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The hybrid Wilson finite volume method for elliptic problems on quadrilateral meshes

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Abstract In this paper, we construct and analyze a nonconforming finite volume method (FVM) for solving the elliptic boundary value problems on quadrilateral meshes: the hybrid Wilson FVM. Under the mesh assumption that the underlying mesh is an h^2-parallelogram mesh, we show that the scheme possesses first order in the mesh-dependent H^1-norm and second order in the L^2-norm error estimates, the same optimal convergence orders as those of the corresponding Wilson finite element method (FEM). Numerical results are presented to demonstrate the theoretical results on the convergence order of the method.

Keywords Nonconforming finite volume methods · L^2 error estimate · Quadrilateral meshes

Mathematics Subject Classification (2010) 65N30 · 65N12

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1 Introduction

The FVM has become one of the major numerical methods for solving partial differential equations in the past several decades. Preserving certain local conservation laws, flexible algorithm constructions, and easy implementation are the most attractive advantages of the FVM. Due to these advantages, the FVM is popular in scientific and engineering computations (cf. [12, 19, 22, 30, 31, 34]). Many researchers have studied this method extensively and obtained some important results. We refer to (cf. [1, 2, 4, 9, 16, 21, 23]) for an incomplete list of references.

In comparing with its wide applications, the development of the theoretical analysis of the FVMs lags far behind. Most existing works focus on the conforming FVM schemes. The literatures [8, 14, 24, 26, 36, 40] studied the inf-sup condition or the uniform ellipticity of the bilinear forms of the FVMs over triangular or rectangular meshes so as to establish the H^1 error estimates. For general quadrilateral meshes, Li and Li [25] studied a bilinear FVM and Yang [37] and Yang et al. [38] studied biquadratic FVMs. They all got optimal H^1 error estimates for the corresponding FVMs under the h^2-parallelogram mesh assumption. Under a weaker mesh condition that the underlying mesh is h^{1+r}-parallelogram ($r > 0$), Zhang and Zou [41] gave a unified proof for the inf-sup condition for any order (bi-k order) FVMs whose dual partitions are based on the Gauss and Lotatto points.

The L^2 error estimate of the FVMs is a challenging task since the difference between the exact and FV solutions is not orthogonal to the trial space with respect to the bilinear form of the corresponding FEM. For triangular meshes, the L^2 error estimate for linear FVMs is studied in [7, 13, 18]. Only very recently, Wang and Li [35] established a unified L^2 error analysis for a class of higher-order Lagrange FVMs provided that the dual partitions satisfy orthogonal conditions. For quadrilateral meshes, Zhang and Zou [40] derived optimal L^2 errors for bi-k order FVMs on rectangular meshes by assuming that the exact solution $u \in H_0^1 \cap H^{k+2}$. Paper [28] proved an optimal L^2 error estimate for a bilinear FVM by assuming that $u \in H_0^1 \cap H^3$ and the underlying mesh is h^2-parallelogram. Under the same mesh assumption as that in [28], papers [29, 38] derived optimal L^2 error estimates for biquadratic FVMs under a strong solution assumption that $u \in H_0^1 \cap H^4$. Recently, Lin et al. [27] gave optimal L^2 error estimates for a class of bi-k order FVMs whose dual partitions are based on Gauss and Lotatto points under a weak regularity assumption of $u \in H_0^1 \cap H^{k+1}$ and $f \in H^k$ and a weak mesh restriction of h^{1+r}-parallelogram, where f is the right-hand side function and $r \geq \frac{1+k}{2k}$.

The hybrid FVM was initially constructed for quadratic FVMs over two-dimensional triangular or rectangular grids in [6] and the optimal rate of convergence in H^1-norm was obtained there. Comparing with the Lagrange or Hermite FVMs, the hybrid FVMs enjoy more simple dual partitions especially for higher-order schemes. To our best knowledge, there are few works about the hybrid or the nonconforming FVMs (cf. [3, 5, 9, 10]). Very recently, the first author of this paper and her cooperator in [39] established a convergence theorem applicable to the nonconforming triangle mesh based FVMs as well as the rectangle mesh based FVMs for solving the second order elliptic boundary problems.

In this paper, the nonconforming hybrid Wilson FVM for solving the elliptic boundary value problems on quadrilateral meshes is considered, whose trial space is the well-known nonconforming Wilson FE space and test space is spanned by the characteristic functions of the control volumes and the nonconforming functions of the trial space. Since the mapping from a reference square to a general quadrilateral is not an affine mapping, the techniques using in the theoretical analysis of [39] no long work for the hybrid Wilson FVM proposed in this paper. The main purpose of this paper is to derive the optimal mesh dependent H^1 and L^2 error estimates for the hybrid Wilson FVM under certain mesh assumptions. Lemma 3.8 is crucial to the establishment of the uniform ellipticity of the bilinear form of the hybrid Wilson FVM. Specifically, by Lemma 3.8, the proof of the positive definiteness of the 6×6 element stiffness matrix with two parameters is reduced to the proof of the positive definiteness of a simple 3×3 matrix with just one parameter. According to this idea, we establish the positive definiteness of the element stiff matrix when the quadrilateral element is a parallelogram. When the quadrilateral is an h^{1+r}-parallelogram ($r > 0$), we treat it as a perturbation of a parallelogram and establish the positive definiteness of its element stiff matrix. The nonconforming error term is properly estimated when $r = 1$. Consequently, the optimal mesh dependent H^1 error estimate of the hybrid Wilson FVM solution can be obtained. By the space decomposition and the Aubin-Nitsche technique, the L^2 error estimate of the scheme is reduced to the analysis of the difference of bilinear forms between the FVM and its corresponding FEM on the conforming part of the trial space, which can be properly treated using the techniques employed in the lower-order FVM schemes.

The rest of this paper is organized as follows. In Section 2, we present the hybrid Wilson FVM for Poisson equations. In Section 3, we establish the mesh dependent H^1 error estimate of our method. In Section 4, we discuss the L^2 error estimate of the method. In the last section, we present numerical examples to confirm the convergence results.

In this paper, the notations of Sobolev spaces and associated norms are the same as those in [11] and C or C_i will denote generic positive constants independent of meshes and may be different at different occurrences, where i is a positive integer.

2 The hybrid wilson FVM

Let Ω be a polygonal domain in $\mathbb{R}^2$ and $f \in L^2(\Omega)$. We consider the Poisson equation with the Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where u is the unknown to be determined.

Let $\mathcal{T} := \{K\}$ be a quadrilateral partition of Ω, where the intersection of any two closed quadrilaterals is either a common side or a common vertex. Let h_K be the diameter of the element K and $h := \max\{h_K | K \in \mathcal{T}\}$. Let ρ_K be the smallest length

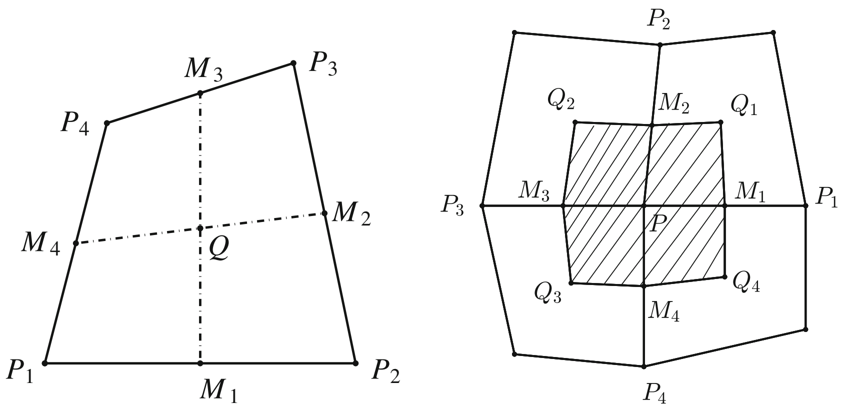

Fig. 1 Left: control volume in K. Right: control volume around P

of the edges and let θ_K be any interior angle of K. We denote by S_K the measure of K.

We shall introduce a dual partition $\mathcal{T}^*$ of $\mathcal{T}$, whose elements are called control volumes. Let $\mathbb{N}_m := \{1, 2, \ldots, m\}$ for a positive integer m. We use $\overline{RS}$ to denote the line segment between R and S and use $\left|\overline{RS}\right|$ to denote its length. We denote by $\Box P_1P_2P_3P_4$ a quadrilateral K with vertices $P_i := (x_i, y_i), i \in \mathbb{N}_4$ in counter clockwise order (see Fig. 1 left). Let $M_i, i \in \mathbb{N}_4$ be the midpoint of the edge $\overline{P_iP_{i+1}}$, where $P_5 := P_1$ and Q be the intersection of $\overline{M_1M_3}$ and $\overline{M_4M_2}$. The control volume in K associated with $P_i, i \in \mathbb{N}_4$ is the subregion $\Box P_iM_iQM_{i-1}$, where $M_0 := M_4$. Then for each vertex P of a quadrilateral in $\mathcal{T}$, we associate a control volume K_P^*, which is built by the union of the above subregions sharing the vertex P (see Fig. 1 right, the region with boundary $M_1Q_1M_2Q_2M_3Q_3M_4Q_4M_1$). The collection of these control volumes makes the dual partition $\mathcal{T}^*$. This dual partition is also used for the bilinear FVMs on quadrilateral meshes (see [24, 25, 32]).

For a quadrilateral K as in Fig. 1, we introduce some notations. Let

$$\vec{a} := (a_1, a_2), \quad \vec{b} := (b_1, b_2), \quad \vec{m} := (m_1, m_2), \tag{2.2}$$

where

$$\begin{aligned} a_1 &:= \tfrac{(x_2+x_3)-(x_1+x_4)}{2}, \ b_1 := \tfrac{(x_3+x_4)-(x_1+x_2)}{2}, \ m_1 := \tfrac{(x_1+x_3)-(x_2+x_4)}{2}, \\ a_2 &:= \tfrac{(y_2+y_3)-(y_1+y_4)}{2}, \ b_2 := \tfrac{(y_3+y_4)-(y_1+y_2)}{2}, \ m_2 := \tfrac{(y_1+y_3)-(y_2+y_4)}{2}. \end{aligned}$$

Let O_1 and O_2 denote the midpoint of $\overline{P_1P_3}$ and $\overline{P_2P_4}$ respectively (see Fig. 2). Note that $\vec{a} = \overrightarrow{M_4M_2}$, $\vec{b} = \overrightarrow{M_1M_3}$ and $\vec{m} = \overrightarrow{O_2O_1}$. We choose the square $\hat{K}$ with vertices $\hat{P}_1 := (-1, -1)$, $\hat{P}_2 := (1, -1)$, $\hat{P}_3 := (1, 1)$ and $\hat{P}_4 := (-1, 1)$ on (ξ, η) plane as a reference element. There exists a mapping $\mathcal{F}_K : \hat{K} \to K$ with $\mathcal{F}_K(\hat{P}_i) = P_i, i \in \mathbb{N}_4$ such that (see Fig. 2)

$$\begin{cases} x = x_0 + \frac{a_1}{2}\xi + \frac{b_1}{2}\eta + \frac{m_1}{2}\xi\eta, \\ y = y_0 + \frac{a_2}{2}\xi + \frac{b_2}{2}\eta + \frac{m_2}{2}\xi\eta, \end{cases} \tag{2.3}$$

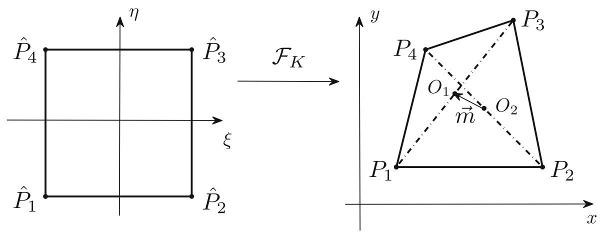

Fig. 2 The mapping $\mathcal{F}_K : \hat{K} \to K$

where $x_0 := \frac{x_1+x_2+x_3+x_4}{4}$, $y_0 := \frac{y_1+y_2+y_3+y_4}{4}$ is the coordinate of the point Q in K. Note that $\mathcal{F}_K$ is nonlinear and $\vec{m}$ measures its the nonlinearity. For parallelograms (including rectangles), O_1 and O_2 overlaps such that $\vec{m} = 0$. Then the mapping $\mathcal{F}_K$ becomes an affine mapping.

We describe the trial and test spaces on the reference element $\hat{K}$ for the hybrid Wilson FVM. The trial space $\mathbb{U}_{\hat{K}}$ on $\hat{K}$ is a space of polynomials of degree less than or equal to 2. The set of degrees of freedom $\hat{\Sigma} := \{\hat{f}_i : i \in \mathbb{N}_6\}$, where

$$\hat{f}_i(w) = w(\hat{P}_i),\ i \in \mathbb{N}_4 \quad \text{and} \quad \hat{f}_{4+j}(w) = \int_{\hat{K}} \partial_{jj} w,\ j \in \mathbb{N}_2. \tag{2.4}$$

There is a basis $\hat{\Phi} := \{\hat{\phi}_i : i \in \mathbb{N}_6\}$ for $\mathbb{U}_{\hat{K}}$ such that

$$\hat{f}_i(\hat{\phi}_j) = \delta_{i,j} := \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j \in \mathbb{N}_6. \tag{2.5}$$

By simple calculation, we get that

$$\begin{aligned} &\hat{\phi}_1 := (1/4)(1-\xi)(1-\eta), \ \hat{\phi}_2 := (1/4)(1+\xi)(1-\eta), \ \hat{\phi}_3 := (1/4)(1+\xi)(1+\eta), \\ &\hat{\phi}_4 := (1/4)(1-\xi)(1+\eta), \ \hat{\phi}_5 := (1/8)(\xi^2-1), \qquad\quad \hat{\phi}_6 := (1/8)(\eta^2-1). \end{aligned} \tag{2.6}$$

Let $\mathbb{U}_{c,\hat{K}} := \text{span}\{\hat{\phi}_i, i \in \mathbb{N}_4\}$ and $\mathbb{U}_{d,\hat{K}} := \text{span}\{\hat{\phi}_{4+i}, i \in \mathbb{N}_2\}$. Then

$$\mathbb{U}_{\hat{K}} = \mathbb{U}_{c,\hat{K}} + \mathbb{U}_{d,\hat{K}}.$$

Let $\hat{M}_1 := (0,-1)$, $\hat{M}_2 := (1,0)$, $\hat{M}_3 := (0,1)$, $\hat{M}_4 := (-1,0)$ and $\hat{Q} := (0,0)$. The dual partition is defined as $\hat{\mathcal{T}}^* := \{\hat{K}_i^* : i \in \mathbb{N}_4\}$, where $\hat{K}_i^*, i \in \mathbb{N}_4$ is the rectangle $\Box \hat{P}_i \hat{M}_i \hat{Q} \hat{M}_{i-1}$ with $\hat{M}_0 := \hat{M}_4$. We use χ_E to denote the characteristic function of $E \subset \mathbb{R}^2$. The test space on $\hat{K}$ is chosen as $\mathbb{V}_{\hat{\mathcal{T}}^*} := \text{span}\, \Psi_{\hat{\mathcal{T}}^*}$, where its basis $\Psi_{\hat{\mathcal{T}}^*}$ consists of

$$\hat{\psi}_i := \chi_{\hat{K}_i^*}, i \in \mathbb{N}_4 \quad \text{and} \quad \hat{\psi}_{4+i} := \hat{\phi}_{4+i}, i \in \mathbb{N}_2. \tag{2.7}$$

Let $\mathbb{V}_{c,\hat{\mathcal{T}}^*} := \text{span}\{\hat{\psi}_i, i \in \mathbb{N}_4\}$. Then

$$\mathbb{V}_{\hat{\mathcal{T}}^*} = \mathbb{V}_{c,\hat{\mathcal{T}}^*} + \mathbb{U}_{d,\hat{K}}.$$

The overall trial space $\mathbb{U}_{\mathcal{T}}$ and test space $\mathbb{V}_{\mathcal{T}^*}$ on Ω are defined as below

$$\mathbb{U}_{\mathcal{T}} := \mathbb{U}_{c,\mathcal{T}} + \mathbb{U}_{d,\mathcal{T}} \quad \text{and} \quad \mathbb{V}_{\mathcal{T}^*} := \mathbb{V}_{c,\mathcal{T}^*} + \mathbb{U}_{d,\mathcal{T}}, \tag{2.8}$$

where

$$
\begin{aligned}
\mathbb{U}_{c,\mathcal{T}} &:= \{u \in H_0^1(\Omega) : u|_K = \hat{u} \circ \mathcal{F}_K^{-1}, \quad \hat{u} \in \mathbb{U}_{c,\hat{K}},\ K \in \mathcal{T}\},\\
\mathbb{U}_{d,\mathcal{T}} &:= \{u \in L^2(\Omega) : u|_K = \hat{u} \circ \mathcal{F}_K^{-1}, \quad \hat{u} \in \mathbb{U}_{d,\hat{K}},\ K \in \mathcal{T}, \quad u|_{\partial\Omega} = 0\},\\
\mathbb{V}_{c,\mathcal{T}^*} &:= \mathrm{span}\{\chi_{K^*} : K^* \in \mathcal{T}^* \ \text{and}\ K^* \cap \partial\Omega = \emptyset\}.
\end{aligned}
$$

We remark that a function w in $\mathbb{U}_{c,\mathcal{T}}$ is uniquely determined by the values of w at the vertices of all $K \in \mathcal{T}$, so that it is a continuous function on $\bar{\Omega}$. Thus, $\mathbb{U}_{c,\mathcal{T}}$ is the conforming part of the trial space $\mathbb{U}_{\mathcal{T}}$. A function w in $\mathbb{U}_{d,\mathcal{T}}$ depends merely on the mean values of the second derivatives on each $K \in \mathcal{T}$ so that it is discontinuous at the inter-element boundaries and thus nonconforming.

In order to establish the FVM scheme, we shall introduce a discrete bilinear form. Associated with $\mathcal{T}$ and $\mathcal{T}^*$, for a positive integer k, we define respectively the space

$$\mathbb{H}^k_{\mathcal{T}}(\Omega) := \{v : v \in L^2(\Omega), v|_K \in H^k(K),\ \text{for all}\ K \in \mathcal{T}\}$$

and the space

$$\mathbb{H}^1_{\mathcal{T}^*}(\Omega) := \{v : v \in L^2(\Omega),\ v|_{K^*} \in H^1(K^*),\ \text{for all}\ K^* \in \mathcal{T}^*,\ \text{and}\ v|_{\partial\Omega} = 0\}.$$

We introduce the discrete bilinear form for $w \in \mathbb{H}^2_{\mathcal{T}}(\Omega)$ and $v \in \mathbb{H}^1_{\mathcal{T}^*}(\Omega)$ by setting

$$a_{\mathcal{T}}(w, v) := \sum_{K \in \mathcal{T}} a_K(w, v) \tag{2.9}$$

where

$$a_K(w, v) := \sum_{K^* \in \mathcal{T}^*} \left\{ \int_{K^* \cap K} \nabla w \cdot \nabla v - \int_{\partial K^* \cap \mathrm{int} K} v \nabla w \cdot \mathbf{n} \right\}$$

with $\mathbf{n}$ being the outward unit normal vector on ∂K^* and $\mathrm{int}K$ being the interior of K. Employing the Green formula on the dual elements, we can show for $w \in H_0^1(\Omega) \cap H^2(\Omega)$ and $v \in \mathbb{H}^1_{\mathcal{T}^*}(\Omega)$ that

$$a_{\mathcal{T}}(w, v) = \int_\Omega (-\Delta w) v.$$

The variational form for (2.1) is written as finding $u \in H_0^1(\Omega) \cap \mathbb{H}^2_{\mathcal{T}}(\Omega)$ such that

$$a_{\mathcal{T}}(u, v) = (f, v), \quad \text{for all}\ \ v \in \mathbb{H}^1_{\mathcal{T}^*}(\Omega). \tag{2.10}$$

The hybrid Wilson FVM for solving (2.1) is a finite-dimensional approximation scheme which finds $u_{\mathcal{T}} \in \mathbb{U}_{\mathcal{T}}$ such that

$$a_{\mathcal{T}}(u_{\mathcal{T}}, v) = (f, v), \quad \text{for all}\ \ v \in \mathbb{V}_{\mathcal{T}^*}. \tag{2.11}$$

From (2.8), we know that $\mathbb{U}_{d,\mathcal{T}} \subset \mathbb{V}_{\mathcal{T}^*}$ and $\mathbb{V}_{c,\mathcal{T}^*} \subset \mathbb{V}_{\mathcal{T}^*}$. When we choose $v \in \mathbb{U}_{d,\mathcal{T}}$ in (2.11), we get that

$$\sum_{K \in \mathcal{T}} \int_K \nabla u_{\mathcal{T}} \cdot \nabla v = \int_\Omega f v.$$

This equation is similar to that of the Wilson FEM ([33]). However, the function $v \in \mathbb{V}_{c,\mathcal{T}^*}$ may have jump between the adjoining control volumes in $\mathcal{T}^*$, the integral along the boundary of $K^* \in \mathcal{T}^*$ can not be ignored in the discrete bilinear form

$a_{\mathcal{T}}(u_{\mathcal{T}}, v)$. This is one of the major differences between FVMs and the corresponding FEMs. Specifically, when we choose $v := \chi_{K^*}$, $K^* \in \mathcal{T}^*$ in (2.11), we get that

$$-\sum_{K\in\mathcal{T}} \int_{\partial K^* \cap \mathrm{int} K} \nabla u_{\mathcal{T}} \cdot \mathbf{n} = \int_{K^*} f.$$

where $\mathbf{n}$ is the outward unit normal vector on ∂K^*.

We introduce an interpolation operator $\Pi_{\mathcal{T}^*} : \mathbb{U}_{\mathcal{T}} \to \mathbb{V}_{\mathcal{T}^*}$. For each $K \in \mathcal{T}$, let

$$\phi_{i,K} := \hat{\phi}_i \circ \mathcal{F}_K^{-1} \quad \text{and} \quad \psi_{i,K} := \hat{\psi}_i \circ \mathcal{F}_K^{-1}.$$

For each $w \in \mathbb{U}_{\mathcal{T}}$ such that $w|_K := \sum_{i\in\mathbb{N}_6} w_{i,K}\phi_{i,K}$, we define

$$\Pi_{\mathcal{T}^*} w|_K := \sum_{i\in\mathbb{N}_6} w_{i,K}\psi_{i,K}.$$

Then the hybrid Wilson FVM can be rewritten as finding $u_{\mathcal{T}} \in \mathbb{U}_{\mathcal{T}}$ such that

$$a_{\mathcal{T}}(u_{\mathcal{T}}, \Pi_{\mathcal{T}^*} w) = (f, \Pi_{\mathcal{T}^*} w), \quad \text{for all} \quad w \in \mathbb{U}_{\mathcal{T}}.$$

3 Mesh dependent H^1 error estimate

In this section, we shall establish the uniform boundedness and ellipticity of the discrete bilinear forms so as to derive the optimal mesh dependent H^1 error estimate for the hybrid Wilson FVM.

For a $K \in \mathcal{T}$, let $\nabla\mathcal{F}_K$ denote the Jacobin matrix of the mapping (2.3) and $\nabla\mathcal{F}_K^{-1}$ denote its inverse matrix. From (2.3), we get that

$$\nabla\mathcal{F}_K = \begin{bmatrix} \frac{a_1}{2} + \frac{m_1}{2}\eta & \frac{b_1}{2} + \frac{m_1}{2}\xi \\ \frac{a_2}{2} + \frac{m_2}{2}\eta & \frac{b_2}{2} + \frac{m_2}{2}\xi \end{bmatrix} \quad \text{and} \quad \nabla\mathcal{F}_K^{-1} = \frac{1}{J_K}\begin{bmatrix} \frac{b_2}{2} + \frac{m_2}{2}\xi & -\frac{b_1}{2} - \frac{m_1}{2}\xi \\ -\frac{a_2}{2} - \frac{m_2}{2}\eta & \frac{a_1}{2} + \frac{m_1}{2}\eta \end{bmatrix}. \tag{3.1}$$

Let $J_K(\xi, \eta)$ be the determinant of $\nabla\mathcal{F}_K$. In the following, we often write J_K instead of $J_K(\xi, \eta)$ for simplicity. We assume that the partition $\mathcal{T}$ is regular, that is, there exist positive constants σ and γ such that for all $K \in \mathcal{T}$

$$h_K/\rho_K \le \sigma, \qquad |\cos\theta_K| \le \gamma < 1. \tag{3.2}$$

It is known from [33] that under the regularity condition (3.2), the mapping $\mathcal{F}_K$ is invertible and there holds

$$C_1 h_K^2 \le J_K \le C_2 h_K^2 \tag{3.3}$$

Let $\mathbf{M}_K := J_K \nabla\mathcal{F}_K^{-1}\nabla\mathcal{F}_K^{-T}$. From (3.1), we get that

$$\mathbf{M}_K = \frac{1}{4J_K}\begin{bmatrix} m_{11}(\xi) & m_{12}(\xi,\eta) \\ m_{21}(\xi,\eta) & m_{22}(\eta) \end{bmatrix},$$

where

$$m_{11}(\xi) := |\vec{b} + \vec{m}\xi|^2,\ m_{12}(\xi,\eta) = m_{21}(\xi,\eta) := -(\vec{a} + \vec{m}\eta)\cdot(\vec{b} + \vec{m}\xi),\ m_{22}(\eta) := |\vec{a} + \vec{m}\eta|^2.$$

The next lemma describes the property of the eigenvalues of the matrix $\mathbf{M}_K$.

Lemma 3.1 *If the regularity condition (3.2) holds, then there exist positive constants λ_1 and λ_2 such that for all $K \in \mathcal{T}$, the eigenvalues of $\mathbf{M}_K$ satisfy*

$$\lambda_1 \le \lambda(\mathbf{M}_K) \le \lambda_2. \tag{3.4}$$

Proof Let $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ denote respect the maximum and minimum eigenvalues of the matrix $\mathbf{A}$. Note the fact that for a matrix $\mathbf{A}$, $\lambda_{\max}(\mathbf{A}\mathbf{A}^T) \le \|\mathbf{A}\|_F^2$, where $\|\mathbf{A}\|_F$ is the Frobenius norm of $\mathbf{A}$. Using this fact and (3.1), we get that

$$\lambda_{\max}(\nabla\mathcal{F}_K^{-1}\nabla\mathcal{F}_K^{-T}) \le \frac{1}{4J_K^2}(|\vec{a}+\vec{m}\eta|^2 + |\vec{b}+\vec{m}\xi|^2) \le \frac{Ch_K^2}{J_K^2} \tag{3.5}$$

and

$$\lambda_{\max}(\nabla\mathcal{F}_K^{T}\nabla\mathcal{F}_K) \le \frac{1}{4}(|\vec{a}+\vec{m}\eta|^2 + |\vec{b}+\vec{m}\xi|^2) \le Ch_K^2. \tag{3.6}$$

Since (3.2) holds, by (3.3) and (3.5), we get that there exists a positive constant λ_2 such that

$$\lambda_{\max}(\mathbf{M}_K) = J_K\lambda_{\max}(\nabla\mathcal{F}_K^{-1}\nabla\mathcal{F}_K^{-T}) \le \lambda_2. \tag{3.7}$$

By (3.3) and (3.6), there exists a positive constant λ_1 such that

$$\lambda_{\min}(\mathbf{M}_K) = \frac{1}{\lambda_{\max}(\mathbf{M}_K^{-1})} = \frac{J_K}{\lambda_{\max}(\nabla\mathcal{F}_K^{T}\nabla\mathcal{F}_K)} \ge \lambda_1 \tag{3.8}$$

The desired result of this lemma is derived immediately from (3.7) and (3.8). □

To each function w defined on K, we associate a function $\hat{w}$ on $\hat{K}$ by

$$\hat{w} := w \circ \mathcal{F}_K. \tag{3.9}$$

An application of (3.3) immediately gives that there exist positive constants C_1 and C_2 such that for all $K \in \mathcal{T}$, all $w \in L^2(K)$ and its associated $\hat{w} \in L^2(\hat{K})$

$$C_1 h_K \|\hat{w}\|_{0,\hat{K}} \le \|w\|_{0,K} \le C_2 h_K \|\hat{w}\|_{0,\hat{K}}. \tag{3.10}$$

The next lemma proves the equivalence of $|w|_{1,K}$ and $|\hat{w}|_{1,\hat{K}}$.

Lemma 3.2 *If the regularity condition (3.2) holds, then there exist positive constants C_1 and C_2 such that for all $K \in \mathcal{T}$, all $w \in H^1(K)$ and its associated $\hat{w} \in H^1(\hat{K})$*

$$C_1|\hat{w}|_{1,\hat{K}} \le |w|_{1,K} \le C_2|\hat{w}|_{1,\hat{K}}. \tag{3.11}$$

Proof By changing variables, we get that

$$|w|_{1,K}^2 = \int_{\hat{K}} (\nabla\hat{w})^T \mathbf{M}_K \nabla\hat{w}\, d\xi\, d\eta.$$

This combined with Lemma 3.1 yields the desired result of this lemma. □

For a quadrilateral $K \in \mathcal{T}$ with notations as in Fig. 1, we define the dual grid lines as follows

$$L_K^* := \{\overline{QM_i} : i \in \mathbb{N}_4\}.$$

Lemma 3.3 *If the regularity condition* (3.2) *holds, then for any quadrilateral* $K \in \mathcal{T}$ *with notations as in* Fig. 1 *and for all* $w \in H^2(K)$

$$\int_{\overline{QM_i}} |\nabla w|^2 ds \leq C h_K^{-1}(|w|_{1,K}^2 + h_K^2 |w|_{2,K}^2).$$

Proof Let $\varphi_1 := \frac{\partial w}{\partial x}, \quad \varphi_2 := \frac{\partial w}{\partial y}$. Then

$$\int_{\overline{QM_i}} |\nabla w|^2 ds = \int_{\overline{QM_i}} \left(|\varphi_1|^2 + |\varphi_2|^2\right) ds. \tag{3.12}$$

By the trace theorem, we get that

$$\int_{\overline{QM_i}} |\varphi_1|^2 ds = \frac{\left|\overline{QM_i}\right|}{\left|\overline{\hat{Q}\hat{M}_i}\right|} \int_{\overline{\hat{Q}\hat{M}_i}} |\hat{\varphi}_1|^2 d\hat{s} \leq h_K \int_{\overline{\hat{Q}\hat{M}_i}} |\hat{\varphi}_1|^2 d\hat{s} \leq C h_K \|\hat{\varphi}_1\|_{1,\hat{K}}^2. \tag{3.13}$$

Substituting (3.10) and (3.11) into (3.13), we derive

$$\int_{\overline{QM_i}} |\varphi_1|^2 ds \leq C h_K (h_K^{-2} \|\varphi_1\|_{0,K}^2 + |\varphi_1|_{1,K}^2). \tag{3.14}$$

Similarly, we can derive that

$$\int_{\overline{QM_i}} |\varphi_2|^2 ds \leq C h_K (h_K^{-2} \|\varphi_2\|_{0,K}^2 + |\varphi_2|_{1,K}^2). \tag{3.15}$$

Combining (3.12) with (3.14) and (3.15) yields the desired result of this lemma. □

By virtue of the decomposition (2.8), each function $w \in \mathbb{U}_{\mathcal{T}}$ consists of two parts

$$w = w_1 + w_2, \tag{3.16}$$

where $w_1 \in \mathbb{U}_{c,\mathcal{T}}$ and $w_2 \in \mathbb{U}_{d,\mathcal{T}}$. The following lemma is derived from [33].

Lemma 3.4 *If the regularity condition* (3.2) *holds, then there exist positive constants* C_1 *and* C_2 *such that for all* $w \in \mathbb{U}_{\mathcal{T}}$ *and all* $K \in \mathcal{T}$

$$|w_1|_{1,K} \leq C_1 |w|_{1,K}, \quad |w_2|_{1,K} \leq C_2 |w|_{1,K} \tag{3.17}$$

where w_1 *and* w_2 *are the two parts of* w *as defined in* (3.16).

For each $w \in \mathbb{H}^2_{\mathcal{T}}(\Omega)$, we define the semi-norms

$$|w|_{i,\mathcal{T}} := \left(\sum_{K \in \mathcal{T}} |w|_{i,K}^2\right)^{1/2}, \quad i \in \mathbb{N}_2.$$

To introduce a discrete norm on the test space, we define the jump of $v \in \mathbb{V}_{\mathcal{T}^*}$ from a control volume to its neighboring control volume. Let ℓ^* be an interior edge shared by the control volumes K_1^* and K_2^*. We assign one fixed unit normal vector $\mathbf{n}$ on ℓ^* exterior to K_1^* or K_2^*. Then the jump of $v \in \mathbb{V}_{\mathcal{T}^*}$ on ℓ^* is defined by

$$[v](x) := \lim_{\delta \to 0^+} v(x - \delta \mathbf{n}) - \lim_{\delta \to 0^+} v(x + \delta \mathbf{n}), \quad x \in \ell^* \subset \Omega. \tag{3.18}$$

For the boundary edge $\ell^* := \partial K^* \cap \partial\Omega$, let

$$[v](x) = 0, \quad x \in \ell^* \subset \partial\Omega. \tag{3.19}$$

For any $v \in \mathbb{V}_{\mathcal{T}^*}$, define

$$|v|^2_{1,\mathbb{V}_{\mathcal{T}^*},K} := \sum_{K^* \in \mathcal{T}^*} |v|^2_{1,K^*\cap K} + \sum_{\ell^* \in L^*_K} |\ell^*|^{-1} \int_{\ell^*} [v]^2, \quad |v|_{1,\mathbb{V}_{\mathcal{T}^*}} := \left(\sum_{K \in \mathcal{T}} |v|^2_{1,\mathbb{V}_{\mathcal{T}^*},K}\right)^{1/2}. \tag{3.20}$$

Lemma 3.5 *If the regularity condition* (3.2) *holds, then for all* $w \in \mathbb{U}_{\mathcal{T}}$ *there holds*

$$|\Pi_{\mathcal{T}^*} w|_{1,\mathbb{V}_{\mathcal{T}^*}} \le C|w|_{1,\mathcal{T}}.$$

Proof Let $w^* := \Pi_{\mathcal{T}^*} w$. By (3.16) and the definition of $\Pi_{\mathcal{T}^*}$, we have that $w = w_1 + w_2$ and $w^* = w_1^* + w_2$, where $w_1^* := \Pi_{\mathcal{T}^*} w_1 \in \mathbb{V}_{c,\mathcal{T}^*}$. To derive the desired inequality of this lemma, it suffices to prove that

$$|w_1^*|^2_{1,\mathbb{V}_{\mathcal{T}^*},K} \le C|w|^2_{1,K}, \quad |w_2|^2_{1,\mathbb{V}_{\mathcal{T}^*},K} \le C|w|^2_{1,K}. \tag{3.21}$$

For each K with the notations as in Fig. 1, we begin to prove the first inequality of (3.21). Note that

$$|w_1^*|^2_{1,\mathbb{V}_{\mathcal{T}^*},K} = \sum_{\ell^* \in L^*_K} |\ell^*|^{-1} \int_{\ell^*} [w_1^*]^2 = \sum_{i \in \mathbb{N}_4} \left(w_1(P_i) - w_1(P_{i+1})\right)^2,$$

where $P_5 := P_1$. Since $\sum_{i \in \mathbb{N}_4} \left(w_1(P_i) - w_1(P_{i+1})\right)^2$ and $|\hat{w}_1|^2_{1,\hat{K}}$ are nonnegative quadratic forms of $w_1(P_i)$, $i \in \mathbb{N}_4$ and they have the same null space, it follows from [17] that they are equivalent, that is

$$C_1|\hat{w}_1|^2_{1,\hat{K}} \le |w_1^*|^2_{1,\mathbb{V}_{\mathcal{T}^*},K} \le C_2|\hat{w}_1|^2_{1,\hat{K}}. \tag{3.22}$$

Combining (3.22) and Lemma 3.2 gives that

$$|w_1^*|^2_{1,\mathbb{V}_{\mathcal{T}^*},K} \le C|w_1|^2_{1,K}. \tag{3.23}$$

Then, the first inequality of (3.21) is derived from (3.23) and (3.17).

Since w_2 is continuous on each $K \in \mathcal{T}$, we observe that

$$|w_2|^2_{1,\mathbb{V}_{\mathcal{T}^*},K} = \sum_{K^* \in \mathcal{T}^*} |w_2|^2_{1,K^*\cap K} = |w_2|^2_{1,K}. \tag{3.24}$$

The second inequality of (3.21) is derived from (3.24) and (3.17). □

Based on the above preparations, we are now ready to establish the uniform boundedness of the discrete bilinear forms.

Proposition 3.6 *If the condition* (3.2) *holds, then there exists a positive constant* C *such that for all* $w \in \mathbb{H}^2_{\mathcal{T}}(\Omega)$ *and all* $v \in \mathbb{U}_{\mathcal{T}}$

$$|a_{\mathcal{T}}(w, \Pi_{\mathcal{T}^*} v)| \le C(|w|_{1,\mathcal{T}} + h|w|_{2,\mathcal{T}})|v|_{1,\mathcal{T}}. \tag{3.25}$$

Proof Let $v^* := \Pi_{\mathcal{T}^*} v$. We note that

$$a_{\mathcal{T}}\left(w, v^*\right) = a_{c,\mathcal{T}}\left(w, v^*\right) + a_{d,\mathcal{T}}\left(w, v^*\right). \tag{3.26}$$

where

$$a_{c,\mathcal{T}}\left(w, v^*\right) := \sum_{K\in\mathcal{T}} \sum_{K^*\in\mathcal{T}^*} \int_{K^*\cap K} \nabla w^T \nabla v^*, \quad a_{d,\mathcal{T}}\left(w, v^*\right) := -\sum_{K\in\mathcal{T}} \sum_{\ell^*\in L_K^*} \int_{\ell^*} [v^*] \nabla w \cdot \mathbf{n},$$

with $\mathbf{n}$ being a fixed unit normal vector on ℓ^* and $[v^*]$ being defined as in (3.18) and (3.19). We first estimate $a_{c,\mathcal{T}}(w, v^*)$. By virtue of the Cauchy-Schwartz inequality, there holds

$$|a_{c,\mathcal{T}}\left(w, v^*\right)| \le |w|_{1,\mathcal{T}} |v^*|_{1,\mathbb{V}_{\mathcal{T}^*}}. \tag{3.27}$$

Combining (3.27) with Lemma 3.5 yields

$$|a_{c,\mathcal{T}}\left(w, v^*\right)| \le C|w|_{1,\mathcal{T}} |v|_{1,\mathcal{T}}. \tag{3.28}$$

We next estimate $a_{d,\mathcal{T}}(w, v^*)$. An application of the Cauchy-Schwartz inequality gives that

$$|a_{d,\mathcal{T}}\left(w, v^*\right)| \le |v^*|_{1,\mathbb{V}_{\mathcal{T}^*}} \cdot \left(\sum_{K\in\mathcal{T}} \sum_{\ell^*\in L_K^*} |\ell^*| \int_{\ell^*} \left(\nabla w \cdot \mathbf{n}\right)^2 ds \right)^{1/2}. \tag{3.29}$$

It follows from Lemma 3.3 that

$$|\ell^*| \int_{\ell^*} \left(\nabla w \cdot \mathbf{n}\right)^2 ds \le h_K \int_{\ell^*} |\nabla w|^2 ds \le C(|w|_{1,K}^2 + h_K^2 |w|_{2,K}^2). \tag{3.30}$$

Substituting (3.30) into (3.29) and using Lemma 3.5, we obtain

$$|a_{d,\mathcal{T}}\left(w, v^*\right)| \le C\left(|w|_{1,\mathcal{T}} + h|w|_{2,\mathcal{T}}\right) |v|_{1,\mathcal{T}}. \tag{3.31}$$

Combining (3.26) with (3.28) and (3.31) yields the desired result of this proposition. □

To prove the uniform ellipticity of the discrete bilinear forms of the hybrid Wilson FVM, we first present two useful lemmas. For a $w \in \mathbb{U}_{\mathcal{T}}$ and a $K \in \mathcal{T}$ (see Fig. 1), let

$$w_{i,K} := f_{i,K}(w), \ i \in \mathbb{N}_6, \tag{3.32}$$

where $f_{i,K} := \hat{f}_i \circ \mathcal{F}_K^{-1}$ with $\hat{f}_i$ as defined in (2.4). Note that for a $K \in \mathcal{T}$, a $w \in \mathbb{U}_{\mathcal{T}}$ and its associated $\hat{w} \in \mathbb{U}_{\hat{K}}$

$$\hat{f}_i(\hat{w}) = f_{i,K}(w) = w_{i,K}. \tag{3.33}$$

Set

$$\begin{aligned} &z_1 := w_{2,K} - w_{1,K}, \ z_2 := w_{3,K} - w_{4,K}, \ \ z_3 := w_{4,K} - w_{1,K}, \\ &z_4 := w_{3,K} - w_{2,K}, \ z_i := w_{i,K}, \ i = 5, 6. \end{aligned} \tag{3.34}$$

We define a discrete H^1 semi-norm on $\mathbb{U}_{\mathcal{T}}$

$$|w|_{1,\mathbb{U}_{\mathcal{T}},K}^2 := \sum_{i=1}^{6} z_i^2, \quad |w|_{1,\mathbb{U}_{\mathcal{T}}} := \left(\sum_{K\in\mathcal{T}} |w|_{1,\mathbb{U}_{\mathcal{T}},K}^2 \right)^{1/2}.$$

The next lemma provides the equivalence of the semi-norms $|\cdot|_{1,\mathcal{T}}$ and $|\cdot|_{1,\mathbb{U}_\mathcal{T}}$

Lemma 3.7 *If the condition (3.2) holds, then there exist positive constants C_1 and C_2 such that for each $K \in \mathcal{T}$ and each $w \in \mathbb{U}_\mathcal{T}$*

$$C_1|w|_{1,\mathbb{U}_\mathcal{T},K} \le |w|_{1,K} \le C_2|w|_{1,\mathbb{U}_\mathcal{T},K}$$

Proof From Lemma 3.2, it is suffices to prove that there exist positive constants c_1 and c_2 such that for each $K \in \mathcal{T}$, each $w \in \mathbb{U}_\mathcal{T}$ and its associated $\hat{w} \in \mathbb{U}_{\hat{K}}$

$$c_1|w|^2_{1,\mathbb{U}_\mathcal{T},K} \le |\hat{w}|^2_{1,\hat{K}} \le c_2|w|^2_{1,\mathbb{U}_\mathcal{T},K}. \tag{3.35}$$

From (3.33) and (2.5), we know that

$$\hat{w} = \sum_{i\in\mathbb{N}_6} w_{i,K}\hat{\phi}_i.$$

Thus, $|\hat{w}|^2_{1,\hat{K}}$ is a nonnegative quadratic form of $w_{i,K}$, $i \in \mathbb{N}_6$ whose null space is the same with that of $|w|^2_{1,\mathbb{U}_\mathcal{T},K}$. Therefore, from [17], we get that $|w|^2_{1,\mathbb{U}_\mathcal{T},K}$ and $|\hat{w}|^2_{1,\hat{K}}$ are equivalent, that is, (3.35) holds. □

The following lemma is derived from Lemma 3.7 of [37].

Lemma 3.8 *Assume that $\mathcal{A}$ and $\mathcal{B}$ are two $n\times n$ symmetric matrices and the constant $\kappa \ne 0$. Then the matrix $\begin{bmatrix} \mathcal{A} & \kappa\mathcal{B} \\ \kappa\mathcal{B} & \kappa^2\mathcal{A} \end{bmatrix}$ is positive definite if and only if the matrices $\mathcal{A} \pm \mathcal{B}$ are all positive definite.*

We assume that each quadrilateral $K \in \mathcal{T}$ is an h^{1+r}-parallelogram ($r > 0$), which means that (see Fig. 2)

$$|\vec{m}| = |\overrightarrow{O_2O_1}| \le Ch^{1+r}. \tag{3.36}$$

In the next proposition, we establish the uniform ellipticity of the discrete bilinear forms of the hybrid Wilson FVM.

Proposition 3.9 *Suppose that the condition (3.2) holds with $\gamma < \frac{2\sqrt{6}}{5}$ and the condition (3.36) holds. Then there exits a positive constant C such that for all $w \in \mathbb{U}_\mathcal{T}$*

$$a_\mathcal{T}(w, \Pi_{\mathcal{T}^*}w) \ge C|w|^2_{1,\mathcal{T}}.$$

Proof Let $w^* := \Pi_{\mathcal{T}^*}w$. By (3.16) and the definition of $\Pi_{\mathcal{T}^*}$, we have that $w = w_1 + w_2$ and $w^* = w_1^* + w_2$, where $w_1^* := \Pi_{\mathcal{T}^*}w_1 \in \mathbb{V}_{c,\mathcal{T}^*}$. It suffices to prove that for each $K \in \mathcal{T}$ with notations as in Fig. 1

$$a_K(w, w^*) = a_K(w, w_1^*) + a_K(w, w_2) \ge C|w|^2_{1,K} \tag{3.37}$$

Note that

$$\begin{aligned} a_K(w, w_1^*) &= -\sum_{K^*\in\mathcal{T}^*} \int_{\partial K^*\cap \mathrm{int}K} w_1^*\nabla w\cdot\mathbf{n} \\ &= z_1\int_{\overline{M_1Q}}\nabla w\cdot\mathbf{n}_1 + z_2\int_{\overline{QM_3}}\nabla w\cdot\mathbf{n}_2 - z_3\int_{\overline{M_4Q}}\nabla w\cdot\mathbf{n}_3 - z_4\int_{\overline{QM_2}}\nabla w\cdot\mathbf{n}_4 \end{aligned}$$

where $z_i, i \in \mathbb{N}_6$ are as defined in (3.34) and $\mathbf{n}_1$ is the unit normal vector on $\overline{M_1 Q}$ pointing right, $\mathbf{n}_2$ is the unit normal vector on $\overline{QM_3}$ pointing right, $\mathbf{n}_3$ is the unit normal vector on $\overline{M_4 Q}$ pointing down and $\mathbf{n}_4$ is the unit normal vector on $\overline{QM_2}$ pointing down. By changing variables, we get that

$$a_K(w, w_1^*) = a_{K,\eta}(w, w_1^*) + a_{K,\xi}(w, w_1^*) \tag{3.38}$$

where

$$\begin{aligned} a_{K,\eta}(w, w_1^*) &:= z_1 \int_{\overline{\hat{M}_1 \hat{Q}}} (\nabla \hat{w})^T M_K \hat{\mathbf{n}}_1 + z_2 \int_{\overline{\hat{Q}\hat{M}_3}} (\nabla \hat{w})^T M_K \hat{\mathbf{n}}_2 \\ a_{K,\xi}(w, w_1^*) &:= -z_3 \int_{\overline{\hat{M}_4 \hat{Q}}} (\nabla \hat{w})^T M_K \hat{\mathbf{n}}_3 - z_4 \int_{\overline{\hat{Q}\hat{M}_2}} (\nabla \hat{w})^T M_K \hat{\mathbf{n}}_4. \end{aligned}$$

Let $I_1 := [-1, 0]$, $I_2 := [0, 1]$, $\varphi_1(t) := 1-t$ and $\varphi_2(t) := 1+t$. It is easy to see that

$$\nabla \hat{w} = \frac{1}{4} \left(\varphi_1(\eta) z_1 + \varphi_2(\eta) z_2 + \xi z_5, \varphi_1(\xi) z_3 + \varphi_2(\xi) z_4 + \eta z_6\right)^T. \tag{3.39}$$

It is obvious that on $\overline{\hat{M}_1 \hat{Q} \hat{M}_3}$, $\xi = 0$ and $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2 = (1, 0)^T$, and on $\overline{\hat{M}_4 \hat{Q} \hat{M}_2}$, $\eta = 0$ and $\hat{\mathbf{n}}_3 = \hat{\mathbf{n}}_4 = (0, -1)^T$. Thus

$$\begin{aligned} a_{K,\eta}(w, w_1^*) &= z_1 \left\{ \int_{I_1} \left(\tfrac{m_{11}(0)}{16J(0,\eta)} (\varphi_1(\eta) z_1 + \varphi_2(\eta) z_2) + \tfrac{m_{21}(0,\eta)}{16J(0,\eta)} (z_3 + z_4 + z_6 \eta) \right) d\eta \right\} \\ &+ z_2 \left\{ \int_{I_2} \left(\tfrac{m_{11}(0)}{16J(0,\eta)} (\varphi_1(\eta) z_1 + \varphi_2(\eta) z_2) + \tfrac{m_{21}(0,\eta)}{16J(0,\eta)} (z_3 + z_4 + z_6 \eta) \right) d\eta \right\} \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} a_{K,\xi}(w, w_1^*) &= z_3 \left\{ \int_{I_1} \left(\tfrac{m_{12}(\xi,0)}{16J(\xi,0)} (z_1 + z_2 + z_5 \xi) + \tfrac{m_{22}(0)}{16J(\xi,0)} (\varphi_1(\xi) z_3 + \varphi_2(\xi) z_4) \right) d\xi \right\} \\ &+ z_4 \left\{ \int_{I_2} \left(\tfrac{m_{12}(\xi,0)}{16J(\xi,0)} (z_1 + z_2 + z_5 \xi) + \tfrac{m_{22}(0)}{16J(\xi,0)} (\varphi_1(\xi) z_3 + \varphi_2(\xi) z_4) \right) d\xi \right\}. \end{aligned} \tag{3.41}$$

Note that

$$a_K(w, w_2) = \int_K \nabla w \cdot \nabla w_2.$$

By changing variables, we get that

$$a_K(w, w_2) = \int_{\hat{K}} (\nabla \hat{w})^T M_K \nabla \hat{w}_2 d\xi d\eta. \tag{3.42}$$

It is easy to see that

$$\nabla \hat{w}_2 = \frac{1}{4} (\xi z_5, \eta z_6)^T. \tag{3.43}$$

Substituting (3.39) and (3.50) into (3.49) yields that

$$\begin{aligned} a_K(w, w_2) = z_5 \Big\{ \int_{\hat{K}} \Big(&\tfrac{m_{11}(\xi)\xi}{64J(\xi,\eta)} (\varphi_1(\eta) z_1 + \varphi_2(\eta) z_2 + \xi z_5) \\ &+ \tfrac{m_{21}(\xi,\eta)\xi}{64J(\xi,\eta)} (\varphi_1(\xi) z_3 + \varphi_2(\xi) z_4 + \eta z_6) \Big) d\xi d\eta \Big\} \\ + z_6 \Big\{ \int_{\hat{K}} \Big(&\tfrac{m_{12}(\xi,\eta)\eta}{64J(\xi,\eta)} (\varphi_1(\eta) z_1 + \varphi_2(\eta) z_2 + \xi z_5) \\ &+ \tfrac{m_{22}(\eta)\eta}{64J(\xi,\eta)} (\varphi_1(\xi) z_3 + \varphi_2(\xi) z_4 + \eta z_6) \Big) d\xi d\eta \Big\}. \end{aligned} \tag{3.44}$$

Since the grids considered here are almost parallelograms, we first assume that K is a parallelogram. For a matrix $\mathbf{M}$, we use $\tilde{\mathbf{M}} := \frac{\mathbf{M}+\mathbf{M}^T}{2}$ to denote its associated symmetric matrix. Let $\mathbf{B}$ be the element stiffness matrix when K is parallelogram. Let

$\theta := \angle P_4 P_1 P_2$, $\kappa := \frac{|\overrightarrow{P_1 P_4}|}{|\overrightarrow{P_1 P_2}|}$ and $\mathbf{z} := [z_1, z_2, z_5, z_3, z_4, z_6]^T$. From (3.38), (3.40), (3.41) and (3.44), we get that

$$a_K(w, w^*) = \mathbf{z}^T \tilde{\mathbf{B}} \mathbf{z} = \frac{|\overrightarrow{P_1 P_4}|^2}{8 S_K} \mathbf{z}^T \begin{bmatrix} \mathbf{B}_1 & \kappa \mathbf{B}_2 \\ \kappa \mathbf{B}_2 & \kappa^2 \mathbf{B}_1 \end{bmatrix} \mathbf{z} \tag{3.45}$$

where

$$\mathbf{B}_1 := \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_2 := \cos\theta \begin{bmatrix} -2 & -2 & 5/6 \\ -2 & -2 & -5/6 \\ 5/6 & -5/6 & 0 \end{bmatrix}$$

The order principal minor determinants of the matrices $\mathbf{B}_1 \pm \mathbf{B}_2$ are

$$3 \mp 2\cos\theta > 0, \quad 8(1 \mp \cos\theta), \quad 8(1 \mp \cos\theta)(\frac{2}{3} - (\frac{5}{6})^2 \cos^2\theta)$$

When we choose the constant γ in (3.2) such that $\gamma < \frac{2\sqrt{6}}{5} (\approx 0.9798)$, we derive that the above order principal minor determinants are all positive. Thus, the matrices $\mathbf{B}_1 \pm \mathbf{B}_2$ are all positive definite. By Lemma 3.8, we know that the matrix $\begin{bmatrix} \mathbf{B}_1 & \kappa \mathbf{B}_2 \\ \kappa \mathbf{B}_2 & \kappa^2 \mathbf{B}_1 \end{bmatrix}$ is positive definite with minimum eigenvalue $\lambda_{\min}(\kappa, \cos\theta) > 0$. Since $\lambda_{\min}(\kappa, \cos\theta)$ is a continuous function of κ and $\cos\theta$ and (3.2) holds, we conclude that there is a positive constant λ_γ independent of κ and $\cos\theta$ such that

$$\lambda_{\min}(\kappa, \cos\theta) \geq \lambda_\gamma. \tag{3.46}$$

From (3.2), we derive that

$$C h_K^2 \leq S_K \leq h_K^2. \tag{3.47}$$

Substituting (3.46) and (3.47) into (3.45), we get

$$\mathbf{z}^T \tilde{\mathbf{B}} \mathbf{z} \geq C_1 \mathbf{z}^T \mathbf{z}. \tag{3.48}$$

We next turn to the case that K is an h^{1+r}-parallelogram ($r > 0$). Let $\mathbf{A}$ be the element stiffness matrix on K and set $\mathbf{D} = \mathbf{B} - \mathbf{A}$. From (3.40), (3.41), and (3.44), we can immediately get the elements of $\mathbf{D}$. For a $K \in \mathcal{T}$ with notations as in Fig. 1, we denote $S_K^* := |\overrightarrow{P_1 P_4} \times \overrightarrow{P_1 P_2}|$, $m_{11}^* := |\overrightarrow{P_1 P_4}|^2$, $m_{12}^* := -\overrightarrow{P_1 P_4} \cdot \overrightarrow{P_1 P_2}$, $m_{21}^* := -\overrightarrow{P_1 P_4} \cdot \overrightarrow{P_1 P_2}$ and $m_{22}^* := |\overrightarrow{P_1 P_2}|^2$. If K is a parallelogram, then

$$4J_K(\xi, \eta) = S_K^*, \ m_{11}(\xi) = m_{11}^*, \ m_{12}(\xi, \eta) = m_{12}^*, \ m_{21}(\xi, \eta) = m_{21}^*, \ m_{22}(\eta) = m_{22}^*.$$

Under conditions (3.2) and (3.36), we derive from (19) and (25) of [32] that

$$|4J_K(\xi, \eta) - S_K^*| \leq C h^{2+r}, \quad \text{for all } (\xi, \eta) \in \hat{K}, \tag{3.49}$$

Form the definition of S_K^* and (3.2), we get that

$$C h_K^2 \leq S_K^* \leq h_K^2 \tag{3.50}$$

We derive from (3.49), (3.50), and (3.3) that

$$\left| \frac{1}{4J_K(\xi, \eta)} - \frac{1}{S_K^*} \right| = \left| \frac{S_K^* - 4J_K(\xi, \eta)}{4J_K(\xi, \eta) S_K^*} \right| \leq C h^{r-2}. \tag{3.51}$$

Under condition (3.36), by simple calculation, we derive that for all $(\xi, \eta) \in \hat{K}$

$$\begin{aligned} |m_{11}(\xi) - m_{11}^*| \leq Ch^{2+r}, \quad |m_{22}(\eta) - m_{22}^*| \leq Ch^{2+r}, \\ |m_{12}(\xi, \eta) - m_{12}^*| = |m_{21}(\xi, \eta) - m_{21}^*| \leq Ch^{2+r}. \end{aligned} \tag{3.52}$$

By (3.51) and (3.52), we can estimate the elements of $\mathbf{D}$

$$|\mathbf{D}_{ij}| \leq Ch^r, \quad 1 \leq i, j \leq 6.$$

Thus

$$\lambda_{\max}(\tilde{\mathbf{D}}) \leq \|\tilde{\mathbf{D}}\|_\infty \leq Ch^r. \tag{3.53}$$

Therefore, from (3.48) and (3.53), we derive that

$$a_K(w, w^*) = \mathbf{z}^T \tilde{\mathbf{A}} \mathbf{z} = \mathbf{z}^T \tilde{\mathbf{B}} \mathbf{z} - \mathbf{z}^T \tilde{\mathbf{D}} \mathbf{z} \geq C_1 \mathbf{z}^T \mathbf{z} - \lambda_{\max}(\tilde{\mathbf{D}}) \mathbf{z}^T \mathbf{z} \geq C \mathbf{z}^T \mathbf{z},$$

which combined with Lemma 3.7 yields the desired inequality (3.37). □

We introduce an interpolation projection operator $P_{\mathcal{T}}$ to the trial space. For any function $\hat{v} \in \mathbb{H}^2(\hat{K})$, we define the interpolation function $\hat{P}\hat{v} \in \mathbb{U}_{\hat{K}}$ as follows

$$\hat{f}_i(\hat{P}\hat{v}) = \hat{f}_i(\hat{v}), \quad i \in \mathbb{N}_6,$$

where $\hat{f}_i$ are defined as in (2.4). Then, for any function $v \in H^2(K)$, the corresponding function $P_K v$ is defined by

$$\widehat{P_K v} = \hat{P}\hat{v}.$$

For each $v \in H^2(\Omega)$, let the interpolation function $P_{\mathcal{T}} v \in \mathbb{U}_{\mathcal{T}}$ be such that

$$P_{\mathcal{T}} v|_K = P_K v, \quad \text{for any } K \in \mathcal{T}. \tag{3.54}$$

Following [20], we have the following interpolation error estimates.

$$|v - P_{\mathcal{T}} v|_{1,\mathcal{T}} \leq Ch|v|_2, \quad \|v - P_{\mathcal{T}} v\|_0 \leq Ch^2|v|_2, \quad \forall v \in H^2(\Omega). \tag{3.55}$$

Theorem 3.10 *Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of (2.1). If that the condition (3.2) holds with $\gamma < \frac{2\sqrt{6}}{5}$ and the condition (3.36) holds with $r = 1$, then the hybrid Wilson FVM equation (2.11) has a unique solution $u_{\mathcal{T}} \in \mathbb{U}_{\mathcal{T}}$ such that*

$$|u - u_{\mathcal{T}}|_{1,\mathcal{T}} \leq Ch\|u\|_2.$$

Proof By making use of Proposition 3.6 and Proposition 3.9, similar to the the proof of Theorem 2.1 of [39], we find that (2.11) has a unique solution $u_{\mathcal{T}} \in \mathbb{U}_{\mathcal{T}}$ satisfying

$$|u - u_{\mathcal{T}}|_{1,\mathcal{T}} \leq C \left(\inf_{w \in \mathbb{U}_{\mathcal{T}}} \left(|u - w|_{1,\mathcal{T}} + h|u - w|_{2,\mathcal{T}} \right) + \sup_{v \in \mathbb{U}_{\mathcal{T}}} \frac{E_{\mathcal{T}}(u, v)}{|v|_{1,\mathcal{T}}} \right). \tag{3.56}$$

where

$$E_{\mathcal{T}}(u, v) := a_{\mathcal{T}}(u - u_{\mathcal{T}}, \Pi_{\mathcal{T}^*} v) = a_{\mathcal{T}}(u, \Pi_{\mathcal{T}^*} v) - (f, \Pi_{\mathcal{T}^*} v) \tag{3.57}$$

is the nonconforming error term. It follows from (3.55) that

$$\inf_{w \in \mathbb{U}_{\mathcal{T}}} \left(|u - w|_{1,\mathcal{T}} + h|u - w|_{2,\mathcal{T}} \right) \leq Ch|u|_2. \tag{3.58}$$

We next estimate the nonconforming error $E_{\mathcal{T}}(u, v)$. For each $v \in \mathbb{U}_{\mathcal{T}}$, we let $v^* := \Pi_{\mathcal{T}^*} v$. By (3.16) and the definition of $\Pi_{\mathcal{T}^*}$, we have that $v = v_1 + v_2$ and

$v^* = v_1^* + v_2$, where $v_1^* := \Pi_{\mathcal{T}^*} v_1 \in \mathbb{V}_{c,\mathcal{T}^*}$. From (2.10), we note that $a_{\mathcal{T}}(u, v_1^*) = (f, v_1^*)$. So

$$E_{\mathcal{T}}(u, v) = a_{\mathcal{T}}(u, v^*) - (f, v^*) = a_{\mathcal{T}}(u, v_2) - (f, v_2). \tag{3.59}$$

Since $v_2 \in \mathbb{U}_{\mathcal{T}}$ and the condition (3.36) holds with $r = 1$, employing (5.15) of [33] produces

$$|a_{\mathcal{T}}(u, v_2) - (f, v_2)| \le Ch\|u\|_2 |v_2|_{1,\mathcal{T}},$$

which combining with Lemma 3.4 yields

$$|a_{\mathcal{T}}(u, v_2) - (f, v_2)| \le Ch\|u\|_2 |v|_{1,\mathcal{T}}. \tag{3.60}$$

From (3.59) and (3.60), we obtain

$$|E_{\mathcal{T}}(u, v)| \le Ch\|u\|_2 |v|_{1,\mathcal{T}}. \tag{3.61}$$

The desired result is derived from (3.56), (3.58), and (3.61). □

In the Theorem 3.10, we present the mesh-dependent H^1 semi-norm error estimate for the hybrid Wilson FVM, achieving the same optimal convergence order as that of the Wilson FEM in [11, 20, 33]. We see that the nonconforming error term $E_{\mathcal{T}}(u, v)$ is of $O(h)$, which hinders higher convergence order of the hybrid Wilson FVM in the mesh dependent H^1 norm.

4 The L^2 error estimate

In this section, we shall provide the L^2 error analysis of the hybrid Wilson FVM. By the space decomposition (2.8) and the Aubin-Nitsche technique, it can be reduced to the estimation of the difference of bilinear forms between the FVM and its corresponding FEM on the lower-order subspace of the trial space. In all the lemmas of this subsection, we assume that the condition (3.2) holds with $\gamma < \frac{2\sqrt{6}}{5}$ and the condition (3.36) holds with $r = 1$.

According to (3.16), the solution $u_{\mathcal{T}}$ of the hybrid Wilson FVM can be written as the sum

$$u_{\mathcal{T}} = u_{\mathcal{T},c} + u_{\mathcal{T},n}, \tag{4.1}$$

where $u_{\mathcal{T},c} \in \mathbb{U}_{c,\mathcal{T}}$ and $u_{\mathcal{T},n} \in \mathbb{U}_{d,\mathcal{T}}$. And for each $v \in H^2(\Omega)$, the interpolation function $P_{\mathcal{T}} v$ as defined in (3.54) can be written as

$$P_{\mathcal{T}} v = Q_{\mathcal{T}} v + R_{\mathcal{T}} v, \tag{4.2}$$

where $Q_{\mathcal{T}} v \in \mathbb{U}_{c,\mathcal{T}}$ and $R_{\mathcal{T}} v \in \mathbb{U}_{d,\mathcal{T}}$. Following [20], for each $v \in H^2(\Omega)$ we have that

$$|v - Q_{\mathcal{T}} v|_1 \le Ch|v|_2, \quad \|v - Q_{\mathcal{T}} v\|_0 \le Ch^2 |v|_2. \tag{4.3}$$

The next lemma describes the relationship between the nonconforming part of $u_{\mathcal{T}}$ and the exact solution u. Its proof is similar to that of Theorem 4 in [33].

Lemma 4.1 *Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of (2.1). Then there holds*

$$|u_{\mathcal{T},n}|_{1,\mathcal{T}} \le Ch\|u\|_2, \quad \|u_{\mathcal{T},n}\|_0 \le Ch^2 \|u\|_2.$$

Following Lemma 4.1, the L^2 error estimate of the FVM solution $u_{\mathcal{T}}$ and the exact solution u has the form

$$\|u - u_{\mathcal{T}}\|_0 \le \|u - u_{\mathcal{T},c}\|_0 + \|u_{\mathcal{T},n}\|_0 \le \|u - u_{\mathcal{T},c}\|_0 + Ch^2\|u\|_2. \tag{4.4}$$

In the following, we devote ourselves to estimating $\|u - u_{\mathcal{T},c}\|_0$. To this end, we introduce an auxiliary problem: find $\varphi \in H^2(\Omega)$ such that

$$-\Delta\varphi = u - u_{\mathcal{T},c} \quad \text{in } \Omega \quad \text{and} \quad \varphi = 0 \quad \text{on } \partial\Omega. \tag{4.5}$$

It is well-known that (cf. [15])

$$\|\varphi\|_2 \le C\|u - u_{\mathcal{T},c}\|_0. \tag{4.6}$$

For $w, v \in \mathbb{H}^1_{\mathcal{T}}(\Omega)$, we define the bilinear form

$$e_K(w, v) := \int_K \nabla w \cdot \nabla v \quad \text{and} \quad a(w, v) := \sum_{K\in\mathcal{T}} e_K(w, v).$$

Based on (4.5), we give the following important decomposition of the L^2 error estimate presented as a proposition.

Proposition 4.2 *There holds that*

$$\|u-u_{\mathcal{T},c}\|_0^2 = a(u-u_{\mathcal{T},c}, \varphi-Q_{\mathcal{T}}\varphi)+a(u-u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi)+a(u_{\mathcal{T},n}, Q_{\mathcal{T}}\varphi-\varphi)+a(u_{\mathcal{T},n}, \varphi). \tag{4.7}$$

Proof Applying the Green's formula to (4.5), we get that

$$\|u - u_{\mathcal{T},c}\|_0^2 = a(u - u_{\mathcal{T},c}, \varphi). \tag{4.8}$$

Obviously,

$$\begin{aligned} a(u - u_{\mathcal{T},c}, \varphi) &= a(u - u_{\mathcal{T},c}, \varphi - Q_{\mathcal{T}}\varphi) + a(u - u_{\mathcal{T},c}, Q_{\mathcal{T}}\varphi) \\ &= a(u - u_{\mathcal{T},c}, \varphi - Q_{\mathcal{T}}\varphi) + a(u - u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi) + a(u_{\mathcal{T},n}, Q_{\mathcal{T}}\varphi) \\ &= a(u - u_{\mathcal{T},c}, \varphi - Q_{\mathcal{T}}\varphi) + a(u - u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi) + a(u_{\mathcal{T},n}, Q_{\mathcal{T}}\varphi - \varphi) + a(u_{\mathcal{T},n}, \varphi). \end{aligned}$$

Thus, the desired result is proved. □

The first, third, and last terms on the right-hand side of (4.7) are relatively easy to analyze, the estimations of which are given in the following three lemmas.

Lemma 4.3 *There holds*

$$|a(u - u_{\mathcal{T},c}, \varphi - Q_{\mathcal{T}}\varphi)| \le Ch^2\|u\|_2\|u - u_{\mathcal{T},c}\|_0.$$

Proof From (4.3) and (4.6), we obtain that

$$|a(u - u_{\mathcal{T},c}, \varphi - Q_{\mathcal{T}}\varphi)| \le |u - u_{\mathcal{T},c}|_1 \cdot |\varphi - Q_{\mathcal{T}}\varphi|_1 \le Ch|u - u_{\mathcal{T},c}|_1 \cdot \|u - u_{\mathcal{T},c}\|_0. \tag{4.9}$$

It follows from Theorem 3.10 and Lemma 4.1 that

$$|u - u_{\mathcal{T},c}|_1 \le |u - u_{\mathcal{T}}|_{1,\mathcal{T}} + |u_{\mathcal{T},n}|_{1,\mathcal{T}} \le Ch\|u\|_2. \tag{4.10}$$

Substituting (4.10) into (4.9) completes the proof of this lemma. □

Lemma 4.4 *There holds*

$$\left|a(u_{\mathcal{T},n}, Q_{\mathcal{T}}\varphi - \varphi)\right| \leq Ch^2|u|_2\|u - u_{\mathcal{T},c}\|_0.$$

Proof Using Lemma 4.1 and (4.3), we have that

$$\left|a(u_{\mathcal{T},n}, Q_{\mathcal{T}}\varphi - \varphi)\right| \leq |u_{\mathcal{T},n}|_{1,\mathcal{T}}|Q_{\mathcal{T}}\varphi - \varphi|_1 \leq Ch^2\|u\|_2|\varphi|_2.$$

This combined with (4.6) yields the desired result of this lemma. □

Lemma 4.5 *There holds*

$$\left|a(u_{\mathcal{T},n}, \varphi)\right| \leq Ch^2\|u\|_2\|u - u_{\mathcal{T},c}\|_0.$$

Proof By the Green's formula, we get that

$$a(u_{\mathcal{T},n}, \varphi) = \sum_{K\in\mathcal{T}}\int_{\partial K} u_{\mathcal{T},n}\nabla\varphi \cdot \mathbf{n} - \sum_{K\in\mathcal{T}}\int_K u_{\mathcal{T},n}\Delta\varphi \tag{4.11}$$

Since the condition (3.36) holds with $r = 1$ and $u_{\mathcal{T},n} \in \mathbb{U}_{\mathcal{T}}$, similar to the proof of (5.15) in [33], we derive that

$$\left|\sum_{K\in\mathcal{T}}\int_{\partial K} u_{\mathcal{T},n}\nabla\varphi \cdot \mathbf{n}\right| \leq Ch\|\varphi\|_2|u_{\mathcal{T},n}|_{1,\mathcal{T}}. \tag{4.12}$$

An application of the Cauchy-Schwartz inequality gives that

$$\left|\sum_{K\in\mathcal{T}}\int_K u_{\mathcal{T},n}\Delta\varphi\right| \leq |\varphi|_2\|u_{\mathcal{T},n}\|_0. \tag{4.13}$$

Then, from (4.11)–(4.13), we have that

$$\left|a(u_{\mathcal{T},n}, \varphi)\right| \leq C\|\varphi\|_2\left(h|u_{\mathcal{T},n}|_{1,\mathcal{T}} + \|u_{\mathcal{T},n}\|_0\right).$$

Thus, by (4.6) and Lemma 4.1, we get the desired result of this lemma. □

Now, we focus on analyzing the second term on the right-hand side of (4.7), which is the major difficulty for the L^2 error estimate of the hybrid Wilson FVM. For a $g \in C^2([a, b])$, we introduce the following integral formula

$$\int_a^b g(x)dx = (b - a)g(\frac{a + b}{2}) + R(g), \tag{4.14}$$

where $R(g)$ is given by (cf. [28])

$$R(g) = \int_a^b \mathcal{K}(t)g''(t)dt,$$

with the kernel function

$$\mathcal{K}(t) := \begin{cases} \frac{1}{2}(t - a)^2, & t < \frac{a+b}{2}, \\ \frac{1}{2}(t - b)^2, & t \geq \frac{a+b}{2}. \end{cases}$$

When the condition (3.36) holds with $r = 1$, following [28], for all $w \in H^3(K)$ and its associated $\hat{w} \in H^3(\hat{K})$ we have that

$$|\hat{w}|_{2,\hat{K}} \leq Ch(|w|_{1,K} + |w|_{2,K}) \quad \text{and} \quad |\hat{w}|_{3,\hat{K}} \leq Ch^2\|w\|_{3,K}. \tag{4.15}$$

We introduce some notations. For each $K \in \mathcal{T}$, set

$$\varphi_{ij} = (Q_{\mathcal{T}}\varphi)_{i,K} - (Q_{\mathcal{T}}\varphi)_{j,K},\ i, j \in \mathbb{N}_4,$$

and

$$\varphi_{1234} = (Q_{\mathcal{T}}\varphi)_{1,K} - (Q_{\mathcal{T}}\varphi)_{2,K} + (Q_{\mathcal{T}}\varphi)_{3,K} - (Q_{\mathcal{T}}\varphi)_{4,K}.$$

where $(Q_{\mathcal{T}}\varphi)_{i,K}, i \in \mathbb{N}_4$ are defined as in (3.32).

We are ready to estimate the second term on the right-hand side of (4.7) in the next lemma.

Lemma 4.6 *Then there holds*

$$|a(u - u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi)| \leq Ch^2\|u\|_3\|u - u_{\mathcal{T},c}\|_0.$$

Proof Let $Q^*_{\mathcal{T}}\varphi := \Pi_{\mathcal{T}^*}(Q_{\mathcal{T}}\varphi)$. Note that $a_{\mathcal{T}}(u - u_{\mathcal{T}}, Q^*_{\mathcal{T}}\varphi) = 0$. So, we get that

$$a(u - u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi) = a(u - u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi) - a_{\mathcal{T}}(u - u_{\mathcal{T}}, Q^*_{\mathcal{T}}\varphi) = \sum_{K \in \mathcal{T}} I_K, \tag{4.16}$$

where

$$I_K := e_K(u - u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi) - a_K(u - u_{\mathcal{T}}, Q^*_{\mathcal{T}}\varphi). \tag{4.17}$$

For a $K \in \mathcal{T}$ with notations as in Fig. 1, we begin to analyze I_K. By changing variables, we have that

$$e_K(u - u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi) = \int_{\hat{K}} \nabla(\widehat{Q_{\mathcal{T}}\varphi})^T M_K \nabla(\hat{u} - \hat{u}_{\mathcal{T}}).$$

Noticing that

$$\nabla(\widehat{Q_{\mathcal{T}}\varphi}) = \frac{1}{4}((\varphi_{21} + \varphi_{34}) + \varphi_{1234}\eta, (\varphi_{41} + \varphi_{32}) + \varphi_{1234}\xi)^T,$$

and letting

$F_1(\xi, \eta) := (1, 0)M_K\nabla(\hat{u}-\hat{u}_{\mathcal{T}}),\ F_2(\xi, \eta) := (0, 1)M_K\nabla(\hat{u}-\hat{u}_{\mathcal{T}}),\ F_3(\xi, \eta) := (\xi, \eta)M_K\nabla(\hat{u}-\hat{u}_{\mathcal{T}}),$

we derive that

$$\begin{aligned} e_K(u - u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi) = \tfrac{1}{4}\Big(& (\varphi_{21} + \varphi_{34}) \textstyle\int_{\hat{K}} F_1(\xi, \eta) + \\ & (\varphi_{41} + \varphi_{32}) \textstyle\int_{\hat{K}} F_2(\xi, \eta) + \varphi_{1234} \int_{\hat{K}} F_3(\xi, \eta)\Big), \end{aligned} \tag{4.18}$$

By virtue of formula (4.14) with $t = \xi$, we obtain that

$$\int_{\hat{K}} F_1(\xi, \eta)d\xi d\eta = 2\int_{-1}^{1} F_1(0, \eta)d\eta + \int_{-1}^{1} R_1(\eta)d\eta, \tag{4.19}$$

where

$$R_1(\eta) := \frac{1}{2}\left(\int_{-1}^{0}(\xi + 1)^2\frac{\partial^2 F_1(\xi, \eta)}{\partial\xi^2}d\xi + \int_{0}^{1}(\xi - 1)^2\frac{\partial^2 F_1(\xi, \eta)}{\partial\xi^2}d\xi\right).$$

By virtue of formula (4.14) with $t=\eta$, we obtain that

$$\int_{\hat{K}} F_2(\xi,\eta)d\xi d\eta = 2\int_{-1}^{1} F_2(\xi,0)d\xi + \int_{-1}^{1} R_2(\xi)d\xi, \tag{4.20}$$

where

$$R_2(\xi) := \frac{1}{2}\left(\int_{-1}^{0}(\eta+1)^2\frac{\partial^2 F_2(\xi,\eta)}{\partial\eta^2}d\eta + \int_{0}^{1}(\eta-1)^2\frac{\partial^2 F_2(\xi,\eta)}{\partial\eta^2}d\eta\right).$$

Substituting (4.19) and (4.20) into (4.18) produces that

$$\begin{aligned} e_K(u-u_{\mathcal{T}}, Q_{\mathcal{T}}\varphi) &= \tfrac{1}{2}(\varphi_{21}+\varphi_{34})\int_{-1}^{1} F_1(0,\eta)d\eta + \tfrac{1}{2}(\varphi_{41}+\varphi_{32})\int_{-1}^{1} F_2(\xi,0)d\xi \\ &\quad + \tfrac{1}{4}(\varphi_{21}+\varphi_{34})\int_{-1}^{1} R_1(\eta)d\eta + \tfrac{1}{4}(\varphi_{41}+\varphi_{32})\int_{-1}^{1} R_2(\xi)d\xi \\ &\quad + \tfrac{1}{4}\varphi_{1234}\int_{\hat{K}} F_3(\xi,\eta)d\xi d\eta. \end{aligned} \tag{4.21}$$

Note that

$$\begin{aligned} a_K(u-u_{\mathcal{T}}, Q^*_{\mathcal{T}}\varphi) &= \varphi_{21}\int_{\overline{M_1Q}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} + \varphi_{34}\int_{\overline{QM_3}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} \\ &\quad -\varphi_{41}\int_{\overline{M_4Q}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} - \varphi_{32}\int_{\overline{QM_2}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} \\ &= \tfrac{1}{2}(\varphi_{21}+\varphi_{34})\int_{\overline{M_1M_3}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} - \tfrac{1}{2}(\varphi_{41}+\varphi_{32})\int_{\overline{M_4M_2}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} \\ &\quad + \tfrac{1}{2}\varphi_{1234}\Big(\int_{\overline{QM_3}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} + \int_{\overline{M_4Q}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} \\ &\qquad - \int_{\overline{M_1Q}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n} - \int_{\overline{QM_2}}\nabla(u-u_{\mathcal{T}})\cdot\mathbf{n}\Big), \end{aligned}$$

where for a segment $\overline{ST}$, $\mathbf{n}$ is the unit normal vector pointing right when we walk from S to T. By changing variables, we get that

$$a_K(u-u_{\mathcal{T}}, Q^*_{\mathcal{T}}\varphi) = \frac{1}{2}\left((\varphi_{21}+\varphi_{34})\int_{-1}^{1} F_1(0,\eta)d\eta + (\varphi_{41}+\varphi_{32})\int_{-1}^{1} F_2(\xi,0)d\xi + \varphi_{1234}T_K\right), \tag{4.22}$$

where

$$T_K := \int_{0}^{1} F_1(0,\eta)d\eta - \int_{-1}^{0} F_2(\xi,0)d\xi - \int_{-1}^{0} F_1(0,\eta)d\eta + \int_{0}^{1} F_2(\xi,0)d\xi.$$

Combining (4.17) with (4.21) and (4.22) yields that

$$\begin{aligned} I_K &= \tfrac{1}{4}(\varphi_{21}+\varphi_{34})\int_{-1}^{1} R_1(\eta)d\eta + \tfrac{1}{4}(\varphi_{41}+\varphi_{32})\int_{-1}^{1} R_2(\xi)d\xi \\ &\quad + \tfrac{1}{4}\varphi_{1234}\int_{\hat{K}} F_3(\xi,\eta)d\xi d\eta - \tfrac{1}{2}\varphi_{1234}T_K, \end{aligned} \tag{4.23}$$

We next estimate the terms on the right-hand side of (4.23). It follows from Lemma 3.7 that

$$|\varphi_{ij}| \le C|Q_{\mathcal{T}}\varphi|_{1,K}, \quad ij = 21, 34, 41, 32. \tag{4.24}$$

By the Cauchy-Schwartz inequality, we derive that

$$\left|\int_{-1}^{1} R_1(\eta)d\eta\right| \le C\left\|\frac{\partial^2 F_1(\xi,\eta)}{\partial\xi^2}\right\|_{0,\hat{K}}.$$

It follows from (3.3) and (3.36) with $r=1$ that

$$\begin{aligned} \left|\frac{\partial^2 F_1(\xi,\eta)}{\partial\xi^2}\right| &\le C\Big(h^2\left|\frac{\partial(\hat{u}-\hat{u}_{\mathcal{T}})}{\partial\xi}\right| + h\left|\frac{\partial^2(\hat{u}-\hat{u}_{\mathcal{T}})}{\partial\xi^2}\right| + h\left|\frac{\partial^2(\hat{u}-\hat{u}_{\mathcal{T}})}{\partial\xi\partial\eta}\right| \\ &\quad + \left|\frac{\partial^3(\hat{u}-\hat{u}_{\mathcal{T}})}{\partial\xi^3}\right| + \left|\frac{\partial^3(\hat{u}-\hat{u}_{\mathcal{T}})}{\partial\xi^2\partial\eta}\right|\Big). \end{aligned}$$

Thus

$$\left|\int_{-1}^{1} R_1(\eta)d\eta\right| \le C\left(h^2|\hat{u}-\hat{u}_{\mathcal{T}}|_{1,\hat{K}} + h|\hat{u}-\hat{u}_{\mathcal{T}}|_{2,\hat{K}} + |\hat{u}|_{3,\hat{K}}\right).$$

This combined with (3.11) and (4.15) yields that

$$\left|\int_{-1}^{1} R_1(\eta)d\eta\right| \le Ch^2\|u\|_{3,K}. \tag{4.25}$$

Similarly, we get that

$$\left|\int_{-1}^{1} R_2(\eta)d\eta\right| \le Ch^2\|u\|_{3,K}. \tag{4.26}$$

By direct calculation, we obtain that

$$|\varphi_{1234}| = 2|\widehat{Q_{\mathcal{T}}\varphi}|_{2,\hat{K}}.$$

This combined with (4.15) yields that

$$|\varphi_{1234}| \le Ch\|Q_{\mathcal{T}}\varphi\|_{2,K}. \tag{4.27}$$

By the Cauchy-Schwartz inequality and (3.11), we get that

$$\left|\int_{\hat{K}} F_3(\xi,\eta)d\xi d\eta\right| \le C|\hat{u}-\hat{u}_{\mathcal{T}}|_{1,\hat{K}} \le C|u-u_{\mathcal{T}}|_{1,K}. \tag{4.28}$$

By the Trace Theorem, (3.11) and (4.15), we derive that

$$|T_K| \le C\left(|\hat{u}-\hat{u}_{\mathcal{T}}|_{1,\hat{K}} + |\hat{u}-\hat{u}_{\mathcal{T}}|_{2,\hat{K}}\right) \le C\left(|u-u_{\mathcal{T}}|_{1,K} + h|u-u_{\mathcal{T}}|_{2,K}\right). \tag{4.29}$$

Substituting (4.24)-(4.29) into (4.23) produces that

$$|I_K| \le C\left(h^2\|u\|_{3,K} + h|u-u_{\mathcal{T}}|_{1,K}\right)\|Q_{\mathcal{T}}\varphi\|_{2,K}.$$

which combining with (4.16), Theorem 3.10 and (4.6) leads to the desired result. □

From (4.4), Proposition 4.2 and Lemmas 4.3-4.6, we can obtain the following L^2 error estimate for the hybrid Wilson FVM.

Theorem 4.7 *Let $u \in H_0^1(\Omega) \cap H^3(\Omega)$ be the solution of (2.1) and $u_{\mathcal{T}} \in \mathbb{U}_{\mathcal{T}}$ be the solution of (2.11). Suppose that the condition (3.2) holds with $\gamma < \frac{2\sqrt{6}}{5}$ and the*

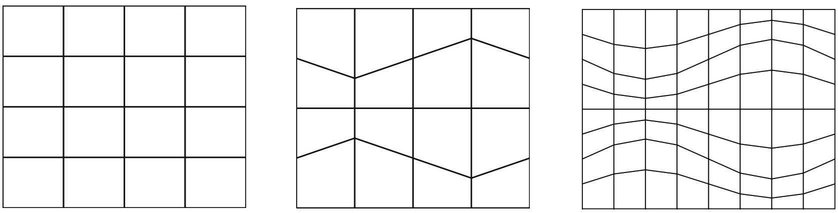

Fig. 3 Left: square, mid: quadrilateral with $N = 4$, right: refined quadrilateral with $N = 8$

Table 1 Error estimates and convergence orders on square mesh

N	$\lvert u - u_{\mathcal{T}}\rvert_{1,\mathcal{T}}$	C.O.	$\lVert u - u_{\mathcal{T}}\rVert_0$	C.O.
4	3.7746e−2		4.2993e−3	
8	1.8710e−2	1.0125	1.2895e−3	1.7374
16	9.3275e−3	1.0043	3.3625e−4	1.9392
32	4.6599e−3	1.0012	8.4909e−5	1.9856
64	2.3294e−3	1.0003	2.1279e−5	1.9965
128	1.1646e−3	1.0001	5.3229e−6	1.9991

condition (3.36) *holds with* $r = 1$. *Then, we have the* L^2 *error estimate for the hybrid Wilson FVM*

$$\|u - u_{\mathcal{T}}\|_0 \le Ch^2 \|u\|_3.$$

5 Numerical examples

In this section, we present numerical results to illustrate the theoretical estimates in the previous sections. The experiments here are performed on a personal computer with 2.70 GHz CPU and 8 Gb RAM and Matlab 7.7 is used as the testing platform.

We consider solving Eq. 2.1 with $f(x, y) := 2(x^2 + y^2 - x - y)$ and $\Omega := (0, 1) \times (0, 1)$. The exact solution of the boundary value problem is given by $u(x, y) = -x(x-1)y(y-1),\ (x, y) \in [0, 1] \times [0, 1]$. In the experiment, we use two families of meshes (see Fig. 3). We subdivide the region $\bar{\Omega}$ into $N \times N$ sub-squares. The left one in Fig. 3 is the square mesh while the mid and right ones are the quadrilateral meshes constructed by the mapping

$$x(i, j) = \frac{i}{N}, \quad y(i, j) = \frac{j}{N} + \frac{1}{10}\sin\left(\frac{2\pi i}{N}\right)\sin\left(\frac{2\pi j}{N}\right), \quad 1 \le i, j \le N - 1,$$

with $N = 4$ and $N = 8$ respectively. It is easy to verify that the above quadrilateral mesh satisfies the condition (3.36) with $r = 1$.

We report the computed $|\cdot|_{1,\mathcal{T}}$ error, $\|\cdot\|_0$ error and their convergence order (C.O.) for the cases of square mesh and quadrilateral mesh respectively in Tables 1 and 2.

Table 2 Error estimates and convergence orders on quadrilateral mesh

N	$\lvert u - u_{\mathcal{T}}\rvert_{1,\mathcal{T}}$	C.O.	$\lVert u - u_{\mathcal{T}}\rVert_0$	C.O.
4	4.2415e−2		9.4073e−3	
8	2.1579e−2	0.9749	3.0516e−3	1.6242
16	1.0918e−2	0.9829	8.2009e−4	1.8957
32	5.5351e−3	0.9800	2.1006e−4	1.9649
64	2.7929e−3	0.9869	5.2980e−5	1.9873
128	1.4036e−3	0.9926	1.3292e−5	1.9949

The numerical results show that the computed convergence order of $|\cdot|_{1,\mathcal{T}}$ error and $\|\cdot\|_0$ error of the hybrid Wilson FVM oscillates around 1 and 2 respectively, which are in agreement with the theoretical results of this paper.

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