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Approximately dual Gabor frames and almost perfect reconstruction based on a class of window functions

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Abstract It is a well-known problem in Gabor analysis how to construct explicitly given dual frames associated with a given frame. In this paper we will consider a class of window functions for which approximately dual windows can be calculated explicitly. The method makes it possible to get arbitrarily close to perfect reconstruction by allowing the modulation parameter to vary. Explicit estimates for the deviation from perfect reconstruction are provided for some of the standard functions in Gabor analysis, e.g., the Gaussian and the two-sided exponential function.

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1 Introduction

Gabor analysis is typically based on a choice of a window function $\varphi \in L^2(\mathbb{R})$ and
suitable parameters $a, b > 0$ such that the system $\{e^{2\pi imbx}\varphi(x - na)\}_{m,n\in\mathbb{Z}}$ forms
a frame for $L^2(\mathbb{R})$. A classical problem is that most applications need information
about the dual window, which might be complicated to calculate explicitly. The pur-
pose of this paper is to obtain explicit constructions of approximately dual windows
for a class of windows $\mathcal{V} \subset L^2(\mathbb{R})$ that cover most of the classical choices in Gabor
analysis. In contrast to the standard Gabor theory, we will allow the modulation
parameter b to change: indeed, we will prove that we can get arbitrarily close to per-
fect reconstruction for any $\varphi \in \mathcal{V}$ using a sufficiently small modulation parameter
and the constructed approximately dual windows. Explicit estimates for the devia-
tion from perfect reconstruction are provided for some of the standard functions in
Gabor analysis, e.g., the Gaussian and the two-sided exponential function. Note that,
in contrast to the approach in [3], the translation parameter will be kept fixed.

In the rest of this section we will set the stage by introducing the relevant class of
window functions and collecting the necessary background information about frames
and Gabor systems. In particular, a crucial ingredient in our method is to provide
estimates for certain Bessel bounds; this is the topic of Section 2. We will indeed
provide two estimates of the Bessel bounds: one that is valid for all functions $\varphi \in$
$\mathcal{V}$, and a sharper estimate for convex functions. The construction of approximately
dual windows for $g \in \mathcal{V}$ is given in Section 3.

A sequence $\{f_k\}_{k\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* if there exist
constants $A, B > 0$ such that

$$A \, ||f||^2 \le \sum_{k\in I} |\langle f, f_k\rangle|^2 \le B \, ||f||^2, \ \forall f \in \mathcal{H}. \tag{1.1}$$

A sequence $\{f_k\}_{k\in I}$ satisfying at least the upper condition in (1.1) is called a *Bessel
sequence with Bessel bound B*. It is well-known (see, e.g., [2]) that $\{f_k\}_{k\in I}$ is a Bessel
sequence with bound B if and only if

$$|| \sum c_k f_k ||^2 \le B \sum |c_k|^2$$

for all finite scalar sequences $\{c_k\}$.

Frames lead to unconditionally convergent series expansions of the elements in
the underlying Hilbert space. Indeed, if $\{f_k\}_{k\in I}$ is a frame for $\mathcal{H}$, then there exists a
frame $\{g_k\}_{k\in I}$ such that

$$f = \sum_{k\in I}\langle f, g_k\rangle f_k = \sum_{k\in I}\langle f, f_k\rangle g_k, \ \forall f \in \mathcal{H}. \tag{1.2}$$

A frame $\{g_k\}_{k\in I}$ that satisfies (1.2) is called a *dual frame* of $\{f_k\}_{k\in I}$; and (1.2) is expressed by saying that the frames $\{f_k\}_{k\in I}$ and $\{g_k\}_{k\in I}$ lead to *perfect reconstruction*.

A weaker concept was introduced in [4]. We say that two Bessel sequences $\{f_k\}_{k\in I}$ and $\{g_k\}_{k\in I}$ form *approximately dual frames* if there exists a constant $\mu < 1$ such that

$$\left\| f - \sum_{k\in I}\langle f, f_k\rangle g_k \right\| \le \mu \|f\|, \quad \forall f \in \mathcal{H}. \tag{1.3}$$

The rationale behind approximately dual frames is that all kinds of implementations involve certain imprecisions; thus, as long as we can control the deviation from perfect reconstruction measured by the parameter μ in (1.3), approximately dual frames can in practice be as good as exact dual frames. We refer to the papers [1, 5–7] for explicit constructions and applications of this concept. We will use the following elementary result from [3] to measure the deviation from perfect reconstruction.

Lemma 1 *Let $\{f_k\}_{k\in I}$ be a frame with bounds A, B and let $\{g_k\}_{k\in I}$ be a dual frame of $\{f_k\}_{k\in I}$ with upper frame bound B_g. If $\{\widetilde{f}_k\}_{k\in I}$ is a sequence in $\mathcal{H}$ and*

$$\left\| \sum_{k\in I} c_k(f_k - \widetilde{f}_k) \right\| \le \mu \left(\sum_{k\in I} |c_k|^2 \right)^{1/2} \tag{1.4}$$

for some $\mu \ge 0$ and all finite sequences $\{c_k\}_{k\in I}$, then

$$\left\| f - \sum_{k\in I}\langle f, \widetilde{f}_k\rangle g_k \right\| \le \mu\sqrt{B_g}\,\|f\|, \quad \forall f \in \mathcal{H}. \tag{1.5}$$

In particular $\{g_k\}_{k\in I}$ and $\{\widetilde{f}_k\}_{k\in I}$ are approximately dual frames if $\mu\sqrt{B_g} < 1$.

For any parameters $a, b \in \mathbb{R}$, define the *translation operator T_a* and *modulation operators E_b* acting on $L^2(\mathbb{R})$ by

$$T_a f(x) = f(x - a), \quad E_b f(x) = e^{2\pi i b x} f(x), \quad x \in \mathbb{R}.$$

Given a function $\varphi \in L^2(\mathbb{R})$ and parameters $a, b > 0$, the associated *Gabor system* is the collection of functions $\{E_{mb}T_{na}\varphi\}_{m,n\in\mathbb{Z}}$. The function φ is called the *window function*.

Throughout this paper we will consider the following class of windows, which contains most of the standard choices in Gabor analysis.

Definition 1 Let $\mathcal{V} \subset L^2(\mathbb{R})$ denote the set consisting of the real-valued continuous functions $\varphi \in L^2(\mathbb{R})$ for which

(i) φ is even and positive on $\mathbb{R}$;
(ii) φ is decreasing on $[0, \infty)$;
(iii) $|\varphi(x)| \le \frac{C}{1+|x|^{1+\sigma}}$ for some $C > 0$ and $\sigma > 0$.

For a fixed function $\varphi \in \mathcal{V}$ and any $N > 0$, define the function $\varphi_N \in C_c(\mathbb{R})$ by

$$\varphi_N(x) := (\varphi(x) - \varphi(N))\, \chi_{[-N,N]}(x). \tag{1.6}$$

The function φ_N in (1.6) can be considered to be a truncated version of φ, and will play a central role throughout this paper. Note that N can be an arbitrary positive number in Definition 1; in Section 3 we will consider a sequence of approximately dual windows indexed by N, so in that case we only consider positive integer-values of N.

In the analysis of a given function $\varphi \in \mathcal{V}$ and its truncations φ_N, we will need the functions $Q_N : (0, \infty) \to \mathbb{R}$ defined by

$$Q_N(x) := (\lfloor 2xN \rfloor + 1)\varphi(N) + 2\sum_{k=0}^{\infty} \varphi\left(N + \frac{k}{x}\right). \tag{1.7}$$

2 Frame properties of functions $\varphi \in \mathcal{V}$ and φ_N

In this section we will discuss the relationship between the frame properties of functions $\varphi \in \mathcal{V}$ and the associated truncations φ_N in (1.6). The results will be based on Proposition 2, where we estimate the Bessel bound for the functions $\varphi - \varphi_N$. We will need the following estimate of the function Q_N defined in (1.7).

Lemma 2 *Let $N > 0$ be fixed. For $\varphi \in \mathcal{V}$ and $C, \sigma > 0$ as in Definition 1, we have*

$$Q_N(x) \leq \frac{C}{N^{1+\sigma}}\left(\lfloor 2xN \rfloor + 1 + 2\left(1 + \frac{xN}{\sigma}\right)\right). \tag{2.1}$$

Proof The decay condition (iii) in Definition 1 implies that

$$\sum_{k=0}^{\infty} \varphi(N + \frac{k}{x}) \leq \sum_{k=0}^{\infty} \frac{C}{\left(N + \frac{k}{x}\right)^{1+\sigma}}$$
$$\leq \frac{C}{N^{1+\sigma}} + \int_0^{\infty} \frac{C}{\left(N + \frac{t}{x}\right)^{1+\sigma}}dt = \frac{C}{N^{1+\sigma}}\left(1 + \frac{xN}{\sigma}\right).$$

Thus

$$Q_N(x) \leq (\lfloor 2xN \rfloor + 1)\frac{C}{N^{1+\sigma}} + \frac{2C}{N^{1+\sigma}}\left(1 + \frac{xN}{\sigma}\right)$$
$$= \frac{C}{N^{1+\sigma}}\left(\lfloor 2xN \rfloor + 1 + 2\left(1 + \frac{xN}{\sigma}\right)\right).$$

$\square$

Using Lemma 2 we can now provide a Bessel bound for the functions

$$\varphi(x) - \varphi_N(x) = \min\{\varphi(N), \varphi(x)\} = \varphi(x)\chi_{[-N,N]^c}(x) + \varphi(N)\chi_{[-N,N]}(x), \ x \in \mathbb{R},$$

whenever $\varphi \in \mathcal{V}$. We will use the fact that for any $g \in L^2(\mathbb{R})$, if

$$B(g, a, b) := \frac{1}{b}\sup_{x \in [0,a]}\sum_{k \in \mathbb{Z}}\left|\sum_{n \in \mathbb{Z}} g(x - na)\overline{g(x - na - k/b)}\right| < \infty,$$

then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with Bessel bound $B(g, a, b)$; see, e.g., [2], Theorem 11.4.2.

Proposition 2 *Let $\varphi \in \mathcal{V}$, and let $a, b > 0$ be given. Given $N > 0$, define the function Q_N by (1.7). Then the following hold:*

(i)   *The Bessel bound $B(\varphi - \varphi_N, a, b)$ can be estimated by*

$$B(\varphi - \varphi_N, a, b) \le \frac{1}{b} Q_N(b) Q_N(\frac{1}{a}). \tag{2.2}$$

(ii)   *If $b \le \frac{1}{2N}$, then*

$$B(\varphi - \varphi_N, a, b) \le \frac{1}{b} Q_N(b) Q_N(\frac{1}{a}) \le \frac{C^2}{bN^{1+2\sigma}} \left(4 + \frac{1}{\sigma}\right) \left(\frac{2}{a}\left(1 + \frac{1}{\sigma}\right) + \frac{3}{N}\right).$$

Proof For notational convenience, let $q_N(x) := \varphi(x) - \varphi_N(x)$. We note that

$$B(q_N, a, b) \le \frac{1}{b} \left(\sup_{x\in[0,a], n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left| q_N(x - na - k/b) \right| \right) \left(\sup_{x\in[0,a]} \sum_{n\in\mathbb{Z}} \left| q_N(x - na) \right| \right) \tag{2.3}$$

and

$$q_N(x) = \begin{cases} \varphi(N), & x \in [-N, N]; \\ \varphi(x), & x \in [-N, N]^c. \end{cases} \tag{2.4}$$

Estimate of $\sum_{k\in\mathbb{Z}} \left| q_N(x - na - k/b) \right|$ Fix $x \in [0, a]$ and $n \in \mathbb{Z}$; then $x - na - k/b$ hits $[-N, N]$ for at most $\lfloor 2bN \rfloor + 1$ values of k, and thus yields a maximal contribution $(\lfloor 2bN \rfloor + 1)\varphi(N)$ to the infinite sum by (2.4). By (2.4) again, the contribution to the infinite sum from $x - na - k/b$ hitting the interval $[N, \infty)$ is at most

$$\sum_{k=0}^{\infty} \left| q_N\left(N + \frac{k}{b}\right) \right| = \sum_{k=0}^{\infty} \varphi\left(N + \frac{k}{b}\right),$$

which converges by (iii) in Definition 1. Including also the contribution from $(-\infty, -N]$ leads to

$$\sum_{k\in\mathbb{Z}} \left| q_N(x - na - k/b) \right| \le (\lfloor 2bN \rfloor + 1)\varphi(N) + 2\sum_{k=0}^{\infty} \varphi\left(N + \frac{k}{b}\right) = Q_N(b). \tag{2.5}$$

Estimate of $\sum_{n\in\mathbb{Z}} \left| q_N(x - na) \right|$ Applying the above result with $b = 1/a$ yields that

$$\sum_{n\in\mathbb{Z}} \left| q_N(x - na) \right| \le Q_N\left(\frac{1}{a}\right) = \left(\left\lfloor \frac{2N}{a} \right\rfloor + 1\right)\varphi(N) + 2\sum_{k=0}^{\infty} \varphi(N + ak). \tag{2.6}$$

Hence (2.3) together with (2.5) and (2.6) implies that the Bessel bound of $\{E_{mb}T_{na}q_N\}_{m,n\in\mathbb{Z}}$ can be estimated as stated in (2.2).

For the proof of (ii), applying (2.1) with $b \leq \frac{1}{2N}$ yields that

$$Q_N(b) \leq \frac{C}{N^{1+\sigma}} \left(\lfloor 2bN \rfloor + 1 + 2 \left(1 + \frac{bN}{\sigma} \right) \right)$$

$$\leq \frac{C}{N^{1+\sigma}} \left(2 + 2 \left(1 + \frac{1}{2\sigma} \right) \right) = \frac{C}{N^{1+\sigma}} \left(4 + \frac{1}{\sigma} \right).$$

Also,

$$Q_N \left(\frac{1}{a} \right) \leq \frac{C}{N^{1+\sigma}} \left(\left\lfloor \frac{2N}{a} \right\rfloor + 1 + 2 \left(1 + \frac{N}{a\sigma} \right) \right)$$

$$\leq \frac{C}{N^{1+\sigma}} \left(\frac{2N}{a} + 1 + 2 \left(1 + \frac{N}{a\sigma} \right) \right) = \frac{C}{N^{\sigma}} \left(\frac{2}{a} \left(1 + \frac{1}{\sigma} \right) + \frac{3}{N} \right).$$

The result in (i) now immediately leads to (ii). □

Proposition 2 implies that if $\{E_{mb}T_{na}\varphi\}_{m,n \in \mathbb{Z}}$ is a frame and $\varphi \in \mathcal{V}$, then also the truncated function φ_N in (1.6) generates a frame whenever $N > 0$ is sufficiently large. This generalizes a result from [3], dealing with the Gaussian.

Corollary 3 *Let $a, b > 0$ and $\varphi \in \mathcal{V}$ be given, and assume that $\{E_{mb}T_{na}\varphi\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Then for $N > 0$ sufficiently large the Gabor system $\{E_{mb}T_{na}\varphi_N\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.*

Proof Fixing $\varphi \in \mathcal{V}$ and $a, b > 0$, Proposition 2 combined with Lemma 2 shows that the Bessel bound $B(\varphi - \varphi_N, a, b)$ tends to zero as $N \to \infty$. Thus, denoting a lower frame bound for $\{E_{mb}T_{na}\varphi\}_{m,n \in \mathbb{Z}}$ by A, we have

$$B(\varphi - \varphi_N, a, b) < A \tag{2.7}$$

for N sufficiently large. The result now follows from standard results in frame perturbation theory (see Corollary 22.1.5 in [2]). □

It is well-known that a continuous function with compact support cannot generate a Gabor Riesz basis (see, e.g., Proposition 13.2.4 in [2]). Since the perturbation condition used in the proof of Corollary 3 preserves the Riesz basis property, the result implies that no function $\varphi \in \mathcal{V}$ can generate a Riesz basis.

Note that the proof of Corollary 3 shows how large N should be chosen in order to guarantee that $\{E_{mb}T_{na}\varphi_N\}_{m,n \in \mathbb{Z}}$ is a frame; we just need to choose N such that (2.7) holds. In order to apply this result it is crucial to have good estimates for the Bessel bound $B(\varphi - \varphi_N, a, b)$. In the following concrete example we will see that based on the estimates in Proposition 2, the frame property of $\{E_{mb}T_{na}\varphi_N\}_{m,n \in \mathbb{Z}}$ is guaranteed for $N = 3/4$, but not for $N = 1/\sqrt{2}$. This motivates the subsequent Proposition 5 where we obtain a sharper estimate for the Bessel bound $B(\varphi - \varphi_N, a, b)$ under the aditional condition that the function φ is convex. Then, in Example 6 we return to the same concrete example and show that $\{E_{mb}T_{na}\varphi_N\}_{m,n \in \mathbb{Z}}$ actually is a frame for $N = 1/\sqrt{2}$.

Example 4 We consider

$$\varphi(x) = 2^{1/4} e^{-\pi x^2}, \ x \in \mathbb{R},$$

and $a = b = 1/\sqrt{2}$. According to (6.7-8) in [8], the optimal frame bounds of $\{E_{mb} T_{na} \varphi\}_{m,n \in \mathbb{Z}}$ are given by

$$A = 2 \sum_{k,\ell \in \mathbb{Z}} (-1)^{k+\ell} e^{-\pi(k^2+\ell^2)} = 2 \left(\sum_{k=-\infty}^{\infty} (-1)^k e^{-\pi k^2} \right)^2$$

and

$$B = 2 \sum_{k,\ell \in \mathbb{Z}} e^{-\pi(k^2+\ell^2)} = 2 \left(\sum_{k=-\infty}^{\infty} e^{-\pi k^2} \right)^2,$$

respectively. By a numerical computation, $A = 1.669253683$, $B = 2.360681198$.

Furthermore, using (2.2) and (1.7),

$$B\left(\varphi - \varphi_N, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \leq \sqrt{2} \, Q_N \left(\frac{1}{\sqrt{2}} \right) Q_N(\sqrt{2})$$

$$\leq 2 \left((\lfloor N\sqrt{2} \rfloor + 1) e^{-\pi N^2} + 2 \sum_{k=0}^{\infty} e^{-\pi(N+k\sqrt{2})^2} \right)$$

$$\times \left((\lfloor 2N\sqrt{2} \rfloor + 1) e^{-\pi N^2} + 2 \sum_{k=0}^{\infty} e^{-\pi(N+\frac{k\sqrt{2}}{2})^2} \right).$$

A numerical inspection shows that for $N = 3/4$ this estimate of $B\left(\varphi - \varphi_N, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ is indeed smaller than the lower frame bound A of $\{E_{mb} T_{na} \varphi\}_{m,n \in \mathbb{Z}}$; thus $\{E_{mb} T_{na} \varphi_N\}_{m,n \in \mathbb{Z}}$ is a frame for $N = 3/4$. However, for $N = 1/\sqrt{2}$ the estimate of $B\left(\varphi - \varphi_N, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ exceeds the lower frame bound A of $\{E_{mb} T_{na} \varphi\}_{m,n \in \mathbb{Z}}$; thus, the proof of Corollary 3 does not guarantee that $\{E_{mb} T_{na} \varphi_N\}_{m,n \in \mathbb{Z}}$ is a frame for $N = 1/\sqrt{2}$.

For windows $\varphi \in \mathcal{V}$ that are convex on $[N, \infty)$ we will now derive the following improvement of Proposition 2.

Proposition 5 *Assume that $\varphi \in \mathcal{V}$ is convex on $[N, \infty)$ for some $N > 0$. Then*

$$B(q_N, a, b) \leq \frac{1}{b} \left(\sum_{k=-\infty}^{\infty} q_N \left(N - \frac{k}{b} \right) \right) \left(\sum_{n=-\infty}^{\infty} q_N \left(N - na \right) \right).$$

Proof By (2.3) and non-negativity of the function $q_N(x) = \min\{\varphi(x), \varphi(N)\}$, we have that

$$B(q_N, a, b) \leq \frac{1}{b} \sup_{x \in [0,1/b]} \left(\sum_{k=-\infty}^{\infty} \left| q_N(x - k/b) \right| \right) \sup_{x \in [0,a]} \left(\sum_{n=\infty}^{\infty} \left| q_N(x - na) \right| \right).$$

We will first show that, under the assumption that φ is convex on $[N, \infty)$,

$$\sup_{x \in [0,a]} \left(\sum_{n \in \mathbb{Z}} q_N(x - na) \right) = \sum_{n=-\infty}^{\infty} q_N(N - na)$$

$$= \left(\left\lfloor \frac{2N}{a} \right\rfloor + 1 \right) \varphi(N) + \sum_{n=1}^{\infty} \varphi(N + na) + \sum_{n=1}^{\infty} \varphi \left(\left\lfloor \frac{2N}{a} \right\rfloor a - N + na \right). \quad (2.8)$$

To show this, let $x \in {]N - a, N]}$; furthermore, let $n(x)$ be the largest integer n such that $x - na \geq -N$, i.e., $n(x) = \lfloor (N + x)/a \rfloor$. We distinguish between the cases that $n(x) = \lfloor 2N/a \rfloor$ and $n(x) = \lfloor 2N/a \rfloor - 1$.

First assume that $n(x) = \lfloor 2N/a \rfloor$. Then

$$N - a \leq \left\lfloor \frac{2N}{a} \right\rfloor a - N \leq x \leq N, \quad (2.9)$$

and

$$\sum_{n=-\infty}^{\infty} q_N(x - na) = \sum_{n=-\infty}^{-1} \varphi(x - na) + \sum_{n=0}^{n(x)} \varphi(N) + \sum_{n=n(x)+1}^{\infty} \varphi(x - na)$$

$$= \left(\left\lfloor \frac{2N}{a} \right\rfloor + 1 \right) \varphi(N) + \sum_{n=1}^{\infty} \varphi(x + na) + \sum_{n=1}^{\infty} \varphi \left(x - \left\lfloor \frac{2N}{a} \right\rfloor a - na \right)$$

$$= \left(\left\lfloor \frac{2N}{a} \right\rfloor + 1 \right) \varphi(N) + \sum_{n=1}^{\infty} \left[\varphi(x + na) + \varphi \left(\left\lfloor \frac{2N}{a} \right\rfloor a - x + na \right) \right], \quad (2.10)$$

where it has been used that φ is even. In particular this proves the second equality sign in (2.8) in the considered case; so we only need to prove that the supremum is attained for $x = N$. If $\frac{2N}{a}$ is integer we have that $\lfloor \frac{2N}{a} \rfloor = \frac{2N}{a}$, in which case (2.9) shows that the only x-value that can be considered under the given conditions is $x = N$. We will therefore now focus on the case where $\frac{2N}{a}$ is not an integer. In this case $2N - \lfloor \frac{2N}{a} \rfloor a > 0$. Now let $n = 1, 2, \cdots$. We have

$$\left\lfloor \frac{2N}{a} \right\rfloor a - N + na \leq x + na \leq N + na,$$

and

$$\left\lfloor \frac{2N}{a} \right\rfloor a - N + na \leq \left\lfloor \frac{2N}{a} \right\rfloor a - x + na \leq N + na.$$

Hence, letting

$$\lambda := \frac{N + na - (x + na)}{N + na - \left(\left\lfloor \frac{2N}{a} \right\rfloor a - N + na \right)} = \frac{N - x}{2N - \left\lfloor \frac{2N}{a} \right\rfloor a} \in [0, 1],$$

the convexity of φ implies that

$$\varphi(x + na) = \varphi\left(\lambda\left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right) + (1 - \lambda)(N + na)\right)$$

$$\leq \lambda\varphi\left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right) + (1 - \lambda)\varphi(N + na). \quad (2.11)$$

Similarly, letting

$$\mu := \frac{N + na - \left(\left\lfloor\frac{2N}{a}\right\rfloor a - x + na\right)}{N + na - \left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right)} = \frac{N - \left\lfloor\frac{2N}{a}\right\rfloor a + x}{2N - \left\lfloor\frac{2N}{a}\right\rfloor a} \in [0, 1],$$

we see that

$$\varphi\left(\left\lfloor\frac{2N}{a}\right\rfloor a - x + na\right) = \varphi\left(\mu\left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right) + (1 - \mu)(N + na)\right)$$

$$\leq \mu\varphi\left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right) + (1 - \mu)\varphi(N + na). \quad (2.12)$$

Observe that $\lambda + \mu = 1$. Thus we get from (2.11) and (2.12) that

$$\varphi(x + na) + \varphi\left(\left\lfloor\frac{2N}{a}\right\rfloor a - x + na\right)$$

$$\leq \lambda\varphi\left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right) + (1 - \lambda)\varphi(N + na)$$

$$+ (1 - \lambda)\varphi\left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right) + \lambda\varphi(N + na)$$

$$= \varphi\left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right) + \varphi(N + na).$$

Therefore, from (2.10),

$$\sum_{n=-\infty}^{\infty} q_N(x - na) \leq \left(\left\lfloor\frac{2N}{a}\right\rfloor + 1\right)\varphi(N)$$

$$+ \sum_{n=1}^{\infty}\left(\varphi(N + na) + \varphi\left(\left\lfloor\frac{2N}{a}\right\rfloor a - N + na\right)\right)$$

$$= \sum_{n=-\infty}^{\infty} q_N(N - na).$$

This proves (2.8) in the case $n(x) = \lfloor 2N/a \rfloor$, as desired. We now consider the case $n(x) = \lfloor 2N/a \rfloor - 1$. Then

$$N - a < x \leq \left\lfloor\frac{2N}{a}\right\rfloor a - N \leq N,$$

and, in particular, $\lfloor \frac{2N}{a} \rfloor a - 2N + a > 0$. With the interpretation that $\sum_{n=0}^{n(x)} \varphi(N) = 0$ if $n(x) = -1$, we now have that

$$\sum_{n=-\infty}^{\infty} q_N(x - na) = \sum_{n=-\infty}^{-1} \varphi(x - na) + \sum_{n=0}^{n(x)} \varphi(N) + \sum_{n=n(x)+1}^{\infty} \varphi(x - na)$$

$$= \left\lfloor \frac{2N}{a} \right\rfloor \varphi(N) + \sum_{n=1}^{\infty} \varphi(x + na) + \sum_{n=0}^{\infty} \varphi\left(x - \left\lfloor \frac{2N}{a} \right\rfloor a - na\right)$$

$$= \left\lfloor \frac{2N}{a} \right\rfloor \varphi(N) + \sum_{n=0}^{\infty} \left[\varphi(x + (n+1)a) + \varphi\left(\left\lfloor \frac{2N}{a} \right\rfloor a - x + na\right)\right].$$

We compare this to the right-hand side of (2.8), which we write as

$$\left\lfloor \frac{2N}{a} \right\rfloor \varphi(N) + \sum_{n=0}^{\infty} \left[\varphi(N + na) + \varphi\left(\left\lfloor \frac{2N}{a} \right\rfloor a - N + (n+1)a\right)\right].$$

Now for $n = 0, 1, \ldots,$

$$N + na < x + (n+1)a \leq \left\lfloor \frac{2N}{a} \right\rfloor a - N + (n+1)a,$$

and

$$N + na \leq \left\lfloor \frac{2N}{a} \right\rfloor a - x + na < \left\lfloor \frac{2N}{a} \right\rfloor a - N + (n+1)a.$$

Now, let

$$\lambda := \frac{\left\lfloor \frac{2N}{a} \right\rfloor a - N - x}{\left\lfloor \frac{2N}{a} \right\rfloor a - 2N + a}, \quad \mu := \frac{x - N + a}{\left\lfloor \frac{2N}{a} \right\rfloor a - 2N + a};$$

then $0 \leq \lambda, \mu \leq 1$ and $\lambda + \mu = 1$. Hence, writing

$$x + (n+1)a = \lambda(N + na) + (1 - \lambda)\left(\left\lfloor \frac{2N}{a} \right\rfloor a - N + (n+1)a\right),$$

and

$$\left\lfloor \frac{2N}{a} \right\rfloor a - x + na = \mu(N + na) + (1 - \mu)\left(\left\lfloor \frac{2N}{a} \right\rfloor a - N + (n+1)a\right),$$

the convexity of φ implies that

$$\varphi(x + (n+1)a) + \varphi\left(\left\lfloor \frac{2N}{a} \right\rfloor a - x + na\right)$$

$$\leq \varphi(N + na) + \varphi\left(\left\lfloor \frac{2N}{a} \right\rfloor a - N + (n+1)a\right).$$

Hence

$$\sum_{n=-\infty}^{\infty} q_N(x - na) \leq \sum_{n=-\infty}^{\infty} q_N(N - na),$$

and the proof of (2.8) is complete.

Now, to finalize the proof we observe that the convexity condition on φ ensuring the validity of (2.8) does not involve a. Hence, using (2.8) once more with $1/b$ replacing a, we get

$$B(q_N, a, b) \leq \frac{1}{b} \left(\sum_{k=-\infty}^{\infty} q_N \left(N - \frac{k}{b} \right) \right) \left(\sum_{n=-\infty}^{\infty} q_N (N - na) \right), \qquad (2.13)$$

as claimed. □

We will now return to the function in Example 4 and show that the sharper estimates in Proposition 5 indeed improves the result.

Example 6 We consider again the function φ in Example 4 and $a = b = \sqrt{2}/2$. We have that $\varphi''(x) \geq 0 \Leftrightarrow |x| \geq 1/\sqrt{2\pi} = 0.398942280$. We take $N = 1/\sqrt{2\pi}$, so that

$$\left\lfloor \frac{2N}{a} \right\rfloor = \left\lfloor \frac{2}{\sqrt{\pi}} \right\rfloor = 1, \ \lfloor 2Nb \rfloor = \left\lfloor \frac{1}{\sqrt{\pi}} \right\rfloor = 0.$$

Using (2.8), the right-hand side of (2.13) can be written as

$$2 \left(2e^{-\pi N^2} + \sum_{n=1}^{\infty} e^{-\pi \left(N + \frac{1}{2} n \sqrt{2} \right)^2} + \sum_{n=1}^{\infty} e^{-\pi \left(-N + \frac{1}{2}(n+1)\sqrt{2} \right)^2} \right)$$

$$\times \left(e^{-\pi N^2} + \sum_{k=1}^{\infty} e^{-\pi \left(N + k\sqrt{2} \right)^2} + \sum_{k=1}^{\infty} e^{-\pi \left(-N + k\sqrt{2} \right)^2} \right)$$

$$= 1.645277057,$$

and this is just less than the lower frame bound A, which we found in Example 4. Thus $\{E_{mb}T_{na}\varphi_N\}_{m,n\in\mathbb{Z}}$ is indeed a frame for $N = 1/\sqrt{2}$, a conclusion that we could not reach based on Proposition 2.

Plots of the functions $\varphi(x)$, $\varphi_{3/4}(x)$ and $\varphi_{1/\sqrt{2\pi}}(x)$ are shown in Fig. 1. Note that the length of the support of the function $\varphi_{1/\sqrt{2\pi}}$ is $2/\sqrt{2\pi}$, which just exceeds $a = \sqrt{2}/2$. Thus Proposition 5 gives a sharp result for this case; indeed, $\{E_{mb}T_{na}\varphi_N\}_{m,n\in\mathbb{Z}}$ ceases to be a Gabor frame when $2N \leq a$.

We will now show that with particular knowledge of certain relations between the parameters $a, b > 0$ and $N > 0$ we can be more specific about the frame properties of the functions φ_N for $\varphi \in \mathcal{V}$, even without assuming that $\{E_{mb}T_{na}\varphi\}_{m,n\in\mathbb{Z}}$ is a frame.

Proposition 7 *Let $\varphi \in \mathcal{V}$. For $N > 0$, consider the function φ_N in (1.6) and let $a \in (0, 2N), b \in (0, \frac{1}{2N}]$. Then $\{E_{mb}T_{na}\varphi_N\}_{m,n\in\mathbb{N}}$ is a frame for $L^2(\mathbb{R})$ with lower frame bound $A(\varphi_N) = \frac{1}{b} \left| \varphi_N \left(\frac{a}{2} \right) \right|^2$, and the canonical dual window is given by*

$$\tilde{\varphi}_N(x) = \frac{b\varphi_N(x)}{\sum_{n\in\mathbb{Z}} |\varphi_N(x - na)|^2}, \ x \in \mathbb{R}. \qquad (2.14)$$
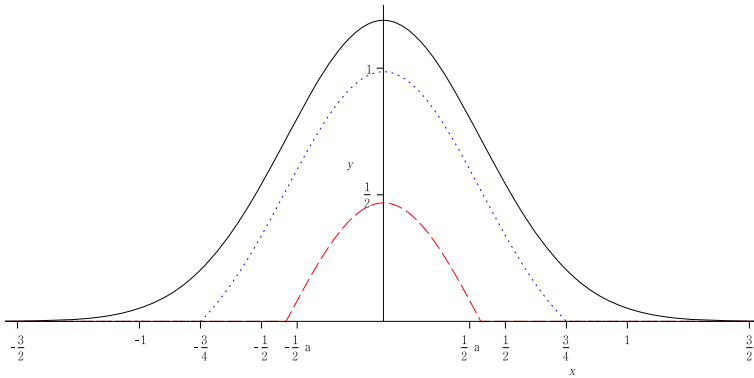
Fig. 1 The function $\varphi(x) = 2^{1/4}e^{-\pi x^2}$ (solid) and the function φ_N for $N = 3/4$ (dotted; see Proposition 2) and for $N = 1/\sqrt{2\pi}$ (dashed; see Proposition 5)

Proof We first show that

$$bA(\varphi_N) \leq \sum_{n \in \mathbb{Z}} |\varphi_N(x - na)|^2 \leq bB(\varphi_N), \ \ x \in \mathbb{R}, \tag{2.15}$$

for some $A(\varphi_N), B(\varphi_N) > 0$. Since φ_N is bounded and compactly supported, the upper bound is clear. Now, since the function $\sum_{n \in \mathbb{Z}} |\varphi_N(x - na)|^2$ is a-periodic, we have

$$\inf_{x \in \mathbb{R}} \sum_{n \in \mathbb{Z}} |\varphi_N(x - na)|^2 = \inf_{x \in \left[-\frac{a}{2}, \frac{a}{2}\right]} \sum_{n \in \mathbb{Z}} |\varphi_N(x - na)|^2 \geq \inf_{x \in \left[-\frac{a}{2}, \frac{a}{2}\right]} |\varphi_N(x)|^2 \,.$$

Since $\varphi \in \mathcal{V}$ is even, it follows that

$$\inf_{x \in \mathbb{R}} \sum_{n \in \mathbb{Z}} |\varphi_N(x - na)|^2 \geq \left|\varphi_N\left(\frac{a}{2}\right)\right|^2 = b\left(\frac{1}{b}\left|\varphi_N\left(\frac{a}{2}\right)\right|^2\right) =: bA(\varphi_N) > 0.$$

Hence φ_N satisfies (2.15). The result now follows by [2, Corollary 11.4.5]. $\qquad\square$

Using Proposition 7 we can now show that for a fixed $a > 0$, any function $\varphi \in \mathcal{V}$ generates a Gabor frame for sufficiently small modulation parameters.

Corollary 8 *Let* $\varphi \in \mathcal{V}$ *and* $a > 0$. *Then* $\{E_{mb}T_{na}\varphi\}_{m,n \in \mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *whenever* $b > 0$ *is sufficiently small.*

Proof Proposition 2 (ii) shows that the Bessel bound $B(\varphi - \varphi_N, a, b)$ tends to zero as $N \to \infty$. On the other hand, by Proposition 7 the function φ_N generates a Gabor frame $\{E_{mb}T_{na}\varphi_N\}_{m,n \in \mathbb{Z}}$ for $N > a/2$, with lower frame bound $A(\varphi_N) = \frac{1}{b}\left|\varphi_N\left(\frac{a}{2}\right)\right|^2$. Since $\varphi_N\left(\frac{a}{2}\right) \to g\left(\frac{a}{2}\right) > 0$, as $N \to \infty$, the result now follows from standard results in frame perturbation theory (see Corollary 22.1.5 in [2]). $\qquad\square$

3 Approximately dual frames for $\varphi \in \mathcal{V}$

Even when a function $\varphi \in \mathcal{V}$ generates a frame $\{E_{mb}T_{na}\varphi\}_{m,n\in\mathbb{Z}}$, information about its dual frames might not be easily available. Thus, we do not have immediate access to apply the perfect reconstruction formula associated with a pair of dual frames. The purpose of this section is to provide a method for obtaining almost perfect reconstruction by using the fact that explicit dual frames associated to the truncated windows φ_N in (1.6) can be calculated for certain parameter values $a, b > 0$, see Proposition 7. Note that Theorem 9 below is formulated via a frame condition on the truncated function φ_N, not on the function $\varphi \in \mathcal{V}$. This formulation matches the actual application to various concrete functions that will be given in Corollary 11.

Theorem 9 *Let a, b be given such that $ab \in (0, 1)$. Given a function $\varphi \in \mathcal{V}$ and $N \in \mathbb{N}$, assume that $\{E_{mb}T_{na}\varphi_N\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ and let $\{E_{mb}T_{na}\varphi_N^\sharp\}_{m,n\in\mathbb{Z}}$ denote a dual frame with upper bound $B(\varphi_N^\sharp)$. Then*

$$\left\| f - \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}\varphi \rangle E_{mb}T_{na}\varphi_N^\sharp \right\| \leq \sqrt{\frac{1}{b} \, Q_N(b)Q_N\left(\frac{1}{a}\right)B(\varphi_N^\sharp)} \, \|f\|, \; \forall f \in L^2(\mathbb{R}).$$

Proof We apply Lemma 1 to the dual frames

$$\{f_k\}_{k\in I} := \{E_{mb}T_{na}\,\varphi_N\}_{m,n\in\mathbb{Z}}, \; \{g_k\}_{k\in I} := \{E_{mb}T_{na}\,\varphi_N^\sharp\}_{m,n\in\mathbb{Z}}.$$

Letting $\{\widetilde{f_k}\}_{k\in I} := \{E_{mb}T_{na}\,\varphi\}_{m,n\in\mathbb{Z}}$, the condition (1.4) is satisfied with $\mu^2 = B(\varphi - \varphi_N, a, b)$. Thus, the factor $\mu\sqrt{B_g}$ in (1.5) corresponds precisely to $\left(B(\varphi - \varphi_N, a, b)B(\varphi_N^\sharp)\right)^{1/2}$. The result now follows by Lemma 1 and Proposition 2. $\square$

Remark 10 Note that if $\varphi \in \mathcal{V}$ generates a Gabor frame $\{E_{mb}T_{na}\,\varphi\}_{m,n\in\mathbb{Z}}$, then Corollary 3 shows that $\{E_{mb}T_{na}\varphi_N\}_{m,n\in\mathbb{Z}}$ is a frame for $N \in \mathbb{N}$ sufficiently large; thus the condition in Theorem 9 is satisfied. In our applications of Theorem 9 we will take $\varphi_N^\sharp$ to be the canonical dual window associated with the window φ_N. In this case $\{E_{mb}T_{na}\varphi_N^\sharp\}_{m,n\in\mathbb{Z}}$ has the upper bound $B(\varphi_N^\sharp) = A(\varphi_N)^{-1}$, where $A(\varphi_N)$ denotes a lower frame bound for the frame $\{E_{mb}T_{na}\varphi_N\}_{m,n\in\mathbb{Z}}$.

For a fixed function $\varphi \in \mathcal{V}$ and fixed parameters $a, b > 0$ and $N \in \mathbb{N}$, Theorem 9 yields an explicit estimate of the deviation from perfect reconstruction that is obtained by doing analysis with the Gabor system $\{E_{mb}T_{na}\,\varphi\}_{m,n\in\mathbb{Z}}$ and synthesis using a dual frame $\{E_{mb}T_{na}\,\varphi_N^\sharp\}_{m,n\in\mathbb{Z}}$. We will now consider the particular case where the synthesis is done using the canonical dual frame $\{E_{mb}T_{na}\,\widetilde{\varphi}_N\}_{m,n\in\mathbb{Z}}$ and show that by allowing the modulation parameter b to vary we can get as close to perfect reconstruction as desired.

Corollary 11 *Let $a \in (0, \infty)$ and let $\{b(N)\}_{N=1}^\infty \subset \mathbb{R}$ be a sequence such that $0 < b(N) \leq \frac{1}{2N}$. Given a function $\varphi \in \mathcal{V}$, consider the functions φ_N in (1.6) and*

$\widetilde{\varphi}_N$ in (2.14) for any integer $N > a/2$ and with $b = b(N)$. Then, with the constants $C, \sigma > 0$ chosen as in Definition 1,

$$\left\| f - \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb(N)} T_{na}\varphi \rangle E_{mb(N)} T_{na}\tilde{\varphi}_N \right\| \leq \frac{C_1(N)}{N^{\frac{1}{2}+\sigma}} \|f\|, \quad \forall f \in L^2(\mathbb{R}), \quad (3.1)$$

where

$$C_1(N) := \frac{2C}{\varphi_N\left(\frac{a}{2}\right)} \sqrt{\left(4 + \frac{1}{\sigma}\right)\left(\frac{2}{a}\left(1 + \frac{1}{\sigma}\right) + \frac{3}{N}\right)}. \qquad (3.2)$$

Proof Proposition 2 (ii) implies that

$$\frac{1}{b(N)} Q_N(b(N)) Q_N(\frac{1}{a}) \leq \frac{C^2}{b(N)N^{1+2\sigma}} \left(4 + \frac{1}{\sigma}\right)\left(\frac{2}{a}\left(1 + \frac{1}{\sigma}\right) + \frac{3}{N}\right). \quad (3.3)$$

Considering now $N \in \mathbb{N}$ such that $N > a/2$ and using that $0 < b(N) \leq \frac{1}{2N}$, Proposition 7 shows that $\{E_{mb(N)} T_{na}\varphi_N\}_{m,n \in \mathbb{Z}}$ is a Gabor frame with the canonical dual frame $\{E_{mb(N)} T_{na}\tilde{\varphi}_N\}_{m,n \in \mathbb{Z}}$. Furthermore, a lower frame bound is $A(\varphi_N) = \frac{|\varphi_N(\frac{a}{2})|^2}{b(N)}$. This together with (3.3) implies that

$$\sqrt{\frac{1}{b(N)} Q(b(N)) Q\left(\frac{1}{a}\right) \frac{1}{A(\varphi_N)}} \leq \frac{C_1(N)}{N^{\frac{1}{2}+\sigma}},$$

where $C_1(N)$ is defined by (3.2). Now (3.1) is a consequence of Theorem 9 and Remark 10. □

Note that Corollary 11 immediately applies to the "canonical functions" in $\mathcal{V}$, i.e.,

$$\varphi(x) = \frac{1}{1 + |x|^{1+\sigma}},$$

for $\sigma > 0$. In the case $\sigma = 1$ this function corresponds (up to a scalar multiple of the variable) to the Fourier transform of the two-sided exponential function $g(x) = e^{-|x|}$, which is known to generate a Gabor frame for all parameter values $a, b > 0$ with $ab < 1$, see [9]. It follows that also the function $\varphi(x) = (1 + x^2)^{-1}$ generates a Gabor frame for all $ab < 1$.

In the next result we improve the estimate in Corollary 11 for two standard functions, namely, the Gaussian and the two-sided exponential function. In order to get as close as desired to perfect reconstruction, the parameter N has to be chosen sufficiently large; the result shows how N depends on the chosen function $\varphi \in \mathcal{V}$.

Corollary 12 Let $a \in (0, \infty)$ and let $\{b(N)\}_{N=1}^{\infty} \subset \mathbb{R}$ be a sequence such that $0 < b(N) \leq \frac{1}{2N}$. Given a function $\varphi \in \mathcal{V}$, consider again the functions φ_N in (1.6) and $\widetilde{\varphi}_N$ in (2.14). for any integer $N > a/2$ and with $b = b(N)$.

(1) Consider $\varphi(x) = e^{-|x|}$, $x \in \mathbb{R}$. Then

$$\left\| f - \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb(N)} T_{na} \varphi \rangle E_{mb(N)} T_{na} \tilde{\varphi}_N \right\| \leq C_2(N) \sqrt{N} e^{-N} \| f \|, \ \forall f \in L^2(\mathbb{R}),$$

where

$$C_2(N) := \frac{e^{\frac{a}{2}}}{\left(1 - e^{-N + \frac{a}{2}}\right)} \sqrt{\left(2 + \frac{2}{1 - e^{-2N}}\right) \left(\frac{2}{a} + \frac{1}{N} \left(\frac{3 - e^{-a}}{1 - e^{-a}}\right)\right)};$$

(2) Consider $\varphi(x) = e^{-x^2}$, $x \in \mathbb{R}$. Then

$$\left\| f - \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb(N)} T_{na} \varphi \rangle E_{mb(N)} T_{na} \tilde{\varphi}_N \right\| \leq C_3(N) \sqrt{N} e^{-N^2} \| f \|, \ \forall f \in L^2(\mathbb{R}),$$

where

$$C_3(N) := \frac{e^{\frac{a^2}{4}}}{\left(1 - e^{-N^2 + \frac{a^2}{4}}\right)} \sqrt{\left(2 + \frac{2}{1 - e^{-4N^2}}\right) \left(\frac{2}{a} + \frac{1}{N} \left(\frac{3 - e^{-2Na}}{1 - e^{-2Na}}\right)\right)}.$$

Proof For (1), applying (1.7) and $b(N) \leq \frac{1}{2N}$ yields that

$$Q_N(b(N)) = (\lfloor 2b(N)N \rfloor + 1)\varphi(N) + 2 \sum_{k=0}^{\infty} \varphi\left(N + \frac{k}{b(N)}\right)$$

$$\leq 2\varphi(N) + 2 \sum_{k=0}^{\infty} \varphi(N + 2Nk) = e^{-N}\left(2 + \frac{2}{1 - e^{-2N}}\right).$$

Similarly,

$$Q_N(\frac{1}{a}) = \left(\left\lfloor \frac{2N}{a} \right\rfloor + 1\right)\varphi(N) + 2 \sum_{k=0}^{\infty} \varphi(N + ak)$$

$$\leq \left(\frac{2N}{a} + 1\right)e^{-N} + \frac{2e^{-N}}{1 - e^{-a}} = Ne^{-N}\left(\frac{2}{a} + \frac{1}{N}\left(\frac{3 - e^{-a}}{1 - e^{-a}}\right)\right).$$

Proposition 7 shows that $\{E_{mb(N)} T_{na} \varphi_N\}_{m,n \in \mathbb{Z}}$ is a frame with canonical dual frame $\{E_{mb(N)} T_{na} \tilde{\varphi}_N\}_{m,n \in \mathbb{Z}}$ and lower frame bound

$$A(\varphi_N) = \frac{\left| \varphi_N\left(\frac{a}{2}\right) \right|^2}{b(N)} = \frac{\left(g\left(\frac{a}{2}\right) - g(N)\right)^2}{b(N)} = \frac{e^{-a}\left(1 - e^{-N + \frac{a}{2}}\right)^2}{b(N)};$$

as in Corollary 11 the result now follows from Theorem 9 and Remark 10.

For (2), we use the estimates

$$Q_N(b(N)) = (\lfloor 2b(N)N \rfloor + 1)\varphi(N) + 2\sum_{k=0}^{\infty} \varphi\left(N + \frac{k}{b(N)}\right)$$

$$\leq 2\varphi(N) + 2\sum_{k=0}^{\infty} \varphi\left(N + 2Nk\right)$$

$$\leq 2e^{-N^2} + 2\sum_{k=0}^{\infty} e^{-N^2 - 4N^2 k} = e^{-N^2}\left(2 + \frac{2}{1 - e^{-4N^2}}\right),$$

and

$$Q_N(\tfrac{1}{a}) = \left(\left\lfloor \frac{2N}{a} \right\rfloor + 1\right)\varphi(N) + 2\sum_{k=0}^{\infty} \varphi\left(N + ak\right)$$

$$\leq \left(\frac{2N}{a} + 1\right)e^{-N^2} + 2\sum_{k=0}^{\infty} e^{-N^2 - 2aNk} = Ne^{-N^2}\left(\frac{2}{a} + \frac{1}{N}\left(\frac{3 - e^{-2Na}}{1 - e^{-2Na}}\right)\right).$$

Again $\{E_{mb(N)}T_{na}\varphi_N\}_{m,n\in\mathbb{Z}}$ is a frame with canonical dual frame $\{E_{mb(N)}T_{na}\tilde{\varphi}_N\}_{m,n\in\mathbb{Z}}$ and lower frame bound

$$A(\varphi_N) = \frac{\left|\varphi_N\left(\frac{a}{2}\right)\right|^2}{b(N)} = \frac{\left(g\left(\frac{a}{2}\right) - g(N)\right)^2}{b(N)} = \frac{e^{-\frac{a^2}{2}}\left(1 - e^{-N^2 + \frac{a^2}{4}}\right)^2}{b(N)},$$

and the conclusion follows as before. □

Remark 13 The 2 standard functions in Corollary 12 are both convex on $[N, \infty)$ when $N \geq 1$, and so the bound $\frac{1}{b}Q_N(b)Q_N(\frac{1}{a})$ for $B(q_N, a, b)$ that is used in the proof of Theorem 9 can be replaced by a sharper bound according to Proposition 5. This yields sharper bounds in Corollary 12 (1) and (2).

References

1. Bui, H., Laugesen, R.: Frequency-Scale Frames and the Solution of the Mexican Hat Problem. Constr. Approx. **33**, 163–189 (2011)
2. Christensen, O.: An Introduction to Frames and Riesz Bases. Second expanded edition. Birkhäuser, Boston (2016)
3. Christensen, O., Kim, H.O., Kim, R.Y.: B-spline approximations of the Gaussian, their Gabor frame properties, and approximately dual frames. J. Fourier Anal. Appl. accepted for publication. https://doi.org/10.1007/s00041-017-9557-3
4. Christensen, O., Laugesen, R.: Approximately dual frames in Hilbert spaces and applications to Gabor frames. Sampl. Theory Signal Image Process. **9**, 77–90 (2011)
5. Dörfler, M., Matusiak, E.: Nonstationary Gabor frames - approximately dual frame and reconstruction errors. Adv. Comput. Math. **41**, 293–316 (2015)
6. Feichtinger, H.G., Grybos, A., Onchis, D.M.: Approximate dual Gabor atoms via the adjoint lattice method. Adv. Comput. Math. **40**, 651–665 (2014)

7. Feichtinger, H.G., Onchis, D.M., Wiesmeyr, C.: Construction of approximate dual wavelet frames. Adv. Comput. Math. **40**, 273–282 (2014)
8. Janssen, A.J.E.M.: Some Weyl-Heisenberg frame bound calculations. Indag. Math. **7**, 165–183 (1996)
9. Janssen, A.J.E.M.: On generating tight Gabor frames at critical density. J. Fourier Anal. Appl. **9**, 175–214 (2003)