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Developing and analyzing fourth-order difference methods for the metamaterial Maxwell's equations

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Abstract In this paper, we develop both a fourth order explicit scheme and a compact implicit scheme for solving the metamaterial Maxwell's equations. A systematic technique is introduced to prove stability and error estimate for both schemes. Numerical results supporting our analysis are presented. To our best knowledge, our convergence theory and stability results are novel, and provide the first error estimate for the fourth order finite difference methods for Maxwell's equations.

Keywords Maxwell's equations · Metamaterial · FDTD method · Fourth order method

Mathematics Subject Classification (2010) 65N30 · 35L15 · 78-08

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1 Introduction

A negative-index metamaterial is an artificial nanomaterial successfully constructed around year 2000, which exhibits a negative index of refraction for a range of frequencies [6]. It has many interesting potential applications including solar cells, subwavelength imaging, invisibility cloaks and reversible Cherenkov radiation. Since 2000, it has been a hot research topic and researchers from various areas have made great progress in the study of metamaterial construction and its applications. Due to the important role of numerical simulation in understanding wave interaction within metamaterials, there has been a growing interest in developing efficient and rigorous numerical methods for solving Maxwell's equations when matematerials are involved. For example, research developing and analyzing finite element methods for metamaterial Maxwell's equations has achieved many interesting results (e.g., papers [2, 3, 8, 10, 15, 16] and the monograph [9]).

Due to its simplicity and robustness, the finite-difference time-domain (FDTD) method, originally proposed by Yee back in 1966 [17], is still one of the most popular numerical methods used especially in the electrical engineering community. Numerous references on the development and application of the FDTD method for metamaterials can be found in the recent monograph on FDTD methods dedicated to metamaterials by Hao and Mittra [6]. Compared to the enormous literature on applications of FDTD methods for solving Maxwell's equations, rigorous analysis of the FDTD methods is quite limited. The first rigorous analysis of the Yee scheme was carried out by Monk and Süli [14] in 1994. In [12], Li, Liang and Lin developed and analyzed a new energy conserving S-FDTD scheme for the Maxwell's equations in metamaterials. In [11], we proved the second order convergence in both time and space for the Yee scheme extended to solve the metamaterial Maxwell's equations on non-uniform rectangular grids. Many studies (e.g.,[1, 4, 5, 13, 19, 20]) show that high order FDTD methods for Maxwell's equations in simple media are much more accurate and efficient than Yee's scheme [7, 18]. In particular high order spatial difference schemes reduce the dispersion error and phase velocity anisotropy error.

Encouraged by the nice properties of high order FDTD methods for Maxwell's equations in simple media, in this paper we extend both the explicit fourth order and compact fourth order difference methods to solve the metamaterial Maxwell's equations. One of the major contributions of our paper is that we manage to prove the stability of both fourth order schemes in an elegant and systematic way. The novelty of our analysis is to transfer the 4th-order implicit compact difference scheme to an equivalent form of 4th-order explicit scheme so that similar analysis to the explicit scheme can be carried over. To the best of our knowledge, this is the first rigorous proof of stability and error estimate for fourth order FDTD methods. We believe that similar ideas and results can be extended to higher order FDTD methods. Details will be explored in our future work. Although applications are often in 3D, it is interesting to consider the 2D case, and the analog can extend to 3D at the expense of more complex notation.

The rest of the paper is organized as follows. In Section 2, we first present the governing equations for wave propagation in Drude metamaterials, one of the most popular metamaterial models. Then we introduce a fourth order spatial difference scheme to discretize those first order spatial derivatives to obtain the fourth order in space and second order in time FDTD method for solving the Drude metamaterial Maxwell's equations. Finally, we establish the stability analysis and convergence rate for the scheme. In Section 3, we first present the fourth order compact scheme for solving the metamaterial Maxwell's equations. Then we carry out a stability analysis of the scheme and conclude with an error estimate. Section 4 is devoted to numerical experiments that demonstrate the effectiveness of the proposed algorithms. Finally, we conclude the paper in Section 5.

2 The explicit fourth order method

2.1 Metamaterial Maxwell's equations

Consider the following 2-D metamaterial model [9] in the rectangular domain $\Omega = [a, b] \times [c, d]$ and time interval $[0, T]$:

$$\epsilon_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - J_x, \tag{1}$$

$$\epsilon_0 \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} - J_y, \tag{2}$$

$$\mu_0 \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} - K_z, \tag{3}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \frac{\partial J_x}{\partial t} + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} J_x = E_x, \tag{4}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \frac{\partial J_y}{\partial t} + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} J_y = E_y, \tag{5}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \frac{\partial K_z}{\partial t} + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} K_z = H_z, \tag{6}$$

subject to the perfect conduct (PEC) boundary condition, which in 2-D becomes:

$$\begin{aligned} E_x(x, c, t) &= E_x(x, d, t) = 0, \quad E_y(a, y, t) = E_y(b, y, t) \\ &= 0, \quad \forall\, x \in [a, b],\ y \in [c, d],\ t \in [0, T], \end{aligned} \tag{7}$$

and the initial conditions

$$E_x(x, y, 0) = E_{x,0}(x, y), \quad E_y(x, y, 0) = E_{y,0}(x, y), \quad H_z(x, y, 0) = H_{z,0}(x, y), \tag{8}$$

$$J_x(x, y, 0) = J_{x,0}(x, y), \quad J_y(x, y, 0) = J_{y,0}(x, y), \quad K_z(x, y, 0) = K_{z,0}(x, y), \tag{9}$$

where $E_{x,0}$, $E_{y,0}$, $H_{z,0}$, $J_{x,0}$, $J_{y,0}$, and $K_{z,0}$ are some properly given functions.

To simplify the notation, in the rest of the paper we denote $H := H_z$ and $K := K_z$. Furthermore, we divide the domain Ω by a uniform rectangular grid

$$a = x_0 < x_1 < \cdots < x_{N_x} = b, \quad c = y_0 < y_1 < \cdots < y_{N_y} = d,$$

and divide the time interval $[0, T]$ into N_t uniform intervals, i.e., we have discrete times $t_k = k\tau, \tau = \frac{T}{N_t}, k = 0, 1, \cdots, N_t$, grid points $x_i = ih_x, h_x = \frac{b-a}{N_x}, i = 0, 1, \cdots, N_x$ in the x-direction, and grid points $y_j = jh_y, h_y = \frac{d-c}{N_y}, j = 0, 1, \cdots, N_y$ in the y-direction. Note that h_x and h_y can be different.

2.2 The fourth order explicit scheme and its analysis

Following the classic Yee scheme, we choose the unknowns E_x (and J_x) at the mid-points of the horizontal edges, E_y (and J_y) at the mid-points of the vertical edges, and H (and K) at the element centers.

We use the following fourth order difference scheme to approximate those partial derivatives $\frac{\partial}{\partial x}$ in the metamaterial model:

$$\delta_x^{(4)} u_{i,j} = \frac{27(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}) - (u_{i+\frac{3}{2},j} - u_{i-\frac{3}{2},j})}{24} = \frac{\partial u}{\partial x}\Big|_{i,j} + \mathcal{O}(h_x^4),$$

where $u_{i,j}$ represents the approximation of u at point (x_i, y_j). On the boundary nodes, we use one-sided difference approximations:

$$\delta_x^{(4)} u_{1,j} = \frac{-22u_{\frac{1}{2},j} + 17u_{\frac{3}{2},j} + 9u_{\frac{5}{2},j} - 5u_{\frac{7}{2},j} + u_{\frac{9}{2},j}}{24},$$
$$\delta_x^{(4)} u_{N_x-1,j} = \frac{-u_{N_x-\frac{9}{2},j} + 5u_{N_x-\frac{7}{2},j} - 9u_{N_x-\frac{5}{2},j} - 17u_{N_x-\frac{3}{2},j} + 22u_{N_x-\frac{1}{2},j}}{24}.$$

Approximation of the derivative $\frac{\partial}{\partial y}$ by the operator $\delta_y^{(4)}$ can be done similarly. Note that these operators can be shifted by half grid points, i.e., i and j can be shifted by $\frac{1}{2}$ in the above formulas.

Application of the fourth order difference operators in (1)–(6) leads to the fourth order explicit scheme:

$$\epsilon_0 \frac{E^{n+1}_{x,i+\frac{1}{2},j} - E^{n}_{x,i+\frac{1}{2},j}}{\tau} = \frac{1}{h_y}\delta_y^{(4)} H^{n+\frac{1}{2}}_{i+\frac{1}{2},j} - J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}, \tag{10}$$

$$\epsilon_0 \frac{E^{n+1}_{y,i,j+\frac{1}{2}} - E^{n}_{y,i,j+\frac{1}{2}}}{\tau} = -\frac{1}{h_x}\delta_x^{(4)} H^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} - J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}, \tag{11}$$

$$\mu_0 \frac{H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}}{\tau} = -\frac{1}{h_x}\delta_x^{(4)} E^{n+1}_{y,i+\frac{1}{2},j+\frac{1}{2}} + \frac{1}{h_y}\delta_y^{(4)} E^{n+1}_{x,i+\frac{1}{2},j+\frac{1}{2}} - K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{12}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2} \frac{J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} - J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}}{\tau} + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} \frac{J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} + J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}}{2} = E^{n+1}_{x,i+\frac{1}{2},j}, \tag{13}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\frac{J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}}-J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}}{\tau}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}}+J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}}{2}=E^{n+1}_{y,i,j+\frac{1}{2}}, \tag{14}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\frac{K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}-K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{\tau}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}+K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{2}=H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{15}$$

To avoid the complexity caused by the PEC boundary condition, in the rest of this subsection, we will establish the discrete stability and error analysis for the scheme (10)–(15) under the following periodic boundary condition assumptions: for any $0 \le i \le N_x - 1,\ \ 0 \le j \le N_y - 1$,

$$E^n_{x,i+\frac{1}{2},0}=E^n_{x,i+\frac{1}{2},N_y},\quad H^{n+\frac{1}{2}}_{i+\frac{1}{2},-\frac{1}{2}}=H^{n+\frac{1}{2}}_{i+\frac{1}{2},N_y-\frac{1}{2}}, \tag{16}$$

$$E^n_{y,0,j+\frac{1}{2}}=E^n_{y,N_x,j+\frac{1}{2}},\quad H^{n+\frac{1}{2}}_{-\frac{1}{2},j+\frac{1}{2}}=H^{n+\frac{1}{2}}_{N_x-\frac{1}{2},j+\frac{1}{2}}, \tag{17}$$

and their periodic shifts.

First, we define some energy norms:

$$||E_x||_*^2=\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}h_xh_y|E_{x,i+\frac{1}{2},j}|^2,\quad ||E_y||_*^2=\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}h_xh_y|E_{y,i,j+\frac{1}{2}}|^2,$$

$$||H||_*^2=\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}h_xh_y|H_{i+\frac{1}{2},j+\frac{1}{2}}|^2,\quad ||J_x||_*^2=\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}h_xh_y|J_{x,i+\frac{1}{2},j}|^2,$$

$$||J_y||_*^2=\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}h_xh_y|J_{y,i,j+\frac{1}{2}}|^2,\quad ||K||_*^2=\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}h_xh_y|K_{i+\frac{1}{2},j+\frac{1}{2}}|^2.$$

Lemma 1 *Denote by $C_v = 1/\sqrt{\epsilon_0\mu_0}$ the wave propagation speed in free space, and let*

$$R_1 := \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}\sum_{0\le n\le N_t-2}h_xh_y\left[\left(H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}-H^{n+\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2}}\right)\left(E^{n+1}_{y,i,j+\frac{1}{2}}+E^n_{y,i,j+\frac{1}{2}}\right)\right.$$
$$\left.+\left(E^{n+1}_{y,i+1,j+\frac{1}{2}}-E^{n+1}_{y,i,j+\frac{1}{2}}\right)\left(H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}+H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\right]. \tag{18}$$

Then we have

$$R_1 \le C_v\left[\mu_0(||H^{N_t-\frac{1}{2}}||_*^2+||H^{\frac{1}{2}}||_*^2)+\epsilon_0(||E_y^{N_t-1}||_*^2+||E_y^0||_*^2)\right].$$

Proof Regrouping the products of R_1, we obtain

$$\begin{aligned}
R_1 = & \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} \sum_{0\le n\le N_t-2} h_x h_y \Bigg[\left(H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i+1,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i,j+\frac{1}{2}}\right) \\
& + \left(H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i+1,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,i+1,j+\frac{1}{2}}\right) \\
& + \left(H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,i+1,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,i,j+\frac{1}{2}}\right) \\
& + \left(H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,i,j+\frac{1}{2}} - H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i,j+\frac{1}{2}}\right)\Bigg] \\
= & \sum_{0\le j\le N_y-1} \sum_{0\le n\le N_t-2} h_x h_y \left(H^{n+\frac{1}{2}}_{N_x-\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,N_x,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{-\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,0,j+\frac{1}{2}}\right) \\
& + \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \left(H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i+1,j+\frac{1}{2}} - H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{0}_{y,i+1,j+\frac{1}{2}}\right) \\
& + \sum_{0\le j\le N_y-1} \sum_{0\le n\le N_t-2} h_x h_y \left(H^{n+\frac{1}{2}}_{N_x-\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,N_x,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{-\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,0,j+\frac{1}{2}}\right) \\
& + \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \left(H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{0}_{y,i,j+\frac{1}{2}} - H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i,j+\frac{1}{2}}\right) \\
= & \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \left(H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i+1,j+\frac{1}{2}} - H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{0}_{y,i+1,j+\frac{1}{2}}\right) \\
& + \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \left(H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{0}_{y,i,j+\frac{1}{2}} - H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i,j+\frac{1}{2}}\right), \qquad (19)
\end{aligned}$$

where the first and the third sums are zero by using periodic boundary conditions.

Using the definition of the energy norms, the Cauchy-Schwarz inequality and Young's inequality, we have

$$\begin{aligned}
\sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i+1,j+\frac{1}{2}} &= C_v \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \sqrt{\mu_0} H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \cdot \sqrt{\epsilon_0} E^{N_t-1}_{y,i+1,j+\frac{1}{2}} \\
&\le \frac{C_v}{2}(\mu_0 ||H^{N_t-\frac{1}{2}}||_*^2 + \epsilon_0 ||E_y^{N_t-1}||_*^2).
\end{aligned}$$

Similarly, we can bound the rest three terms of (19). Substituting these estimates into (19) completes the proof. □

Lemma 2 *Denote by*

$$R_2 := \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \sum_{0\le n\le N_t-2} h_x h_y \left[\left(H^{n+\frac{1}{2}}_{i+\frac{3}{2},j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i-\frac{3}{2},j+\frac{1}{2}}\right)\left(E^{n+1}_{y,i,j+\frac{1}{2}} + E^{n}_{y,i,j+\frac{1}{2}}\right) + \left(E^{n+1}_{y,i+2,j+\frac{1}{2}} - E^{n+1}_{y,i-1,j+\frac{1}{2}}\right)\left(H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\right]. \tag{20}$$

Then we have

$$R_2 \le C_v\left[\mu_0(||H^{N_t-\frac{1}{2}}||_*^2 + ||H^{\frac{1}{2}}||_*^2) + \epsilon_0(||E_y^{N_t-1}||_*^2 + ||E_y^0||_*^2)\right].$$

Proof Expanding and regrouping the products of R_2, we have

$$\begin{aligned} R_2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \sum_{0\le n\le N_t-2} h_x h_y &\left[\left(H^{n+\frac{1}{2}}_{i+\frac{3}{2},j+\frac{1}{2}} E^{n+1}_{y,i,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i-1,j+\frac{1}{2}}\right)\right. \\ &+ \left(H^{n+\frac{1}{2}}_{i+\frac{3}{2},j+\frac{1}{2}} E^{n}_{y,i,j+\frac{1}{2}} - H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i-1,j+\frac{1}{2}}\right) \\ &+ \left(H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i+2,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{n}_{y,i,j+\frac{1}{2}}\right) \\ &\left.+ \left(H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i+2,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{n+1}_{y,i,j+\frac{1}{2}}\right)\right] = \sum_{k=1}^{4} S_k. \end{aligned} \tag{21}$$

Using periodic boundary conditions, we see that

$$S_1 = \sum_{0\le j\le N_y-1} \sum_{0\le n\le N_t-2} h_x h_y \left(H^{n+\frac{1}{2}}_{N_x+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,N_x-1,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,-1,j+\frac{1}{2}}\right) = 0. \tag{22}$$

Similarly, we have

$$\begin{aligned} S_2 &= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \sum_{0\le n\le N_t-2} h_x h_y \left[\left(H^{n+\frac{1}{2}}_{i+\frac{3}{2},j+\frac{1}{2}} E^{n}_{y,i,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,i-1,j+\frac{1}{2}}\right)\right. \\ &\quad \left.+ \left(H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,i-1,j+\frac{1}{2}} - H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i-1,j+\frac{1}{2}}\right)\right] \\ &= \sum_{0\le j\le N_y-1} \sum_{0\le n\le N_t-2} h_x h_y \left[\left(H^{n+\frac{1}{2}}_{N_x+\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,N_x-1,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{\frac{1}{2},j+\frac{1}{2}} E^{n}_{y,-1,j+\frac{1}{2}}\right)\right. \\ &\quad \left.+ \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y \left(H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{0}_{y,i-1,j+\frac{1}{2}} - H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i-1,j+\frac{1}{2}}\right)\right] \\ &= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y \left(H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{0}_{y,i-1,j+\frac{1}{2}} - H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i-1,j+\frac{1}{2}}\right), \end{aligned} \tag{23}$$

$$S_3 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \sum_{0\le n\le N_t-2} h_x h_y \Big[\Big(H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,i+2,j+\frac{1}{2}} - H^{n+\frac{3}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{n+1}_{y,i,j+\frac{1}{2}}\Big)$$
$$+ \Big(H^{n+\frac{3}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{n+1}_{y,i,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{n}_{y,i,j+\frac{1}{2}}\Big)\Big]$$
$$= \sum_{0\le j\le N_y-1} \sum_{0\le n\le N_t-2} h_x h_y \Big[H^{n+\frac{3}{2}}_{N_x-\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,N_x+1,j+\frac{1}{2}} + H^{n+\frac{3}{2}}_{N_x-\frac{3}{2},j+\frac{1}{2}} E^{n+1}_{y,N_x,j+\frac{1}{2}}$$
$$-H^{n+\frac{3}{2}}_{-\frac{3}{2},j+\frac{1}{2}} E^{n+1}_{y,0,j+\frac{1}{2}} - H^{n+\frac{3}{2}}_{-\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,1,j+\frac{1}{2}}\Big]$$
$$+ \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y \Big(H^{N_t-\frac{1}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i,j+\frac{1}{2}} - H^{\frac{1}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{0}_{y,i,j+\frac{1}{2}}\Big)$$
$$= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y \Big(H^{N_t-\frac{1}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{N_t-1}_{y,i,j+\frac{1}{2}} - H^{\frac{1}{2}}_{i-\frac{3}{2},j+\frac{1}{2}} E^{0}_{y,i,j+\frac{1}{2}}\Big). \tag{24}$$

and

$$S_4 = \sum_{0\le j\le N_y-1} \sum_{0\le n\le N_t-2} h_x h_y \Big[\Big(H^{n+\frac{1}{2}}_{N_x-\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,N_x+1,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{-\frac{1}{2},j+\frac{1}{2}} E^{n+1}_{y,1,j+\frac{1}{2}}\Big)$$
$$+ \Big(H^{n+\frac{1}{2}}_{N_x-\frac{3}{2},j+\frac{1}{2}} E^{n+1}_{y,N_x,j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{-\frac{3}{2},j+\frac{1}{2}} E^{n+1}_{y,0,j+\frac{1}{2}}\Big)\Big] = 0, \tag{25}$$

where in the last step we used the periodic boundary conditions.

Using the Cauchy-Schwarz inequality to (23) and (24) completes the proof. □

With the preparatory work of Lemmas 1 and 2, we can now prove the following stability result for the fourth order scheme (10)–(15) with periodic boundary conditions.

Theorem 1 *Under the time step constraint*

$$\tau \le \min\left\{\frac{3h_x}{28C_v}, \frac{3h_y}{28C_v}, \frac{1}{2\omega_{pe}}, \frac{1}{2\omega_{pm}}\right\}, \tag{26}$$

for any $m \in [0, N_t-2]$, *we have*

$$\epsilon_0(||E_x^{m+1}||_*^2 + ||E_y^{m+1}||_*^2) + \mu_0||H^{m+\frac{3}{2}}||_*^2$$
$$+\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{m+\frac{3}{2}}||_*^2 + ||J_y^{m+\frac{3}{2}}||_*^2) + \frac{1}{\mu_0\omega_{pm}^2}||K^{m+2}||_*^2 \tag{27}$$
$$\le 3\left[\epsilon_0(||E_x^0||_*^2 + ||E_y^0||_*^2) + \mu_0||H^{\frac{1}{2}}||_*^2 + \frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{\frac{1}{2}}||_*^2 + ||J_y^{\frac{1}{2}}||_*^2) + \frac{1}{\mu_0\omega_{pm}^2}||K^1||_*^2\right].$$

Proof Multiplying (10)–(15) by $\tau h_x h_y (E^{n+1}_{x,i+\frac{1}{2},j} + E^{n}_{x,i+\frac{1}{2},j})$, $\tau h_x h_y (E^{n+1}_{y,i,j+\frac{1}{2}} + E^{n}_{y,i,j+\frac{1}{2}})$, $\tau h_x h_y (H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}})$, $\tau h_x h_y (J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} + J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j})$, $\tau h_x h_y (J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}} + J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}})$, $\tau h_x h_y (K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}} + K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}})$, respectively, then summing up each resultant over i from 0 to $N_x - 1$, and j from 0 to $N_y - 1$, we have

$$\epsilon_0(||E_x^{n+1}||_*^2 - ||E_x^n||_*^2) = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \left(\frac{1}{h_y}\delta_y^{(4)} H^{n+\frac{1}{2}}_{i+\frac{1}{2},j} - J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}\right) \times \left(E^{n+1}_{x,i+\frac{1}{2},j} + E^{n}_{x,i+\frac{1}{2},j}\right), \tag{28}$$

$$\epsilon_0(||E_y^{n+1}||_*^2 - ||E_y^n||_*^2) = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \left(-\frac{1}{h_x}\delta_x^{(4)} H^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} - J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}\right) \times \left(E^{n+1}_{y,i,j+\frac{1}{2}} + E^{n}_{y,i,j+\frac{1}{2}}\right), \tag{29}$$

$$\mu_0(||H^{n+\frac{3}{2}}||_*^2 - ||H^{n+\frac{1}{2}}||_*^2) = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \left(-\frac{1}{h_x}\delta_x^{(4)} E^{n+1}_{y,i+\frac{1}{2},j+\frac{1}{2}} + \frac{1}{h_y}\delta_y^{(4)} E^{n+1}_{x,i+\frac{1}{2},j+\frac{1}{2}} - K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\left(H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right),$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{n+\frac{3}{2}}||_*^2 - ||J_x^{n+\frac{1}{2}}||_*^2) + \frac{\tau\Gamma_e}{2\epsilon_0\omega_{pe}^2}||J_x^{n+\frac{3}{2}} + J_x^{n+\frac{1}{2}}||_*^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y E^{n+1}_{x,i+\frac{1}{2},j}\left(J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} + J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}\right),$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}(||J_y^{n+\frac{3}{2}}||_*^2 - ||J_y^{n+\frac{1}{2}}||_*^2) + \frac{\tau\Gamma_e}{2\epsilon_0\omega_{pe}^2}||J_y^{n+\frac{3}{2}} + J_y^{n+\frac{1}{2}}||_*^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y E^{n+1}_{y,i,j+\frac{1}{2}}\left(J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}} + J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}\right),$$

$$\frac{1}{\mu_0\omega_{pm}^2}(||K^{n+2}||_*^2 - ||K^{n+1}||_*^2) + \frac{\tau\Gamma_m}{2\mu_0\omega_{pm}^2}||K^{n+2} + K^{n+1}||_*^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\left(K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}} + K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}\right). \tag{30}$$

Adding (29)–(30), then summing up n from 0 to $N_t - 2$, we easily see that the sum of the left hand side (LHS) satisfies the following:

$$LHS \ge \epsilon_0(||E_x^{N_t-1}||_*^2 - ||E_x^0||_*^2) + \epsilon_0(||E_y^{N_t-1}||_*^2 - ||E_y^0||_*^2) + \mu_0(||H^{N_t-\frac{1}{2}}||_*^2 - ||H^{\frac{1}{2}}||_*^2) + \frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{N_t-\frac{1}{2}}||_*^2 - ||J_x^{\frac{1}{2}}||_*^2) + \frac{1}{\epsilon_0\omega_{pe}^2}(||J_y^{N_t-\frac{1}{2}}||_*^2 - ||J_y^{\frac{1}{2}}||_*^2) + \frac{1}{\mu_0\omega_{pm}^2}(||K^{N_t}||_*^2 - ||K^1||_*^2), \tag{31}$$

and the sum of the right hand side (RHS) is:

$$
\begin{aligned}
RHS = &-\frac{\tau}{h_x}\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[\delta_x^{(4)}H^{n+\frac12}_{i,j+\frac12}\left(E^{n+1}_{y,i,j+\frac12}+E^{n}_{y,i,j+\frac12}\right)\right.\\
&\left.+\delta_x^{(4)}E^{n+1}_{y,i+\frac12,j+\frac12}\left(H^{n+\frac32}_{i+\frac12,j+\frac12}+H^{n+\frac12}_{i+\frac12,j+\frac12}\right)\right]\\
&+\frac{\tau}{h_y}\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[\delta_y^{(4)}H^{n+\frac12}_{i+\frac12,j}\left(E^{n+1}_{x,i+\frac12,j}+E^{n}_{x,i+\frac12,j}\right)\right.\\
&\left.+\delta_y^{(4)}E^{n+1}_{x,i+\frac12,j+\frac12}\left(H^{n+\frac32}_{i+\frac12,j+\frac12}+H^{n+\frac12}_{i+\frac12,j+\frac12}\right)\right]\\
&+\tau\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[E^{n+1}_{x,i+\frac12,j}J^{n+\frac32}_{x,i+\frac12,j}-E^{n}_{x,i+\frac12,j}J^{n+\frac12}_{x,i+\frac12,j}\right]\\
&+\tau\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[E^{n+1}_{y,i,j+\frac12}J^{n+\frac32}_{y,i,j+\frac12}-E^{n}_{y,i,j+\frac12}J^{n+\frac12}_{y,i,j+\frac12}\right]\\
&+\tau\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[H^{n+\frac32}_{i+\frac12,j+\frac12}K^{n+2}_{i+\frac12,j+\frac12}-H^{n+\frac12}_{i+\frac12,j+\frac12}K^{n+1}_{i+\frac12,j+\frac12}\right]\\
=&\sum_{k=1}^{5}RHS_k. \qquad (32)
\end{aligned}
$$

By the definition of R_1 and Lemmas 1 and 2, we have

$$
\begin{aligned}
RHS_1 &= -\frac{\tau}{h_x}\left(\frac{27}{24}R_1-\frac{1}{24}R_2\right)\\
&\le \frac{\tau C_v}{h_x}\cdot\frac{28}{24}\left[\mu_0(||H^{N_t-\frac12}||_*^2+||H^{\frac12}||_*^2)+\epsilon_0(||E_y^{N_t-1}||_*^2+||E_y^0||_*^2)\right]. \qquad (33)
\end{aligned}
$$

By symmetry, we can show that

$$
RHS_2 \le \frac{\tau C_v}{h_y}\cdot\frac{28}{24}\left[\mu_0(||H^{N_t-\frac12}||_*^2+||H^{\frac12}||_*^2)+\epsilon_0(||E_x^{N_t-1}||_*^2+||E_x^0||_*^2)\right]. \qquad (34)
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
RHS_3 &= \tau\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[E^{N_t-1}_{x,i+\frac12,j}J^{N_t-\frac12}_{x,i+\frac12,j}-E^{0}_{x,i+\frac12,j}J^{\frac12}_{x,i+\frac12,j}\right]\\
&\le \frac{\tau\omega_{pe}}{2}\left[\epsilon_0(||E_x^{N_t-1}||_*^2+||E_x^0||_*^2)+\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{N_t-\frac12}||_*^2+||J_x^{\frac12}||_*^2)\right]. \qquad (35)
\end{aligned}
$$

Similarly, we can show that

$$RHS_4 = \tau \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \left[E^{N_t-1}_{y,i,j+\frac{1}{2}} J^{N_t-\frac{1}{2}}_{y,i,j+\frac{1}{2}} - E^{0}_{y,i,j+\frac{1}{2}} J^{\frac{1}{2}}_{y,i,j+\frac{1}{2}} \right]$$

$$\le \frac{\tau\omega_{pe}}{2}\left[\epsilon_0(||E_y^{N_t-1}||_*^2 + ||E_y^0||_*^2) + \frac{1}{\epsilon_0\omega_{pe}^2}(||J_y^{N_t-\frac{1}{2}}||_*^2 + ||J_y^{\frac{1}{2}}||_*^2)\right], \quad (36)$$

and

$$RHS_5 = \tau \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \left[H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} K^{N_t}_{i+\frac{1}{2},j+\frac{1}{2}} - H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} K^{1}_{i+\frac{1}{2},j+\frac{1}{2}} \right]$$

$$\le \frac{\tau\omega_{pm}}{2}\left[\mu_0(||H^{N_t-\frac{1}{2}}||_*^2 + ||H^{\frac{1}{2}}||_*^2) + \frac{1}{\mu_0\omega_{pm}^2}(||K^{N_t}||_*^2 + ||K^1||_*^2)\right]. \quad (37)$$

Substituting the estimates (33)–(37) into (32), we obtain

$$\begin{aligned} RHS \le & \left(\frac{7\tau C_v}{6h_x} + \frac{7\tau C_v}{6h_y} + \frac{\tau\omega_{pm}}{2}\right)\cdot \mu_0(||H^{N_t-\frac{1}{2}}||_*^2 + ||H^{\frac{1}{2}}||_*^2) \\ & + \left(\frac{7\tau C_v}{6h_x} + \frac{\tau\omega_{pe}}{2}\right)\cdot \epsilon_0(||E_y^{N_t-1}||_*^2 + ||E_y^0||_*^2) \\ & + \left(\frac{7\tau C_v}{6h_y} + \frac{\tau\omega_{pe}}{2}\right)\cdot \epsilon_0(||E_x^{N_t-1}||_*^2 + ||E_x^0||_*^2) \\ & + \frac{\tau\omega_{pe}}{2}\cdot\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{N_t-\frac{1}{2}}||_*^2 + ||J_x^{\frac{1}{2}}||_*^2) \\ & + \frac{\tau\omega_{pe}}{2}\cdot\frac{1}{\epsilon_0\omega_{pe}^2}(||J_y^{N_t-\frac{1}{2}}||_*^2 + ||J_y^{\frac{1}{2}}||_*^2) \\ & + \frac{\tau\omega_{pm}}{2}\cdot\frac{1}{\mu_0\omega_{pm}^2}(||K^{N_t}||_*^2 + ||K^1||_*^2). \end{aligned} \quad (38)$$

Using the constraint (26), we conclude the proof by combining the estimates (31) and (38). □

To consider the convergence of the scheme (10)–(15), we define the following solution errors:

$$\mathcal{E}^n_{x,i+\frac{1}{2},j} = E_x(x_{i+\frac{1}{2}}, y_j, t_n) - E^n_{x,i+\frac{1}{2},j}, \quad \mathcal{E}^n_{y,i,j+\frac{1}{2}} = E_y(x_i, y_{j+\frac{1}{2}}, t_n) - E^n_{y,i,j+\frac{1}{2}},$$
$$\mathcal{H}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} = H(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}) - H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}, \quad \mathcal{J}^n_{x,i+\frac{1}{2},j} = J_x(x_{i+\frac{1}{2}}, y_j, t_n) - J^n_{x,i+\frac{1}{2},j},$$
$$\mathcal{J}^n_{y,i,j+\frac{1}{2}} = J_y(x_i, y_{j+\frac{1}{2}}, t_n) - J^n_{y,i,j+\frac{1}{2}}, \quad \mathcal{K}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} = K(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}) - K^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}.$$

Theorem 2 *Suppose that the initial errors*

$$\mathcal{E}^0_{x,i+\frac{1}{2},j} = \mathcal{E}^0_{y,i,j+\frac{1}{2}} = \mathcal{H}^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} = \mathcal{J}^0_{x,i+\frac{1}{2},j} = \mathcal{J}^0_{y,i,j+\frac{1}{2}} = \mathcal{K}^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} = 0$$

are all zero. Then under the time step constraint (26)*, we have the following optimal error estimate*

$$\max_{0\le n\le N_t-1}\Bigg[\epsilon_0(||\mathcal{E}_x^n||_*^2+||\mathcal{E}_y^n||_*^2)+\mu_0||\mathcal{H}^{n+\frac{1}{2}}||_*^2 + \frac{1}{\mu_0\omega_{pm}^2}||\mathcal{K}^{n+1}||_*^2+\frac{1}{\epsilon_0\omega_{pe}^2}(||\mathcal{J}_x^{n+\frac{1}{2}}||_*^2+||\mathcal{J}_y^{n+\frac{1}{2}}||_*^2)\Bigg]^{1/2} \le CT(\tau^2+h_x^4+h_y^4),$$

where the constant $C > 0$ is independent of T, τ, h_x and h_y.

Proof Using the solution error definition and scheme (10), we have

$$\begin{aligned}
&\epsilon_0\frac{\mathcal{E}^{n+1}_{x,i+\frac{1}{2},j}-\mathcal{E}^{n}_{x,i+\frac{1}{2},j}}{\tau} - \frac{1}{h_y}\delta_y^{(4)}\mathcal{H}^{n+\frac{1}{2}}_{i+\frac{1}{2},j} + \mathcal{J}^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}\\
&= \epsilon_0\frac{E_x(x_{i+\frac{1}{2}},y_j,t_{n+1})-E_x(x_{i+\frac{1}{2}},y_j,t_n)}{\tau} - \frac{1}{h_y}\delta_y^{(4)}H(x_{i+\frac{1}{2}},y_j,t_{n+\frac{1}{2}})\\
&\quad + J_x(x_{i+\frac{1}{2}},y_j,t_{n+\frac{1}{2}})\\
&= \left[\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\partial_y H(x_{i+\frac{1}{2}},y_j,t)dt - \frac{1}{h_y}\delta_y^{(4)}H(x_{i+\frac{1}{2}},y_j,t_{n+\frac{1}{2}})\right]\\
&\quad + \left[J_x(x_{i+\frac{1}{2}},y_j,t_{n+\frac{1}{2}}) - \frac{1}{\tau}\int_{t_n}^{t_{n+1}} J_x(x_{i+\frac{1}{2}},y_j,t)dt\right] = Err_1+Err_2,
\end{aligned} \tag{39}$$

where in the last step we used the integration of (1) at point $(x_{i+\frac{1}{2}}, y_j)$ with respect to t from t_n to t_{n+1}.

By the Taylor expansion, we easily see that

$$Err_1 = O(h_y^4)||\partial_{y^5}H||_\infty, \quad Err_2 = O(\tau^2)||\partial_{t^2}J_x||_\infty.$$

By exactly the same technique, from scheme (11) and integration of (2) at point $(x_i, y_{j+\frac{1}{2}})$ from $t = t_n$ to $t = t_{n+1}$, we can obtain the second error equation:

$$\begin{aligned}
&\epsilon_0\frac{\mathcal{E}^{n+1}_{y,i,j+\frac{1}{2}}-\mathcal{E}^{n}_{y,i,j+\frac{1}{2}}}{\tau} + \frac{1}{h_x}\delta_x^{(4)}\mathcal{H}^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} + \mathcal{J}^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}\\
&= \left[-\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\partial_x H(x_i,y_{j+\frac{1}{2}},t)dt + \frac{1}{h_x}\delta_x^{(4)}H(x_i,y_{j+\frac{1}{2}},t_{n+\frac{1}{2}})\right]\\
&\quad + \left[J_y(x_i,y_{j+\frac{1}{2}},t_{n+\frac{1}{2}}) - \frac{1}{\tau}\int_{t_n}^{t_{n+1}} J_y(x_i,y_{j+\frac{1}{2}},t)dt\right]\\
&= Err_3+Err_4 = O(h_x^4)||\partial_{x^5}H||_\infty + O(\tau^2)||\partial_{t^2}J_y||_\infty.
\end{aligned} \tag{40}$$

Next, using scheme (12) and integration of (3) we can obtain the third error equation:

$$\mu_0 \frac{\mathcal{H}^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \mathcal{H}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}}{\tau} + \frac{1}{h_x}\delta_x^{(4)}\mathcal{E}^{n+1}_{y,i+\frac{1}{2},j+\frac{1}{2}} - \frac{1}{h_y}\delta_y^{(4)}\mathcal{E}^{n+1}_{x,i+\frac{1}{2},j+\frac{1}{2}} + \mathcal{K}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{41}$$

$$\begin{aligned}
&= \frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}} (-\partial_x E_y + \partial_y E_x - K)(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t)dt \\
&\quad + \frac{1}{h_x}\delta_x^{(4)} E_y(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t_{n+1}) - \frac{1}{h_y}\delta_y^{(4)} E_x(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t_{n+1}) \\
&\quad + K(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t_{n+1}) \\
&= O(h_x^4)||\partial_{x^5} E_y||_\infty + O(h_y^4)||\partial_{y^5} E_x||_\infty + O(\tau^2)||\partial_{t^2} K||_\infty.
\end{aligned} \tag{42}$$

Again, using scheme (13) and integration of (4) we can obtain the fourth error equation:

$$\frac{1}{\epsilon_0\omega_{pe}^2} \frac{\mathcal{J}^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} - \mathcal{J}^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}}{\tau} + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} \frac{\mathcal{J}^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} + \mathcal{J}^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}}{2} - \mathcal{E}^{n+1}_{x,i+\frac{1}{2},j}, \tag{43}$$

$$\begin{aligned}
&= \frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}} \left(E_x - \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} J_x\right)(x_{i+\frac{1}{2}}, y_j, t)dt \\
&\quad + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} \frac{J_x(x_{i+\frac{1}{2}}, y_j, t_{n+\frac{3}{2}}) + J_x(x_{i+\frac{1}{2}}, y_j, t_{n+\frac{1}{2}})}{2} - E_x(x_{i+\frac{1}{2}}, y_j, t_{n+1}) \\
&= O(\tau^2)||\partial_{t^2} E_x||_\infty + O(\tau^2)||\partial_{t^2} J_x||_\infty.
\end{aligned} \tag{44}$$

Similarly, using scheme (14) and integration of (5) we can obtain the fifth error equation:

$$\frac{1}{\epsilon_0\omega_{pe}^2} \frac{\mathcal{J}^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}} - \mathcal{J}^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}}{\tau} + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} \frac{\mathcal{J}^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}} + \mathcal{J}^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}}{2} - \mathcal{E}^{n+1}_{y,i,j+\frac{1}{2}}, \tag{45}$$

$$\begin{aligned}
&= \frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}} \left(E_y - \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} J_y\right)(x_i, y_{j+\frac{1}{2}}, t)dt \\
&\quad + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} \frac{J_y(x_i, y_{j+\frac{1}{2}}, t_{n+\frac{3}{2}}) + J_y(x_i, y_{j+\frac{1}{2}}, t_{n+\frac{3}{2}})}{2} - E_y(x_i, y_{j+\frac{1}{2}}, t_{n+1}) \\
&= O(\tau^2)||\partial_{t^2} E_y||_\infty + O(\tau^2)||\partial_{t^2} J_y||_\infty.
\end{aligned} \tag{46}$$

Finally, using scheme (15) and integration of (6) we can obtain the sixth error equation:

$$\frac{1}{\mu_0\omega_{pm}^2}\frac{\mathcal{K}^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}-\mathcal{K}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{\tau}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{\mathcal{K}^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}+\mathcal{K}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{2}-\mathcal{H}^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \quad (47)$$

$$=\frac{1}{\tau}\int_{t_{n+1}}^{t_{n+2}}(H-\frac{\Gamma_m}{\mu_0\omega_{pm}^2}K)(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},t)dt$$
$$+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{K(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},t_{n+2})+K(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},t_{n+1})}{2}-H(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},t_{n+\frac{3}{2}})$$
$$=O(\tau^2)||\partial_{t^2}K||_\infty+O(\tau^2)||\partial_{t^2}H||_\infty. \quad (48)$$

The rest of the proof follows the stability proof of Theorem 3 by multiplying (39)–(48) by $\tau h_x h_y(\mathcal{E}^{n+1}_{x,i+\frac{1}{2},j}+\mathcal{E}^{n}_{x,i+\frac{1}{2},j})$, $\tau h_x h_y(\mathcal{E}^{n+1}_{y,i,j+\frac{1}{2}}+\mathcal{E}^{n}_{y,i,j+\frac{1}{2}})$, $\tau h_x h_y(\mathcal{H}^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}+\mathcal{H}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}})$, $\tau h_x h_y(\mathcal{J}^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j}+\mathcal{J}^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j})$, $\tau h_x h_y(\mathcal{J}^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}}+\mathcal{J}^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}})$, $\tau h_x h_y(\mathcal{K}^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}+\mathcal{K}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}})$, respectively, then summing up each resultant over i from 0 to N_x-1, j from 0 to N_y-1, and n from 0 to N_t-2, we obtain

$$\epsilon_0(||\mathcal{E}_x^{N_t-1}||_*^2-||\mathcal{E}_x^0||_*^2)+\epsilon_0(||\mathcal{E}_y^{N_t-1}||_*^2-||\mathcal{E}_y^0||_*^2)+\mu_0(||\mathcal{H}^{N_t-\frac{1}{2}}||_*^2-||\mathcal{H}^{\frac{1}{2}}||_*^2)$$
$$+\frac{1}{\epsilon_0\omega_{pe}^2}(||\mathcal{J}_x^{N_t-\frac{1}{2}}||_*^2-||\mathcal{J}_x^{\frac{1}{2}}||_*^2)+\frac{1}{\epsilon_0\omega_{pe}^2}(||\mathcal{J}_y^{N_t-\frac{1}{2}}||_*^2-||\mathcal{J}_y^{\frac{1}{2}}||_*^2)$$
$$+\frac{1}{\mu_0\omega_{pm}^2}(||\mathcal{K}^{N_t}||_*^2-||\mathcal{K}^1||_*^2)$$
$$\le\left(\frac{7\tau C_v}{6h_x}+\frac{7\tau C_v}{6h_y}+\frac{\tau\omega_{pm}}{2}\right)\cdot\mu_0(||\mathcal{H}^{N_t-\frac{1}{2}}||_*^2+||\mathcal{H}^{\frac{1}{2}}||_*^2)$$
$$+\left(\frac{7\tau C_v}{6h_x}+\frac{\tau\omega_{pe}}{2}\right)\cdot\epsilon_0(||\mathcal{E}_y^{N_t-1}||_*^2+||\mathcal{E}_y^0||_*^2)+\left(\frac{7\tau C_v}{6h_y}+\frac{\tau\omega_{pe}}{2}\right)\cdot\epsilon_0(||\mathcal{E}_x^{N_t-1}||_*^2+||\mathcal{E}_x^0||_*^2)$$
$$+\frac{\tau\omega_{pe}}{2}\cdot\frac{1}{\epsilon_0\omega_{pe}^2}(||\mathcal{J}_x^{N_t-\frac{1}{2}}||_*^2+||\mathcal{J}_x^{\frac{1}{2}}||_*^2)+\frac{\tau\omega_{pe}}{2}\cdot\frac{1}{\epsilon_0\omega_{pe}^2}(||\mathcal{J}_y^{N_t-\frac{1}{2}}||_*^2+||\mathcal{J}_y^{\frac{1}{2}}||_*^2)$$
$$+\frac{\tau\omega_{pm}}{2}\cdot\frac{1}{\mu_0\omega_{pm}^2}(||\mathcal{K}^{N_t}||_*^2+||\mathcal{K}^1||_*^2)$$
$$+\frac{CT}{\delta}(\tau^2+h_x^4+h_y^4)^2+\delta\tau\sum_{n=0}^{N_t-2}\Big[\epsilon_0(||\mathcal{E}_x^n||_*^2+||\mathcal{E}_x^{n+1}||_*^2+||\mathcal{E}_y^n||_*^2+||\mathcal{E}_y^{n+1}||_*^2)$$
$$+\mu_0(||\mathcal{H}^{n+\frac{1}{2}}||_*^2+||\mathcal{H}^{n+\frac{3}{2}}||_*^2)+\frac{1}{\mu_0\omega_{pm}^2}(||\mathcal{K}^{n+1}||_*^2+||\mathcal{K}^{n+2}||_*^2)$$
$$+\frac{1}{\epsilon_0\omega_{pe}^2}(||\mathcal{J}_x^{n+\frac{1}{2}}||_*^2+||\mathcal{J}_x^{n+\frac{3}{2}}||_*^2+||\mathcal{J}_y^{n+\frac{1}{2}}||_*^2+||\mathcal{J}_y^{n+\frac{3}{2}}||_*^2)\Big], \quad (49)$$

where the last three lines are obtained by using the Cauchy-Schwarz inequality to those extra Err_i terms.

Note that the last three lines can be bounded further by choosing the maximum over all n:

$$LAST_3 \le \frac{CT}{\delta}(\tau^2+h_x^4+h_y^4)^2+2T\delta \max_{0\le n\le N_t-1}\Big[\epsilon_0(||\mathcal{E}_x^n||_*^2+||\mathcal{E}_y^n||_*^2)+\mu_0||\mathcal{H}^{n+\frac{1}{2}}||_*^2 + \frac{1}{\mu_0\omega_{pm}^2}||\mathcal{K}^{n+1}||_*^2 + \frac{1}{\epsilon_0\omega_{pe}^2}(||\mathcal{J}_x^{n+\frac{1}{2}}||_*^2 + ||\mathcal{J}_y^{n+\frac{1}{2}}||_*^2)\Big]. \tag{50}$$

Substituting (50) into (49), then using the time step constraint (26) so that the rest terms above the last three lines of (49) can be bounded by corresponding terms on the left hand side of (49), then taking the maximum of left hand side terms and choosing δT small enough, we complete the proof. □

3 The fourth order compact scheme and its analysis

Considering the Taylor expansions:

$$\frac{1}{24}\left(\frac{\partial u}{\partial x}\Big|_{i-1,j} + 22\frac{\partial u}{\partial x}\Big|_{i,j} + \frac{\partial u}{\partial x}\Big|_{i+1,j}\right) = \frac{\partial u}{\partial x}\Big|_{i,j} + \frac{h_x^2}{24}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j} + \mathcal{O}(h_x^4)$$

and

$$\frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{h_x} = \frac{\partial u}{\partial x}\Big|_{i,j} + \frac{h_x^2}{24}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j} + \mathcal{O}(h_x^4),$$

we obtain

$$\bar{\delta}_x\left(\frac{\partial u}{\partial x}\right)\Big|_{i,j} = \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{h_x} + \mathcal{O}(h_x^4), \tag{51}$$

where we denote by $\bar{\delta}_x u_{i,j} = \frac{1}{24}(u_{i-1,j} + 22u_{i,j} + u_{i+1,j})$. Equation (51) means that a tridiagonal linear system has to be solved in order to obtain the derivatives $\frac{\partial u}{\partial x}\Big|_{i,j}$ at the grid points.

Applying the above implicit scheme for approximating the derivatives in the metamaterial Maxwell's equations (1)–(6), we obtain the following fourth order compact difference scheme:

$$\epsilon_0\frac{E^{n+1}_{x,i+\frac{1}{2},j} - E^n_{x,i+\frac{1}{2},j}}{\tau} = \frac{\partial H}{\partial y}\Big|^{n+\frac{1}{2}}_{i+\frac{1}{2},j} - J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}, \tag{52}$$

$$\epsilon_0\frac{E^{n+1}_{y,i,j+\frac{1}{2}} - E^n_{y,i,j+\frac{1}{2}}}{\tau} = -\frac{\partial H}{\partial x}\Big|^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} - J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}, \tag{53}$$

$$\mu_0\frac{H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}}{\tau} = -\frac{\partial E_y}{\partial x}\Big|^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{\partial E_x}{\partial y}\Big|^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{54}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\frac{J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j}-J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}}{\tau}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j}+J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}}{2}=E^{n+1}_{x,i+\frac{1}{2},j},\tag{55}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\frac{J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}}-J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}}{\tau}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}}+J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}}{2}=E^{n+1}_{y,i,j+\frac{1}{2}},\tag{56}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\frac{K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}-K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{\tau}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}+K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{2}=H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}},\tag{57}$$

where all the spatial derivatives have to be obtained from (51) or some shifts in i and/or j.

To establish the stability analysis of the compact scheme (52)–(57) by a technique similar to that developed for the explicit scheme (10)–(15), we can rewrite (52)–(57) into the following equivalent form:

$$\epsilon_0\frac{\bar{\delta}_x\bar{\delta}_y E^{n+1}_{x,i+\frac{1}{2},j}-\bar{\delta}_x\bar{\delta}_y E^{n}_{x,i+\frac{1}{2},j}}{\tau}=\frac{\bar{\delta}_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}-\bar{\delta}_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}}{h_y}-\bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j},\tag{58}$$

$$\epsilon_0\frac{\bar{\delta}_x\bar{\delta}_y E^{n+1}_{y,i,j+\frac{1}{2}}-\bar{\delta}_x\bar{\delta}_y E^{n}_{y,i,j+\frac{1}{2}}}{\tau}=-\frac{\bar{\delta}_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}-\bar{\delta}_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}}{h_x}-\bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}},\tag{59}$$

$$\begin{aligned}\mu_0\frac{\bar{\delta}_x\bar{\delta}_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}-\bar{\delta}_x\bar{\delta}_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}}{\tau}&=-\frac{\bar{\delta}_y E^{n+1}_{y,i+1,j+\frac{1}{2}}-\bar{\delta}_y E^{n+1}_{y,i,j+\frac{1}{2}}}{h_x}\\&\quad+\frac{\bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j+1}-\bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j}}{h_y}-\bar{\delta}_x\bar{\delta}_y K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}},\end{aligned}\tag{60}$$

$$\begin{aligned}&\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\bar{\delta}_x\bar{\delta}_y J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j}-\bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}}{\tau}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{\bar{\delta}_x\bar{\delta}_y J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j}+\bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}}{2}\\&=\bar{\delta}_x\bar{\delta}_y E^{n+1}_{x,i+\frac{1}{2},j},\end{aligned}\tag{61}$$

$$\begin{aligned}&\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\bar{\delta}_x\bar{\delta}_y J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}}-\bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}}{\tau}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{\bar{\delta}_x\bar{\delta}_y J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}}+\bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}}{2}\\&=\bar{\delta}_x\bar{\delta}_y E^{n+1}_{y,i,j+\frac{1}{2}},\end{aligned}\tag{62}$$

$$\begin{aligned}&\frac{1}{\mu_0\omega_{pm}^2}\frac{\bar{\delta}_x\bar{\delta}_y K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}-\bar{\delta}_x\bar{\delta}_y K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{\tau}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{\bar{\delta}_x\bar{\delta}_y K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}}+\bar{\delta}_x\bar{\delta}_y K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{2}\\&=\bar{\delta}_x\bar{\delta}_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}.\end{aligned}\tag{63}$$

To prove the discrete stability for scheme (52)–(57), we introduce some new energy norms:

$$||E_x||_{\#}^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x \bar{\delta}_y E_{x,i+\frac{1}{2},j}|^2, \quad ||E_y||_{\#}^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x \bar{\delta}_y E_{y,i,j+\frac{1}{2}}|^2,$$

$$||H||_{\#}^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x \bar{\delta}_y H_{i+\frac{1}{2},j+\frac{1}{2}}|^2, \quad ||J_x||_{\#}^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x \bar{\delta}_y J_{x,i+\frac{1}{2},j}|^2,$$

$$||J_y||_{\#}^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x \bar{\delta}_y J_{y,i,j+\frac{1}{2}}|^2, \quad ||K||_{\#}^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x \bar{\delta}_y K_{i+\frac{1}{2},j+\frac{1}{2}}|^2.$$

First, we like to show that the energy norm $||u||_{\#}^2$ is equivalent to the energy norm $||u||_*^2$ for any index function $u_{i,j}$.

Lemma 3 *Denote $C_* = \frac{32}{81}$ and $C^* = \frac{32}{15}$. Then for any periodic index function $u_{i,j}, 0 \le i \le N_x - 1, 0 \le j \le N_y - 1$, we have*

$$C_* \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_y u_{i,j}|^2 \le \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |u_{i,j}|^2 \le C^* \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_y u_{i,j}|^2, \quad (64)$$

$$C_* \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x u_{i,j}|^2 \le \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |u_{i,j}|^2 \le C^* \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x u_{i,j}|^2. \quad (65)$$

Furthermore, from (64) and (65)*, we have*

$$\begin{aligned} ||u||_*^2 &= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |u_{i,j}|^2 \le C^* \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_x u_{i,j}|^2 \\ &\le (C^*)^2 \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_y \bar{\delta}_x u_{i,j}|^2 = (C^*)^2 ||u||_{\#}^2, \\ ||u||_{\#}^2 &= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_y \bar{\delta}_x u_{i,j}|^2 \le \frac{1}{C_*} \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |\bar{\delta}_y u_{i,j}|^2 \\ &\le \frac{1}{C_*^2} \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y |u_{i,j}|^2 = \frac{1}{C_*^2} ||u||_*^2. \end{aligned}$$

Proof By the Cauchy-Schwarz inequality, we easily have

$$\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |\bar{\delta}_y u_{i,j}|^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |\frac{22}{24}u_{i,j} + \frac{1}{24}u_{i,j+1} + \frac{1}{24}u_{i,j-1}|^2$$

$$\le 3 \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \left[\left(\frac{22}{24}\right)^2 |u_{i,j}|^2 + \left(\frac{1}{24}\right)^2 |u_{i,j+1}|^2 + \left(\frac{1}{24}\right)^2 |u_{i,j-1}|^2\right]$$

$$= 3\cdot\frac{22^2+2}{24^2} \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |u_{i,j}|^2 = \frac{81}{32} \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |u_{i,j}|^2, \tag{66}$$

where we used periodic boundary conditions in the last step. This concludes the proof of the first part of (64).

Similarly, by the Cauchy-Schwarz inequality, we easily have

$$\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |u_{i,j}|^2 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |\bar{\delta}_y u_{i,j} + \left(\frac{2}{24}u_{i,j} - \frac{1}{24}u_{i,j+1} - \frac{1}{24}u_{i,j-1}\right)|^2$$

$$\le 2 \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |\bar{\delta}_y u_{i,j}|^2 + 2 \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |\frac{2}{24}u_{i,j} - \frac{1}{24}u_{i,j+1} - \frac{1}{24}u_{i,j-1}|^2. \tag{67}$$

By the Cauchy-Schwarz inequality again, we have

$$2 \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |\frac{2}{24}u_{i,j} - \frac{1}{24}u_{i,j+1} - \frac{1}{24}u_{i,j-1}|^2$$

$$\le 6 \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \left[\left(\frac{2}{24}\right)^2 |u_{i,j}|^2 + \left(\frac{1}{24}\right)^2 |u_{i,j+1}|^2 + \left(\frac{1}{24}\right)^2 |u_{i,j-1}|^2\right]$$

$$= \frac{36}{(24)^2} \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |u_{i,j}|^2 = \frac{1}{16} \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} |u_{i,j}|^2, \tag{68}$$

where we used periodic boundary conditions in the last step.

Substituting the estimate (68) into (67), we conclude the proof of the second part of (64).

By symmetry, (65) holds true. □

First, we give a bound for a partial sum appearing in the stability proof of the scheme (58)–(63).

Lemma 4 *Denote by*

$$R_1 := \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \sum_{0\le n\le N_t-2} h_x h_y \Bigg[\left(\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)\left(\bar\delta_x\bar\delta_y E^{n+1}_{x,i+\frac{1}{2},j} + \bar\delta_x\bar\delta_y E^{n}_{x,i+\frac{1}{2},j}\right)$$
$$+\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1} - \bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j}\right)\left(\bar\delta_x\bar\delta_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \bar\delta_x\bar\delta_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\Bigg]. \quad (69)$$

Then we have

$$R_1 \le \sqrt{C^*}C_v\left[\mu_0(||H^{N_t-\frac{1}{2}}||^2_\# + ||H^{\frac{1}{2}}||^2_\#) + \epsilon_0(||E_x^{N_t-1}||^2_\# + ||E_x^0||^2_\#)\right].$$

Proof Expanding the products of R_1 and regrouping them, we obtain

$$R_1 = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \sum_{0\le n\le N_t-2} h_x h_y \Bigg[\left(\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)\Bigg(\frac{22}{24}\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j} + \frac{1}{24}\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1}$$
$$+\frac{1}{24}\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j-1} + \frac{22}{24}\bar\delta_x E^{n}_{x,i+\frac{1}{2},j} + \frac{1}{24}\bar\delta_x E^{n}_{x,i+\frac{1}{2},j+1} + \frac{1}{24}\bar\delta_x E^{n}_{x,i+\frac{1}{2},j-1}\Bigg)$$
$$+\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1} - \bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j}\right)\Bigg(\frac{22}{24}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{1}{24}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{3}{2}} + \frac{1}{24}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}$$
$$+\frac{22}{24}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{1}{24}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}} + \frac{1}{24}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\Bigg)\Bigg]$$
$$= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \sum_{0\le n\le N_t-2} h_x h_y \Bigg[\frac{22}{24}\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x E^{n}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)$$
$$+\frac{22}{24}\left(\bar\delta_x E^{n}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x E^{n}_{x,i+\frac{1}{2},j}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$
$$+\frac{22}{24}\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j}\bar\delta_x H_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$
$$+\frac{22}{24}\left(\bar\delta_x E^{n}_{x,i+\frac{1}{2},j}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)$$
$$+\frac{1}{24}\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{3}{2}} - \bar\delta_x E^{n}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}}\right)$$
$$+\frac{1}{24}\left(\bar\delta_x E^{n}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}} - \bar\delta_x E^{n}_{x,i+\frac{1}{2},j-1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$
$$+\frac{1}{24}\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}} - \bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j-1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$
$$+\frac{1}{24}\left(\bar\delta_x E^{n}_{x,i+\frac{1}{2},j-1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j-1}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)$$
$$+\frac{1}{24}\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j-1}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{3}{2}}\right)$$
$$+\frac{1}{24}\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j-1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}}\right)$$
$$+\frac{1}{24}\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j-\frac{1}{2}} - \bar\delta_x E^{n}_{x,i+\frac{1}{2},j+1}\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$

$$+\frac{1}{24}\left(\bar{\delta}_x E^{n}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)$$
$$+\frac{1}{24}\left(\bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j}\bar{\delta}_x H^{n+\frac{3}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$
$$+\frac{1}{24}\left(\bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j}\bar{\delta}_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right) = \sum_{k=1}^{14} S_k. \quad (70)$$

Note that terms in S_2, S_3, S_9, S_{10}, S_{13} and S_{14} are different by one in j, hence all these sums are zero by using periodic boundary conditions. By the same arguments, the sums S_6 and S_7 are also zero since those terms differ by two in j.

The rest of the terms in (70) differ by one in n, and can be summed up as follows:

$$S_1 = \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \cdot \frac{22}{24}\left(\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x E^{0}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right),$$

$$S_4 = \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \cdot \frac{22}{24}\left(\bar{\delta}_x E^{0}_{x,i+\frac{1}{2},j}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right),$$

$$S_5 = \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \cdot \frac{1}{24}\left(\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}} - \bar{\delta}_x E^{0}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}}\right),$$

$$S_8 = \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \cdot \frac{1}{24}\left(\bar{\delta}_x E^{0}_{x,i+\frac{1}{2},j-1}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j-1}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right),$$

$$S_{11} = \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \cdot \frac{1}{24}\left(\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}} - \bar{\delta}_x E^{0}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right),$$

$$S_{12} = \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \cdot \frac{1}{24}\left(\bar{\delta}_x E^{0}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right).$$

Substituting the above estimates into (70), we have

$$R_1 = \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \Bigg[\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1}\left(\frac{22}{24}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{1}{24}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}} + \frac{1}{24}\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$
$$-\bar{\delta}_x E^{0}_{x,i+\frac{1}{2},j+1}\left(\frac{22}{24}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{1}{24}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{3}{2}} + \frac{1}{24}\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$

$$
\begin{aligned}
&+\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \left(\frac{22}{24}\bar{\delta}_x E^0_{x,i+\frac{1}{2},j} + \frac{1}{24}\bar{\delta}_x E^0_{x,i+\frac{1}{2},j+1} + \frac{1}{24}\bar{\delta}_x E^0_{x,i+\frac{1}{2},j-1}\right) \\
&-\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \left(\frac{22}{24}(\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j} + \frac{1}{24}\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1} + \frac{1}{24}\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j-1}\right)\Bigg] \\
&= \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \Bigg(\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1} \bar{\delta}_y\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x E^0_{x,i+\frac{1}{2},j+1}\bar{\delta}_y\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \\
&+\bar{\delta}_x H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\bar{\delta}_y\bar{\delta}_x E^0_{x,i+\frac{1}{2},j} - \bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\bar{\delta}_y\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j}\Bigg) = \sum_{k=1}^{4} R_{1,k}.
\end{aligned}
\tag{71}
$$

Using the Cauchy-Schwarz inequality, the definition of the energy norms and Lemma 3, we have

$$
\begin{aligned}
R_{1,1} &= \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1} \cdot \bar{\delta}_y\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \\
&= C_v\sqrt{C^*} \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} h_x h_y \frac{\sqrt{\epsilon_0}}{\sqrt{C^*}}\bar{\delta}_x E^{N_t-1}_{x,i+\frac{1}{2},j+1} \cdot \sqrt{\mu_0}\bar{\delta}_y\bar{\delta}_x H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \\
&\le \frac{C_v\sqrt{C^*}}{2}(\epsilon_0||E_x^{N_t-\frac{1}{2}}||_\#^2 + \mu_0||H^{N_t-1}||_\#^2).
\end{aligned}
$$

Similarly, we can bound the rest three terms of (71). Substituting these estimates into (71) completes the proof. □

With the above preparation, we can obtain the following stability for the compact difference scheme (58)–(63) with periodic boundary conditions.

Theorem 3 *Under the time step constraint*

$$
\tau \le \min\left\{\frac{3h_x}{28C_v\sqrt{C^*}}, \frac{3h_y}{28C_v\sqrt{C^*}}, \frac{1}{2\omega_{pe}}, \frac{1}{2\omega_{pm}}\right\},
\tag{72}
$$

then for any $m \in [0, N_t - 2]$, *we have*

$$
\begin{aligned}
&\epsilon_0(||E_x^{m+1}||_\#^2 + ||E_y^{m+1}||_\#^2) + \mu_0||H^{m+\frac{3}{2}}||_\#^2 \\
&\quad+\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{m+\frac{3}{2}}||_\#^2 + ||J_y^{m+\frac{3}{2}}||_\#^2) + \frac{1}{\mu_0\omega_{pm}^2}||K^{m+2}||_\#^2 \\
&\le 3\left[\epsilon_0(||E_x^0||_\#^2 + ||E_y^0||_\#^2) + \mu_0||H^{\frac{1}{2}}||_\#^2 + \frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{\frac{1}{2}}||_\#^2 + ||J_y^{\frac{1}{2}}||_\#^2) + \frac{1}{\mu_0\omega_{pm}^2}||K^1||_\#^2\right].
\end{aligned}
\tag{73}
$$

Proof Multiplying (58)–(63) by $\tau h_x h_y(\bar{\delta}_x\bar{\delta}_y E^{n+1}_{x,i+\frac{1}{2},j} + \bar{\delta}_x\bar{\delta}_y E^n_{x,i+\frac{1}{2},j})$, $\tau h_x h_y(\bar{\delta}_x\bar{\delta}_y E^{n+1}_{y,i,j+\frac{1}{2}} + \bar{\delta}_x\bar{\delta}_y E^n_{y,i,j+\frac{1}{2}})$, $\tau h_x h_y(\bar{\delta}_x\bar{\delta}_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \bar{\delta}_x\bar{\delta}_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}})$,

$\tau h_x h_y(\bar{\delta}_x\bar{\delta}_y J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} + \bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j})$, $\tau h_x h_y(\bar{\delta}_x\bar{\delta}_y J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}} + \bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}})$, $\tau h_x h_y(\bar{\delta}_x\bar{\delta}_y K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}} + \bar{\delta}_x\bar{\delta}_y K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}})$, respectively, then summing up each resultant over i from 0 to $N_x - 1$, and j from 0 to $N_y - 1$, we have

$$\begin{aligned}
&\epsilon_0(||E_x^{n+1}||_\#^2 - ||E_x^n||_\#^2) \\
&= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \left(\frac{\bar{\delta}_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}}{h_y} - \bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}\right) \\
&\times \left(\bar{\delta}_x\bar{\delta}_y E^{n+1}_{x,i+\frac{1}{2},j} + \bar{\delta}_x\bar{\delta}_y E^{n}_{x,i+\frac{1}{2},j}\right),
\end{aligned} \tag{74}$$

$$\begin{aligned}
&\epsilon_0(||E_y^{n+1}||_\#^2 - ||E_y^n||_\#^2) \\
&= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \left(-\frac{\bar{\delta}_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_y H^{n+\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}}}{h_x} - \bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}\right) \\
&\times \left(\bar{\delta}_x\bar{\delta}_y E^{n+1}_{y,i,j+\frac{1}{2}} + \bar{\delta}_x\bar{\delta}_y E^{n}_{y,i,j+\frac{1}{2}}\right),
\end{aligned} \tag{75}$$

$$\begin{aligned}
&\mu_0(||H^{n+\frac{3}{2}}||_\#^2 - ||H^{n+\frac{1}{2}}||_\#^2) = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \left(-\frac{\bar{\delta}_y E^{n+1}_{y,i+1,j+\frac{1}{2}} - \bar{\delta}_y E^{n+1}_{y,i,j+\frac{1}{2}}}{h_x}\right. \\
&\quad \left. + \frac{\bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j+1} - \bar{\delta}_x E^{n+1}_{x,i+\frac{1}{2},j}}{h_y} - \bar{\delta}_x\bar{\delta}_y K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}\right) \\
&\times \left(\bar{\delta}_x\bar{\delta}_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \bar{\delta}_x\bar{\delta}_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right),
\end{aligned} \tag{76}$$

$$\begin{aligned}
&\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{n+\frac{3}{2}}||_\#^2 - ||J_x^{n+\frac{1}{2}}||_\#^2) + \frac{\tau\Gamma_e}{2\epsilon_0\omega_{pe}^2}||J_x^{n+\frac{3}{2}} + J_x^{n+\frac{1}{2}}||_\#^2 \\
&\quad = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \bar{\delta}_x\bar{\delta}_y E^{n+1}_{x,i+\frac{1}{2},j}\left(\bar{\delta}_x\bar{\delta}_y J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} + \bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}\right),
\end{aligned} \tag{77}$$

$$\begin{aligned}
&\frac{1}{\epsilon_0\omega_{pe}^2}(||J_y^{n+\frac{3}{2}}||_\#^2 - ||J_y^{n+\frac{1}{2}}||_\#^2) + \frac{\tau\Gamma_e}{2\epsilon_0\omega_{pe}^2}||J_y^{n+\frac{3}{2}} + J_y^{n+\frac{1}{2}}||_\#^2 \\
&\quad = \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \bar{\delta}_x\bar{\delta}_y E^{n+1}_{y,i,j+\frac{1}{2}}\left(\bar{\delta}_x\bar{\delta}_y J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}} + \bar{\delta}_x\bar{\delta}_y J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}\right),
\end{aligned} \tag{78}$$

$$\frac{1}{\mu_0\omega_{pm}^2}(||K^{n+2}||_\#^2 - ||K^{n+1}||_\#^2) + \frac{\tau\Gamma_m}{2\mu_0\omega_{pm}^2}||K^{n+2} + K^{n+1}||_\#^2$$
$$= \sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} \tau h_x h_y \bar\delta_x\bar\delta_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \left(\bar\delta_x\bar\delta_y K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}} + \bar\delta_x\bar\delta_y K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}\right). \quad (79)$$

Adding (74)–(79), then summing up n from 0 to $N_t - 2$, we easily see that the sum of the left hand side (LHS) satisfies the following:

$$LHS \ge \epsilon_0(||E_x^{N_t-1}||_\#^2 - ||E_x^0||_\#^2) + \epsilon_0(||E_y^{N_t-1}||_\#^2 - ||E_y^0||_\#^2)$$
$$+\mu_0(||H^{N_t-\frac{1}{2}}||_\#^2 - ||H^{\frac{1}{2}}||_\#^2) + \frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{N_t-\frac{1}{2}}||_\#^2 - ||J_x^{\frac{1}{2}}||_\#^2)$$
$$+\frac{1}{\epsilon_0\omega_{pe}^2}(||J_y^{N_t-\frac{1}{2}}||_\#^2 - ||J_y^{\frac{1}{2}}||_\#^2) + \frac{1}{\mu_0\omega_{pm}^2}(||K^{N_t}||_\#^2 - ||K^1||_\#^2), \quad (80)$$

and the sum of the right hand side (RHS) is:

$$RHS = -\frac{\tau}{h_x}\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[\left(\bar\delta_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_y H^{n+\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2}}\right)\right.$$
$$\times\left(\bar\delta_x\bar\delta_y E^{n+1}_{y,i,j+\frac{1}{2}} + \bar\delta_x\bar\delta_y E^{n}_{y,i,j+\frac{1}{2}}\right)$$
$$\left.+\left(\bar\delta_y E^{n+1}_{y,i+1,j+\frac{1}{2}} - \bar\delta_y E^{n}_{y,i+1,j+\frac{1}{2}}\right)\left(\bar\delta_x\bar\delta_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \bar\delta_x\bar\delta_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\right]$$
$$+\frac{\tau}{h_y}\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[\left(\bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x H^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}}\right)\right.$$
$$\times\left(\bar\delta_x\bar\delta_y E^{n+1}_{x,i+\frac{1}{2},j} + \bar\delta_x\bar\delta_y E^{n}_{x,i+\frac{1}{2},j}\right)$$
$$\left.+\left(\bar\delta_x E^{n+1}_{x,i+\frac{1}{2},j+1} - \bar\delta_x E^{n}_{x,i+\frac{1}{2},j}\right)\left(\bar\delta_x\bar\delta_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + \bar\delta_x\bar\delta_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\right]$$
$$+\tau\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[\bar\delta_x\bar\delta_y E^{n+1}_{x,i+\frac{1}{2},j}\cdot\bar\delta_x\bar\delta_y J^{n+\frac{3}{2}}_{x,i+\frac{1}{2},j} - \bar\delta_x\bar\delta_y E^{n}_{x,i+\frac{1}{2},j}\cdot\bar\delta_x\bar\delta_y J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}\right]$$
$$+\tau\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[\bar\delta_x\bar\delta_y E^{n+1}_{y,i,j+\frac{1}{2}}\cdot\bar\delta_x\bar\delta_y J^{n+\frac{3}{2}}_{y,i,j+\frac{1}{2}} - \bar\delta_x\bar\delta_y E^{n}_{y,i,j+\frac{1}{2}}\cdot\bar\delta_x\bar\delta_y J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}\right]$$
$$+\tau\sum_{0\le n\le N_t-2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}} h_x h_y\left[\bar\delta_x\bar\delta_y H^{n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\cdot\bar\delta_x\bar\delta_y K^{n+2}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar\delta_x\bar\delta_y H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\cdot\bar\delta_x\bar\delta_y K^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}\right]$$
$$= \sum_{k=1}^{5} RHS_k. \quad (81)$$

By the definition of R_1 and Lemmas 4, we have

$$RHS_2 = \frac{\tau}{h_y} R_1 \\ \leq \frac{\tau\sqrt{C^*}C_v}{h_y}\left[\mu_0(||H^{N_t-\frac{1}{2}}||_\#^2 + ||H^{\frac{1}{2}}||_\#^2) + \epsilon_0(||E_x^{N_t-1}||_\#^2 + ||E_x^0||_\#^2)\right]. \quad (82)$$

By symmetry, we can show that

$$RHS_1 \leq \frac{\tau\sqrt{C^*}C_v}{h_x}\left[\mu_0(||H^{N_t-\frac{1}{2}}||_\#^2 + ||H^{\frac{1}{2}}||_\#^2) + \epsilon_0(||E_y^{N_t-1}||_\#^2 + ||E_y^0||_\#^2)\right]. \quad (83)$$

By the Cauchy-Schwarz inequality, we easily have

$$RHS_3 = \tau \sum_{\substack{0\leq i\leq N_x-1\\ 0\leq j\leq N_y-1}} h_x h_y \left[\bar{\delta}_x\bar{\delta}_y E^{N_t-1}_{y,i,j+\frac{1}{2}} \bar{\delta}_x\bar{\delta}_y J^{N_t-\frac{1}{2}}_{y,i,j+\frac{1}{2}} - \bar{\delta}_x\bar{\delta}_y E^{0}_{y,i,j+\frac{1}{2}} \bar{\delta}_x\bar{\delta}_y J^{\frac{1}{2}}_{y,i,j+\frac{1}{2}}\right] \\ \leq \frac{\tau\omega_{pe}}{2}\left[\epsilon_0(||E_x^{N_t-1}||_\#^2 + ||E_x^0||_\#^2) + \frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{N_t-\frac{1}{2}}||_\#^2 + ||J_x^{\frac{1}{2}}||_\#^2)\right]. \quad (84)$$

Similarly, we can show that

$$RHS_4 = \tau \sum_{\substack{0\leq i\leq N_x-1\\ 0\leq j\leq N_y-1}} h_x h_y \left[\bar{\delta}_x\bar{\delta}_y E^{N_t-1}_{y,i,j+\frac{1}{2}} \bar{\delta}_x\bar{\delta}_y J^{N_t-\frac{1}{2}}_{y,i,j+\frac{1}{2}} - \bar{\delta}_x\bar{\delta}_y E^{0}_{y,i,j+\frac{1}{2}} \bar{\delta}_x\bar{\delta}_y J^{\frac{1}{2}}_{y,i,j+\frac{1}{2}}\right] \\ \leq \frac{\tau\omega_{pe}}{2}\left[\epsilon_0(||E_y^{N_t-1}||_\#^2 + ||E_y^0||_\#^2) + \frac{1}{\epsilon_0\omega_{pe}^2}(||J_y^{N_t-\frac{1}{2}}||_\#^2 + ||J_y^{\frac{1}{2}}||_\#^2)\right], \quad (85)$$

and

$$RHS_5 = \tau \sum_{\substack{0\leq i\leq N_x-1\\ 0\leq j\leq N_y-1}} h_x h_y \left[\bar{\delta}_x\bar{\delta}_y H^{N_t-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \bar{\delta}_x\bar{\delta}_y K^{N_t}_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{\delta}_x\bar{\delta}_y H^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \bar{\delta}_x\bar{\delta}_y K^{1}_{i+\frac{1}{2},j+\frac{1}{2}}\right] \\ \leq \frac{\tau\omega_{pm}}{2}\left[\mu_0(||H^{N_t-\frac{1}{2}}||_\#^2 + ||H^{\frac{1}{2}}||_\#^2) + \frac{1}{\mu_0\omega_{pm}^2}(||K^{N_t}||_\#^2 + ||K^1||_\#^2)\right]. \quad (86)$$

Substituting the estimates (82)–(86) into (81), we obtain

$$
\begin{aligned}
RHS \leq & \left(\frac{7\tau C_v\sqrt{C^*}}{6h_x}+\frac{7\tau C_v\sqrt{C^*}}{6h_y}+\frac{\tau\omega_{pm}}{2}\right)\cdot\mu_0(||H^{N_t-\frac{1}{2}}||_\#^2+||H^{\frac{1}{2}}||_\#^2)\\
&+\left(\frac{7\tau C_v\sqrt{C^*}}{6h_x}+\frac{\tau\omega_{pe}}{2}\right)\cdot\epsilon_0(||E_y^{N_t-1}||_\#^2+||E_y^0||_\#^2)\\
&+\left(\frac{7\tau C_v\sqrt{C^*}}{6h_y}+\frac{\tau\omega_{pe}}{2}\right)\cdot\epsilon_0(||E_x^{N_t-1}||_\#^2+||E_x^0||_\#^2)\\
&+\frac{\tau\omega_{pe}}{2}\cdot\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{N_t-\frac{1}{2}}||_\#^2+||J_x^{\frac{1}{2}}||_\#^2)+\frac{\tau\omega_{pe}}{2}\cdot\frac{1}{\epsilon_0\omega_{pe}^2}(||J_y^{N_t-\frac{1}{2}}||_\#^2+||J_y^{\frac{1}{2}}||_\#^2)\\
&+\frac{\tau\omega_{pm}}{2}\cdot\frac{1}{\mu_0\omega_{pm}^2}(||K^{N_t}||_\#^2+||K^1||_\#^2). \qquad (87)
\end{aligned}
$$

Using the constraint (72), we conclude the proof by combining the estimates (80) and (87). □

Remark 1 Using Lemma 3 and Theorem 3, we can obtain the other form of stability for the scheme (52) and (57): For any $m \in [0, N_t - 2]$, we have

$$
\begin{aligned}
&\epsilon_0(||E_x^{m+1}||_*^2+||E_y^{m+1}||_*^2)+\mu_0||H^{m+\frac{3}{2}}||_*^2\\
&\quad+\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{m+\frac{3}{2}}||_*^2+||J_y^{m+\frac{3}{2}}||_*^2)+\frac{1}{\mu_0\omega_{pm}^2}||K^{m+2}||_*^2 \qquad (88)\\
&\leq 3\left(\frac{C^*}{C_*}\right)^2\left[\epsilon_0(||E_x^0||_*^2+||E_y^0||_*^2)+\mu_0||H^{\frac{1}{2}}||_*^2+\frac{1}{\epsilon_0\omega_{pe}^2}(||J_x^{\frac{1}{2}}||_*^2+||J_y^{\frac{1}{2}}||_*^2)+\frac{1}{\mu_0\omega_{pm}^2}||K^1||_*^2\right].
\end{aligned}
$$

Using similar techniques to those developed for the error analysis in Theorem 2 and the stability analysis developed for the compact scheme, we can prove the following optimal error estimate for the compact scheme: For zero initial errors

$$
\mathcal{E}^0_{x,i+\frac{1}{2},j}=\mathcal{E}^0_{y,i,j+\frac{1}{2}}=\mathcal{H}^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}=\mathcal{J}^0_{x,i+\frac{1}{2},j}=\mathcal{J}^0_{y,i,j+\frac{1}{2}}=\mathcal{K}^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}=0,
$$

we have

$$
\begin{aligned}
\max_{0\leq n\leq N_t-1}&\left[\epsilon_0(||\mathcal{E}_x^n||_*^2+||\mathcal{E}_y^n||_*^2)+\mu_0||\mathcal{H}^{n+\frac{1}{2}}||_*^2\right.\\
&\left.+\frac{1}{\epsilon_0\omega_{pe}^2}(||\mathcal{J}_x^{n+\frac{1}{2}}||_*^2+||\mathcal{J}_y^{n+\frac{1}{2}}||_*^2)+\frac{1}{\mu_0\omega_{pm}^2}||\mathcal{K}^{n+1}||_*^2\right]^{1/2}\leq CT(\tau^2+h_x^4+h_y^4).
\end{aligned}
$$

4 Numerical results

To justify our theoretical analysis, we have implemented both fourth order schemes to solve the model equations (1)–(6) with added source terms g_x, g_y, f:

$$\begin{cases} \epsilon_0 \dfrac{\partial E_x}{\partial t} = \dfrac{\partial H}{\partial y} - J_x + g_x, \\ \epsilon_0 \dfrac{\partial E_y}{\partial t} = -\dfrac{\partial H}{\partial x} - J_y + g_y, \\ \mu_0 \dfrac{\partial H}{\partial t} = -\dfrac{\partial E_y}{\partial x} + \dfrac{\partial E_x}{\partial y} - K + f, \\ \dfrac{\partial J_x}{\partial t} = -\Gamma_e J_x + \epsilon_0 \omega_{pe}^2 E_x, \\ \dfrac{\partial J_y}{\partial t} = -\Gamma_e J_y + \epsilon_0 \omega_{pe}^2 E_y, \\ \dfrac{\partial K}{\partial t} = -\Gamma_m K + \mu_0 \omega_{pm}^2 H. \end{cases} \tag{89}$$

In our test, we choose the physical domain $\Omega = [0, 2]^2$, and coefficients as follows:

$$\epsilon_0 = \mu_0 = 1, \quad \Gamma_m = \Gamma_e = \pi, \quad \omega_{pm} = \omega_{pe} = \pi,$$

so that (89) has the exact solution:

$$\begin{aligned} &\boldsymbol{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \cos(4\pi x)\sin(4\pi y)e^{-\pi t} \\ -\sin(4\pi x)\cos(4\pi y)e^{-\pi t} \end{pmatrix}, \\ &H = \cos(4\pi x)\cos(4\pi y)e^{-\pi t}, \\ &\boldsymbol{J} = \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \pi^2 t\cos(4\pi x)\sin(4\pi y)e^{-\pi t} \\ -\pi^2 t\sin(4\pi x)\cos(4\pi y)e^{-\pi t} \end{pmatrix}, \\ &K = \pi^2 t\cos(4\pi x)\cos(4\pi y)e^{-\pi t}, \end{aligned} \tag{90}$$

which leads to the source terms as follows:

$$\begin{aligned} g_x &= (\pi^2 t + 3\pi)\cos(4\pi x)\sin(4\pi y)e^{-\pi t}, \\ g_y &= -(\pi^2 t + 3\pi)\sin(4\pi x)\cos(4\pi y)e^{-\pi t}, \\ f &= (-9\pi + \pi^2 t)\cos(4\pi x)\cos(4\pi y)e^{-\pi t}. \end{aligned} \tag{91}$$

First, to demonstrate the convergence rate under the periodic boundary conditions, we use $h_x = h_y = h$ with varying h from $1/8$ to $1/256$ and a fixed time step $\tau = h^2$ to solve the model problem till $T = 2$ by both the fourth order explicit and compact difference schemes. The obtained errors for fields E_x, E_y, H_z at $T = 2$ in discrete energy norms are presented in Tables 1 and 2 for the explicit and compact difference schemes, respectively. Tables 1 and 2 clearly justify the optimal convergence rate $O(\tau^2 + h^4)$ as we proved.

Second, we recalculate our model problem to check the convergence rate under the practical PEC boundary condition with the same meshes as used above for the periodic boundary condition case. The obtained errors for fields E_x, E_y, H_z at $T = 2$ in discrete energy norms are presented in Tables 3 and 4 for the explicit and compact

Table 1 The errors obtained by the fourth order explicit scheme with periodic boundary conditions and $\tau = h^2$

h	$\|E_x - E_{x,h}\|_*$	$rate$	$\|E_y - E_{y,h}\|_*$	$rate$	$\|H - H_h\|_*$	$rate$
1/8	9.0071×10^{-3}	–	9.0071×10^{-3}	–	2.6508×10^{-2}	–
1/16	1.4801×10^{-3}	2.6053	1.4801×10^{-3}	2.6053	8.2667×10^{-4}	5.0030
1/32	9.8937×10^{-5}	3.9031	9.8937×10^{-5}	3.9031	4.8686×10^{-5}	4.0857
1/64	6.2658×10^{-6}	3.9809	6.2658×10^{-6}	3.9809	3.0727×10^{-6}	3.9859
1/128	3.9283×10^{-7}	3.9955	3.9283×10^{-7}	3.9955	1.9284×10^{-7}	3.9940
1/256	2.4571×10^{-8}	3.9989	2.4571×10^{-8}	3.9989	1.2066×10^{-8}	3.9984

Table 2 The errors obtained by the fourth order compact scheme with periodic boundary conditions and $\tau = h^2$

h	$\|E_x - E_{x,h}\|_*$	$rate$	$\|E_y - E_{y,h}\|_*$	$rate$	$\|H - H_h\|_*$	$rate$
1/8	9.7266×10^{-3}	–	9.7266×10^{-3}	–	1.5724×10^{-2}	–
1/16	9.7586×10^{-4}	3.3172	9.7586×10^{-4}	3.3172	5.1118×10^{-4}	4.9430
1/32	6.2924×10^{-5}	3.9550	6.2924×10^{-5}	3.9550	3.0795×10^{-5}	4.0531
1/64	3.9546×10^{-6}	3.9920	3.9546×10^{-6}	3.9920	1.9369×10^{-6}	3.9909
1/128	2.4747×10^{-7}	3.9982	2.4747×10^{-7}	3.9982	1.2136×10^{-7}	3.9963
1/256	1.5472×10^{-8}	3.9996	1.5472×10^{-8}	3.9996	7.5904×10^{-9}	3.9990

Table 3 The errors obtained by the fourth order explicit scheme with PEC boundary conditions and $\tau = h^2$

h	$\|E_x - E_{x,h}\|_*$	$rate$	$\|E_y - E_{y,h}\|_*$	$rate$	$\|H - H_h\|_*$	$rate$
1/8	4.3416×10^{-2}	–	4.3416×10^{-2}	–	7.0865×10^{-2}	–
1/16	2.5276×10^{-3}	4.1024	2.5276×10^{-3}	4.1024	3.5087×10^{-3}	4.3361
1/32	1.0501×10^{-4}	4.5892	1.0501×10^{-4}	4.5892	1.3862×10^{-4}	4.6617
1/64	5.8430×10^{-6}	4.1676	5.8430×10^{-6}	4.1676	4.6463×10^{-6}	4.8989
1/128	3.7091×10^{-7}	3.9776	3.7091×10^{-7}	3.9776	1.9511×10^{-7}	4.5737
1/256	2.3744×10^{-8}	3.9655	3.7091×10^{-8}	3.9655	1.1137×10^{-8}	4.1308

Table 4 The errors obtained by the fourth order compact scheme with PEC boundary conditions and $\tau = h^2$

h	$\|E_x - E_{x,h}\|_*$	$rate$	$\|E_y - E_{y,h}\|_*$	$rate$	$\|H - H_h\|_*$	$rate$
1/8	2.4513×10^{-2}	–	2.4513×10^{-2}	–	6.6895×10^{-2}	–
1/16	9.3067×10^{-4}	4.7191	9.3067×10^{-4}	4.7191	1.2040×10^{-3}	5.7959
1/32	6.6841×10^{-5}	3.7995	6.6841×10^{-5}	3.7995	9.8290×10^{-5}	3.6147
1/64	3.7749×10^{-6}	4.1462	3.7749×10^{-6}	4.1462	3.7535×10^{-6}	4.7107
1/128	2.3163×10^{-7}	4.0265	2.3163×10^{-7}	4.0265	1.4206×10^{-7}	4.7237
1/256	1.4792×10^{-8}	3.9690	1.4792×10^{-8}	3.9690	7.1359×10^{-9}	4.3153

difference schemes, respectively. Tables 3 and 4 clearly show the optimal convergence rate $O(\tau^2 + h^4)$, though how to prove this rigorously is still open. We will pursue this further in our future work. Also in the future we plan to extend these fourth order schemes to some perfectly matched layer models [9, Ch.8] and carry out simulations for practical wave propagation in metamaterials.

5 Conclusion

For the first time, we established the theoretical analysis showing that the fourth order methods can achieve optimal convergence when applied to metamaterial Maxwell's equations. Note that the Drude metamaterial model is more complicated than the simple Maxwell system, which brings more challenges in analyzing the fourth-order scheme for the Drude model. Exactly the same results and analysis can be extended to the standard Maxwell system in free space. We believe that similar ideas and results can be extended to higher order FDTD methods and 3D Maxwell's equations at the expense of more complex notation. Interesting simulations of wave propagation in metamaterials will be explored by using high order difference methods in the future.

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