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Spectral collocation method for system of weakly singular Volterra integral equations

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Abstract
Based on our previous research, we investigate spectral collocation method for system of weakly singular Volterra integral equations. The provided convergence analysis shows that global convergence order is related to regularity of the solution to this system, and the local convergence order on collocation points only depends on the regularity of kernel functions. Numerical experiments are carried out to confirm these theoretical results. Numerical methods are developed to solve nonlinear system of weakly singular Volterra integral equations and high-order weakly singular Volterra integro-differential equations.

Keywords Spectral collocation method · System of weakly singular VIEs · Convergence analysis · Numerical experiments

Mathematics Subject Classification (2010) 65M70 · 45D05

1 Introduction

Systems of weakly singular Volterra integral equations (VIEs) appear for example in the spatial discretization of partial VIEs [8]. Weakly singular equations are widely applied in fractional calculus [9, 13, 14]. Many high-order weakly singular Volterra integro-differential equations (VIDEs) can be transformed to be system of weakly singular VIEs [3]. Waveform and time point relaxation methods were developed to solve large systems of nonlinear systems of weakly singular VIDEs [4, 15]. Discontinuous piecewise polynomial collocation methods were investigated for solving system of weakly singular VIEs of the first kind [12]. A hybrid collocation method

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[6] was developed for weakly singular VIEs. Collocation method was proposed to solve fractional differential equations involving non-singular kernel [1]. Spectral methods [5] are the numerical methods with high precision, which are widely used to solve Volterra-type integral equations [7, 18, 19]. In [2], spectral method was proposed to solve a system of fractional differential equations within a fractional derivative involving the Mittag-Leffler kernel. In our previous research, we have investigated the spectral collocation methods for weakly singular VIE with proportional delay [11], and the system of VIEs with smooth kernel functions [10]. Based on the findings, we investigate a spectral collocation method for a system of weakly singular VIEs in this paper.

The system of weakly singular VIEs considered in this paper is

$$\mathbf{y}(t) = \mathbf{g}(t) + \mathscr{V}(\mathbf{y})(t), t \in [0, 1], \tag{1}$$

where

$$\mathbf{y}(t) := [y_1(t), y_2(t), \cdots, y_M(t)]^T, \tag{2}$$

$$\mathbf{g}(t) := [g_1(t), g_2(t), \cdots, g_M(t)]^T. \tag{3}$$

where A^T means the transposed matrix of A. The integral operator $\mathscr{V}$ is defined as

$$\begin{aligned}\mathscr{V}(\mathbf{y})(t) &:= \int_0^t [\mathbf{h}(t-s) \cdot \mathbf{K}(t,s)]\mathbf{y}(s)ds \\ &:= \left[\sum_{q=1}^{M} V_{1q}(y_q)(t), \sum_{q=1}^{M} V_{2q}(y_q)(t), \cdots, \sum_{q=1}^{M} V_{Mq}(y_q)(t)\right]^T,\end{aligned} \tag{4}$$

where

$$\mathbf{h}(t-s) := [(t-s)^{-\mu_{pq}}]_{M\times M}, 0 \le \mu_{pq} < 1, 1 \le p, q \le M, \tag{5}$$

and we assume that at least one of μ_{pp}, $1 \le p \le M$ is not zero. The kernel functions are

$$\mathbf{K}(t,s) := [K_{pq}(t,s)]_{M\times M}, (t,s) \in \Delta := \{(t,s) : 0 \le s \le t \le 1\}. \tag{6}$$

We define

$$\mathbf{h}(t-s) \cdot \mathbf{K}(t,s) := [(t-s)^{-\mu_{pq}} K_{pq}(t,s)]_{M\times M}. \tag{7}$$

The integral operator V_{pq} is defined as

$$V_{pq}(y_q)(t) := \int_0^t (t-s)^{-\mu_{pq}} K_{pq}(t,s) y_q(s)ds, 1 \le p, q \le M. \tag{8}$$

If the given functions are smooth on their own definition domain, then the solution to system (1) is continuous but not smooth on their own definition domain (see Lemma 2 in this paper).

In this paper, we assume that the given functions possess continuous derivatives of order at least m. Chebyshev Gauss-Lobatto points of order N are selected as collocation points. We provide convergence analysis to show that the global convergence order is $\log(N)N^{\mu-1}$ where $\mu = \max\{\mu_{pp} : 1 \le p \le M\}$, and the local convergence order on collocation points is $\log(N)N^{-m}$. If the solution to system (1) is smooth, namely, $\mathbf{y}(t)$ possesses continuous derivatives of order at least m, then global convergence order is $\log(N)N^{1-r-m}, \forall\, 0 < r < 1$. These convergence analysis results show that the global convergence order depends on the regularity of the solution to system (1) and the local convergence order depends on the regularity of given functions (especially kernel functions). It is worth noting that N and m are independent of each other. We carry out numerical experiments to confirm these theoretical results. We also investigate the numerical experiments for high-order VIDEs with weakly singular kernels which can be transformed to be a system of weakly singular VIEs. Numerical schemes are developed for the nonlinear system of weakly singular VIEs and nonlinear high-order VIDEs.

This paper is organized as follows. Numerical methods are demonstrated in Section 2. Useful lemmas for convergence analysis are prepared in Section 3. Convergence analysis is presented in Section 4. Numerical experiments are carried out in Section 5. Finally, we end with conclusion and future work in Section 6.

2 Spectral collocation method

Numerical scheme for system (1) is derived in this section.

Let

$$t_i := \frac{1}{2}(x_i + 1), i = 0, 1, \cdots, N, \tag{9}$$

where $\{x_i\}_{i=0}^N$ is the set of Chebyshev Gauss-Lobatto points of order N in standard interval $[-1, 1]$. System (1) holds at t_i,

$$\mathbf{y}(t_i) = \mathbf{g}(t_i) + \mathscr{V}(\mathbf{y})(t_i), i = 0, 1, \cdots, N. \tag{10}$$

Approximate $y_p(t_i)$ by y_{pi}. Then, $\mathbf{y}(t_i)$ can be approximated by

$$\mathbf{y}_i := [y_{1i}, y_{2i}, \cdots, y_{Mi}]^T. \tag{11}$$

Note that

$$\mathbf{y}_i L_i(t) = [y_{1i}, y_{2i}, \cdots, y_{Mi}]^T L_i(t) = [y_{1i} L_i(t), y_{2i} L_i(t), \cdots, y_{Mi} L_i(t)]^T, \tag{12}$$

where $L_j(t)$ is the jth Lagrange interpolation basic function associated with points $\{t_i\}_{i=0}^N$. Let

$$\mathbf{y}^N(t) := \sum_{j=0}^{N} \mathbf{y}_j L_j(t) \approx \mathbf{y}_p(t). \tag{13}$$

Then,

$$\mathbf{y}^N(t) := \sum_{j=0}^{N} \mathbf{y}_j L_j(t) = \left[y_1^N(t), y_2^N(t), \cdots, y_M^N(t)\right]^T \approx \mathbf{y}(t). \tag{14}$$

Approximate (10) by

$$\mathbf{y}_i \approx \mathbf{g}(t_i) + \mathscr{V}(\mathbf{y}^N)(t_i), i = 0, 1, \cdots, N. \tag{15}$$

Let

$$s(t_i, v) := \frac{t_i}{2}(v + 1), v \in [-1, 1], \tag{16}$$

which can change the interval $[0, t_i]$ to the standard interval $[-1, 1]$. Use Gauss quadrature formula to approximate integral terms in (15),

$$\mathscr{V}^N(\mathbf{y}^N)(t_i) := \left[\sum_{q=1}^{M} V_{1q}^N(y_q^N)(t_i), \sum_{q=1}^{M} V_{2q}^N(y_q^N)(t_i), \cdots, \sum_{q=1}^{M} V_{Mq}^N(y_q^N)(t_i)\right]^T \approx \mathscr{V}(\mathbf{y}^N)(t_i), \quad (17)$$

where

$$V_{pq}^N\left(y_q^N\right)(t_i) := \left(\frac{t_i}{2}\right)^{1-\mu_{pq}} \sum_{k=1}^{N} K_{pq}\left(t_i, s\left(t_i, v_k^{pq}\right)\right) y_q^N\left(s\left(t_i, v_k^{pq}\right)\right) w_k^{pq} \approx V_{pq}(y_q)(t_i), \quad (18)$$

$v_k^{pq}, w_k^{pq}, k = 0, 1, \cdots, N$ are Gauss points and weights of order N, corresponding to weight function $\omega^{0,-\mu_{pq}}(x) := (1 + x)^0(1 - x)^{-\mu_{pq}}, x \in [-1, 1]$. Finally, we obtain a discrete system with unknown elements $y_{pi}, p = 1, 2, \cdots, M, i = 0, 1, \cdots, N$,

$$\mathbf{y}_i = \mathbf{g}(t_i) + \mathscr{V}^N(\mathbf{y}^N)(t_i), i = 0, 1, \cdots, N. \quad (19)$$

Solving this system, we can obtain the numerical solution $\mathbf{y}^N(t)$ to approximate $\mathbf{y}(t)$.

In order to solve discrete system (19) easily by computer, we write it in matrix form. Let

$$\mathbf{y}^N := [y_{10}, y_{11}, \cdots, y_{1N}, y_{20}, \cdots, y_{2N}, \cdots, y_{M0}, \cdots, y_{MN}]^T, \quad (20)$$

and

$$\mathbf{g}^N := [g_1(t_0), g_1(t_1), \cdots, g_1(t_N), g_2(t_0), \cdots, g_2(t_N), \cdots, g_M(t_0), \cdots, g_M(t_N)]^T. \quad (21)$$

Our goal is to write (19) into matrix form

$$\mathbf{y}^N = \mathbf{g}^N + \mathbf{K}^N\mathbf{y}^N. \quad (22)$$

The row of $\mathbf{K}^N$ corresponding to y_{pi} and $g_p(t_i)$ is

$$[a_{p10}, a_{p11}, \cdots, a_{p1N}, a_{p20}, \cdots, a_{p2N}, \cdots, a_{pM0}, \cdots, a_{pMN}],$$

where

$$a_{pqj} = \left(\frac{t_i}{2}\right)^{1-\mu_{pq}} \sum_{k=0}^{N} K_{pq}(t_i, s(t_i, v_{pqk}) L_j(s(t_i, v_{pqk}) w_{pqk},$$

and $v_{pqk}, w_{pqk}, k = 0, 1, \cdots, N$ are Gauss points and weights of order N, corresponding to weight function $\omega^{0,-\mu_{pq}}(x) := (1 + x)^0(1 - x)^{-\mu_{pq}}$.

3 Lemmas

Useful lemmas for convergence analysis are prepared in this section.

Lemma 1 [3] *Assume that $v(t)$ is nonnegative function on* $[0, 1]$*, and $u(t)$ satisfies*

$$u(t) \leq v(t) + \lambda \int_0^t (t-s)^{-\mu} u(s) ds, t \in [0, 1], \lambda > 0. \tag{23}$$

Then, there exists constant C independent of $u(t)$ and $v(t)$ such that

$$u(t) \leq C v(t). \tag{24}$$

Here after, denote by $C(0, 1)$ the space of continuous functions on $[0, 1]$, and $C^m(0, 1)$ the space of functions on $[0, 1]$ possessing continuous derivatives of order at least m.

Assume that $A(t) := [a_{pq}(t)]_{P\times Q}$ is a matrix of P rows and Q columns, whose element at pth row and qth column is $a_{pq}(t)$. $A(t) \in C(0, 1)$ means that $a_{pq}(t) \in C(0, 1), 1 \leq p \leq P, 1 \leq q \leq Q$. For $A(t) \in C(0, 1)$, define

$$|A(t)| := \max\{|a_{pq}(t)| : 1 \leq p \leq P, 1 \leq q \leq Q\}. \tag{25}$$

It is clear that $|A(t)|$ is a continuous function on $[0, 1]$. Let

$$\|A\|_{L^\infty(0,1)} := \max\{|A(t)| : t \in [0, 1]\}. \tag{26}$$

It is worth noting that

$$|a_{pq}(t)| \leq |A(t)|, t \in [0, 1], 1 \leq p \leq P, 1 \leq q \leq Q. \tag{27}$$

The space of functions $C^{m,r}(0, 1)$, where m is a nonnegative integer and $0 < r < 1$, is equipped the norm

$$\|u\|_{C^{m,r}(0,1)} := \max_{0\leq k\leq m} \sup_{0\leq t\leq 1} |\partial_t^k u(t)| \max_{0\leq k\leq m} \sup_{0\leq \tau_1,\tau_2\leq 1} \frac{|\partial_t^k u(\tau_1) - \partial_t^k u(\tau_2)|}{|\tau_1 - \tau_2|^r}. \tag{28}$$

Specially,

$$\|u\|_{C^{0,r}(0,1)} := \|u\|_{L^\infty(0,1)} + \sup_{0\leq \tau_1,\tau_2\leq 1} \frac{|u(\tau_1) - u(\tau_2)|}{|\tau_1 - \tau_2|^r}. \tag{29}$$

If $u(t) \in C^{m+1}(0, 1)$, then for $\forall\ 0 < r < 1$,

$$\|u\|_{C^{m,r}(0,1)} = \max_{0\leq k\leq m} \sup_{0\leq t\leq 1} |\partial_t^k u(t)| + \max_{0\leq k\leq m} \sup_{0\leq \tau_1,\tau_2\leq 1} \frac{|\partial_t^k u(\tau_1) - \partial_t^k u(\tau_2)|}{|\tau_1 - \tau_2|^r} < \infty, \tag{30}$$

which means that $u(t) \in C^{m,r}(0, 1)$. Therefore, we conclude that

$$C^{m+1}(0, 1) \subseteq C^{m,r}(0, 1) \text{ holds for } \forall\ 0 < r < 1. \tag{31}$$

For $\mathbf{u}(t) = [u_1(t), \cdots, u_M(t)]^T$, let $\mathbf{u}(t) \in (C^{0,r}(0, 1))^M$ means $u_p(t) \in C^{0,r_p}(0, 1)$, $p = 1, \cdots, M$, where $r := \min\{r_1, r_2, \cdots, r_M\}$. Define

$$\|\mathbf{u}\|_{(C^{0,r}(0,1))^M} := \|\mathbf{u}\|_{L^\infty(0,1)} + \max_{1\leq p\leq M} \sup_{0\leq \tau_1,\tau_2\leq 1} \{\frac{|u_p(\tau_1) - u_p(\tau_2)|}{|\tau_1 - \tau_2|^r}\}. \tag{32}$$

It is worth noting that

$$\|u_p\|_{C^{0,r_p}(0,1)} \leq \|\mathbf{u}\|_{(C^{0,r}(0,1))^M}, p = 1, \cdots, M. \tag{33}$$

Lemma 2 *Assume that* $\mathbf{v}(t) = [v_1(t), \cdots, v_M(t)]^T \in C(0,1)$*, and*

$$\mathbf{u}(t) = \mathbf{v}(t) + \mathscr{V}(\mathbf{u})(t), \tag{34}$$

where the integral operator $\mathscr{V}$ *is defined by* (4) *with continuous kernels* $\mathbf{K}(t,s) \in C(\Delta)$*. Then, there exists constant* C *independent of* $\mathbf{v}(t)$ *and* $\mathbf{u}(t)$ *such that*

$$|\mathbf{u}(t)| \leq C|\mathbf{v}(t)|, \tag{35}$$

and $\mathbf{u}(t) \in (C^{0,1-\mu}(0,1))^M$*, where* $\mu := \max\{\mu_{pp} : 1 \leq p \leq M\}$.

Proof From (34),

$$\begin{aligned} &u_p(t) = v_p(t) + \sum_{q=1}^{M} \int_0^t (t-s)^{-\mu_{pq}} K_{pq}(t,s) u_q(s) ds, \\ &p = 1, 2, \cdots, M. \end{aligned} \tag{36}$$

Then,

$$\begin{aligned} |u_p(t)| \; &\leq \; |v_p(t)| + \sum_{q=1}^{M} \int_0^t (t-s)^{-\mu_{pq}} |K_{pq}(t,s)||u_q(s)| ds, \\ p \; &= \; 1, 2, \cdots, M. \end{aligned} \tag{37}$$

Let

$$\tilde{\mu} := \max\{\mu_{pq} : 1 \leq p, q \leq M\}, \tag{38}$$

and

$$|\mathbf{K}(t,s)| := \max\{|K_{pq}(t,s)| : 1 \leq p, q \leq M\}. \tag{39}$$

In view of (27), from (37), we have

$$\begin{aligned} &|u_p(t)| \leq |v_p(t)| + M \int_0^t (t-s)^{-\tilde{\mu}} |\mathbf{K}(t,s)||\mathbf{u}(s)| ds, \\ &p = 1, 2, \cdots, M. \end{aligned} \tag{40}$$

Then,

$$\max_{1 \leq p \leq M} |u_p(t)| \leq \max_{1 \leq p \leq M} |v_p(t)| + M \int_0^t (t-s)^{-\tilde{\mu}} |\mathbf{K}(t,s)||\mathbf{u}(s)| ds, \tag{41}$$

namely,

$$|\mathbf{u}(t)| \leq |\mathbf{v}(t)| + M \int_0^t (t-s)^{-\tilde{\mu}} |\mathbf{K}(t,s)||\mathbf{u}(s)| ds. \tag{42}$$

By Lemma 1, we obtain (35).

From [3] (page 432), we have $u_p(t) \in C(0,1)$. From (1),

$$u_p(t) = v_p(t) + \sum_{i \neq p}^{M} \int_0^t (t-s)^{-\mu_{pi}} K_{pi}(t,s) u_i(s) ds + \int_0^t (t-s)^{-\mu_{pp}} K_{pp}(t,s) u_p(s) ds. \tag{43}$$

It is worth noting that $v_p(t) + \sum_{i \neq p}^{M} \int_0^t (t-s)^{-\mu_{pi}} K_{pi}(t,s) u_i(s) ds$ is continuous on $[0,1]$. From [3] (page 348), we have $u_p(t) \in C^{0,1-\mu_{pp}}(0,1)$, $p = 1, 2, \cdots, M$. Then, $\mathbf{u}(t) \in (C^{0,1-\mu}(0,1))^M$. □

The interpolation operator I_N is defined by

$$I_N(u)(t) := \sum_{i=0}^{N} u(t_i)L_i(t), t \in [0, 1], \tag{44}$$

where $u(t) \in C(0, 1)$. For the matrix $\mathbf{u}(t) := [u_1(t), u_2(t), \cdots, u_M(t)]^T$,

$$I_N(\mathbf{u})(t) := [I_N(u_1)(t), I_N(u_2)(t), \cdots, I_N(u_M)(t)]^T. \tag{45}$$

The identity operator is denoted by I.

Lemma 3 [16, 17] *Assume that* $u \in C^{m,r}(0, 1)$. *Then, there exists polynomial* $P_N(u)(t)$ *of degree not exceeding* N *such that*

$$\|(I - P_N)(u)\|_{L^\infty(0,1)} \leq CN^{-m-r}\|u\|_{C^{m,r}(0,1)}, \tag{46}$$

where constant C *is independent of* $u(t)$.

Lemma 4 *(I)* [5]*If* $u(t) \in C(0, 1)$*, then*

$$\|I_N(u)\|_{L^\infty(0,1)} \leq C\log(N)\|u\|_{L^\infty(0,1)}. \tag{47}$$

(II) If $u(t) \in C^{m,r}(0, 1)$*, then*

$$\|(I - I_N)(u)\|_{L^\infty(0,1)} \leq C\log(N)N^{-r-m}\|u\|_{C^{m,r}(0,1)}. \tag{48}$$

(III) If $u(t) \in C^m(0, 1)$*, then*

$$\|(I - I_N)(u)\|_{L^\infty(0,1)} \leq C\log(N)N^{1-r-m}\|u\|_{C^{m-1,r}(0,1)}, \forall\, 0 < r < 1. \tag{49}$$

Proof It is clear that

$$\|(I - I_N)(u)\|_{L^\infty(0,1)} \leq \|(I - P_N)(u)\|_{L^\infty(0,1)} + \|(P_N - I_N)(u)\|_{L^\infty(0,1)}. \tag{50}$$

Note that $I_N(P_N(u)) = P_N(u)$. Then,

$$\|(P_N - I_N)(u)\|_{L^\infty(0,1)} = \|I_N(P_N - I)(u)\|_{L^\infty(0,1)}. \tag{51}$$

By (47),

$$\|I_N(P_N - I)(u)\|_{L^\infty(0,1)} \leq C\log(N)\|(P_N - I)(u)\|_{L^\infty(0,1)}. \tag{52}$$

By Lemma 3, if $u(t) \in C^{m,r}(0, 1)$, then

$$\|(I - P_N)(u)\|_{L^\infty(0,1)} \leq CN^{-r-m}\|u\|_{C^{m,r}(0,1)}. \tag{53}$$

Combining (50) with (51), (52), and (53) yields (48).

If $u(t) \in C^m(0, 1)$, by (31), we have $u(t) \in C^{m-1,r}(0, 1), \forall\, 0 < r < 1$. By (48), we obtain (49). □

Lemma 5 *Assume that* $\mathbf{u}(t) = [u_1(t), u_2(t), \cdots, u_M(t)]^T$.
(I) If $\mathbf{u}(t) \in (C(0, 1))^M$*, then*

$$\|I_N(\mathbf{u})\|_{L^\infty(0,1)} \leq C\log(N)\|\mathbf{u}\|_{L^\infty(0,1)}. \tag{54}$$

(II) If $u_p(t) \in C^{m,r_p}(0,1)$, $p = 1, 2, \cdots, M$, *then*

$$\|(I - I_N)(\mathbf{u})\|_{L^\infty(0,1)} \le C \log(N) N^{-r-m} \|\mathbf{u}\|_{(C^{m,r}(0,1))^M}, \tag{55}$$

where $r = \min\{r_p : p = 1, 2, \cdots, M\}$.

(III) If $\mathbf{u}(t) \in (C^m(0,1))^M$, *then*

$$\|(I - I_N)(\mathbf{u})\|_{L^\infty(0,1)} \le C \log(N) N^{1-r-m} \|\mathbf{u}\|_{(C^{m-1,r}(0,1))^M}, \forall\, 0 < r < 1. \tag{56}$$

Proof By (47),

$$|I_N(u_p)(t)| \le \|I_N(u_p)\|_{L^\infty(0,1)} \le C \log(N) \|u_p\|_{L^\infty(0,1)} \le C \log(N) \|\mathbf{u}\|_{L^\infty(0,1)}.$$

Then,

$$\max_{1 \le p \le M} |I_N(u_p)(t)| \le C \log(N) \|\mathbf{u}\|_{L^\infty(0,1)},$$

which leads to (54).

By (48),

$$\begin{aligned} |(I - I_N)(\mathbf{u})(t)| &= \max_{1 \le p \le M} |(I - I_N)(u_p)(t)| \\ &\le \max_{1 \le p \le M} C \log(N) N^{-r_p - m} \|u_p\|_{C^{m,r_p}(0,1)} \\ &\le C \log(N) N^{-r-m} \max_{1 \le p \le M} \|u_p\|_{C^{m,r}(0,1)} \\ &\le C \log(N) N^{-r-m} \|\mathbf{u}\|_{(C^{m,r}(0,1))^M}, \end{aligned} \tag{57}$$

which leads to (55).

Assume that $\mathbf{u}(t) \in (C^m(0,1))^M$, namely, $u_p(t) \in C^m(0,1)$, $p = 1, 2, \cdots, M$. By (49), for $\forall\, 0 < r < 1$,

$$\begin{aligned} |(I - I_N)(\mathbf{u})(t)| &= \max_{1 \le p \le M} |(I - I_N)(u_p)(t)| \\ &\le \max_{1 \le p \le M} C \log(N) N^{1-r-m} \|u_p\|_{C^{m-1,r}(0,1)} \\ &\le C \log(N) N^{1-r-m} \max_{1 \le p \le M} \|u_p\|_{C^{m-1,r}(0,1)} \\ &\le C \log(N) N^{1-r-m} \|\mathbf{u}\|_{(C^{m-1,r}(0,1))^M}, \end{aligned} \tag{58}$$

which leads to (56). □

Lemma 6 [11] *Let*

$$u(t) = \int_0^t (t-s)^{-r} R(t,s) v(s) ds, \; t \in [0,1], \tag{59}$$

where $v(t)$, $R(t,s)$ *are continuous on their own definition domains. Then,* $u(t) \in C^{0,1-r}(0,1)$ *and* $\|u\|_{C^{0,1-r}(0,1)} \le C \|v\|_{L^\infty(0,1)}$.

Lemma 7 *Let* $\mathbf{v}(t) = [v_1(t), v_2(t), \cdots, v_M(t)]^T \in C(0,1)$ *and* $[u_p(t), u_2(t), \cdots, u_p(t)]^T = \mathbf{u}(t) = \mathscr{V}(\mathbf{v})(t)$ *with kernel functions* $\mathbf{K}(t,s) \in C(\Delta)$. *Then,* $\mathbf{u}(t) \in (C^{0,1-\tilde{\mu}}(0,1))^M$ *and* $\|\mathbf{u}\|_{(C^{0,1-\tilde{\mu}}(0,1))^M} \le C \|\mathbf{v}\|_{L^\infty(0,1)}$, *where* $\tilde{\mu} = \max\{\mu_{pq} : 1 \le p, q \le M\}$.

Proof Let $u_{pq}(t) := V_{pq}(v_q)(t)$. By Lemma 6,

$$u_{pq}(t) \in C^{0,1-\mu_{pq}}(0,1), \text{ and } \|u_{qp}\|_{C^{0,1-\mu_{pq}}(0,1)} \le C\|v_q\|_{L^\infty(0,1)}. \tag{60}$$

On the one hand,

$$\begin{aligned} |u_p(t)| &= \sum_{q=1}^{M} \int_0^t (t-s)^{-\mu_{pq}} |K_{pq}(t,s)||v_q(s)|ds \\ &\le \sum_{q=1}^{M} \|v_q\|_{L^\infty(0,1)} \int_0^t (t-s)^{-\mu_{pq}} |K_{pq}(t,s)|ds \\ &\le C\|\mathbf{v}\|_{L^\infty(0,1)}. \end{aligned} \tag{61}$$

Then,

$$\|u_p\|_{L^\infty(0,1)} \le C\|\mathbf{v}\|_{L^\infty(0,1)}. \tag{62}$$

Let $\mu_p := \max\{\mu_{pq} : q = 1, 2, \cdots, M\}$. On the other hand,

$$\begin{aligned} &\sup_{0\le\tau_1,\tau_2\le 1} \left\{\frac{|u_p(\tau_1)-u_p(\tau_2)|}{|\tau_1-\tau_2|^{1-\mu_p}}\right\} \\ &\le \sup_{0\le\tau_1,\tau_2\le 1} \left\{\sum_{q=1}^{M} \frac{|u_{pq}(\tau_1)-u_{pq}(\tau_2)|}{|\tau_1-\tau_2|^{1-\mu_p}}\right\} \\ &\le \sup_{0\le\tau_1,\tau_2\le 1} \left\{\sum_{q=1}^{M} \frac{|u_{pq}(\tau_1)-u_{pq}(\tau_2)|}{|\tau_1-\tau_2|^{1-\mu_{pq}}}\right\} \\ &\le M\|u_{pq}\|_{C^{0,1-\mu_{pq}}(0,1)} \le M\|v_q\|_{L^\infty(0,1)} \le M\|\mathbf{v}\|_{L^\infty(0,1)}. \end{aligned} \tag{63}$$

Combining (62) with (63), by the definition of $\|\cdot\|_{C^{0,1-\mu_p}(0,1)}$, we have

$$\|u_p\|_{C^{0,1-\mu_p}} = \|u_p\|_{L^\infty(0,1)} + \sup_{0\le\tau_1,\tau_2\le 1}\left\{\frac{|u_p(\tau_1)-u_p(\tau_2)|}{|\tau_1-\tau_2|^{1-\mu_p}}\right\} \le C\|\mathbf{v}\|_{L^\infty(0,1)}, \tag{64}$$

which means that $u_p(t) \in C^{0,1-\mu_p}(0,1)$. Then, we have $\mathbf{u}(t) \in (C^{0,1-\tilde{\mu}}(0,1))^M$.

From (61),

$$|\mathbf{u}(t)| = \max\{|u_p(t)| : p = 1, 2, \cdots, M\} \le C\|\mathbf{v}\|_{L^\infty(0,1)}. \tag{65}$$

Then,

$$\|\mathbf{u}\|_{L^\infty(0,1)} \le C\|\mathbf{v}\|_{L^\infty(0,1)}. \tag{66}$$

From (63),

$$\max_{0\le p\le M} \sup_{0\le\tau_1,\tau_2\le 1}\left\{\frac{|u_p(\tau_1)-u_p(\tau_2)|}{|\tau_1-\tau_2|^{1-\mu_p}}\right\} \le C\|\mathbf{v}\|_{L^\infty(0,1)}. \tag{67}$$

Combine (66) with (67),

$$\|\mathbf{u}\|_{(C^{0,1-\tilde{\mu}}(0,1))^M} = \|\mathbf{u}\|_{L^\infty(0,1)} + \max_{0\le p\le M} \sup_{0\le\tau_1,\tau_2\le 1}\left\{\frac{|u_p(\tau_1)-u_p(\tau_2)|}{|\tau_1-\tau_2|^{1-\mu_p}}\right\} \le C\|\mathbf{v}\|_{L^\infty(0,1)}. \tag{68}$$

□

Lemma 8 [5]*Assume that $\partial_t^m R(t)$ is continuous on its own definition domain, and $P_N(t)$ is polynomial of order not exceeding N. Then,*

$$\left|\int_{-1}^{1} \omega^{\alpha,\beta}(v) R(v) P_N(v) dv - \sum_{k=0}^{N} R\left(v_k^{\alpha,\beta}\right) P_N\left(v_k^{\alpha,\beta}\right) w_k^{\alpha,\beta}\right| \\ \leq CN^{-m}\|\partial_t^m R\|_{L^2_{\omega^{\alpha,\beta}}(-1,1)}\|P_N\|_{L^\infty(0,1)}, \tag{69}$$

where $v_k^{\alpha,\beta}, w_k^{\alpha,\beta}, k = 0, 1, \cdots, N$ are Gauss points and weights of order N corresponding to weight function $\omega^{\alpha,\beta}(v)$ on interval $[-1, 1]$.

Lemma 9 *Assume that $\mathbf{u}^N(t) = [u_1^N(t), u_2^N(t), \cdots, u_M^N(t)]^T$, where $u_i^N(t), i = 1, 2, \cdots, M$ are polynomials of order not exceeding N. Then,*

$$\|(\mathscr{V} - \mathscr{V}^N)(\mathbf{u}^N)\|_{L^\infty(0,1)} \leq CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{u}^N\|_{L^\infty(0,1)}, \tag{70}$$

where $\partial_s^m \mathbf{K}(t, s) := [\partial_s^m K_{pq}(t, s)]_{M\times M} \in C(\Delta)$, and $\mathscr{V}$ and $\mathscr{V}^N$ are defined by (4) and *respectively, .*

Proof Note that

$$(\mathscr{V}-\mathscr{V}^N)(\mathbf{u}^N)(t)=\left[\sum_{q=1}^{N}\left(V_{1q}-V_{1q}^N\right)\left(u_q^N\right)(t), \sum_{q=1}^{N}\left(V_{2q}-V_{2q}^N\right)\left(u_q^N\right)(t), \cdots, \sum_{p=1}^{N}\left(V_{Mq}-V_{Mq}^N\right)\left(u_q^N\right)(t)\right]^T. \tag{71}$$

By Lemma 8,

$$\begin{aligned} |(V_{pq} - V_{pq}^N)(u_q^N)(t)| &\leq CN^{-m}\|\partial_v^m K_{pq}(t, s(t,\cdot)\|_{L^2_{\omega^{0,-\mu_{pq}}}(-1,1)}\|u_q^N(s(t,\cdot))\|_{L^\infty(-1,1)} \\ &\leq CN^{-m}\|\partial_v^m K_{pq}(t, s(t,\cdot)\|_{L^\infty(-1,1)}\|u_q^N\|_{L^\infty(0,t)} \\ &\leq CN^{-m}\|\partial_s^m K_{pq}(t,\cdot)\|_{L^\infty(0,t)}\|u_q^N\|_{L^\infty(0,t)} \\ &\leq CN^{-m}\|\partial_s^m K_{pq}\|_{L^\infty(\Delta)}\|u_q^N\|_{L^\infty(0,1)} \\ &\leq CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{u}^N\|_{L^\infty(0,1)}. \end{aligned} \tag{72}$$

Then,

$$\begin{aligned} \left|\sum_{q=1}^{M}\left(V_{pq} - V_{pq}^N\right)\left(u_q^N\right)(t)\right| &\leq \sum_{q=1}^{M}\left|\left(V_{pq} - V_{pq}^N\right)\left(u_q^N\right)(t)\right| \\ &\leq \sum_{q=1}^{M} CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{u}^N\|_{L^\infty(0,1)} \\ &\leq CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{u}^N\|_{L^\infty(0,1)}. \end{aligned} \tag{73}$$

Finally,

$$\begin{aligned} \|(\mathscr{V} - \mathscr{V}^N)(\mathbf{u}^N)\|_{L^\infty(0,1)} &= \max_{0\leq t\leq 1}|(\mathscr{V} - \mathscr{V}^N)(\mathbf{u}^N)(t)| \\ &= \max_{1\leq p\leq M}\max_{0\leq t\leq 1}|\sum_{q=1}^{N}(V_{pq} - V_{pq}^N)(u_q^N)(t)| \\ &\leq CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{u}^N\|_{L^\infty(0,1)}. \end{aligned} \tag{74}$$

□

4 Convergence analysis

In this section, we provide convergence analysis for the numerical method in Section 2.

Theorem 1 *Assume that* $\mathbf{y}(t)$ *is the solution to system* (1)*, and the corresponding numerical solution* $\mathbf{y}^N(t)$ *obtained by numerical method in* Section 2 *exists and is unique.*

(I) If $\mathbf{g}(t) \in (C(0,1))^M$ *and* $\partial_s^m \mathbf{K} \in C(\Delta)$*, then*

$$\|\mathbf{y} - \mathbf{y}^N\|_{L^\infty(0,1)} \le C \log(N) N^{\mu-1} \|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M} (1 + N^{1-\mu-m} \|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}), \tag{75}$$

and

$$\max_{0 \le i \le N} \|(\mathbf{y} - \mathbf{y}^N)(t_i)\| \le C \log(N) N^{-m} \|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M} \|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}. \tag{76}$$

(II) If $\mathbf{y}(t) \in (C^m(0,1))^M$ *and* $\partial_s^m \mathbf{K}(t,s) \in C(\Delta)$*, then*

$$\|\mathbf{y} - \mathbf{y}^N\|_{L^\infty(0,1)} \le C \log(N) N^{1-r-m} \|\mathbf{y}\|_{(C^{m-1,r}(0,1))^M} (1 + N^{r-1} \|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}), \forall\, 0 < r < 1. \tag{77}$$

Convergence analysis result (75) shows that global convergence order is $\log(N)N^{\mu-1}$ which is related to the regularity of $\mathbf{y}(t)$. Convergence analysis result (77) implies that better regularity of $\mathbf{y}(t)$ will enhance the global convergence order to $\log(N)N^{1-r-m}$, $\forall\, 0 < r < 1$. Local convergence order on collocation points is $\log(N)N^{-m}$ which is only related to the regularity of $\mathbf{K}(t,s)$ with respect to variable s. It is worth noting that N and m are independent of each other.

Proof Subtract (19) from (10),

$$\mathbf{y}(t_i) - \mathbf{y}_i = \mathscr{V}(\mathbf{y})(t_i) - \mathscr{V}^N(\mathbf{y}^N)(t_i), i = 0, 1, \cdots, N. \tag{78}$$

Change the right-hand side of (78) as

$$\mathscr{V}(\mathbf{y})(t_i) - \mathscr{V}^N(\mathbf{y}^N)(t_i) = \mathbf{E}_1(t_i) + \mathbf{E}_2(t_i), \tag{79}$$

where

$$\mathbf{E}_0(t) := \mathbf{y}(t) - \mathbf{y}^N(t), \tag{80}$$

$$\mathbf{E}_1(t) := \mathscr{V}(\mathbf{E}_0)(t), \tag{81}$$

$$\mathbf{E}_2(t) := (\mathscr{V} - \mathscr{V}^N)(\mathbf{y}^N)(t). \tag{82}$$

Multiply $L_i(t)$ to both side of (78) and sum up for $i = 0$ to $i = N$,

$$I_N(\mathbf{y})(t) - \mathbf{y}^N(t) = I_N(\mathscr{V}(\mathbf{E}_0))(t) + I_N(\mathbf{E}_1)(t), \tag{83}$$

which can be written as follows:

$$\mathbf{E}_0(t) = (I - I_N)(\mathbf{y})(t) + (I_N - I)(\mathscr{V}(\mathbf{E}_0))(t) + I_N(\mathbf{E}_1)(t) + \mathscr{V}(\mathbf{E}_0)(t). \tag{84}$$

By Lemma 2,

$$|\mathbf{E}_0(t)| \le C(|(I - I_N)(\mathbf{y})(t)| + |(I_N - I)(\mathscr{V}(\mathbf{E}_0))(t)| + |I_N(\mathbf{E}_1)(t)|). \tag{85}$$

Then,

$$\begin{aligned}\|\mathbf{E}_0\|_{L^\infty(0,1)} \le\ & C(\|(I - I_N)(\mathbf{y})\|_{L^\infty(0,1)} \\ &+ \|(I_N - I)(\mathscr{V}(\mathbf{E}_0))\|_{L^\infty(0,1)} + \|I_N(\mathbf{E}_1)\|_{L^\infty(0,1)}).\end{aligned} \tag{86}$$

From Lemma 2, $\mathbf{y} \in (C^{0,1-\mu}(0,1))^M$ where $\mu = \max\{\mu_{pp} : 1 \le p \le M\}$. By Lemma 5,

$$\|(I - I_N)(\mathbf{y})\|_{L^\infty(0,1)} \le C\log(N)N^{\mu-1}\|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M}. \tag{87}$$

From Lemma 7, $\mathscr{V}(\mathbf{E}_0) \in (C^{0,1-\tilde{\mu}}(0,1))^M$ where $\tilde{\mu} := \{u_{pq} : 1 \le p, q \le M\}$. By Lemma 5,

$$\|(I_N - I)(\mathscr{V}(\mathbf{E}_0))\|_{L^\infty(0,1)} \le C\log(N)N^{\tilde{\mu}-1}\|\mathscr{V}(\mathbf{E}_0)\|_{(C^{0,1-\tilde{\mu}}(0,1))^M}. \tag{88}$$

By Lemma 7,

$$\|\mathscr{V}(\mathbf{E}_0)\|_{(C^{0,1-\tilde{\mu}}(0,1))^M} \le C\|\mathscr{V}(\mathbf{E}_0)\|_{L^\infty(0,1)}. \tag{89}$$

By (66),

$$\|\mathscr{V}(\mathbf{E}_0)\|_{L^\infty(0,1)} \le C\|\mathbf{E}_0\|_{L^\infty(0,1)}. \tag{90}$$

Combine (88) with (89) and (90),

$$\|(I_N - I)(\mathscr{V}(\mathbf{E}_0))\|_{L^\infty(0,1)} \le C\log(N)N^{\tilde{\mu}-1}\|\mathbf{E}_0\|_{L^\infty(0,1)}. \tag{91}$$

By Lemma 5,

$$\|I_N(\mathbf{E}_1)\|_{L^\infty(0,1)} \le C\log(N)\|\mathbf{E}_1\|_{L^\infty(0,1)}. \tag{92}$$

By Lemma 9,

$$\begin{aligned}\|\mathbf{E}_1\|_{L^\infty(0,1)} &= \|(\mathscr{V} - \mathscr{V}^N)(\mathbf{y}^N)\|_{L^\infty(0,1)} \\ &\le CN^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{y}^N\|_{L^\infty(0,1)} \\ &\le CN^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{y}\|_{L^\infty(0,1)} + \|\mathbf{E}_0\|_{L^\infty(0,1)}).\end{aligned} \tag{93}$$

Combine (92) with (93),

$$\|I_N(\mathbf{E}_1)\|_{L^\infty(0,1)} \le C\log(N)N^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{y}\|_{L^\infty(0,1)} + \|\mathbf{E}_0\|_{L^\infty(0,1)}). \tag{94}$$

Combine (86) with (87), (88), (91), and (94),

$$\begin{aligned}\|\mathbf{E}_0\|_{L^\infty(0,1)} \le\ & C\log(N)N^{\mu-1}\|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M} + C\log(N)N^{\tilde{\mu}-1}\|\mathbf{E}_0\|_{L^\infty(0,1)} \\ &+ C\log(N)N^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{y}\|_{L^\infty(0,1)} + \|\mathbf{E}_0\|_{L^\infty(0,1)}).\end{aligned} \tag{95}$$

For sufficient large N,

$$\|\mathbf{E}_0\|_{L^\infty(0,1)} \le C\log(N)N^{\mu-1}(\|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M} + N^{1-\mu-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{y}\|_{L^\infty(0,1)}), \tag{96}$$

which together with the inequality $\|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M} \ge \|\mathbf{y}\|_{L^\infty(0,1)}$ yields (75).

From (85),

$$|\mathbf{E}_0(t_i)| \le C(|(I - I_N)(\mathbf{y})(t_i)| + |(I_N - I)(\mathscr{V}(\mathbf{E}_0))(t_i)| + |I_N(\mathbf{E}_1)(t_i)|). \tag{97}$$

Note that $(I - I_N)(\mathbf{u})(t_i) = \mathbf{0}$ for $\mathbf{u} = \mathbf{y}$ and $\mathscr{V}(\mathbf{E}_0)$. Then,

$$|\mathbf{E}_0(t_i)| \le C|I_N(\mathbf{E}_1)(t_i)|. \tag{98}$$

By (94),

$$|I_N(\mathbf{E}_1)(t_i)| \leq C\log(N)N^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{y}\|_{L^\infty(0,1)} + \|\mathbf{E}_0\|_{L^\infty(0,1)}). \tag{99}$$

In view of (75) and $\|\mathbf{y}\|_{L^\infty(0,1)} \leq \|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M}$,

$$\begin{aligned}|I_N(\mathbf{E}_1)(t_i)| \leq\ & C\log(N)N^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M}\\ & (1 + C\log(N)N^{\mu-1}(1+N^{1-\mu-m}\|\partial_s^m\mathbf{K}(t,\cdot)\|_{L^\infty(0,1)})).\end{aligned} \tag{100}$$

For sufficient large N,

$$|I_N(\mathbf{E}_1)(t_i)| \leq C\log(N)N^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{y}\|_{(C^{0,1-\mu}(0,1))^M}, \tag{101}$$

which together with (98) yields (76).

If $\mathbf{y}(t) \in C^m(0,1)$, then by (56),

$$\|(I - I_N)(\mathbf{y})\|_{L^\infty(0,1)} \leq C\log(N)N^{1-r-m}\|\mathbf{y}\|_{(C^{m-1,r}(0,1))^M}, \forall\, 0 < r < 1. \tag{102}$$

Combine (86) with (91), (94), and (102),

$$\begin{aligned}\|\mathbf{E}_0\|_{L^\infty(0,1)} \leq\ & C\log(N)N^{1-r-m}\|\mathbf{y}\|_{(C^{m-1,r}(0,1))^M} + C\log(N)N^{\mu-1}\|\mathbf{E}_0\|_{L^\infty(0,1)}\\ & + C\log(N)N^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{y}\|_{L^\infty(0,1)} + \|\mathbf{E}_0\|_{L^\infty(0,1)}).\end{aligned} \tag{103}$$

For sufficient large N,

$$\|\mathbf{E}_0\|_{L^\infty(0,1)} \leq C\log(N)N^{1-r-m}\|\mathbf{y}\|_{(C^{m-1,r}(0,1))^M}(1 + N^{r-1}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}). \tag{104}$$

This is (77). □

5 Numerical experiments

We carry out numerical experiments in this section.

Example 1 Let

$$\mathbf{g}(t) = [g_1(t), g_1(t), g_1(t)]^T,$$

where $g_1(t) = g_2(t) = g_3(t) = 1 + 2t^{1/2} + 3t^{2/3} + 4t^{3/4}$, and

$$\mathbf{K}(t,s) = \begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}, \mathbf{h}(t-s) = \begin{bmatrix} (t-s)^{-1/2} & (t-s)^{-1/3} & (t-s)^{-1/4}\\ (t-s)^{-1/3} & (t-s)^{-1/4} & (t-s)^{-1/2}\\ (t-s)^{-1/4} & (t-s)^{-1/4} & (t-s)^{-1/3}\end{bmatrix}. \tag{105}$$

The corresponding exact solution is $\mathbf{y}(t) = [1, 1, 1]^T$. We investigate the global numerical errors

$$\xi(N) := \|\mathbf{y} - \mathbf{y}^N\|_{L^\infty(0,1)},$$

where $\mathbf{y}^N(t)$ defined by (14) is the numerical solution to VIEs (1). It is worth noting that the solution $\mathbf{y}(t) \in C^m(0,1)$ for sufficient large m. The convergence result (77) shows that the global convergence order is $\log(N)N^{1-r-m}$, $\forall\, 0 < r < 1$. Numerical errors versus N are demonstrated in Table 1 and plotted in Fig. 1 which shows that numerical solution possesses very high precision. Increasing N, numerical errors stay near the level 10^{-10}.

Table 1 Example 1: Error versus N

N	2	6	10	14	16	20
$\xi(N)$	1.33e−15	9.52e−13	5.91e−12	4.91e−11	3.11e−11	2.35e−11

In general case, it is impossible to obtain the expression of the exact solution to VIEs (1). In order to effectively investigate the performance of numerical errors, we define global numerical error

$$\epsilon(N) := \max_{t\in[0,1]} |\mathbf{y}^N(t) - \mathbf{g}(t) - \mathscr{V}^{30}(\mathbf{y}^N)(t)|. \tag{106}$$

The corresponding local error is

$$\delta(N) := \max_{t\in\{t_i\}_{i=0}^N} |\mathbf{y}^N(t) - \mathbf{g}(t) - \mathscr{V}^{30}(\mathbf{y}^N)(t)|. \tag{107}$$

Example 2 Consider system of weakly singular VIEs (1) with

$$\mathbf{g}(t) = [\sin t, \cos t, e^t]^T, \tag{108}$$

and

$$\mathbf{K}(t,s) = \begin{bmatrix} \sin(t+s) & \cos(t+s) & e^{t-s} \\ \cos(t+s) & \sin(t+0.5s) & e^{t+s} \\ \sin(t+s+1) & \cos(t+s-1) & e^{t-0.5s} \end{bmatrix}. \tag{109}$$

Case I:

$$\mathbf{h}(t-s) = \begin{bmatrix} (t-s)^{-0.1} & (t-s)^{-0.1} & (t-s)^{-0.1} \\ (t-s)^{-0.1} & (t-s)^{-0.1} & (t-s)^{-0.1} \\ (t-s)^{-0.1} & (t-s)^{-0.1} & (t-s)^{-0.1} \end{bmatrix}. \tag{110}$$

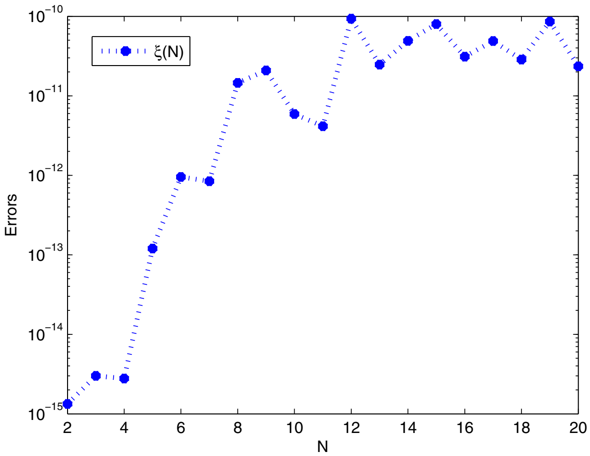

Fig. 1 Example 1: Error versus N

Table 2 Example 2: Errors versus N

N	2	6	10	14	16	20
$\epsilon(N)$ for Case I	5.21e−00	7.67e−02	1.02e−03	5.13e−04	4.85e−04	2.08e−04
$\delta(N)$ for Case I	1.81e−03	4.63e−11	1.71e−13	3.69e−13	8.53e−14	1.99e−13
$\epsilon(N)$ for Case II	1.27e+01	7.64e−00	2.72e−00	2.51e−00	3.03e−00	1.90e−00
$\delta(N)$ for Case II	6.37e−04	5.31e−10	1.07e−14	1.87e−14	4.89e−15	6.44e−15

Case II:

$$\mathbf{h}(t-s) = \begin{bmatrix} (t-s)^{-0.9} & (t-s)^{-0.9} & (t-s)^{-0.9} \\ (t-s)^{-0.9} & (t-s)^{-0.9} & (t-s)^{-0.9} \\ (t-s)^{-0.9} & (t-s)^{-0.9} & (t-s)^{-0.9} \end{bmatrix}. \tag{111}$$

This example is to provide to underline the role of μ in the performance of errors decaying. From the convergence result, global convergence order is $\log(N)N^{\mu-1}$ and the local convergence order is $\log(N)N^{-m}$. In other words, if μ is smaller, global errors decay faster. The local convergence order on collocation points is independent of μ but depends on m, the regularity index of given functions especially kernel functions. Numerical errors versus N are recorded in Table 2 and plotted in Fig. 2 which shows that global errors corresponding to $\mu = 0.1$ decay faster than the one corresponding to $\mu = 0.9$, while there is no significant difference for local errors between two cases.

Example 3 Consider VIEs (1) with

$$\mathbf{g}(t) = [\sin t, \cos t]^T, \mathbf{h}(t-s) = \begin{bmatrix} (t-s)^{-0.1} & (t-s)^{-0.2} \\ (t-s)^{-0.2} & (t-s)^{-0.1} \end{bmatrix}.$$

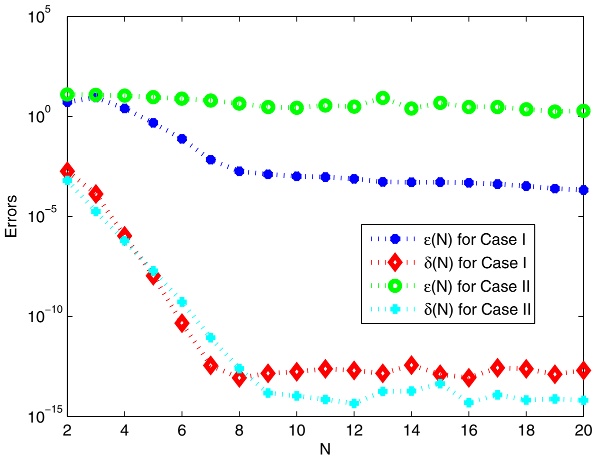

Fig. 2 Example 2: Errors versus N

Table 3 Example 3: Error versus N

N	2	6	10	14	16	20
$\epsilon(N)$ for Case I	3.14e−01	2.08e−03	5.13e−04	2.60e−04	2.45e−04	1.05e−04
$\delta(N)$ for Case I	2.01e−04	2.61e−13	4.44e−15	5.33e−15	1.78e−15	4.44e−15
$\epsilon(N)$ for Case II	3.46e−01	2.07e−03	5.12e−04	2.59e−04	2.45e−04	1.05e−04
$\delta(N)$ for Case II	2.78e−03	2.49e−04	6.56e−05	2.45e−05	1.58e−05	6.83e−06

Case I:

$$\mathbf{K}(t,s) = \begin{bmatrix} t^{1/2}+s & t^{1/2}-s \\ \sin(t^{1/2}+s) & \cos(t^{1/2}-s) \end{bmatrix}. \tag{112}$$

Case II:

$$\mathbf{K}(t,s) = \begin{bmatrix} t+s^{1/2} & t-s^{1/2} \\ \sin(t+s^{1/2}) & \cos(t-s^{1/2}) \end{bmatrix}. \tag{113}$$

The convergence analysis result (76) shows that local convergence order $\log(N)N^{-m}$ depends on m, the regularity index of $\mathbf{K}(t,s)$ with respect to variable s. In order to confirm this theoretical result, we carry out numerical experiments for two cases. The kernel functions in Case I possess better regularity with respect variable s than t, while it is opposite in Case II. Numerical errors versus N recorded in Table 3 and plotted in Fig. 3 show that, if $\mathbf{K}(t,s)$ possess better regularity with respect to variable s, local convergence order on collocation points will be higher. Global errors of Case I and Case II perform similarly to each other. This confirm the theoretical result that global convergence order depends on the regularity of $\mathbf{y}(t)$.

Base on the common sense that high-order weakly singular Volterra integro-differential equations can be transformed to be system (1), we consider the following

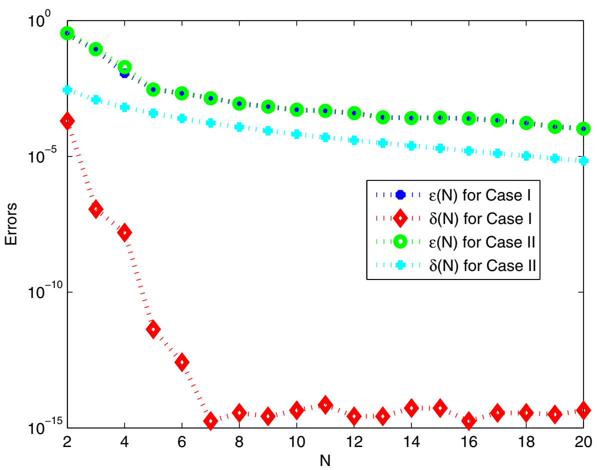

Fig. 3 Example 3: Errors versus N

high-order VIDE

$$y^{(n)}(t) = g(t) + \sum_{i=0}^{n-1} a_j(t) y^{(j)}(t) + \sum_{i=0}^{n} \int_0^t (t-s)^{-\mu_i} K_i(t,s) y^{(i)}(s) ds, \tag{114}$$

with $y^{(i)}(0) = c_i$, $i = 0, 1, \cdots, n-1$. Let $y_i(t) := y^{(i)}(t)$, then

$$y_i(t) = c_i + \int_0^t y_{i+1}(s) ds, i = 0, 1, \cdots, n-1. \tag{115}$$

The system of VIEs corresponding to (114) is

$$\begin{aligned} y_0(t) &= c_0 + \int_0^t y_1(s) ds, \\ y_1(t) &= c_1 + \int_0^t y_2(s) ds, \\ &\cdots \\ y_{n-1}(t) &= c_{n-1} + \int_0^t y_n(s) ds, \\ y_n(t) &= f(t) + \sum_{i=0}^{n-1} a_i(t) c_i \\ &\quad + \sum_{i=0}^{n-1} a_i(t) \int_0^t y_{i+1}(s) ds, \\ &\quad + \sum_{i=0}^{n} \int_0^t (t-s)^{-\mu_i} K_i(t,s) y_i(s) ds. \end{aligned} \tag{116}$$

Write them in matrix form,

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_0^t \mathbf{H}(t,s) \mathbf{y}(s) ds, \tag{117}$$

where

$$\mathbf{g}(t) := [c_0, c_1, \cdots, c_{n-1}, f(t) + \sum_{i=0}^{n-1} a_i(t) c_i]^T, \tag{118}$$

and

$$\mathbf{H}(t,s) := \begin{bmatrix} 0 & 1 & 0 & \cdots 0 \\ 0 & 0 & 1 & \cdots 0 \\ \vdots & \vdots & \vdots & \ddots \vdots \\ 0 & 0 & 0 & \cdots 1 \\ H_0(t,s) & H_1(t,s) & H_2(t,s) & \cdots & H_N(t,s) \end{bmatrix}, \tag{119}$$

$$H_0(t,s) = (t-s)^{-\mu_0} K_0(t,s),$$

$$H_i(t,s) := a_i(t) + (t-s)^{-\mu_{i+1}} K_{i+1}(t,s), i = 1, 2 \cdots, N.$$

The discrete system for (117) is

$$\mathbf{y}_i = \mathbf{g}(t_i) + \widetilde{\mathscr{V}}^N(\mathbf{y}^N)(t_i), i = 0, 1, \cdots, N, \tag{120}$$

where $\widetilde{\mathscr{V}}^N(\mathbf{y}^N)(t_i)$ is a matrix of dimension $(n+1)\times 1$, whose jth $(1\le j\le n)$ element is

$$\frac{t_i}{2}\sum_{p=0}^{N} y_{jp}\sum_{k=0}^{N} L_p(s(t_i, v_k^{(0,0)}))w_k^{(0,0)} \approx \int_0^{t_i} y_j(s)ds,$$

where $v_k^{(\alpha,\beta)}$ and $w_k^{(\alpha,\beta)}$ are N-order Jacobi Gauss points and weights in interval $[-1, 1]$ corresponding to the weight function $\omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$. The $n+1$-th element of $\widetilde{\mathscr{V}}^N(\mathbf{y}^N)(t_i)$ is

$$\sum_{p=0}^{n}\widetilde{V}^N(y_p)(t_i),$$

where

$$\widetilde{V}^N(y_0)(t_i)=\left(\frac{t_i}{2}\right)^{1-\mu_0}\sum_{j=0}^{N} y_{0j}\sum_{k=0}^{N} K_0(t_i, s(t_i, v_k^{(-\mu_0,0)}))L_j(s(t_i, v_k^{(-\mu_0,0)}))w_k^{(-\mu_0,0)}$$
$$\approx\int_0^{t_i}(t-s)^{-\mu_0}K_0(t_i, s)y_0(s)ds, \tag{121}$$

and

$$\begin{aligned}\widetilde{V}^N(y_p)(t_i) =& a_{p-1}(t_i)\tfrac{t_i}{2}\sum_{j=0}^{N} y_{pj}\sum_{k=0}^{N} L_j(s(t_i, v_k^{(0,0)}))w_k^{(0,0)}\\ &+(\tfrac{t_i}{2})^{1-\mu_0}\sum_{j=0}^{N} y_{pj}\sum_{k=0}^{N} K_p(t_i, s(t_i, v_k^{(-\mu_p,0)}))L_j(s(t_i, v_k^{(-\mu_p,0)}))w_k^{(-\mu_p,0)}\\ &\approx \textstyle\int_0^{t_i}(a_{p-1}(t_i)+(t_i-s)^{-\mu_p}K_p(t_i, s)y_p(s))ds,\\ &p=1, 2, \cdots, n.\end{aligned} \tag{122}$$

Example 4 Consider high-order VIDE (114) for the following two cases:
Case I:

$$\begin{aligned}y''(t) =& -\cos t - 2t^{1/2} + \cos t y(t) + e^t y'(t) + \textstyle\int_0^t((t-s)^{-1/2}y(s)\\ &+(t-s)^{-1/3}\cos(t+s)y'(s) + (t-s)^{-1/4}e^{t-s}y''(s))ds,\end{aligned} \tag{123}$$

with $y(0) = 1$, $y'(0) = 0$. The corresponding exact solution is $y(t) = 1$, $t\in[0, 1]$.
Case II:

$$\begin{aligned}y''(t) =& \sin(t) + \cos t y(t) + e^t y'(t) + \textstyle\int_0^t((t-s)^{-1/2}\sin(t+s)y(s)\\ &+(t-s)^{-1/3}\cos(t+s)y'(s) + (t-s)^{-1/4}e^{t-s}y''(s))ds,\end{aligned} \tag{124}$$

with $y(0) = 1$, $y'(0) = 1$.

It is worth noting that kernel functions and exact solution to Case I are sufficient smooth. According to the convergence analysis result (77), global convergence order is $\log(N)N^{1-r-m}$, $\forall\ 0 < r < 1$. It implies that a high precision numerical solution will be obtained. The given functions in Case II are sufficient smooth. Then, the solution to Case II lies in $C^{0,3/4}(0, 1)$. The corresponding global convergence order is $\log(N)N^{-3/4}$ which is slower than $\log(N)N^{1-r-m}$ for Case I. The local convergence order for Case II is $\log(N)N^{-m}$ which is similar to global convergence order for

Table 4 Example 4: Error versus N

N	2	6	10	14	16	20
$\xi(N)$ for Case I	3.11e−15	1.11e−15	1.00e−15	2.22e−15	2.22e−15	3.00e−15
$\epsilon(N)$ for Case II	1.99e−00	1.34e−01	8.74e−03	5.32e−03	5.27e−03	2.38e−03
$\delta(N)$ for Case II	1.26e−03	2.41e−10	5.68e−14	7.11e−14	8.53e−14	4.97e−14

Case II. Numerical errors recorded in Table 4 and plotted in Fig. 4 confirm these theoretical results.

Nonlinear system of weakly singular VIEs is

$$\begin{cases} y_1(t) = g_1(t) + \int_0^t \sum_{j=1}^{M} (t-s)^{-\mu_{1j}} K_{1j}(t, s, y_1(s), \cdots, y_M(s))ds, \\ y_2(t) = g_2(t) + \int_0^t \sum_{j=1}^{M} (t-s)^{-\mu_{2j}} K_{2j}(t, s, y_1(s), \cdots, y_M(s))ds, \\ \cdots \\ y_M(t) = g_M(t) + \int_0^t \sum_{j=1}^{M} (t-s)^{-\mu_{Mj}} K_{Mj}(t, s, y_1(s), \cdots, y_M(s))ds. \end{cases} \tag{125}$$

Corresponding numerical scheme is

$$\begin{cases} y_{1i} = g_1(t_i) + S_1(N, t_i), \\ y_{2i} = g_2(t_i) + S_2(N, t_i), \\ \cdots \\ y_{Mi} = g_M(t_i) + S_M(N, t_i), \end{cases} , i = 0, 1, \cdots, N, \tag{126}$$

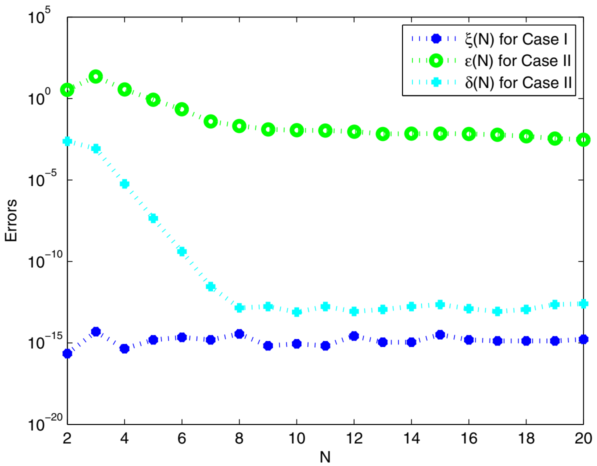

Fig. 4 Example 4: Errors versus N

Table 5 Example 5: Error versus N

N	2	6	10	14	16	20
$\epsilon(N)$	1.25e−01	1.84e−02	5.13e−03	3.46e−03	3.54e−03	1.66e−03
$\delta(N)$	5.59e−03	1.01e−04	1.33e−06	3.81e−07	2.07e−07	8.61e−08

where $S_p(N, t_i)$, $p = 1, 2, \cdots, M$ are defined as

$$S_p(N, t_i) := \sum_{j=1}^{M}\sum_{k=0}^{N}(t_i/2)^{1-\mu_{pj}} K_{pj}(t_i, s(t_i, v_k^{pj}), y_1^N(s(t_i, v_k^{pj})), \cdots, y_M^N(s(t_i, v_k^{pj}))) * w_k^{pj},$$

v_k^{pj}, w_k^{pj} are $N+1$-order Jacobi Gauss points in interval $[-1, 1]$ corresponding to weight function $\omega^{(-\mu_{pj},0)}(x) := (1-x)^{-\mu_{pj}}(1+x)^0$.

Example 5 Consider nonlinear VIEs (1) as follows:

$$\begin{aligned}
y_1(t) = \sin t + \textstyle\int_0^t ((t-s)^{-1/2}\sin(t-s+y_1(s)+y_2(s)-y_3(s))\\
+(t-s)^{-1/3}\cos(t+s+y_1(s)-y_2(s)+y_3(s))\\
+(t-s)^{-1/4}\cos(ts+y_1(s)-y_2(s)-y_3(s))ds/5,\\
y_2(t) = \cos t + \textstyle\int_0^t ((t-s)^{-1/4}\cos(t+s+y_1(s)y_2(s)+y_2(s)y_3(s)+y_1(s)y_3(s))\\
+(t-s)^{-1/3}\sin(t-s+y_1^2(s)+y_2^2(s)+y_3^2(s))\\
+(t-s)^{-1/2}\cos(ts+y_1^2(s)+y_2^2(s)-y_3^2(s)))ds/5,\\
y_3(t) = \sin 0.5t + \textstyle\int_0^t ((t-s)^{-1/3}\sin(t*s+y_1(s)y_2(s)-y_2(s)y_3(s)+y_1(s)y_3(s))\\
+(t-s)^{-1/2}\cos(ts+y_1^2(s)-y_2^2(s)+y_3^2(s))\\
+(t-s)^{-1/4}\cos(t-s+y_1^2(s)-y_2^2(s)-y_3^2(s)))ds/5.
\end{aligned} \tag{127}$$

Numerical errors versus N recorded in Table 5 and plotted in Fig. 5 show that global numerical errors decay slower that the local errors. This is similar to the linear case.

The nonlinear high-order weakly singular VIDE is

$$y^{(n)} = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t), \int_0^t (t-s)^{-\mu}K(t, s, Y(s))ds), \tag{128}$$

with $y^{(i)}(0) = c_i$, $i = 0, 1, \cdots, n-1$, where

$$Y(s) := [y(s), y'(s), \cdots, y^{(n)}(s)].$$

The corresponding system of VIEs is

$$\begin{aligned}
&y_0(t) = c_0 + \textstyle\int_0^t y_1(s)ds,\\
&\cdots\\
&y_{n-1}(t) = c_{n-1} + \textstyle\int_0^t y_n(s)ds,\\
&y_n(t) = f(t, y_0(t), y_1(t), \cdots, y_{n-1}(t), \textstyle\int_0^t (t-s)^{-\mu}K(t, s, \tilde{Y}(s))ds),
\end{aligned} \tag{129}$$

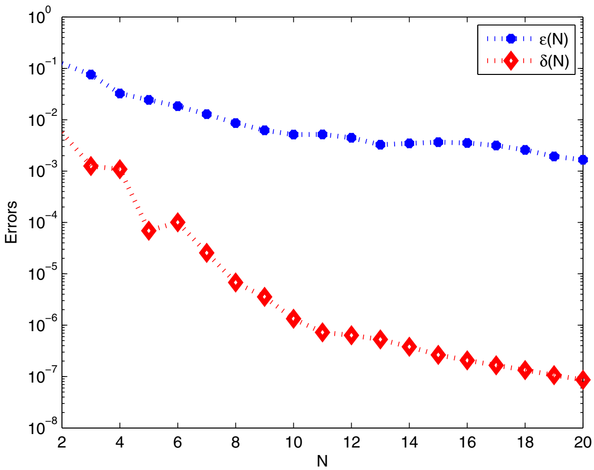

Fig. 5 Example 5: Errors versus N

where $y_i(t) := y^{(i)}(t)$, $\tilde{Y}(s) := [y_0(s), y_1(s), \cdots, y_n(s)]$. Then, numerical scheme is

$$\begin{aligned} y_{0i} &= c_0 + \tfrac{t_i}{2} \sum_{k=0}^{N} y_1^N(s(t_i, v_k^{(0,0)}))w_k^{(0,0)}, \\ &\cdots \\ y_{n-1,i} &= c_{n-1} + \tfrac{t_i}{2} \sum_{k=0}^{N} y_n^N(s(t_i, v_k^{(0,0)}))w_k^{(0,0)}, \\ y_{ni} &= f(t, y_0^N(t_i), y_1^N(t_i), \cdots, y_{n-1}^N(t_i), S(N, t_i)), \end{aligned} \tag{130}$$

where

$$S(N, t_i) := \left(\frac{t_i}{2}\right)^{1-\mu} \sum_{k=0}^{N} K(t, s(t_i, v_k^{(-\mu,0)}), \tilde{Y}^N(s(t_i, v_k^{(-\mu,0)})))w_k^{(-\mu,0)},$$

$$\tilde{Y}^N(s(t_i, v_k^{(-\mu,0)})) := \left[y_0^N\left(s\left(t_i, v_k^{(-\mu,0)}\right)\right), y_1^N\left(s\left(t_i, v_k^{(-\mu,0)}\right)\right), \cdots, y_n^N\left(s\left(t_i, v_k^{(-\mu,0)}\right)\right)\right].$$

Example 6 Consider nonlinear VIDE (128) as follows:

$$y''(t) = \sin\left(t + y(t) + y'(t)\right) + \frac{1}{5}\cos\left(\int_0^t (t-s)^{-1/2}\cos(t+s+y(s)+y'(s)-(y''(s))^2)\,ds\right). \tag{131}$$

Table 6 Example 6: Errors versus N

N	2	6	10	14	16	20
$\epsilon(N)$	8.31e−02	1.78e−02	2.05e−03	2.09e−04	1.25e−04	2.23e−05
$\delta(N)$	6.66e−16	3.33e−16	4.44e−16	1.72e−15	3.33e−16	5.00e−16

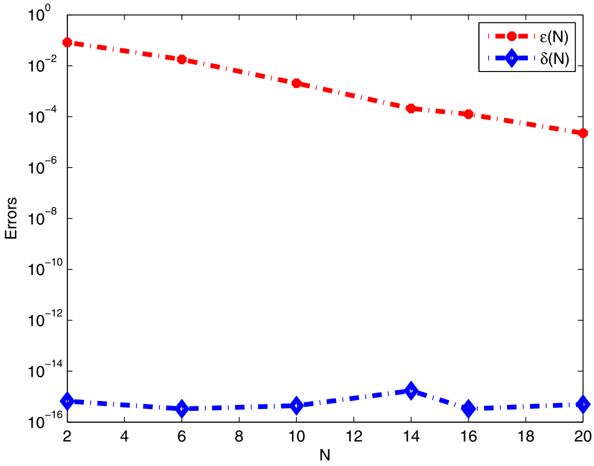

Fig. 6 Example 6: Errors versus N

Numerical errors versus N recorded in Table 6 and plotted in Fig. 6 show that global errors decay slower than local errors. This is also similar to the linear case.

6 Conclusion and future work

In [11], we have investigated the spectral collocation method for a weakly singular VIE with proportional delay. In [10], we investigated the spectral collocation method for a system of VIEs with smooth kernels. Based on the valuable research findings in these work, we investigate the spectral collocation method for the system of weakly singular VIEs in the present paper. The convergence analysis results are that the global convergence order depends on the regularity of the solution lying in this system, and the local convergence order only depends on the regularity of given functions especially the kernel functions with respect to variable s. Numerical experiments are carried out to confirm these theoretical results. Numerical examples include system of linear and nonlinear weakly singular VIEs and linear and nonlinear high-order VIDEs.

The convergence analysis result (76) shows that the high convergence order at collocation points is obtained, only if that kernel functions possess better regularity with respect to variable s. Based on this result, our future work will focus on the piecewise fractional polynomial collocation method for weakly singular VIEs.

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