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Left Lie reduction for curves in homogeneous spaces

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Abstract Let H be a closed subgroup of a connected finite-dimensional Lie group G, where the canonical projection $\pi : G \to G/H$ is a Riemannian submersion with respect to a bi-invariant Riemannian metric on G. Given a C^∞ curve $x : [a, b] \to G/H$, let $\tilde{x} : [a, b] \to G$ be the horizontal lifting of x with $\tilde{x}(a) = e$, where e denotes the identity of G. When (G, H) is a Riemannian symmetric pair, we prove that the *left Lie reduction* $V(t) := \tilde{x}(t)^{-1}\dot{\tilde{x}}(t)$ of $\dot{\tilde{x}}(t)$ for $t \in [a, b]$ can be identified with the *parallel pullback* $P(t)$ of the velocity vector $\dot{x}(t)$ from $x(t)$ to $x(a)$ along x. Then left Lie reductions are used to investigate Riemannian cubics, Riemannian cubics in tension and elastica in homogeneous spaces G/H. Simplifications of reduced equations are found when (G, H) is a Riemannian symmetric pair. These equations are compared with equations known for curves in Lie groups, focusing on the special case of Riemannian cubics in the 3-dimensional unit sphere S^3.

Keywords Lie reduction · Homogeneous space · Symmetric space · Cubics · Cubics in tension · Elastica

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1 Introduction

Variational curves in Riemannian manifolds M, including Riemannian cubics [2, 11], Riemannian cubics in tension [21], and elastica [28],[1] have applications in engineering, computer graphics, and quantum computing. Riemannian cubics in $SO(3)$ are used in trajectory planning for rigid body motion [1–3]; Riemannian cubics in tension are applied for interpolating figures [4, 6]; elastic curves serve as interpolating curves in computer vision [7]; subRiemannian geodesics and subRiemannian cubics are used to assist in the design of quantum circuits [8]. Most research on such curves has been carried out for the special case where M is a Lie group G, using the notion of *Lie reduction* for curves in groups. The present paper extends Lie reduction to the case where M is a Riemannian homogeneous space, with special attention to when M is also a symmetric space.

Riemannian cubics in symmetric spaces are already studied in [3] using the so-called *parallel pullback*. From a computational point of view there are some advantages in using left Lie reduction. The two approaches are compared, and related by Theorem 1.2 below.

Let M be a finite-dimensional connected Riemannian manifold with a Riemannian metric $\langle\cdot,\cdot\rangle$. The associated Levi-Civita connection is denoted by ∇, and its curvature tensor field R is defined according to the convention

$$R(X,Y)Z = \left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right)Z$$

where X, Y, Z are vector fields on M, $[\cdot,\cdot]$ is the Lie bracket of vector fields.

1.1 Riemannian homogeneous spaces

Let $\mathfrak{g}$ denote the Lie algebra of a connected finite-dimensional Lie group G. Given a bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on G, let H be a closed Lie subgroup with Lie algebra $\mathfrak{h}$, and let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. For $g \in G$ the *vertical subspace* of TG_g is the kernel of $d\pi_g : TG_g \to TM_{\pi(g)}$. The *horizontal subspace* of TG_g is the orthogonal complement of the vertical subspace with respect to the Riemannian metric on G. Orthogonal projections of tangent spaces of G to their horizontal and vertical subspaces are denoted by $\mathscr{H}$ and $\mathscr{V}$. Using left-invariance of $\langle\cdot,\cdot\rangle$, a Riemannian metric on M is defined by requiring $d\pi_g$ to be a linear isometry from the horizontal subspace of TG_g onto $TM_{\pi(g)}$. In particular $d\pi_e$ restricts to a linear isometry from $\mathfrak{m}$ onto the tangent space $TM_{\pi(e)}$, and G acts by isometries on the left of $G/H = M$. Then $\pi : G \to M := G/H$ is a *Riemannian submersion* in the sense of [16, 17], and M is called a *Riemannian homogeneous space*.

[1] A variational curve is the curve governed by a variational principle. For instance, Riemannian cubics are critical points of the functional of total squared norm of the angular acceleration, which have prescribed initial and final positions and velocities. In addition, if the functional is added by proportional total energy, such critical points are called Riemannian cubics in tension. Elastica are solutions of the problem: find a unit-speed curve with fixed length interpolating two prescribed points and velocities, which minimises the total squared geodesic curvature. Specific definitions and equations for these variational curves are given at the beginning of Section 2,3,4, respectively.

The vector space of smooth vector fields on manifold N is denoted by $\mathscr{X}(N)$. Let $\tilde{X}, \tilde{Y} \in \mathscr{X}(G)$ be the horizontal liftings of $X, Y \in \mathscr{X}(M)$, $\widetilde{\nabla_X Y}$ is the horizontal lifting of $\nabla_X Y$. The relationship between the Levi-Civita covariant derivative $\tilde{\nabla}$ on G and its counterpart ∇ on M is given by the following lemma (Page 186 in [10] or Theorem 1 in [17]).

Lemma 1.1 $\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}\mathscr{V}([\tilde{X}, \tilde{Y}])$.

Proof Let $\tilde{V} \in \mathscr{X}(M)$ be a vertical field and $Z \in \mathscr{X}(G/H)$ a horizontal field, by observing,

$$\langle \tilde{X}, \tilde{V}\rangle = \langle \tilde{Y}, \tilde{V}\rangle = 0,\ d\pi([\tilde{X}, \tilde{V}]) = [d\pi(\tilde{Y}), d\pi(\tilde{V})]) = 0,\ \tilde{V}\langle \tilde{X}, \tilde{Y}\rangle = 0,$$
$$\tilde{X}\langle \tilde{Y}, \tilde{Z}\rangle = X\langle Y, Z\rangle \circ \pi,\ d\pi([\tilde{X}, \tilde{Y}]) = [X, Y] \circ \pi,\ \langle [\tilde{X}, \tilde{Y}], \tilde{Z}\rangle = \langle [X, Y], Z\rangle \circ \pi,$$

and using the formula for the Riemannian connection as a function of the Riemannian metric, we conclude that

$$\begin{aligned}
2\langle \tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{V}\rangle &= \tilde{X}\langle \tilde{Y}, \tilde{V}\rangle + \tilde{Y}\langle \tilde{V}, \tilde{X}\rangle - \tilde{V}\langle \tilde{X}, \tilde{Y}\rangle + \langle [\tilde{X}, \tilde{Y}], \tilde{V}\rangle + \langle [\tilde{V}, \tilde{X}], \tilde{Y}\rangle - \langle [\tilde{Y}, \tilde{V}], \tilde{X}\rangle \\
&= \langle [\tilde{X}, \tilde{Y}], \tilde{V}\rangle, \\
2\langle \tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}\rangle &= (X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle) \circ \pi \\
&= \langle \nabla_X Y, Z\rangle \circ \pi,
\end{aligned}$$

which proves the lemma. □

1.2 Riemannian symmetric spaces

The pair (G, H) is called *symmetric* if there exists an involutive analytic automorphism σ of G such that $(H_\sigma)_0 \subseteq H \subseteq H_\sigma$, where H_σ is the set of fixed points of σ and $(H_\sigma)_0$ is the identity component of H_σ. In addition, (G, H) is said to be a *Riemannian symmetric pair* if $Ad_G(H)$ is compact. Then $M := G/H$ is called a *Riemannian symmetric space* and we have

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h},\ [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m},\ [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}. \tag{1.1}$$

In [3], Crouch and Silva Leite use parallel translation to study Riemannian cubics when M is a Riemannian symmetric space. A Riemannian cubic $x : [a, b] \to M$ starting at $x_0 := \pi(e)$ may be characterised by a *parallel pullback* $P = P_{\dot{x}} : [a, b] \to TM_{x_0}$ of $\dot{x}$. In general, the parallel pullback of any vector field X along x is defined to be the the curve $P_X : [a, b] \to TM_{x_0}$ whose value at $t \in [a, b]$ is the parallel translation $P_X(t)$ of $X(t)$ along x from $x(t)$ to x_0. Evidently $P_{\nabla_t^3 \dot{x}} = \dddot{P}$, and combining with Theorem 10.3 in [9], the Euler-Lagrange equation for Riemannian cubics

$$\nabla^3_{d/dt}\dot{x}(t) + R(\nabla_{d/dt}\dot{x}(t), \dot{x}(t))\dot{x}(t) = \mathbf{0} \tag{1.2}$$

is equivalent to equation (46) in [3]:

$$\dddot{P}(t) + [P(t), [\dot{P}(t), P(t)]] = \mathbf{0}. \tag{1.3}$$

In practise it can be extremely difficult to use this equation to construct x from P. This is because $\dot{x}(t)$ and $P(t)$ are related by parallel translation. The operation of parallel translation is almost always complicated, and dependent on the underlying curve x for which we are trying to solve. Usually parallel translation is defined by means of a linear differential equation with variable coefficients that are dependent on x[5]. A separate solution is required to compute $P(t)$ for each value of t, making determination of $\dot{x}(t)$ for every t extremely time-consuming. An alternative characterisation of x is by what we call a *left Lie reduction* V, generalising the previously studied notion of left Lie reduction for curves in Lie groups. The *horizontal lifting* $\tilde{x} : [a, b] \to G$ of x to G is defined to be the unique C^∞ curve satisfying $\pi \circ \tilde{x} = x$ and $\tilde{x}(a) = e$, with $\dot{\tilde{x}}(t)$ everywhere horizontal on G. Then $V : [a, b] \to \mathfrak{m}$ is defined by

$$V(t) := dL(\tilde{x}(t))_e^{-1}\dot{\tilde{x}}(t), \tag{1.4}$$

where $L(g) : G \to G$ denotes left multiplication by $g \in G$. Note that x is readily recoverable from V, as $\pi \circ \tilde{x}$, where $\tilde{x}$ is the solution of the linear ODE

$$\dot{\tilde{x}}(t) = dL(\tilde{x}(t))_e V(t).$$

Both parallel translation, and the derivative of left multiplication by an element of G, are isometries. It therefore follows that $\|P(t)\| = \|V(t)\|$ for all $t \in [a, b]$.

By contrast, the Lie reduction for curves in Lie groups G is defined as follows [2, 14, 18–20, 25, 26, 29]. Let $L_g : G \to G$ be the left translation by $g \in G$. For any C^∞ curve $\tilde{x} : [a, b] \to G$, the *Lie reduction* $U : [a, b] \to \mathfrak{g}$ of the velocity vector field $\dot{\tilde{x}}$ is defined by

$$U(t) := (dL_{\tilde{x}(t)^{-1}})_{\tilde{x}(t)}\dot{\tilde{x}}(t), \tag{1.5}$$

where $(dL_{g_1})_{g_2} : T_{g_2}G \to T_{g_1 g_2}G$ is the derivative of L_{g_1} at $g_2 \in G$.

So our left Lie reduction V for curves x in G/H is actually the standard left Lie reduction U of the horizontal lift $\tilde{x}$, where now U is considered as a curve in $\mathfrak{m}$. If H is taken to be trivial, then $G/H = G$, π is the identity map, and $V = U$.

Theorem 1.2 *Let G/H be a Riemannian symmetric space. Then $d\pi_e \circ V = P$.*

Proof Let $s \mapsto Y_t(s)$ be a parallel vector field along the curve x in M, that is

$$\nabla_s Y_t(s) = \mathbf{0}$$

satisfying $Y_t(t) = \dot{x}(t)$ and $Y_t(a) = P(t)$. Then the curvature formula gives

$$\mathbf{0} = \nabla_s \nabla_t Y_t(s) = \nabla_t \nabla_s Y_t(s) + \nabla_{[\partial s, \partial t]} Y_t(s) + R(\partial s, \partial t) Y_t(s).$$

By Lemma 1.1, the horizontal lifting of $\nabla_t Y_t(s)$ is $\mathscr{H}\tilde{\nabla}_t \tilde{Y}_t(s)$, then

$$d\pi(\tilde{\nabla}_s(\mathscr{H}\tilde{\nabla}_t \tilde{Y}_t(s))) = d\pi(\tilde{\nabla}_s \widetilde{\nabla_t Y_t(s)}) = \nabla_s \nabla_t Y_t(s) = \mathbf{0}.$$

This means

$$dL(\tilde{x}(s))_e \mathscr{H}\left(\frac{\partial W(s, t)}{\partial s} + \frac{1}{2}[V(s), W(s, t)]\right) = \tilde{x}(s)\frac{\partial W(s, t)}{\partial s} = \mathbf{0},$$

where $W(s,t) = dL(\tilde{x}(s))_e^{-1}\mathscr{H}(\tilde{\nabla}_t\tilde{Y}_t(s))$, and (1.1) is used to show that the Lie bracket is vertical. Thus,

$$W(s,t) = W(a,t) = W(t,t).$$

So

$$\dot{P}(t) = \nabla_t Y_t(a) = d\pi_e(\tilde{\nabla}_t\tilde{Y}_t(a)) = d\pi_e(dL(\tilde{x}(t))_e^{-1}\tilde{\nabla}_t\tilde{Y}_t(t)) = d\pi_e(\dot{V}(t)).$$

Combining with $P(a) = \dot{x}(a) = d\pi_e(\dot{\tilde{x}}(a)) = d\pi_e(V(a))$, the proof is completed by uniqueness of solutions of smooth ODEs. □

Next we discuss some variational curves in Riemannian homogeneous spaces G/H, with special attention to symmetric spaces. This is organised as follows. In Section 2, we consider Riemannian cubics in the space G/H. Following the discussion in Section 2, we study Riemannian cubics in tension in Section 3, and elastic curves in Section 4. Finally, we comment on the differences between equations for such curves in the Lie group G and in the symmetric space G/H.

2 Riemannian cubics in homogeneous spaces

For fixed $y_a, y_b \in M$, $v_a \in T_{y_a}M$ and $v_b \in T_{y_b}M$, let $\mathcal{C}_{y_a,y_b}^{v_a,v_b}$ be the space of all curves $y : [a,b] \to M$ satisfying $y(a) = y_a$, $y(b) = y_b$, $\dot{y}(a) = v_a$ and $\dot{y}(b) = v_b$. Define a functional Ψ_1 on $\mathcal{C}_{y_a,y_b}^{v_a,v_b}$ by

$$\Psi_1(y) := \int_a^b \|\nabla_{d/dt}\dot{y}(t)\|^2 dt, \tag{2.6}$$

where $\|\cdot\|$ is the norm induced from the metric $\langle\cdot,\cdot\rangle$.

The critical points of the functional (2.6), namely *Riemannian cubics*, are widely studied [2, 11–15] and references therein. Initially, the Euler-Lagrange equations for cubics on Riemannian manifolds were established by Gabriel and Kajiya [11] and Noakes et al. [2] in the following theorem.

Theorem 2.1 *$y \in \mathcal{C}_{y_a,y_b}^{v_a,v_b}$ is a critical point of Ψ_1 if and only if y satisfies the Euler-Lagrange equation*

$$\nabla_{d/dt}^3\dot{y}(t) + R(\nabla_{d/dt}\dot{y}(t), \dot{y}(t))\dot{y}(t) = \boldsymbol{0} \tag{2.7}$$

for all $t \in [a,b]$.

In this section M is first taken to be a homogeneous space G/H, specialising later to a Riemannian symmetric space.

2.1 *M* is a homogeneous space

In order to write the Euler-Lagrange Eq. 2.7 for Riemannian cubics in G/H in terms of lifted curves in G, we relate covariant derivatives of vector fields in $M = G/H$ and the Riemannian curvature of M with their counterparts in G. The approach is based on O'Neill's work [16, 17] on Riemannian submersions.

Following Barrett O'Neill, a tensor field[2] A of type (1,2) on G is defined by

$$A_{\tilde{X}}\tilde{Y} = \mathscr{H}\tilde{\nabla}_{\mathscr{H}\tilde{X}}(\mathscr{V}\tilde{Y}) + \mathscr{V}\tilde{\nabla}_{\mathscr{H}\tilde{X}}(\mathscr{H}\tilde{Y}), \tag{2.8}$$

where $\tilde{X}, \tilde{Y} \in \mathscr{X}(G)$. If $\tilde{X}$ and $\tilde{Y}$ are horizontal then by Lemma 1.1 or Lemma 2 in [16]:

$$A_{\tilde{X}}\tilde{Y} = \frac{1}{2}\mathscr{V}[\tilde{X}, \tilde{Y}]. \tag{2.9}$$

When $\tilde{X}$ is *basic*, namely the horizontal lift of a vector field X on G/H, we have

$$\mathscr{H}\tilde{\nabla}_{\tilde{V}}\tilde{X} = A_{\tilde{X}}\tilde{V}, \tag{2.10}$$

for any vertical vector field $\tilde{V}$ (Part of Lemma 3 in [16]).

From now on, we fix the following notations: Given a C^∞ curve $x : [a, b] \to M$, let $\tilde{x} : [a, b] \to G$ be its horizontal lifting with $\tilde{x}(a) = e$, $\widetilde{\nabla_t \dot{x}}$, $\widetilde{\nabla_t^2 \dot{x}}$, $\widetilde{\nabla_t^3 \dot{x}}$ denote the horizontal liftings of $\nabla_t \dot{x}$, $\nabla_t^2 \dot{x}$, $\nabla_t^3 \dot{x}$, respectively. Then

Theorem 2.2 (1) $\mathscr{H}(\tilde{\nabla}_t \dot{\tilde{x}}) = \widetilde{\nabla_t \dot{x}}$, $\mathscr{V}(\tilde{\nabla}_t \dot{\tilde{x}}) = \boldsymbol{0}$;
(2) $\mathscr{H}(\tilde{\nabla}_t^2 \dot{\tilde{x}}) = \widetilde{\nabla_t^2 \dot{x}}$, $\mathscr{V}(\tilde{\nabla}_t^2 \dot{\tilde{x}}) = A_{\dot{\tilde{x}}}\widetilde{\nabla_t \dot{x}}$;
(3) $\mathscr{H}(\tilde{\nabla}_t^3 \dot{\tilde{x}}) = \widetilde{\nabla_t^3 \dot{x}} + A_{\dot{\tilde{x}}}A_{\dot{\tilde{x}}}\widetilde{\nabla_t \dot{x}}$, $\mathscr{V}(\tilde{\nabla}_t^3 \dot{\tilde{x}}) = \mathscr{V}\tilde{\nabla}_t(A_{\dot{\tilde{x}}}\widetilde{\nabla_t \dot{x}}) + A_{\dot{\tilde{x}}}\widetilde{\nabla_t^2 \dot{x}}$.

Proof The first-order covariant derivative[3] (1) comes from Theorem 1 in [17] because of the property (2.9) and $T = 0$ for the Riemannian submersion π. (2) and (3) follow from the definition of the tensor A. □

Now we turn to relations between the Riemannian curvature $\tilde{R}$ of G with R of M. Let $\widetilde{R(X_1, X_2)X_3}$ be the horizontal lift of the curvature $R(X_1, X_2)X_3$, where $X_i \in \mathscr{X}(M)$ for $i = 1, 2, 3$.

Theorem 2.3 *If $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ are basic vector fields on G, then*

$$\mathscr{H}(\tilde{R}(\tilde{X}_1, \tilde{X}_2)\tilde{X}_3) = \widetilde{R(X_1, X_2)X_3} + A_{\tilde{X}_1}A_{\tilde{X}_2}\tilde{X}_3 - A_{\tilde{X}_2}A_{\tilde{X}_1}\tilde{X}_3 - 2A_{\tilde{X}_3}A_{\tilde{X}_1}\tilde{X}_2. \tag{2.11}$$

Proof By the definition of the curvature R, we compute each horizontal part of

$$\tilde{R}(\tilde{X}_1, \tilde{X}_2)\tilde{X}_3 = \tilde{\nabla}_{\tilde{X}_1}\tilde{\nabla}_{\tilde{X}_2}\tilde{X}_3 - \tilde{\nabla}_{\tilde{X}_2}\tilde{\nabla}_{\tilde{X}_1}\tilde{X}_3 - \tilde{\nabla}_{[\tilde{X}_1, \tilde{X}_2]}\tilde{X}_3.$$

Recall the definition of tensor A and its properties, we have

$$\tilde{\nabla}_{\tilde{X}_2}\tilde{X}_3 = \mathscr{H}\tilde{\nabla}_{\tilde{X}_2}\tilde{X}_3 + A_{\tilde{X}_2}\tilde{X}_3,$$

[2]Reversing $\mathscr{H}$ and $\mathscr{V}$ in A, defines another tensor T that vanishes for our particular Riemannian submersion $\pi : G \to M$.

[3]We can get the conclusion from Lemma 1.1 directly.

then

$$\mathscr{H}(\tilde{\nabla}_{\tilde{X}_1}\tilde{\nabla}_{\tilde{X}_2}\tilde{X}_3) = \mathscr{H}\tilde{\nabla}_{\tilde{X}_1}(\mathscr{H}\tilde{\nabla}_{\tilde{X}_2}\tilde{X}_3) + A_{\tilde{X}_1}A_{\tilde{X}_2}\tilde{X}_3 \tag{2.12}$$

and the horizontal part of the second term in $\tilde{R}$ can be obtained by reversing $\tilde{X}_1$ and $\tilde{X}_2$ in (2.12).

Based on the property (2.9), $\mathscr{V}[\tilde{X}_1, \tilde{X}_2] = 2A_{\tilde{X}_1}\tilde{X}_2$, which yields

$$\mathscr{H}(\nabla_{\mathscr{V}[\tilde{X}_1,\tilde{X}_2]}\tilde{X}_3) = \mathscr{H}(2\tilde{\nabla}_{A_{\tilde{X}_1}\tilde{X}_2}\tilde{X}_3) = 2A_{\tilde{X}_3}A_{\tilde{X}_1}\tilde{X}_2. \tag{2.13}$$

Combining equations above, we finally get the relation (2.11). □

Set $\tilde{X}_1 = \tilde{\nabla}_t\dot{\tilde{x}}$ and $\tilde{X}_2 = \tilde{X}_3 = \dot{\tilde{x}}$, we have the following corollary.

Corollary 2.4 *In the setting of Theorem 2.2, the relation between the curvature $\tilde{R}(\tilde{\nabla}_t\dot{\tilde{x}}, \dot{\tilde{x}})\dot{\tilde{x}}$ of G and the curvature $R(\nabla_t\dot{x}, \dot{x})\dot{x}$ of M is given by*

$$\mathscr{H}(\tilde{R}(\tilde{\nabla}_t\dot{\tilde{x}}, \dot{\tilde{x}})\dot{\tilde{x}}) = R(\widetilde{\nabla_t\dot{x}, \dot{x}})\dot{x} + 3A_{\dot{\tilde{x}}}A_{\dot{\tilde{x}}}\tilde{\nabla}_t\dot{\tilde{x}}. \tag{2.14}$$

Then, Theorems 2.1, 2.2 and Corollary 2.4 give rise to the following theorem.

Theorem 2.5 *If x is a Riemannian cubic in M, then its horizontal lifting $\tilde{x}$ satisfies the following equation*

$$\mathscr{H}(\tilde{\nabla}_t^3\dot{\tilde{x}} + \tilde{R}(\tilde{\nabla}_t\dot{\tilde{x}}, \dot{\tilde{x}})\dot{\tilde{x}}) - 4A_{\dot{\tilde{x}}}A_{\dot{\tilde{x}}}\tilde{\nabla}_t\dot{\tilde{x}} = \mathbf{0}. \tag{2.15}$$

2.2 (G, H) is a Riemannian symmetric pair

Theorem 2.6 *In the situation of Theorem 2.5,* Eq. 2.15 *is equivalent to*

$$\mathscr{H}\left(-3\tilde{\nabla}_t^3\dot{\tilde{x}}(t) + \tilde{R}(\tilde{\nabla}_t\dot{\tilde{x}}(t), \dot{\tilde{x}}(t))\dot{\tilde{x}}(t) + 4\tilde{\nabla}_t(\mathscr{H}\tilde{\nabla}_t^2\dot{\tilde{x}}(t))\right) = \mathbf{0}, \tag{2.16}$$

where $\tilde{\nabla}$ is the Levi-Civita connection and $\tilde{R}$ is the Riemannian curvature on G.

(1) *If H is trivial, using Lie reduction* (1.5)*, we have*

$$\dddot{U}(t) + \left[U(t), \ddot{U}(t)\right] = \mathbf{0}. \tag{2.17}$$

(2) *If H is nontrivial and M is a Riemannian symmetric space, using reduction* (1.4)*, we obtain*

$$\ddot{V}(t) + \left[V(t), \left[\dot{V}(t), V(t)\right]\right] = \mathbf{0}. \tag{2.18}$$

Proof Equation 2.16 comes from the definition of the tensor A directly. Since Eq. 2.17 appears in [14, 19, 20] for Riemannian cubics in Lie groups, we only have to prove the second case. Using Lemma 4.2 in [3] or Lemma 2.1 in [18], we get

$$(dL_{\tilde{x}(t)^{-1}})_{\tilde{x}(t)}\tilde{\nabla}_t^2\dot{\tilde{x}}(t)) = \ddot{V}(t)+\frac{1}{2}\left[V(t),\dot{V}(t)\right],$$

$$(dL_{\tilde{x}(t)^{-1}})_{\tilde{x}(t)}\tilde{\nabla}_t^3\dot{\tilde{x}}(t)) = \dddot{V}(t)+\left[V(t),\ddot{V}(t)\right]+\frac{1}{4}\left[V(t),\left[V(t),\dot{V}(t)\right]\right],$$

$$(dL_{\tilde{x}(t)^{-1}})_{\tilde{x}(t)}\tilde{R}(\tilde{\nabla}_t\dot{\tilde{x}}(t),\dot{\tilde{x}}(t))\dot{\tilde{x}}(t) = -\frac{1}{4}\left[V(t),\left[V(t),\dot{V}(t)\right]\right].$$

Then, the condition (1.1) implies

$$\mathscr{H}\left((dL_{\tilde{x}(t)^{-1}})_{\tilde{x}(t)}\tilde{\nabla}_t^2\dot{\tilde{x}}(t))\right) = \ddot{V}(t),$$

$$\mathscr{H}\left((dL_{\tilde{x}(t)^{-1}})_{\tilde{x}(t)}\tilde{\nabla}_t^3\dot{\tilde{x}}(t))\right) = \dddot{V}(t)+\frac{1}{4}\left[V(t),\left[V(t),\dot{V}(t)\right]\right],$$

$$\mathscr{H}\left((dL_{\tilde{x}(t)^{-1}})_{\tilde{x}(t)}\tilde{R}(\tilde{\nabla}_t\dot{\tilde{x}}(t),\dot{\tilde{x}}(t))\dot{\tilde{x}}(t)\right) = -\frac{1}{4}\left[V(t),\left[V(t),\dot{V}(t)\right]\right],$$

which results in the Eq. 2.18. □

The method used in the horizontal subspace of Lie group G can be viewed as a natural generalisation of Lie reduction for curves in the whole group G. On the other hand, it is worth pointing out that our Eq. 2.18 is exactly the same as equation (46) in [3] (also see Eq. 1.3 in the introduction). The former is obtained by Lie reduction in horizontal subspace and the latter is achieved by parallel translation. As we proved in Theorem 1.2, our Lie reduction V can be identified with the parallel pullback defined in [3], however, our method has greater generality in the sense that G/H can be a Riemannian homogeneous space and our definition leads easily to recovering the Riemannian cubic x from V.

3 Riemannian cubics in tension in G/H

Let $\mathcal{C}^{v_a,v_b}_{y_a,y_b}$ be the space defined at the beginning of Section 2. We define a functional Ψ_2 over $\mathcal{C}^{v_a,v_b}_{y_a,y_b}$ for $\tau > 0$ by

$$\Psi_2(y) := \int_a^b \|\nabla_{d/dt}\dot{y}(t)\|^2 + \tau\|\dot{y}(t)\|^2 dt. \tag{3.19}$$

The critical points of the functional (3.19) are called *Riemannian cubics in tension*. In [21], cubics in tension are called elastic curves. We won't adopt this terminology, in order to avoid possible confusion with *elastica* studied in [15, 22–24] and in the following section. Originally, Silva Leite, Camarinha and Crouch in [21] proved the following theorem.

Theorem 3.1 *$y \in \mathcal{C}^{v_a,v_b}_{y_a,y_b}$ is a critical point of Ψ_2 if and only if*

$$\nabla^3_{d/dt}\dot{y}(t) + R(\nabla_{d/dt}\dot{y}(t),\dot{y}(t))\dot{y}(t) - \tau\nabla_{d/dt}\dot{y}(t) = \boldsymbol{0} \tag{3.20}$$

for all $t \in [a, b]$.

Now we focus on Riemannian cubics in tension in the Riemannian homogeneous space G/H. If x is a Riemannian cubic in tension, then $\mathscr{V}(\tilde{\nabla}_t \dot{\tilde{x}}) = \mathbf{0}$ by Theorem 2.2, where $\tilde{x}$ is the horizontal lifting of x, $\tilde{\nabla}$ and ∇ are Levi-Civita connections on G and G/H, respectively.

Therefore, we have

Theorem 3.2 *If x is a Riemannian cubic in tension in the homogeneous space G/H, then its horizontal lifting $\tilde{x}$ satisfies*

$$\mathscr{H}(\tilde{\nabla}_t^3 \dot{\tilde{x}} + \tilde{R}(\tilde{\nabla}_t \dot{\tilde{x}}, \dot{\tilde{x}})\dot{\tilde{x}}) - 4A_{\dot{\tilde{x}}}A_{\dot{\tilde{x}}}\tilde{\nabla}_t \dot{\tilde{x}} - \tau \tilde{\nabla}_t \dot{\tilde{x}} = \mathbf{0}. \tag{3.21}$$

(1) *If H is trivial,* (3.21) *can be simplified as*

$$\dddot{U}(t) + [U(t), \ddot{U}(t)] - \tau \dot{U}(t) = \mathbf{0}, \tag{3.22}$$

where U is defined in (1.5).

(2) *If H is nontrivial and M is a Riemannian symmetric space,* (3.21) *can be simplified as*

$$\dddot{V}(t) + [V(t), [\dot{V}(t), V(t)]] - \tau \dot{V}(t) = \mathbf{0}, \tag{3.23}$$

where V is defined in (1.4).

The Eq. 3.22 was discussed in [25, 26] and references therein. Further, focusing on the Eq. 3.23, we have

Proposition 3.3 *Since G is a bi-invariant Lie group, then*

(1) $\frac{d^2}{dt^2}\|V(t)\|^2 - 3\|\dot{V}(t)\|^2 = \tau\|V(t)\|^2 + C_1$*;*
(2) $\|\ddot{V}(t)\|^2 + \|[V(t), \dot{V}(t)]\|^2 = \tau\|\dot{V}(t)\|^2 + C_2$*;*
(3) $\frac{d^2}{dt^2}\|\dot{V}(t)\|^2 + 2\|[V(t), \dot{V}(t)]\|^2 = 2\tau\|\dot{V}(t)\|^2 + 2\|\ddot{V}(t)\|^2$*, for some constant* $C_1, C_2 \in \mathbb{R}$.

Proof The left hand side of (1) evaluates to $2\langle V(t), \dddot{V}(t)\rangle$. Substitute for $\dddot{V}(t)$ by Eq. 3.23 and apply the bi-invariant condition. For (2), differentiate the left side, giving

$$2\langle \ddot{V}(t), \dddot{V}(t)\rangle + 2\langle [V(t), \ddot{V}(t)], [V(t), \dot{V}(t)]\rangle,$$

which is $2\tau\langle \ddot{V}(t), \dot{V}(t)\rangle$ by bi-invariance. For (3), differentiate $\|\dot{V}(t)\|^2$ twice and use bi-invariance. □

Example 3.1 Consider Riemannian cubics or Riemannian cubics in tension on the unit 2-dimensional sphere S^2 with $G = SO(3)$ and $H = SO(2)$. The Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ is Lie isomorphic to the Euclidean 3-space E^3 with the Lie bracket given by the cross product $\times$. Then, under such isomorphism, the horizontal subspace of $\mathfrak{so}(3)$ is isomorphic to $E^2 \times \{0\}$, and

$$[V(t), [\dot{V}(t), V(t)]] = V(t)\times(\dot{V}(t)\times V(t)) = \dot{V}(t)\langle V(t), V(t)\rangle - V(t)\langle \dot{V}(t), V(t)\rangle.$$

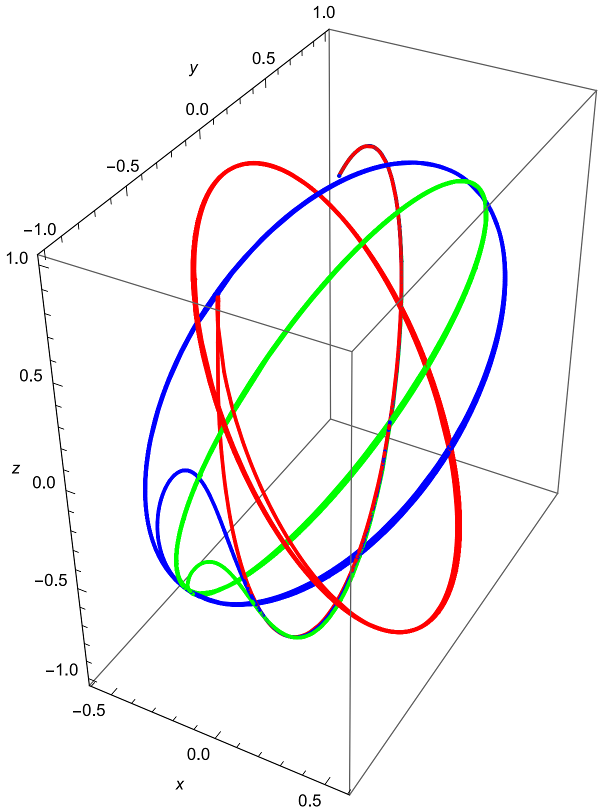

Fig. 1 The red curve is a cubic on S^2, the blue curve is a cubic in tension with $\tau = 1$ and the green one is a cubic in tension with $\tau = 2$. (Online version in colour.)

To display Riemannian cubic on S^2, initial conditions rather than boundary conditions are given for simplicity. Taking $x(0) = (0, 0, 1)$, $\dot{x}(0) = (-1, 4, 0)$, $\ddot{x}(0) = (0.5, -0.3, -17)$, $\dddot{x}(0) = (19, -69, 5.1)$, then, by Theorem 2.2, we have $V(0) = \dot{x}(0)$, $\dot{V}(0) = (0.5, -0.3, 0)$, $\ddot{V}(0) = (2, -1, 0)$. Mathematica's NDsolve presents the solution curve x of (2.18) or (3.23) on a 2.7GHz Intel Core i5 Mac with 8GB RAM (See Fig. 1).

In Fig. 1, asymptotics of Riemannian cubics and Riemannian cubics in tension on S^2 appear to be great circles. Taking cubic curve as an example, suppose $V(t) = r(t)e^{i\theta(t)}$. By (2) in Proposition 3.3, we know $r(t)^2\dot{\theta}(t)$, $r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t)$, $\ddot{r}(t) - r(t)\dot{\theta}(t)^2$ are bounded. When $r(t) \to \infty$,[4] then $\dot{\theta}(t) \to 0$, $\ddot{r}(t) \to 0$, which means asymptotics of Riemannian cubics are great circles on the sphere.

[4]If we choose suitable initial conditions, it's possible to make C_1 in Proposition 3.3 non-negative, which definitely makes $r(t)$ approach infinity as t runs to infinity.

4 Elastica in homogeneous space G/H

In the space $\mathcal{C}_{y_a,y_b}^{v_a,v_b}$ defined in previous sections, if v_a, v_b are both unit vectors, the minimiser of the functional (2.6) over curves $y \in \mathcal{C}_{y_a,y_b}^{v_a,v_b}$ subject to

$$\|\dot{y}(t)\|^2 = 1 \tag{4.24}$$

is said to be an *elastica* or *elastic curve*.

Theorem 4.1 *If a C^∞ curve $y : [a,b] \to M$ is an elastic curve, then*

$$\nabla^3_{d/dt}\dot{y}(t) + R(\nabla_{d/dt}\dot{y}(t), \dot{y}(t))\dot{y}(t) + \nabla_{d/dt}\left(\left(\frac{3}{2}\|\nabla_{d/dt}\dot{y}(t)\|^2 + c\right)\dot{y}(t)\right) = \mathbf{0} \tag{4.25}$$

for some constant $c \in \mathbb{R}$ and all $t \in [a,b]$.

Theorem 4.1 is proved in [18, 21–24]. Because of (4.24), $\|\nabla_{d/dt}\dot{y}(t)\|$ in (4.25) is the geodesic curvature of y.

For the homogeneous space G/H, (1) in Theorem 2.2 gives

$$\|\tilde{\nabla}_t\dot{\tilde{x}}\| = \|\nabla_t\dot{x}\| = \|\dot{V}\|,$$

where the second equality holds because the metric on G is left invariant.

Combining with results from the previous section, we have the following theorem

Theorem 4.2 *If a C^∞ curve $x : [a,b] \to G/H$ is an elastic curve, then its horizontal lifting $\tilde{x}$ satisfies*

$$\mathscr{H}(\tilde{\nabla}^3_t\dot{\tilde{x}} + \tilde{R}(\tilde{\nabla}_t\dot{\tilde{x}}, \dot{\tilde{x}})\dot{\tilde{x}}) - 4A_{\dot{\tilde{x}}}A_{\dot{\tilde{x}}}\tilde{\nabla}_t\dot{\tilde{x}} + \frac{d}{dt}\left(\left(\frac{3}{2}\|\nabla_t\dot{x}(t)\|^2 + c\right)\tilde{\nabla}_t\dot{\tilde{x}}\right) = \mathbf{0} \tag{4.26}$$

for some constant $c \in \mathbb{R}$.

(1) *If H is trivial,* (4.26) *can be reduced to*

$$\ddot{U}(t) + [U(t), \dot{U}(t)] + \left(\frac{3}{2}\|\dot{U}(t)\|^2 + c\right)U(t) = \tilde{c}, \tag{4.27}$$

where $\tilde{c} \in \mathfrak{g}$ is constant.

(2) *If H is nontrivial and (G, H) is a Riemannian symmetric pair,* (4.26) *reduces to*

$$\dddot{V}(t) + [V(t), [\dot{V}(t), V(t)]] + \frac{d}{dt}\left(\left(\frac{3}{2}\|\dot{V}(t)\|^2 + c\right)V(t)\right) = \mathbf{0}. \tag{4.28}$$

Equation 4.27 for elastica in Lie groups was investigated by Popiel and Noakes [18].

Example 4.1 In quantum computation, the control of quantum states leads to an interpolation problem in complex projective space $\mathbb{C}P^n$ [27]. In this example, we consider elastica in the simplest complex projective space $\mathbb{C}P^1 \cong S^2$ with $G = U(2)$ and $H = U(1) \times U(1)$.

Here, $U(2)$ is connected and compact with the bi-invariant metric

$$\langle X, Y\rangle_I := \mathrm{tr}(X^*Y)$$

for tangent vectors X, Y on $U(2)$ at the identity I, where X^* is the conjugate transpose of X. Choose a basis of the Lie algebra $\mathfrak{u}(2)$ of $U(2)$ as

$$X_1 = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, X_3 = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix},$$

where X_1, X_2 form a basis of $\mathfrak{m}$ and X_3 generates the Lie algebra of H.

Suppose x is an elastic curve in $\mathbb{C}P^1$, the left Lie reduction of the horizontal lifting $\tilde{x}$ of x is denoted by V. Then, under the constraint $\|V(t)\|^2 = 1$, a straightforward calculation gives rise to

$$[V(t), [\dot{V}(t), V(t)]] = 2\dot{V}(t).$$

Further, integration turns Eq. 4.28 into

$$\begin{cases} \ddot{V}(t) + (\|\dot{V}(t)\| + \langle V(t), \tilde{C}\rangle)V(t) = \tilde{C}, \\ \|V(t)\| = 1, \end{cases} \tag{4.29}$$

where $\tilde{C} \in \mathfrak{m}$ is constant. Since V is a curve on the unit circle S^1, we suppose $V(t) = e^{i\theta(t)}$, then (4.29) turns out to be the equation for a simple pendulum

$$\ddot{\theta}(t) + \theta_0 \sin(\theta(t)) = 0, \tag{4.30}$$

where θ_0 is constant. Consequently, the elastic curve on $\mathbb{C}P^1$ is given by

$$x(t) = \pi(\tilde{x}(t)) = e^{\int_0^t V(s)ds} \cdot \begin{bmatrix} U(1) & 0 \\ 0 & U(1) \end{bmatrix}, \tag{4.31}$$

where $t \in [a, b]$, $U(1)$ is the unitary group. Elastica on S^2 was studied also by Jurdjevic [28] using different methods.

5 Comparison between curves in Lie groups and Riemannian symmetric spaces

Based on Theorems 2.6, 3.2 and 4.2, the main difference between Riemannian cubics/cubics in tension/elastic curves in Lie groups and Riemannian symmetric spaces is that one has the Lie bracket $[U(t), \ddot{U}(t)]$ whilst the other has the double Lie bracket $[V(t), [\dot{V}(t), V(t)]]$. So what happens for curves in a Riemannian manifold, which is a Lie group as well as a symmetric space? Cubic curves in the unit 3-dimensional sphere S^3 will be taken as an example.

It is well known that S^3 is isomorphic to the group Q of all unit quaternions, and to the special unitary group $SU(2)$, and S^3 is a Riemannian symmetric space $S^3 = SO(4)/SO(3) = U(2)/U(1)$ as well. Even though S^3 acts on itself isometrically, it's impossible to regard S^3 as the symmetric space S^3/id. This is because (G, H) is not a symmetric pair. Thus, the reduction (1.5) and (1.4) are actually conducted in different spaces $\mathfrak{g}$ and $\mathfrak{m}$ when the Riemannian symmetric space $M = G/H$ is also a Lie group.

For the 3-dimensional unit sphere S^3, we choose the identity connected component $SO(4)$ of the group of all isometries, which is double covered by $SU(2) \times SU(2)$. The relationships between them are displayed in the following diagram,

$$\begin{array}{ccc} SU(2) & \xrightarrow{-\phi_1} & SO(3) \\ \downarrow & & \downarrow \\ SU(2) \times SU(2) & \xrightarrow{-\phi_2} & SO(4) \\ \downarrow & & \downarrow \\ SU(2) & \xrightarrow{-\phi} & S^3 \end{array}$$

where $\phi_1 : SU(2) \to SO(3)$, $\phi_2 : SU(2) \times SU(2) \to SO(4)$ are both double covers (details can be found in [28]) and $\phi : SU(2) \to S^3$ is given by

$$\phi\left(\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}\right) := z + w\mathbf{j}, \tag{4.1}$$

where $z, w \in \mathbb{C}$, $|z|^2 + |w|^2 = 1$. Thus, the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ and the Lie algebra $\mathfrak{so}(4)$ of $SO(4)$ is isomorphic to the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ of $SU(2) \times SU(2)$. Since $\mathfrak{so}(3)$ is isomorphic to the Euclidean 3-space E^3 equipped with the Lie bracket $\times$, $\mathfrak{so}(4)$ and $E^3 \oplus E^3$ are isomorphic.

Numerical examples show that Eq. 2.17 and (2.18) give exactly the same Riemannian cubic on the sphere S^3 if same boundary conditions are given.

6 Conclusions

Variational curves including Riemannian cubics, Riemannian cubics in tension and elastica are widely used in engineering and computer science. In this paper, we consider these curves in Riemannian homogeneous space of type G/H, where G is a connected finite-dimensional Lie group, H is a closed subgroup and $G \to G/H$ is a Riemannian submersion. Instead of investigating equations for curves in G/H, we discuss equations for their horizontal lifting curves in G. It is proved that parallel pullback of $\dot{x}$ along x is equivalent to the left Lie reduction $\tilde{x}^{-1}\dot{\tilde{x}}$ of $\dot{\tilde{x}}$, where $\tilde{x}$ is the horizontal lifting of x. Even though we obtain the same equation as Crouch et al. in [3] for cubics in Riemannian symmetric spaces, our method is more amenable to recovery of Riemannian cubics, and is a natural generalisation of a method that is standard for Lie groups.

The applications are not limited to curves discussed in this manuscript, for instance, high-order cubics [29, 30], Jupp-Kent cubics [31] can be discussed by the method proposed in this paper. Discussions of asymptotics, Jacobi fields, and Lie quadratics of curves in homogeneous space, are left for future work.

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