.75\}$ on a square and an L-shaped domain. On the square, we use a coarse mesh of 2674 elements, resulting in 358 admissible and 591 non-admissible blocks, and a fine mesh of 17,130 elements, resulting in 5234 admissible and 5486 non-admissible blocks. On the L-shaped domain, we use a mesh of 6560 elements, resulting in 640 admissible and 1332 non-admissible blocks. Note that for a fixed mesh and cluster tree, Theorem 2.5 predicts $\|\mathbf{A}^{-1} - \mathbf{B}^r_{\mathcal{H}}\|_2 \lesssim \exp(-br^{1/4})$. However, in our experiments, we observe that the error behaves like $\|\mathbf{A}^{-1} - \mathbf{B}^r_{\mathcal{H}}\|_2 \sim \exp(-10r^{1/3})$. Hence, we will plot the error logarithmically over the third root of the block rank r and include the reference curve $\exp(-10r^{1/3})$. Recalling the proof of Lemma 4.2, one discerns a possible reason for the discrepancy between the proved convergence $O(\exp(-br^{1/4}))$ and the observed $O(\exp(-br^{1/3}))$: Lemma 4.2 constructs approximations by function defined on $\mathbb{R}^{d+1}_+$ and later, traces on $\mathbb{R}^d$ are taken (Figs. 1, 2, 3, and 4).

6 Conclusions and extensions

We have shown that the inverse $\mathbf{A}^{-1}$ of the stiffness matrix $\mathbf{A}$ of a Galerkin discretization of the fractional Laplacian can be approximated at an exponential rate in the block rank by $\mathcal{H}$-matrices, using the standard admissibility criterion (2.11). The following extensions are possible:

– We restricted our analysis to the discretization by piecewise linears. However, the analysis generalizes to approximation by piecewise polynomials of fixed degree p.
– We focused on the approximability of $\mathbf{A}^{-1}$ in the $\mathcal{H}$-matrix format. Computationally attractive are also factorizations such as $\mathcal{H}$-LU or $\mathcal{H}$-Cholesky factorizations. The ability to find an approximate $\mathbf{A} \approx L_{\mathcal{H}} U_{\mathcal{H}}$ has been shown for (classical) FEM discretizations in [7, 22] and for non-local BEM matrices in [23, 24] with techniques that generalize to the present case of the fractional Laplacian.
– Related to $\mathcal{H}$-matrices is the format of $\mathcal{H}^2$-matrices discussed in [10, 11, 36, 37]. Using the techniques employed in [10, 22–24], one may also show that $\mathbf{A}^{-1}$ can be approximated by $\mathcal{H}^2$-matrices at an exponential rate in the block rank.

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