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A fractional spectral method with applications to some singular problems

Dianming Hou[1] · **Chuanju Xu**[1]

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Abstract In this paper we propose and analyze fractional spectral methods for a class of integro-differential equations and fractional differential equations. The proposed methods make new use of the classical fractional polynomials, also known as Müntz polynomials. We first develop a kind of fractional Jacobi polynomials as the approximating space, and derive basic approximation results for some weighted projection operators defined in suitable weighted Sobolev spaces. We then construct efficient fractional spectral methods for some integro-differential equations which can achieve spectral accuracy for solutions with limited regularity. The main novelty of the proposed methods is that the exponential convergence can be attained for any solution $u(x)$ with $u(x^{1/\lambda})$ being smooth, where λ is a real number between 0 and 1 and it is supposed that the problem is defined in the interval $(0, 1)$. This covers a large number of problems, including integro-differential equations with weakly singular kernels, fractional differential equations, and so on. A detailed convergence analysis is carried out, and several error estimates are established. Finally a series of numerical examples are provided to verify the efficiency of the methods.

Keywords Integro-differential equations · Fractional differential equations · Singularity · Müntz polynomials · Fractional Jacobi polynomials · Weighted Sobolev spaces · Spectral accuracy

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✉ Chuanju Xu
cjxu@xmu.edu.cn

1 School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling and High Performance Scientific Computing, Xiamen University, Xiamen, 361005, China

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1 Introduction

Spectral methods are essentially discretization methods for approximating solution of partial-differential equations. The most attractive property of spectral methods may be that when the solution of the problem is infinitely smooth, the convergence of the spectral method is exponential. Due to this advantage, spectral methods have achieved great success and become popular in the scientific computing community. On the other side, classical spectral methods meet also some limits, such as loss of global accuracy when facing problems with non-smooth/singular solution. Such problems can be found, for example, in integro-differential equations with singular kernels, fractional partial differential equations, traditional equations with singular data, and so on.

Integro-differential equations arise in mathematical models of many disciplines such as electromagnetic scattering, biological, and etc. They have been subject of many theoretical and numerical investigations; see, e.g., [8, 10–12, 14, 15, 37, 42–45, 47–49]. Some of these work address spectral-like methods, such as polynomial spline collocation methods for Volterra integro-differential equations [10, 12, 37, 45], spectral methods for Volterra integral equations [14, 15, 44], Sinc functions for linear Volterra integro-differential equations [49], and Legendre spectral methods for second order Volterra integro-differential equations [47, 48]. Very few works have touched spectral approximations to integro-differential equations with weakly singular kernels [10, 12, 14, 48]. It is known that the solutions of the weakly singular integro-differential equations have limited regularity at the end points, and so far there are no spectral methods constructed for such problems with spectral accuracy for non-smooth solutions.

Singular solutions appear also in fractional partial differential equations (FPDEs), which are generalizations of the integer-order models, based on fractional calculus. They are becoming increasingly popular as a flexible modeling tool for diffusive processes associated with the so-called anomalous diffusion. FPDEs have been applied to diverse fields, including control theory, biology, electrochemical processes, porous media, viscoelastic materials, polymer, finance, and etc; see, e.g., [1, 2, 4–6, 20, 23, 27, 28, 33, 35, 36] and the references therein. The main feature/difficulty of FPDEs lies in three facts: 1) the solutions of the associated problems have limited regularity at the end points; 2) fractional derivative is non-local operator; 3) the kernel function in fractional derivative is singular. Not mention advances in theoretical aspects and other numerical methods (mostly finite differences and finite elements), there exist limited but promising efforts in developing spectral methods for FPDEs; see, e.g., [13, 29–31, 40, 50, 51]. In particular, Zayernouri and Karniadakis [50, 51] introduced the so-called polyfractonomials, which are eigenfunctions of a fractional Sturm-Liouville problem, and proposed to use these functions as approximation to the solution. Applications to some model equations showed the efficiency of this new approximation, but there is no theoretical analysis available for the approximation error. More recently, Chen et al. [13] considered a class of generalized Jacobi functions related to fractional calculus, and proposed to use these functions to approximate the solutions of a class of fractional boundary value problems. The proposed scheme was based on Petrov-Galerkin methods, and leads to sparse linear systems using suitable orthogonal basis functions. Error estimates with convergence rate only depending on the smoothness of data were derived therein. The limitation of their approach is that it requires an anticipated knowledge of the solution form, i.e., the exact solution must behave like $(1-x)^{\alpha}g(x)$ in $[-1,1]$, where $g(x)$ is a smooth function. More precisely, in order to get exponential convergence, the fractional power α must be exactly known in advance, which is needed in the construction of approximation spaces.

This paper aims at providing a fractional spectral method that is capable to handle a family of the aforementioned problems in a more efficient way. We will develop a kind of fractional Jacobi polynomials as the approximating space, which achieve spectral convergence for the solution with limited regularity at the end points. More precisely, we will consider at this first phase applications of the proposed method to the following three types of equations:

1) integro-differential equation $u_t = a_1u(t) + a_2\, {}_0I_t^{\mu}u(t) + f(t)$, $\mu > 0$;
2) fractional differential equation $bu(x) - D_x^{\rho}u(x) = f(x)$, $1 < \rho < 2$;
3) classical elliptic equation with singular forcing $-\partial_x^2 u(x) = f(x)$.

The advantage of our approach is that the exponential convergence can be guaranteed for any solution $u(x)$ with $u(x^{1/\lambda})$ being smooth enough, where $0 < \lambda \le 1$ is a parameter related to the approximating space. We are going to see that by suitably choosing λ, our method can solve a large number of problems with mixed singularities of distinct types. The proposed fractional spectral method makes use of the fractional polynomials, also known as Müntz polynomials, that have originally appeared in approximation theory [9, 32, 34]. A particular class of Müntz polynomials, i.e., fractional Jacobi polynomials, has been used in a few work. In [3, 19, 26] the authors used a fractional Jacobi polynomial spectral Tau method for fractional differential equations and constructed operational matrix for fractional derivatives. However, there is neither numerical analysis available for the approximation quality of the proposed schemes therein, nor discussion about the significant advantages of the fractional Jacobi polynomial based methods. Very recently, Shen and Wang [39] considered a kind of Müntz functions to construct a Galerkin method to solve the Poisson equation and obtained optimal error estimates.

The purpose of this paper is to set up a framework for solving a kind of integro-differential equations and some related problems using the fractional polynomials. The main ingredients of the paper are:

- In the first part, we derive necessary error estimates for approximation in suitably weighted spaces using fractional polynomials. In particular, we introduce some weighted projection operators in the fractional Jacobi polynomial spaces, and establish approximation results in related norms. Some complementary properties of the fractional Jacobi polynomials are also provided.

– In the second part, we construct efficient fractional spectral methods for a class of integro-differential equations with singular kernels and fractional differential equations. Error estimates for the proposed approaches are derived by employing the results obtained in the first part for the fractional polynomial approximations. The obtained estimates show that spectral accuracy can be achieved for a large number of integro-differential equations with singular kernels.
– Finally, a series of numerical experiments are carried out to verify the theoretical claims. Moreover, we numerically demonstrate that the proposed fractional spectral method can equally treat non smooth solutions for some classical (integer order) equations with spectral accuracy.

The paper is organized as follows: In the next section, we give minimum preparatory materials of fractional polynomials, and present necessary properties of fractional Jacobi polynomials. In Section 3, we establish the approximation results for several projection operators defined in fractional Jacobi polynomial spaces. In Section 4, we construct fractional spectral methods for a class of integro-differential equations, fractional differential equations, as well as classical elliptical equations. Error analysis are conducted for integro-differential equations. Some implementation details are presented in Section 5. The numerical examples are given in Section 6. Finally we give some concluding remarks in Section 7.

2 Notations and fractional polynomials

The well-known Weierstrass theorem states that every continuous function on a compact interval can be uniformly approximated by algebraic polynomials. This result was generalized by Müntz, who proved that the Müntz polynomials of the form $\sum_{k=0}^{n} a_k x^{\lambda_k}$ with real coefficients are dense in $L^2[0,1]$ if and only if $\sum_{k=0}^{\infty} \lambda_k^{-1} = +\infty$, where $\{\lambda_0, \lambda_1, \lambda_2, \ldots\}$ is a sequence of distinct positive numbers such that $0 \le \lambda_0 < \lambda_1 < \ldots \to \infty$. This generalization is usually called Müntz-Szász theorem [18, 34]. If the constant function 1 belongs to the system, that is if $\lambda_0 = 0$, the same result holds for $C[0,1]$ with the uniform norm.

2.1 Fractional integrals and derivatives

Firstly, we review some basic definitions of fractional calculus [17, 35].

Left and right Riemann-Liouville fractional integrals of order $\mu \in (0, \infty)$, denoted by ${}_aI_x^\mu$ and ${}_xI_b^\mu$ for $a < x < b$ respectively, are defined by

$$
\begin{aligned}
{}_aI_x^\mu v(x) &= \frac{1}{\Gamma(\mu)} \int_a^x (x-s)^{\mu-1} v(s)ds, \\
{}_xI_b^\mu v(x) &= \frac{1}{\Gamma(\mu)} \int_x^b (s-x)^{\mu-1} v(s)ds, \qquad (2.1)
\end{aligned}
$$

where $\Gamma(\cdot)$ is the Gamma function.

Left and right Riemann-Liouville fractional derivatives of order $\mu \in (0, \infty)$, denoted by ${}_a^{RL}D_x^{\mu}$ and ${}_x^{RL}D_b^{\mu}$ respectively, are defined as

$$ {}_a^{RL}D_x^{\mu} v(x) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_a^x (x-s)^{n-\mu-1} v(s) ds, \tag{2.2} $$

$$ {}_x^{RL}D_b^{\mu} v(x) = \frac{(-1)^n}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_x^b (s-x)^{n-\mu-1} v(s) ds, \tag{2.3} $$

where n is the integer number such that $n-1 \leq \mu < n$.

Left and right Caputo fractional derivatives of order $\mu \in (0, \infty)$, denoted by ${}_a^{C}D_x^{\mu}$ and ${}_x^{C}D_b^{\mu}$ respectively, are defined as

$$ {}_a^{C}D_x^{\mu} v(x) = \frac{1}{\Gamma(n-u)} \int_a^x (x-s)^{n-\mu-1} v^{(n)}(s) ds, \tag{2.4} $$

$$ {}_x^{C}D_b^{\mu} v(x) = \frac{(-1)^n}{\Gamma(n-\mu)} \int_x^b (s-x)^{n-\mu-1} v^{n}(s) ds, \tag{2.5} $$

where n is the integer number such that $n-1 \leq \mu < n$.

Riemann-Liouville fractional derivatives are not equal to Caputo derivatives, however they are linked by the following relationship

$$ {}_a^{C}D_x^{\mu} v(x) = {}_a^{RL}D_x^{\mu} v(x) - \sum_{k=0}^{n-1} \frac{v^{(k)}(a)}{\Gamma(k+1-\mu)} x^{k-\mu}. \tag{2.6} $$

2.2 Recurrence of orthogonal polynomials

Let $\omega = \omega(x)$ be a weight function on the interval (a, b), which is a non-negative integrable function defined in (a, b). Define the inner product:

$$ (u, v)_{\omega} = \int_a^b u(x) v(x) \omega(x) dx. \tag{2.7} $$

The numbers

$$ M_r := \int_a^b x^r \omega(x) dx, \quad r = 0, 1, 2, ..., \tag{2.8} $$

are called the moments related to the weight function $\omega(x)$. For any positive weight function $\omega \in L^1(a, b)$, there exist uniquely determined monic polynomials p_k of degree k, which are orthogonal to each other with respect to the weighted inner product (2.7). These polynomials satisfy a three-term recurrence relation

$$ p_{k+1}(x) = (t - \alpha_k) p_k(x) - \beta_k p_{k-1}(x), \quad k = 0, 1, 2, \ldots $$

with $p_{-1} = 0, \quad p_0 = 1$, and the coefficients

$$ \alpha_k = \frac{(t p_k, p_k)_{\omega}}{(p_k, p_k)_{\omega}}, \quad k = 0, 1, 2, \ldots, $$

$$ \beta_k = \frac{(p_k, p_k)_{\omega}}{(p_{k-1}, p_{k-1})_{\omega}}, \quad k = 1, 2, \ldots. $$

The above three-term recurrence relation gives a convenient way to numerically calculate the orthogonal polynomials. For the classical orthogonal polynomials, e.g. Jacobi, Laguerre, and Hermite polynomials, formulae for the coefficients α_k and β_k are known in closed form [22, 41]. For the nonclassical weight functions, their recurrence coefficients are not explicitly known. In this case, numerical techniques such as Stieltjes procedure or Chebyshev algorithm are used to evaluate the coefficients [21]. In this paper, we use the Chebyshev algorithm to calculate the desired coefficients from the moments of the underlying weight function $\omega(x)$. For the readers convenience, the Chebyshev algorithm is presented below [22].

Chebyshev Algorithm

1. Initialization:

$$\begin{aligned}
&\alpha_0 = \frac{M_1}{M_0}, \quad \beta_0 = M_0, \\
&\sigma_{-1,l} = 0, \quad l = 1, 2, \ldots, 2n-2, \\
&\sigma_{0,l} = M_l, \quad l = 0, 1, \ldots, 2n-1.
\end{aligned}$$

2. Loop for $n > 1$: for $k = 1, 2, \ldots, n-1$, do

$$\begin{aligned}
&\sigma_{k,l} = \sigma_{k-1,l+1} - \alpha_{k-1}\sigma_{k-1,l} - \beta_{k-1}\sigma_{k-2,l}, \quad l = k, k+1, \ldots, 2n-k-1, \\
&\alpha_k = \frac{\sigma_{k,k+1}}{\sigma_{k,k}} - \frac{\sigma_{k-1,k}}{\sigma_{k-1,k-1}}, \\
&\beta_k = \frac{\sigma_{k,k}}{\sigma_{k-1,k-1}}.
\end{aligned}$$

The complexity of the algorithm is $O(n^2)$.

2.3 Jacobi polynomials

The well-known Jacobi polynomials $\left\{J_k^{\alpha,\beta}(x)\right\}_{k=0}^{\infty}$ are orthogonal with respect to the weight function $\omega^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}$ with $\alpha, \beta > -1$ over $\Lambda := (-1, 1)$:

$$\int_{-1}^{1} \omega^{\alpha,\beta}(x) J_n^{\alpha,\beta}(x) J_m^{\alpha,\beta}(x) dx = \gamma_n^{\alpha,\beta} \delta_{m,n}, \tag{2.9}$$

where

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

The special case $\alpha = \beta = 0$ corresponds to the Legendre polynomials, denoted by $\{L_n(x)\}_{n=0}^{\infty}$ hereafter, and $\alpha = \beta = -\frac{1}{2}$ yields Chebyshev polynomials up to a constant depending on n.

A very useful property of the Jacobi polynomials is the following Sturm-Liouville equation:

$$(\omega^{\alpha,\beta}(x))^{-1} \frac{d}{dx} \left\{(1-x^2)\omega^{\alpha,\beta}(x) \frac{d}{dx} J_n^{\alpha,\beta}(x)\right\} = -n(n+\alpha+\beta+1) J_n^{\alpha,\beta}(x). \tag{2.10}$$

From (2.9) and (2.10), we have

$$\int_{-1}^{1} \omega^{\alpha+1,\beta+1}(x)(J_n^{\alpha,\beta}(x))'(J_m^{\alpha,\beta}(x))'dx = \sigma_n^{\alpha.\beta}\gamma_n^{\alpha,\beta}\delta_{m,n}, \tag{2.11}$$

where

$$\sigma_n^{\alpha,\beta} = n(n+\alpha+\beta+1). \tag{2.12}$$

We define the operator A by

$$A\phi = -(\omega^{\alpha,\beta}(x))^{-1}\frac{d}{dx}\left\{(1-x^2)\omega^{\alpha,\beta}(x)\frac{d}{dx}\phi\right\}. \tag{2.13}$$

By using integration by parts it is seen that A is a self-adjoint operator, i.e.,

$$(A\phi,\varphi)_{\omega^{\alpha,\beta}(x)} = (\phi, A\varphi)_{\omega^{\alpha,\beta}(x)}, \quad \forall \phi,\varphi \in \{v : v, Av \in L^2_{\omega^{\alpha,\beta}(x)}(\Lambda)\}, \tag{2.14}$$

and it satisfies the following stability inequality:

$$\|A\phi\|_{0,\omega^{\alpha,\beta}(x)} \le c\|\phi\|_{2,\omega^{\alpha,\beta}(x)}, \quad \forall \phi \in H^2_{\omega^{\alpha,\beta}(x)}(\Lambda). \tag{2.15}$$

In practice, one can compute the Jacobi polynomials using the following three-term recurrence relation:

$$\begin{aligned} J_0^{\alpha,\beta}(x) &= 1, \quad J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta), \\ J_{n+1}^{\alpha,\beta}(x) &= (a_n^{\alpha,\beta}x - b_n^{\alpha,\beta})J_n^{\alpha,\beta}(x) - c_n^{\alpha,\beta}J_{n-1}^{\alpha,\beta}(x), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} a_n^{\alpha,\beta} &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \\ b_n^{\alpha,\beta} &= \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \\ c_n^{\alpha,\beta} &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \end{aligned}$$

We list below some more properties to be used in what follows.

Lemma 2.1 (see [41] p62-67)

$$\begin{aligned} J_n^{\alpha,\beta}(x) &= (-1)^n J_n^{\beta,\alpha}(-x); \\ J_n^{\alpha,\beta}(1) &= \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}; \\ J_n^{\alpha,\beta}(-1) &= (-1)^n\frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(n+1)}. \end{aligned} \tag{2.17}$$

Lemma 2.2 (see [38] Theorem 3.19) *The Jacobi polynomials satisfy*

$$(1-x)J_n^{\alpha+1,\beta}(x) = \frac{2}{2n+\alpha+\beta+2}\left[(n+\alpha+1)J_n^{\alpha,\beta}(x) - (n+1)J_{n+1}^{\alpha,\beta}(x)\right],$$

$$(1+x)J_n^{\alpha,\beta+1}(x) = \frac{2}{2n+\alpha+\beta+2}\left[(n+\beta+1)J_n^{\alpha,\beta}(x) + (n+1)J_{n+1}^{\alpha,\beta}(x)\right].$$

Lemma 2.3 ([51] Lemma 3.2 and [13] Lemma 2.4) *Let $\mu > 0, x \in \Lambda$ and n be nonnegative integer number.*
(i) For $\alpha \in R, \beta > -1$, it holds

$$ {}_{-1}I_x^{\mu}\{(1+x)^{\beta} J_n^{\alpha,\beta}(x)\} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\mu+1)}(1+x)^{\beta+\mu} J_n^{\alpha-\mu,\beta+\mu}(x). \tag{2.18} $$

(ii) For $\alpha > -1, \beta \in R$, it holds

$$ {}_xI_1^{\mu}\{(1-x)^{\alpha} J_n^{\alpha,\beta}(x)\} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\mu+1)}(1-x)^{\alpha+\mu} J_n^{\alpha+\mu,\beta-\mu}(x). \tag{2.19} $$

2.4 Fractional Jacobi polynomials

Definition 2.1 The fractional Jacobi polynomials of degree n are defined on $I := [0, 1]$ as

$$ J_n^{\alpha,\beta,\lambda}(x) = J_n^{\alpha,\beta}(2x^{\lambda}-1), \ \forall x \in I, \tag{2.20} $$

where $J_n^{\alpha,\beta}(x)$ denotes Jacobi polynomial of degree n, and $\alpha, \beta > -1, 0 < \lambda \le 1$.

When $\lambda = 1$, the polynomials $\{J_n^{\alpha,\beta,1}(x)\}_{n=0}^{\infty}$ are called shifted Jacobi polynomials, which are orthogonal polynomials with the weight $(1-x)^{\alpha}x^{\beta}$.

From Lemma 2.1, we have

$$ \begin{aligned} J_n^{\alpha,\beta,\lambda}(0) &= J_n^{\alpha,\beta}(-1) = (-1)^n \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(n+1)}, \\ J_n^{\alpha,\beta,\lambda}(1) &= J_n^{\alpha,\beta}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}. \end{aligned} \tag{2.21} $$

Let's denote

$$ \omega^{\alpha,\beta,\lambda}(x) := \lambda(1-x^{\lambda})^{\alpha} x^{(\beta+1)\lambda-1}. \tag{2.22} $$

Lemma 2.4 *The fractional Jacobi polynomials $J_n^{\alpha,\beta,\lambda}(x)$ are orthogonal with respect to the weight function $\omega^{\alpha,\beta,\lambda}(x)$, $\alpha, \beta > -1, 0 < \lambda \le 1$ over I, i.e.,*

$$ \int_0^1 \omega^{\alpha,\beta,\lambda}(x) J_n^{\alpha,\beta,\lambda}(x) J_m^{\alpha,\beta,\lambda}(x)dx = \hat{\gamma}_n^{\alpha,\beta}\delta_{m,n}, \tag{2.23} $$

where

$$ \hat{\gamma}_n^{\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}. $$

The special case $\alpha = 0, \beta = \frac{1}{\lambda} - 1$ yields the Müntz Legendre polynomials, which have been defined in a different way in [32] and [9].

In virtue of (2.10) and (2.20), we have the following lemma.

Lemma 2.5 *The fractional Jacobi polynomials* $\left\{J_n^{\alpha,\beta,\lambda}\right\}_{n=0}^{\infty}$ *satisfy the following Sturm-Liouville problem:*

$$(\omega^{\alpha,\beta,\lambda}(x))^{-1}\frac{d}{dx}\left\{\lambda^{-1}(1-x^{\lambda})^{\alpha+1}x^{\beta\lambda+1}\frac{d}{dx}J_n^{\alpha,\beta,\lambda}(x)\right\}=-\sigma_n^{\alpha,\beta}J_n^{\alpha,\beta,\lambda}(x), \tag{2.24}$$

where $\sigma_n^{\alpha,\beta}=n(n+\alpha+\beta+1)$.

Proof It can be directly verified by using (2.10) and (2.20). □

Lemma 2.6 *The derivative of fractional Jacobi polynomials are orthogonal fractional polynomials with respect to the weight* $\hat{\omega}^{\alpha,\beta,\lambda}(x)=\lambda^{-1}(1-x^{\lambda})^{\alpha+1}x^{\beta\lambda+1}$, *i.e.,*

$$\int_0^1\hat{\omega}^{\alpha,\beta,\lambda}(x)(J_n^{\alpha,\beta,\lambda}(x))'(J_m^{\alpha,\beta,\lambda}(x))'dx=\sigma_n^{\alpha.\beta}\hat{\gamma}_n^{\alpha,\beta}\delta_{m,n}. \tag{2.25}$$

Proof It is a direct consequence of using (2.23), (2.24), and integration by parts. □

3 Some projectors and error estimates

In this section, we will introduce some projection operators in different weighted Sobolev spaces and derive error estimates for these operators. These results play key role in the error estimation for fractional spectral methods to be constructed hereafter for integro-differential equations with weakly singular kernels.

Firstly, we introduce the following differential operators:

$$D_\lambda^0:=I_d,\quad D_\lambda:=\frac{d}{dx^\lambda}:=\frac{d}{\lambda x^{\lambda-1}dx},\quad D_\lambda^2:=D_\lambda D_\lambda,\cdots,$$
$$D_\lambda^k:=\overbrace{D_\lambda D_\lambda\cdots D_\lambda}^{k},\quad k=0,1,\cdots,$$

and define the non-uniform fractional Jacobi-weighted Sobolev spaces:

$$B^m_{\omega^{\alpha,\beta,\lambda}}(I):=\left\{v: D_\lambda^k v\in L^2_{\omega^{\alpha+k,\beta+k,\lambda}}(I),\,0\le k\le m\right\},\qquad m=0,1,2,\cdots, \tag{3.1}$$

equipped with the inner product, norm and semi-norm respectively as follows:

$$(u,v)_{B^m_{\omega^{\alpha,\beta,\lambda}}}=\sum_{k=0}^m(D_\lambda^k u,D_\lambda^k)_{\omega^{\alpha+k,\beta+k,\lambda}},$$

$$\|v\|_{m,\omega^{\alpha,\beta,\lambda}}=(v,v)^{1/2}_{B^m_{\omega^{\alpha,\beta,\lambda}}},\ |v|_{m,\omega^{\alpha,\beta,\lambda}}=\|D_\lambda^m v\|_{0,\omega^{\alpha+m,\beta+m,\lambda}}.$$

The special case $\lambda=1$ gives the classical non-uniform Jacobi-weighted Sobolev spaces:

$$B^m_{\omega^{\alpha,\beta,1}}(I):=\left\{v:\partial_x^k v\in L^2_{\omega^{\alpha+k,\beta+k,1}}(I),\,0\le k\le m\right\},\qquad m=0,1,2,\cdots. \tag{3.2}$$

We then generalize the definition of the fractional Jacobi polynomials defined in (2.20) to $\alpha = -1$ or/and $\beta = -1$:

$$J_{i+1}^{\alpha,-1,\lambda}(x) := \frac{i+\alpha+1}{i+1} x^{\lambda} J_i^{\alpha,1,\lambda}(x), i = 0, 1, 2, \cdots, \tag{3.3}$$

$$J_{i+1}^{-1,\beta,\lambda}(x) := \frac{i+\beta+1}{i+1}(1-x^{\lambda}) J_i^{1,\beta,\lambda}(x), i = 0, 1, 2, \cdots, \tag{3.4}$$

$$J_{i+2}^{-1,-1,\lambda}(x) := (1-x^{\lambda}) x^{\lambda} J_i^{1,1,\lambda}(x), i = 0, 1, 2, \cdots, \tag{3.5}$$

where $\alpha, \beta > -1$. Define the fractional polynomial spaces for $\alpha, \beta \geq -1$:

$$S_{N,\lambda}^{\alpha,\beta}(I) := \text{span}\left\{J_{i+l}^{\alpha,\beta,\lambda}(x), i = 0, 1, 2, \cdots, N\right\} \tag{3.6}$$

with

$$l = \begin{cases} 0, & \alpha, \beta > -1, \\ 1, & \alpha = -1, \beta > -1 \text{ or } \alpha > -1, \beta = -1, \\ 2, & \alpha = \beta = -1. \end{cases}$$

3.1 $L^2_{\omega^{\alpha,-1,\lambda}}(I)$ and $L^2_{\omega^{-1,\beta,\lambda}}(I)$-orthogonal projectors with $\alpha, \beta > -1$

In this subsection, we will discuss the properties of $\{J_{i+1}^{\alpha,-1,\lambda}(x)\}_{i=0}^{\infty}$ and $\{J_{j+1}^{-1,\beta,\lambda}(x)\}_{j=0}^{\infty}$ and related $L^2_{\omega^{\alpha,-1,\lambda}}$- and $L^2_{\omega^{-1,\beta,\lambda}}$-projection operators. We will only present the results for the $L^2_{\omega^{\alpha,-1,\lambda}}$-projector, since $L^2_{\omega^{-1,\beta,\lambda}}$-projector can be analyzed in a similar way.

Lemma 3.1 *The fractional polynomials $\{J_{i+1}^{\alpha,-1,\lambda}(x)\}_{i=0}^{\infty}$ defined in (3.3) have the following orthogonality*

$$\int_0^1 \omega^{\alpha,-1,\lambda}(x) J_{i+1}^{\alpha,-1,\lambda}(x) J_{j+1}^{\alpha,-1,\lambda}(x) dx = \hat{\gamma}_{i+1}^{\alpha,-1} \delta_{ij}, \tag{3.7}$$

where $\omega^{\alpha,-1,\lambda}(x) = \lambda(1-x^{\lambda})^{\alpha} x^{-1}$, and $\hat{\gamma}_{i+1}^{\alpha,-1} = \frac{i+\alpha+1}{(i+1)(2i+\alpha+2)}$; see also (2.23). Furthermore, it holds:

$$-(\omega^{\alpha,-1,\lambda}(x))^{-1} \partial_x \left\{\lambda^{-1}(1-x^{\lambda})^{\alpha+1} x^{1-\lambda} \partial_x J_{i+1}^{\alpha,-1,\lambda}(x)\right\} = \sigma_{i+1}^{\alpha,-1} J_{i+1}^{\alpha,-1,\lambda}(x), \tag{3.8}$$

where $\sigma_{i+1}^{\alpha,-1} = (i+1)(i+\alpha+1)$; see also (2.24).

Proof We derive from a direct calculation using (3.3) and (2.23)

$$\begin{aligned} &\int_0^1 \omega^{\alpha,-1,\lambda}(x) J_{i+1}^{\alpha,-1,\lambda}(x) J_{j+1}^{\alpha,-1,\lambda}(x) dx \\ &= \frac{(i+\alpha+1)(j+\alpha+1)}{(i+1)(j+1)} \int_0^1 \omega^{\alpha,1,\lambda}(x) J_i^{\alpha,1,\lambda}(x) J_j^{\alpha,1,\lambda}(x) dx \\ &= \frac{i+\alpha+1}{(i+1)(2i+\alpha+2)} \delta_{ij}. \end{aligned}$$

This proves (3.7). Using Lemma 2.3, we have

$$(1+x)J_i^{\alpha,1}(x) = (i+1)\int_{-1}^{x} J_i^{\alpha+1,0}(s)ds,$$
$$(1-x)^{\alpha+1}J_i^{\alpha+1,0}(x) = (i+\alpha+1)\int_{x}^{1}(1-s)^{\alpha}J_i^{\alpha,1}(s)ds.$$

By change of variable $x \to 2x^{\lambda}-1$, we obtain

$$x^{\lambda}J_i^{\alpha,1,\lambda}(x) = \lambda(i+1)\int_0^x s^{\lambda-1}J_i^{\alpha+1,0,\lambda}(s)ds,$$
$$(1-x^{\lambda})^{\alpha+1}J_i^{\alpha+1,0,\lambda}(x) = \lambda(i+\alpha+1)\int_x^1(1-s^{\lambda})^{\alpha}s^{\lambda-1}J_i^{\alpha,1,\lambda}(s)ds. \quad (3.9)$$

Thus

$$\begin{aligned}
&-(\omega^{\alpha,-1,\lambda}(x))^{-1}\partial_x\left\{\lambda^{-1}(1-x^{\lambda})^{\alpha+1}x^{1-\lambda}\partial_x J_{i+1}^{\alpha,-1,\lambda}(x)\right\}\\
&= -\lambda^{-1}(1-x^{\lambda})^{-\alpha}x\partial_x\left\{(1-x^{\lambda})^{\alpha+1}x^{1-\lambda}\partial_x\left[\frac{i+\alpha+1}{i+1}x^{\lambda}J_i^{\alpha,1,\lambda}(x)\right]\right\}\\
&= -\lambda^{-1}(1-x^{\lambda})^{-\alpha}x\partial_x\left\{(i+1)(1-x^{\lambda})^{\alpha+1}J_i^{\alpha+1,0,\lambda}(x)\right\}\\
&= (i+\alpha+1)^2x^{\lambda}J_i^{\alpha,1,\lambda}(x) = (i+1)(i+\alpha+1)J_{i+1}^{\alpha,-1,\lambda}(x).
\end{aligned}$$

The proof is completed. □

Now we define the $L^2_{\omega^{\alpha,-1,\lambda}}(I) \to S_{N,\lambda}^{\alpha,\beta}(I)$ orthogonal projector $\pi_{N,\omega^{\alpha,-1,\lambda}}$: for all $v \in L^2_{\omega^{\alpha,-1,\lambda}}(I)$, $\pi_{N,\omega^{\alpha,-1,\lambda}}v \in S_{N,\lambda}^{\alpha,-1}(I)$ such that

$$(v-\pi_{N,\omega^{\alpha,-1,\lambda}}v,\ v_N)_{\omega^{\alpha,-1,\lambda}} = 0, \quad \forall v_N \in S_{N,\lambda}^{\alpha,-1}(I). \quad (3.10)$$

Then define the dual fractional polynomial space of $S_{N,\lambda}^{\alpha,-1}(I)$ as follows:

$$V_{N,\lambda}^{-\alpha-1,0}(I) := \text{span}\left\{(1-x^{\lambda})^{\alpha+1}J_j^{\alpha+1,0,\lambda}(x),\ j=0,1,2,\cdots,N\right\}$$

Remark 3.1 It is observed from (3.9) that $(1-x^{\lambda})^{-\alpha}x\partial_x v_N \in S_{N,\lambda}^{\alpha,-1}(I)$ if $v_N \in V_{N,\lambda}^{-\alpha-1,0}(I)$.

Lemma 3.2 *The projector $\pi_{N,\omega^{\alpha,-1,\lambda}}$ satisfies the following property:*

$$(\partial_x(v-\pi_{N,\omega^{\alpha,-1,\lambda}}v), v_N) = 0, \quad \forall v \in B^1_{\omega^{\alpha,-1,\lambda}}(I),\ \forall v_N \in V_{N,\lambda}^{-\alpha-1,0}(I).$$

Proof Using the fact that $(v-\pi_{N,\omega^{\alpha,-1,\lambda}}v)(x)v_N(x)|_{x=0,1} = 0$ for all $v \in B^1_{\omega^{\alpha,-1,\lambda}}(I)$ and all $v_N \in V_{N,\lambda}^{-\alpha-1,0}(I)$, we have

$$\begin{aligned}
(\partial_x(v-\pi_{N,\omega^{\alpha,-1,\lambda}}v), v_N) &= -(v-\pi_{N,\omega^{\alpha,-1,\lambda}}v,\ \partial_x v_N)\\
&= -\frac{1}{\lambda}(v-\pi_{N,\omega^{\alpha,-1,\lambda}}v, (1-x^{\lambda})^{-\alpha}x\partial_x v_N)_{\omega^{\alpha,-1,\lambda}}.
\end{aligned}$$

From the definition (3.10), the right hand side term vanishes since, according to Remark 3.1, $(1-x^\lambda)^{-\alpha} x\partial_x v_N \in S^{\alpha,-1}_{N,\lambda}(I)$. This concludes the lemma. □

Theorem 3.1 (see Theorem 2.2 in [24] for case $\lambda = 1$) *The following approximation result holds for the projector $\pi_{N,\omega^{\alpha,-1,1}}$:*

$$\|\partial_x^l (v - \pi_{N,\omega^{\alpha,-1,1}} v)\|_{0,\omega^{l+\alpha,l-1,1}} \le cN^{l-m} \|\partial_x^m v\|_{0,\omega^{m+\alpha,m-1,1}},$$
$$0 \le l \le m, \ \forall v \in B^m_{\omega^{\alpha,-1,1}}(I).$$

Theorem 3.2 (case $0 < \lambda \le 1$) *For any $v(x)$ such that $v(x^{\frac{1}{\lambda}}) \in B^m_{\omega^{\alpha,-1,1}}(I)$, $m \ge 1$, its orthogonal projection $\pi_{N,\omega^{\alpha,-1,\lambda}} v$ admits the following error estimates:*

$$\|v - \pi_{N,\omega^{\alpha,-1,\lambda}} v\|_{0,\omega^{\alpha,-1,\lambda}} \le cN^{-m} \|\partial_x^m v(x^{\frac{1}{\lambda}})\|_{0,\omega^{m+\alpha,m-1,1}}, \tag{3.11}$$

$$\|\partial_x (v - \pi_{N,\omega^{\alpha,-1,\lambda}} v)\|_{0,\omega^{\alpha+1,2/\lambda-2,\lambda}} \le cN^{1-m} \|\partial_x^m v(x^{\frac{1}{\lambda}})\|_{0,\omega^{m+\alpha,m-1,1}}. \tag{3.12}$$

Proof Any $L^2_{\omega^{\alpha,-1,\lambda}}(I)$ function v can be expressed as

$$v(x) = \sum_{i=0}^{\infty} \hat{v}_i J^{\alpha,-1,\lambda}_{i+1}(x), \quad \text{with } \hat{v}_i = \frac{\left(v(x), J^{\alpha,-1,\lambda}_{i+1}(x)\right)_{\omega^{\alpha,-1,\lambda}}}{\|J^{\alpha,-1,\lambda}_{i+1}(x)\|^2_{0,\omega^{\alpha,-1,\lambda}}}.$$

By making variable change $y = x^\lambda$, we have

$$\begin{aligned}\left(v(x), J^{\alpha,-1,\lambda}_{i+1}(x)\right)_{\omega^{\alpha,-1,\lambda}} &= \frac{i+\alpha+1}{i+1}\int_0^1 \lambda(1-x^\lambda)^\alpha x^{\lambda-1} v(x) J^{\alpha,1}_i(2x^\lambda - 1)dx \\ &= \frac{i+\alpha+1}{i+1}\int_0^1 (1-y)^\alpha v(y^{\frac{1}{\lambda}}) J^{\alpha,1}_i(2y-1)dy \\ &= (v(x^{\frac{1}{\lambda}}), J^{\alpha,-1,1}_{i+1}(x))_{\omega^{\alpha,-1,1}}.\end{aligned}$$

Furthermore, it follows from Lemma 3.1

$$\|J^{\alpha,-1,\lambda}_{i+1}(x)\|^2_{0,\omega^{\alpha,-1,\lambda}} = \|J^{\alpha,-1,1}_{i+1}(x)\|^2_{0,\omega^{\alpha,-1,1}}.$$

Therefore, we obtain

$$\hat{v}_i = \frac{\left(v(x^{\frac{1}{\lambda}}), J^{\alpha,-1,1}_{i+1}(x)\right)_{\omega^{\alpha,-1,1}}}{\|J^{\alpha,-1,1}_{i+1}(x)\|^2_{0,\omega^{\alpha,-1,1}}}. \tag{3.13}$$

Consequently,

$$\begin{aligned}\|v - \pi_{N,\omega^{\alpha,-1,\lambda}} v\|^2_{0,\omega^{\alpha,-1,\lambda}} &= \sum_{i=N+1}^{\infty} \hat{v}_i^2 \|J^{\alpha,-1,\lambda}_{i+1}(x)\|^2_{0,\omega^{0,-1,\lambda}} \\ &= \sum_{i=N+1}^{\infty} \frac{\left(v(x^{\frac{1}{\lambda}}), J^{\alpha,-1,1}_{i+1}(x)\right)^2_{\omega^{\alpha,-1,1}}}{\|J^{\alpha,-1,1}_{i+1}(x)\|^2_{0,\omega^{\alpha,-1,1}}} \\ &= \|v(x^{\frac{1}{\lambda}}) - \pi_{N,\omega^{\alpha,-1,1}} v(x^{\frac{1}{\lambda}})\|^2_{0,\omega^{\alpha,-1,1}}.\end{aligned}$$

Finally, using the approximation results already established in Theorem 3.1, we obtain
$$\|v-\pi_{N,\omega^{\alpha,-1,\lambda}}v\|_{0,\omega^{\alpha,-1,\lambda}} \le cN^{-m}\|\partial_x^m v(x^{\frac{1}{\lambda}})\|_{0,\omega^{m+\alpha,m-1,1}}.$$
This proves (3.11). Next, we shall prove (3.12). Noticing
$$\begin{aligned}\|v(x)\|^2_{0,\omega^{\alpha,-1,\lambda}} &= \int_0^1 \lambda(1-x^\lambda)^\alpha x^{-1}v^2(x)dx = \int_0^1 (1-t)^\alpha t^{-1}v^2(t^{1/\lambda})dt\\ &= \|v(x^{1/\lambda})\|^2_{0,\omega^{\alpha,-1,1}},\end{aligned}$$
we see $v(x)\in L^2_{\omega^{\alpha,-1,\lambda}}(I)$ if and only if $v(x^{1/\lambda})\in L^2_{\omega^{\alpha,-1,1}}(I)$. Thus for any $v(x)\in L^2_{\omega^{\alpha,-1,\lambda}}(I)$, we have
$$v(x^{\frac{1}{\lambda}})=\sum_{i=0}^{\infty}\hat{v}_i J_{i+1}^{\alpha,-1,1}(x),\quad \pi_{N,\omega^{\alpha,-1,1}}v(x^{\frac{1}{\lambda}})=\sum_{i=0}^{N}\tilde{v}_i J_{i+1}^{\alpha,-1,1}(x),$$
where
$$\hat{v}_i=\frac{\left(v(x^{\frac{1}{\lambda}}),\, J_{i+1}^{\alpha,-1,1}(x)\right)_{\omega^{\alpha,-1,1}}}{\|J_{i+1}^{\alpha,-1,1}(x)\|^2_{0,\omega^{\alpha,-1,1}}}.$$
It then follows from (3.3) and (3.9),
$$\begin{aligned}\left\|\partial_x(v(x^{\frac{1}{\lambda}})-\pi_{N,\omega^{\alpha,-1,1}}v(x^{\frac{1}{\lambda}}))\right\|^2_{0,\omega^{\alpha+1,0,1}} &= \sum_{i=N+1}^{\infty}\hat{v}_i^2\left\|\partial_x J_{i+1}^{\alpha,-1,1}(x)\right\|^2_{0,\omega^{\alpha+1,0,1}}\\ &= \sum_{i=N+1}^{\infty}(i+\alpha+1)^2\hat{v}_i^2\left\|J_i^{\alpha+1,0,1}(x)\right\|^2_{0,\omega^{\alpha+1,0,1}},\\ \left\|\partial_x(v-\pi_{N,\omega^{\alpha,-1,\lambda}}v)\right\|^2_{0,\omega^{\alpha+1,2/\lambda-2,\lambda}} &= \sum_{i=N+1}^{\infty}\hat{v}_i^2\left\|\partial_x J_{i+1}^{\alpha,-1,\lambda}(x)\right\|^2_{0,\omega^{\alpha+1,2/\lambda-2,\lambda}}\\ &= \sum_{i=N+1}^{\infty}\lambda^2(i+\alpha+1)^2\hat{v}_i^2\left\|x^{\lambda-1}J_i^{\alpha+1,0}(2x^\lambda-1)\right\|^2_{0,\omega^{\alpha+1,2/\lambda-2,\lambda}}\\ &= \sum_{i=N+1}^{\infty}\lambda^2(i+\alpha+1)^2\hat{v}_i^2\left\|J_i^{\alpha+1,0,\lambda}(x)\right\|^2_{0,\omega^{\alpha+1,0,\lambda}}.\end{aligned}$$
Furthermore, using Lemma 2.4 gives
$$\left\|J_i^{\alpha+1,0,1}(x)\right\|_{0,\omega^{\alpha+1,0,1}}=\left\|J_i^{\alpha+1,0,\lambda}(x)\right\|_{0,\omega^{\alpha+1,0,\lambda}}.$$
Therefore,
$$\begin{aligned}\left\|\partial_x(v-\pi_{N,\omega^{\alpha,-1,\lambda}}v)\right\|_{0,\omega^{\alpha+1,2/\lambda-2,\lambda}} &= \lambda\left\|\partial_x(v(x^{\frac{1}{\lambda}})-\pi_{N,\omega^{\alpha,-1,1}}v(x^{\frac{1}{\lambda}}))\right\|_{0,\omega^{\alpha+1,0,1}}\\ &\le c\lambda N^{1-m}\left\|\partial_x^m v(x^{\frac{1}{\lambda}})\right\|_{0,\omega^{m+\alpha,m-1,1}}.\end{aligned}$$
The proof is completed. □

Remark 3.2 It is seen from the proof of Theorem 3.2 that
$$\left\|v(x)-\pi_{N,\omega^{\alpha,-1,\lambda}}v(x)\right\|_{0,\omega^{\alpha,-1,\lambda}}=\left\|v(x^{\frac{1}{\lambda}})-\pi_{N,\omega^{\alpha,-1,1}}v(x^{\frac{1}{\lambda}})\right\|_{0,\omega^{\alpha,-1,1}},$$
$$\left\|\partial_x(v(x)-\pi_{N,\omega^{\alpha,-1,\lambda}}v(x))\right\|_{0,\omega^{\alpha+1,2/\lambda-2,\lambda}}=\lambda\left\|\partial_x(v(x^{\frac{1}{\lambda}})-\pi_{N,\omega^{\alpha,-1,1}}v(x^{\frac{1}{\lambda}}))\right\|_{0,\omega^{\alpha+1,0,1}}.$$

This means that the approximation error of the fractional polynomial projector $\pi_{N,\omega^{\alpha,-1,\lambda}}$ of a given function $v(x)$ is related to the classical polynomial projection $\pi_{N,\omega^{\alpha,-1,1}}$ of $v(x^{1/\lambda})$.

3.2 $L^2_{\omega^{-1,-1,\lambda}}(I)$-orthogonal projection

In this subsection, we will investigate the properties of $\{J^{-1,-1,\lambda}_{i+2}(x)\}_{i=0}^{\infty}$ and the related $L^2_{\omega^{-1,-1,\lambda}}$-projection operator.

Lemma 3.3 *The fractional polynomials* $\left\{J^{-1,-1,\lambda}_{i+2}(x)\right\}_{i=0}^{\infty}$ *defined in* (3.5) *satisfy the following equality:*

$$J^{-1,-1,\lambda}_{i+2}(x) = -\lambda(i+1)\int_0^x s^{\lambda-1}L_{i+1}(2s^{\lambda}-1)ds, \tag{3.14}$$

where $L_{i+1}(x)$ *is Legendre polynomial of degree* $i+1$.

Proof According to Lemma 2.1 with $\alpha = \beta = -1$ in [24], we have

$$\partial_x J^{-1,-1,1}_{i+2}(x) = -(i+1)L_{i+1}(2x-1).$$

Then we obtain

$$\partial_x J^{-1,-1,\lambda}_{i+2}(x) = \partial_x J^{-1,-1,1}_{i+2}(x^{\lambda}) = -\lambda(i+1)x^{\lambda-1}L_{i+1}(2x^{\lambda}-1).$$

This gives the desired result using the fact that $J^{-1,-1,\lambda}_{i+2}(0) = 0$. □

Similar to Lemma 3.1, we can prove the following properties for the fractional polynomials $\left\{J^{-1,-1,\lambda}_{i+2}(x)\right\}_{i=0}^{\infty}$.

Lemma 3.4 *The fractional polynomials* $\left\{J^{-1,-1,\lambda}_{i+2}(x)\right\}_{i=0}^{\infty}$ *defined in* (3.5) *satisfy the following orthogonality*

$$\int_0^1 \omega^{-1,-1,\lambda}(x)J^{-1,-1,\lambda}_{i+2}(x)J^{-1,-1,\lambda}_{j+2}(x)dx = \hat{\gamma}^{-1,-1}_{i+2}\delta_{ij},$$

where $\omega^{-1,-1,\lambda}(x) = \lambda(1-x^{\lambda})^{-1}x^{-1}$, $\hat{\gamma}^{-1,-1}_{i+2} = \dfrac{i+1}{(2i+3)(i+2)}$. *Furthermore,* $\left\{J^{-1,-1,\lambda}_{i+2}(x)\right\}_{i=0}^{\infty}$ *satisfy the Sturm-Liouville equation as follows:*

$$-(\omega^{-1,-1,\lambda}(x))^{-1}\partial_x\left\{\lambda^{-1}x^{1-\lambda}\partial_x J^{-1,-1,\lambda}_{i+2}(x)\right\} = \sigma^{-1,-1}_{i+2}J^{-1,-1,\lambda}_{i+2}(x),$$

where $\sigma^{-1,-1}_{i+2} = (i+1)(i+2)$.

Now, let $\pi_{N,\omega^{-1,-1,\lambda}} : L^2_{\omega^{-1,-1,\lambda}}(I) \to S^{-1,-1}_{N,\lambda}(I)$ be the $L^2_{\omega^{-1,-1,\lambda}}(I)$-orthogonal projector: for all $v \in L^2_{\omega^{-1,-1,\lambda}}(I)$, $\pi_{N,\omega^{-1,-1,\lambda}} v \in S^{-1,-1}_{N,1}(I)$ such that

$$(v - \pi_{N,\omega^{-1,-1,\lambda}} v,\ v_N)_{\omega^{-1,-1,\lambda}} = 0, \quad \forall v_N \in S^{-1,-1}_{N,\lambda}(I). \tag{3.15}$$

Theorem 3.3 *For any $v(x^{\frac{1}{\lambda}}) \in B^m_{\omega^{-1,-1,1}}(I), m \geq 1$, the projection operator $\pi_{N,\omega^{-1,-1,\lambda}}$ admits the following error estimate:*

$$\|v - \pi_{N,\omega^{-1,-1,\lambda}} v\|_{0,\omega^{-1,-1,\lambda}} \leq cN^{-m} \|\partial_x^m v(x^{\frac{1}{\lambda}})\|_{0,\omega^{m-1,m-1,1}}, \tag{3.16}$$

$$\|\partial_x (v - \pi_{N,\omega^{-1,-1,\lambda}} v)\|_{0,\omega^{0,2/\lambda-2,\lambda}} \leq cN^{1-m} \|\partial_x^m v(x^{\frac{1}{\lambda}})\|_{0,\omega^{m-1,m-1,1}}. \tag{3.17}$$

Proof First we state a known result in the special case $\lambda = 1$, which was given in [7, 38, 46] for the $\pi_{N,\omega^{-1,-1,1}}$ orthogonal projector

$$\begin{aligned} &\|\partial_x^l (v - \pi_{N,\omega^{-1,-1,1}} v)\|_{0,\omega^{l-1,l-1,1}} \leq cN^{l-m} \|\partial_x^m v\|_{0,\omega^{m-1,m-1,1}}, \\ &\quad \forall v \in B^m_{\omega^{-1,-1,1}}(I), l = 0, 1. \end{aligned} \tag{3.18}$$

Similar to the proof of Theorem 3.2, we can obtain

$$\begin{aligned} \left\|v - \pi_{N,\omega^{-1,-1,\lambda}} v\right\|_{0,\omega^{-1,-1,\lambda}} &= \left\|v(x^{\frac{1}{\lambda}}) - \pi_{N,\omega^{-1,-1,1}} v(x^{\frac{1}{\lambda}})\right\|_{0,\omega^{-1,-1,1}}, \\ \|\partial_x (v - \pi_{N,\omega^{-1,-1,\lambda}} v)\|_{0,\omega^{0,2/\lambda-2,\lambda}} &= \lambda \|\partial_x (v(x^{\frac{1}{\lambda}}) - \pi_{N,\omega^{-1,-1,1}} v(x^{\frac{1}{\lambda}}))\|_{0,\omega^{0,0,1}}. \end{aligned} \tag{3.19}$$

Putting (3.18) and (3.19) together completes the proof. □

4 Fractional spectral methods

This section is devoted to developing efficient fractional spectral methods for several kinds of problems with weakly singular solutions, and derive error estimates using the approximation results established in the previous sections.

4.1 A integro-differential equation

Firstly, we consider the integro-differential problem:

$$\begin{cases} u_t = a_1 u(t) + a_2\, {}_0I_t^{\mu} u(t) + f(t), \quad t \in I, \\ u(0) = 0, \end{cases} \tag{4.1}$$

where a_1 and a_2 are real bounded constants, and $\mu > 0$.

This equation or its variants have been investigated in a number of papers; see, e.g., [10, 43] and the references therein. We consider the following Petrov-Galerkin based fractional spectral method: Find $u_N \in S^{0,-1}_{N,\lambda}(I)$, such that

$$(u_N', v_N) = a_1(u_N, v_N) + a_2({}_0I_t^{\mu} u_N, v_N) + (f, v_N), \quad \forall v_N \in V^{-1,0}_{N,\lambda}(I). \tag{4.2}$$

Notice that for any $v_N \in S_{N,\lambda}^{0,-1}(I)$ we have $\omega^{1,\frac{1}{\lambda}-2,\lambda} v_N = \lambda t^{-\lambda}(1-t^\lambda) v_N \in V_{N,\lambda}^{-1,0}(I)$. Thus the problem (4.2) is equivalent to the weighted Galerkin form as follows: Find $u_N \in S_{N,\lambda}^{0,-1}(I)$, such that

$$(u_N', v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} = a_1(u_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^\mu u_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + (f, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}},\quad \forall v_N \in S_{N,\lambda}^{0,-1}(I). \tag{4.3}$$

The error estimation of the numerical solution will make use of a series of lemmas, which are stated below.

Lemma 4.1 ([25] Theorem 329) *If $p > 1$, $r > 0$, and $g \in L^p(0,+\infty)$, then it holds*

$$\int_0^\infty (t^{-r} {}_0I_t^r g(t))^p dt < \left\{\frac{\Gamma(1-1/p)}{\Gamma(r+1-1/p)}\right\}^p \int_0^\infty g^p(t)dt,$$

unless $g \equiv 0$.

A direct extension of this lemma for the case that g is only defined a bounded interval is given in the next lemma.

Lemma 4.2 *If $r > 0$, and $g \in L^2(I)$, then we have*

$$\int_0^1 (t^{-r} {}_0I_t^r g(t))^2 dt < \left\{\frac{\Gamma(1/2)}{\Gamma(r+1/2)}\right\}^2 \int_0^1 g^2(t)dt,$$

unless $g \equiv 0$.

Proof It can be readily derived from the zero extension of g outside $(0, 1)$. □

Lemma 4.3 *If $0 < \lambda < 1$, $\gamma > 0$. Then it holds*

$$(1-t^\lambda)t^\gamma < \frac{\lambda}{\gamma e}, \quad (1-t^\lambda)t^{1-\lambda} < \lambda, \quad \forall t \in I,$$

where e is the nature number.

Proof Let $h(t) = (1-t^\lambda)t^\gamma$, $0 < t < 1$. Then

$$h'(t) = t^{\gamma-1}(\gamma - (\lambda+\gamma)t^\lambda).$$

Note that $h(0) = h(1) = 0$, $h(t) > 0$ for all $0 < t < 1$, and $h'\left(\left(\frac{\gamma}{\lambda+\gamma}\right)^{1/\lambda}\right) = 0$, we have

$$h(t) \le h\left(\left(\frac{\gamma}{\lambda+\gamma}\right)^{1/\lambda}\right) = \frac{\lambda}{\lambda+\gamma}\left(\frac{\gamma}{\lambda+\gamma}\right)^{\frac{\gamma}{\lambda}} = \frac{\lambda}{\gamma}\frac{1}{\left(\frac{\lambda}{\gamma}+1\right)^{\gamma/\lambda+1}}. \tag{4.4}$$

Furthermore, using the fact $(x+1)^{\frac{1}{x}+1} > e$ for all $x > 0$, we obtain $h(t) < \frac{\lambda}{\gamma e}$ for all $t \in I$. This proves the first inequality.

Substituting $\gamma = 1 - \lambda$ into the both sides of (4.4) yields

$$(1-t^{\lambda})t^{1-\lambda} \le h\left(\left(\frac{\gamma}{\lambda+\gamma}\right)^{1/\lambda}\right) = \lambda(1-\lambda)^{\frac{1}{\lambda}-1}.$$

Finally noticing $\frac{1}{e} < (1-x)^{\frac{1}{x}-1} < 1$ for all $0 < x < 1$, we get

$$(1-t^{\lambda})t^{1-\lambda} < \lambda.$$

This proves the second inequality. □

Lemma 4.4 *For all $u \in L^2(I)$, $v \in L^2_{\omega^{0,-2,\lambda}}(I)$, $r > 0$, we have*

$$({}_0I_t^r u, v)_{\omega^{1,1/\lambda-2,\lambda}} \le \frac{\lambda\Gamma(1/2)}{\sqrt{2re}\,\Gamma(r+1/2)}\left(\int_0^1 \lambda u^2 dt\right)^{1/2} \|v\|_{0,\omega^{0,-2,\lambda}}.$$

Proof By using Cauchy-Schwarz inequality, Lemma 4.2, and Lemma 4.3, we obtain

$$\begin{aligned}
&({}_0I_t^r u, v)_{\omega^{1,1/\lambda-2,\lambda}}\\
&\le \lambda\left(\int_0^1 (1-t^{\lambda})({}_0I_t^r u(t))^2 dt\right)^{1/2}\left(\int_0^1 (1-t^{\lambda})t^{-2\lambda}v^2(t)dt\right)^{1/2}\\
&= \lambda\left(\int_0^1 (1-t^{\lambda})t^{2r}(t^{-r}{}_0I_t^r u(t))^2 dt\right)^{1/2}\left(\int_0^1 (1-t^{\lambda})t^{1-\lambda}t^{-\lambda-1}v^2(t)dt\right)^{1/2}\\
&\le \frac{\lambda}{\sqrt{2re}}\left(\int_0^1 \lambda(t^{-r}{}_0I_t^r u(t))^2 dt\right)^{1/2}\left(\int_0^1 \lambda t^{-\lambda-1}v^2(t)dt\right)^{1/2}\\
&\le \frac{\lambda\Gamma(1/2)}{\sqrt{2re}\,\Gamma(r+1/2)}\left(\int_0^1 \lambda u^2(t)dt\right)^{1/2} \|v\|_{0,\omega^{0,-2,\lambda}}.
\end{aligned}$$

The proof is completed. □

Theorem 4.1 *If the coefficients a_1 and a_2 satisfy*

$$a_1 \le 0,\ |a_2| < \frac{\sqrt{2\mu e}\,\Gamma(\mu+1/2)}{2\Gamma(1/2)}, \tag{4.5}$$

or

$$a_1 > 0,\ \frac{a_1}{e} + \frac{|a_2|\Gamma(1/2)}{\sqrt{2\mu e}\,\Gamma(\mu+1/2)} < \frac{1}{2}. \tag{4.6}$$

Then the fractional spectral discrete problem (4.2) admits a unique solution. Furthermore, if the exact solution of (4.1) $u(t^{\frac{1}{\lambda}}) \in B^m_{\omega^{0,-1,1}}(I)$, the following error estimate holds:

$$\|u - u_N\|_{0,\omega^{0,-1,\lambda}} \le cN^{-m}\|\partial_t^m u(t^{\frac{1}{\lambda}})\|_{0,\omega^{m,m-1,1}}. \tag{4.7}$$

Proof In order to prove that the problem (4.2) admits a unique solution, it suffices to prove that the problem (4.2) with $f = 0$ admits only the trivial solution $u_N = 0$. To this end, by taking $v_N = u_N$ in (4.3) and letting $f = 0$, we obtain

$$(\partial_t u_N, u_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} = a_1(u_N, u_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^{\mu} u_N, u_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}}. \tag{4.8}$$

Now we estimate the above equation term by term. For the term in the left hand side, we have

$$\begin{aligned}(\partial_t u_N, u_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} &= \frac{\lambda}{2}\int_0^1 (1-t^{\lambda})t^{-\lambda}\partial_t u_N^2 dt = -\frac{\lambda}{2}\int_0^1 ((1-t^{\lambda})t^{-\lambda})' u_N^2 dt \\ &= \frac{\lambda^2}{2}\int_0^1 t^{-\lambda-1}u_N^2 dt = \frac{\lambda}{2}\|u_N\|^2_{0,\omega^{0,-2,\lambda}}. \end{aligned}\tag{4.9}$$

For the second term in the right hand side, applying Lemma 4.4 yields

$$\begin{aligned}({}_0I_t^{\mu} u_N, u_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} &\le \frac{\lambda\Gamma(1/2)}{\sqrt{2\mu e}\,\Gamma(\mu+1/2)}\left(\int_0^1 \lambda u_N^2 dx\right)^{1/2}\|u_N\|_{0,\omega^{0,-2,\lambda}} \\ &\le \frac{\lambda\Gamma(1/2)}{\sqrt{2\mu e}\,\Gamma(\mu+1/2)}\|u_N\|^2_{0,\omega^{0,-2,\lambda}}. \end{aligned}\tag{4.10}$$

Combining (4.8), (4.9) and (4.10) gives

$$\frac{\lambda}{2}\|u_N\|^2_{0,\omega^{0,-2,\lambda}} \le a_1(u_N, u_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + \frac{|a_2|\lambda\Gamma(1/2)}{\sqrt{2\mu e}\,\Gamma(\mu+1/2)}\|u_N\|^2_{0,\omega^{0,-2,\lambda}}. \tag{4.11}$$

In the case $a_1 \le 0$, it follows from (4.11)

$$\|u_N\|^2_{0,\omega^{0,-2,\lambda}} \le \frac{2|a_2|\Gamma(1/2)}{\sqrt{2\mu e}\,\Gamma(\mu+1/2)}\|u_N\|^2_{0,\omega^{0,-2,\lambda}}.$$

If furthermore $|a_2| < \frac{\sqrt{2\mu e}\,\Gamma(\mu+1/2)}{2\Gamma(1/2)}$, then necessarily $u_N \equiv 0$.
In the case $a_1 > 0$, by using Lemma 4.3, we have

$$(u_N, u_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} = \int_0^1 \lambda(1-t^{\lambda})t\cdot t^{-\lambda-1}(u_N(t))^2 dt \le \frac{\lambda}{e}\|u_N\|^2_{0,\omega^{0,-2,\lambda}}.$$

Combining this estimate with (4.11) leads to

$$\|u_N\|^2_{0,\omega^{0,-2,\lambda}} \le 2\left(\frac{a_1}{e} + \frac{|a_2|\Gamma(1/2)}{\sqrt{2\mu e}\,\Gamma(\mu+1/2)}\right)\|u_N\|^2_{0,\omega^{0,-2,\lambda}}.$$

Thus, if $2\left(\frac{a_1}{e} + \frac{|a_2|\Gamma(1/2)}{\sqrt{2\mu e}\,\Gamma(\mu+1/2)}\right) < 1$, then $u_N \equiv 0$. This proves the well-posedness of the problem (4.2) under condition (4.5) or (4.6).
Next we derive the error estimate (4.7). Let $\hat{e}_N = \pi_{N,\omega^{0,-1,\lambda}}u - u_N$ and $e_N = u - u_N$. On one side, we deduce from (4.1) and (4.3)

$$(\partial_t e_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} = a_1(e_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^{\mu} e_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}}, \quad \forall v_N \in S^{0,-1}_{N,\lambda}(I).$$

On the other side, in virtue of Lemma 3.2 and the fact that $\omega^{1,\frac{1}{\lambda}-2,\lambda}v_N \in V_{N,\lambda}^{-\alpha-1,0}(I)$ if $v_N \in S_{N,\lambda}^{0,-1}(I)$, we have

$$\begin{aligned}(\partial_t e_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} &= (\partial_t(u-\pi_{N,\omega^{0,-1,\lambda}}u), v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + (\partial_t \hat{e}_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &= (\partial_t \hat{e}_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}}, \qquad \forall v_N \in S_{N,\lambda}^{0,-1}(I).\end{aligned}$$

Hence, we obtain

$$(\partial_t \hat{e}_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} = a_1(e_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^{\mu} e_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}}, \quad \forall v_N \in S_{N,\lambda}^{0,-1}(I).$$

Taking $v_N = \hat{e}_N \in S_{N,\lambda}^{0,-1}(I)$ in the above equation gives

$$\begin{aligned}&(\partial_t \hat{e}_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &= a_1(e_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^{\mu} e_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &= a_1(u-\pi_{N,\omega^{0,-1,\lambda}}u+\hat{e}_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^{\mu}(u-\pi_{N,\omega^{0,-1,\lambda}}u+\hat{e}_N), \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &= a_1(\hat{e}_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^{\mu} \hat{e}_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_1(u-\pi_{N,\omega^{0,-1,\lambda}}u, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &\quad + a_2({}_0I_t^{\mu}(u-\pi_{N,\omega^{0,-1,\lambda}}u), \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}}.\end{aligned}$$

Thus,

$$\begin{aligned}(\partial_t \hat{e}_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} &= a_1(\hat{e}_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^{\mu} \hat{e}_N, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &\quad + a_1(u-\pi_{N,\omega^{0,-1,\lambda}}u, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &\quad + a_2({}_0I_t^{\mu}(u-\pi_{N,\omega^{0,-1,\lambda}}u), \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}}.\end{aligned} \tag{4.12}$$

Estimations of the left hand side term and the first two terms in the right hand side are similar to (4.9)–(4.11). For the remaining terms in (4.12), we first use the Cauchy-Schwarz inequality and Lemma 4.3 to get

$$\begin{aligned}&a_1(u-\pi_{N,\omega^{0,-1,\lambda}}u, \hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &\le |a_1| \|u-\pi_{N,\omega^{0,-1,\lambda}}u\|_{0,\omega^{1,\frac{1}{\lambda}-2,\lambda}} \|\hat{e}_N\|_{0,\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &= |a_1| \left(\int_0^1 \lambda(1-t^{\lambda})t^{1-\lambda}t^{-1}(u-\pi_{N,\omega^{0,-1,\lambda}}u)^2 dt\right)^{1/2} \\ &\quad \times \left(\int_0^1 \lambda(1-t^{\lambda})t\cdot t^{-\lambda-1}\hat{e}_N^2(t)dt\right)^{1/2} \\ &\le \frac{|a_1|\lambda}{\sqrt{e}} \|u-\pi_{N,\omega^{0,-1,\lambda}}u\|_{0,\omega^{0,-1,\lambda}} \|\hat{e}_N\|_{0,\omega^{0,-2,\lambda}},\end{aligned} \tag{4.13}$$

then apply Lemma 4.4 to obtain

$$a_2({}_0I_t^{\mu}(u-\pi_{N,\omega^{0,-1,\lambda}}u),\hat{e}_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \le \frac{\lambda|a_2|\Gamma(1/2)}{\sqrt{2\mu e}\Gamma(\mu+1/2)} \times \left(\int_0^1 \lambda(u-\pi_{N,\omega^{0,-1,\lambda}}u)^2(t)dt\right)^{1/2} \|\hat{e}_N\|_{0,\omega^{0,-2,\lambda}}$$
$$\le \frac{\lambda|a_2|\Gamma(1/2)}{\sqrt{2\mu e}\Gamma(\mu+1/2)} \times \|u-\pi_{N,\omega^{0,-1,\lambda}}u\|_{0,\omega^{0,-1,\lambda}}\|\hat{e}_N\|_{0,\omega^{0,-2,\lambda}}. \quad (4.14)$$

Similar to the proof of the well-posedness, we know that, under the conditions (4.5) or (4.6) on a_1 and a_2, the first two terms in the right hand side of (4.12) can be controlled by the left hand side term of the same equation. This fact, together with the above estimates, leads to

$$\|\hat{e}_N\|^2_{0,\omega^{0,-2,\lambda}} \le c\left(\frac{|a_1|}{\sqrt{e}}+\frac{|a_2|\Gamma(1/2)}{\sqrt{2\mu e}\Gamma(\mu+1/2)}\right)\|u-\pi_{N,\omega^{0,-1,\lambda}}u\|_{\omega^{0,-1,\lambda}}\|\hat{e}_N\|_{0,\omega^{0,-2,\lambda}}.$$

Thus,

$$\|\hat{e}_N\|_{0,\omega^{0,-2,\lambda}} \le c\|u-\pi_{N,\omega^{0,-1,\lambda}}u\|_{\omega^{0,-1,\lambda}},$$

where c is a constant depending on $|a_1|$ and $|a_2|$. Furthermore, noticing $\omega^{0,-1,\lambda}(t)=\lambda t^{-1}<\lambda t^{-\lambda-1}=\omega^{0,-2,\lambda}(t)$, we have

$$\|\hat{e}_N\|_{0,\omega^{0,-1,\lambda}} \le \|\hat{e}_N\|_{0,\omega^{0,-2,\lambda}} \le c\|u-\pi_{N,\omega^{0,-1,\lambda}}u\|_{\omega^{0,-1,\lambda}}.$$

Finally by using the triangle inequality and the approximation result established in Theorem 3.2 with $\alpha=0$, we obtain

$$\|e_N\|_{0,\omega^{0,-1,\lambda}} \le \|u-\pi_{N,\omega^{0,-1,\lambda}}u\|_{\omega^{0,-1,\lambda}}+\|\hat{e}_N\|_{0,\omega^{0,-1,\lambda}} \le cN^{-m}\|\partial_x^m u(t^{\frac{1}{\lambda}})\|_{0,\omega^{m,m-1,1}}.$$

The proof is completed. □

4.2 A fractional differential equation

Next, we consider the fractional differential equation with Caputo derivative:

$$\begin{cases} bu(x)-{}_0^CD_x^{\rho}u(x)=f(x), & x\in I, 1<\rho<2, \\ u(0)=0, u_x(0)=u_1, \end{cases} \quad (4.15)$$

where b and u_1 are real bounded constants. Some theoretical results concerning existence and regularity of the solution of this kind problems have been given in [16]. In particular, it has been proved that smooth solution can't be expected even if the data is smooth. In fact, if we take $f\equiv-\Gamma(5/2)$, then it can be verified that the function $u(x)=x^{3/2}$ is the unique solution of the problem (4.15) with $b=u_1=0$ and $\rho=3/2$. This simple example shows that even for $f\in C^{\infty}(\bar{I})$ it may happen that $u\notin C^2(\bar{I})$. Although precise structure of the solution to the fractional differential equation like (4.15) is unknown, the boundary singularity is believed to be one of the main features of boundary value problems associated to fractional differential equations. We are going to see below that the fractional spectral method proposed here is well adapted to such problems, and the exponential convergence can be reached in all our numerical examples by suitably choosing the parameter λ.

Applying Riemann-Liouville integral of order $\rho - 1$ to the both sides of (4.15), and noticing that

$$ {}_0I_x^{\rho-1}\,{}_0^CD_x^{\rho}u(x) = {}_0I_x^{\rho-1}\,{}_0I_x^{2-\rho}u_{xx} = {}_0I_x^{1}u_{xx} = u_x - u_x(0) = u_x - u_1, \quad (4.16) $$

we get the following equivalent integro-differential equation:

$$ \begin{cases} u_x = b\,{}_0I_x^{\rho-1}u(x) - {}_0I_x^{\rho-1}f(x) + c_0, & x \in I, 1 < \rho < 2, \\ u(0) = 0. \end{cases} \quad (4.17) $$

Therefore the fractional spectral method constructed in Section 4.1 for the integro-differential equation can be directly applied. It is worthwhile to emphasize that the above method is also applicable to fractional differential equations based on the Riemann-Liouville definition, using the relationship between Riemann-Liouville and Caputo derivatives given in (2.6). The efficiency of the proposed method will be demonstrated by mean of the numerical experiments presented in the next section.

4.3 Classical elliptic problems

In this subsection, we aim at demonstrating that the fractional spectral method can be equally constructed for traditional integer order differential equations. The main benefit of such method is its capability to produce numerical solutions with exponential accuracy for problems whose solutions have limited regularity at the boundaries.

Let's consider the classical elliptic problem with homogeneous Dirichlet boundary condition:

$$ \begin{cases} -\partial_x^2 u(x) = f(x), \ x \in I, \\ u(0) = u(1) = 0. \end{cases} \quad (4.18) $$

The method to be proposed is based on the following weak form: For $f \in L^2_{\omega^{1,4/\lambda-3,\lambda}}(I)$, find $u \in B^1_{\omega^{-1,-1,\lambda}}(I)$, such that

$$ \mathcal{A}(u, v) = \mathcal{F}(v), \quad \forall v \in B^1_{\omega^{-1,-1,\lambda}}(I), \quad (4.19) $$

where the bilinear form $\mathcal{A}(\cdot, \cdot)$ is defined by

$$ \mathcal{A}(u, v) = \left(\partial_x u(x), \partial_x\{\omega^{0,2/\lambda-2,\lambda}(x)v(x)\}\right), $$

and the functional $\mathcal{F}(\cdot)$ is given by

$$ \mathcal{F}(v) = (f(x), v(x))_{\omega^{0,2/\lambda-2,\lambda}}. $$

Then the fractional spectral method to (4.19) reads: Find $u_N \in B^1_{\omega^{-1,-1,\lambda}}(I) \cap S^{-1,-1}_{N,\lambda}(I)$, such that

$$ \mathcal{A}(u_N, v_N) = \mathcal{F}(v_N), \quad \forall v_N \in B^1_{\omega^{-1,-1,\lambda}}(I) \cap S^{-1,-1}_{N,\lambda}(I). \quad (4.20) $$

The proof of the well-posedness of the problems (4.19) and (4.20) will make use of the following Hardy inequality (see, e.g., [38] p428).

Lemma 4.5 *Let $a < b$ be two real numbers and $\gamma < 1$. Then for any $\phi \in L^2_\omega(a,b)$ with $\omega = (x-a)^\gamma$, it holds:*

$$\int_a^b \left(\frac{1}{x-a}\int_a^x \phi(y)dy\right)^2 (x-a)^\gamma dx \le \frac{4}{1-\gamma}\int_a^b \phi^2(x)(x-a)^\gamma dx. \tag{4.21}$$

Similarly, for any $\phi \in L^2_\omega(a,b)$ with $\omega = (b-x)^\gamma$, we have

$$\int_a^b \left(\frac{1}{b-x}\int_x^b \phi(y)dy\right)^2 (b-x)^\gamma dx \le \frac{4}{1-\gamma}\int_a^b \phi^2(x)(b-x)^\gamma dx. \tag{4.22}$$

We introduce a non-uniformly shifted Jacobi-weighted Sobolev space:

$$B^1_{0,\omega^{\alpha,\beta,1}}(I) = \left\{u : \partial_x^k u \in L^2_{\omega^{\alpha+k,\beta+k,1}}(I), u(0) = u(1) = 0, k = 0, 1\right\},$$

equipped with the inner product, norm and semi-norm:

$$\begin{aligned}(u,v)_{B^1_{0,\omega^{\alpha,\beta,1}}} &= (u,v)_{\omega^{\alpha,\beta,1}} + (\partial_x u, \partial_x v)_{\omega^{\alpha+1,\beta+1,1}},\\ \|v\|_{1,\omega^{\alpha,\beta,1}} &:= (v,v)^{1/2}_{B^1_{0,\omega^{\alpha,\beta,1}}}, \qquad |v|_{1,\omega^{\alpha,\beta,1}} := \|\partial_x v\|_{0,\omega^{\alpha+1,\beta+1,1}}\\ &:= (\partial_x v, \partial_x v)^{1/2}_{\omega^{\alpha+1,\beta+1,1}}.\end{aligned}$$

In the special case $\alpha = \beta = -1$, we have $B^1_{0,\omega^{-1,-1,1}}(I) = B^1_{\omega^{-1,-1,1}}(I)$, where the latter was defined in (3.2).

Lemma 4.6 *If $\alpha < 0, \beta < 0$, we have*

$$\|v\|_{0,\omega^{\alpha-1,\beta-1,1}} \le c\|\partial_x v\|_{0,\omega^{\alpha+1,\beta+1}}, \quad \forall v \in B^1_{0,\omega^{\alpha,\beta,1}}(I),$$

which implies the Poincaré-like inequality in $B^1_{0,\omega^{\alpha,\beta,1}}(I)$:

$$\|v\|_{0,\omega^{\alpha,\beta,1}} \le c\|\partial_x v\|_{0,\omega^{\alpha+1,\beta+1,1}}, \quad \forall v \in B^1_{0,\omega^{\alpha,\beta,1}}(I).$$

Proof For any given $v \in B^1_{0,\omega^{\alpha,\beta,1}}(I)$, applying Lemma 4.5 to $\partial_x v$ with $a = \frac{1}{2}, b = 1, \gamma = \alpha + 1$ gives

$$\int_{1/2}^1 v^2(x)(1-x)^{\alpha-1}dx \le \frac{4}{-\alpha}\int_{1/2}^1 (\partial_x v(x))^2(1-x)^{\alpha+1}dx,\ \alpha < 0.$$

Using the inequalities: for all $x \in [\frac{1}{2}, 1]$,

$$1 \le x^{\beta-1} \le 2^{1-\beta}, \forall \beta < 0; \quad 1 \le x^{\beta+1} \le 2^{-\beta-1},$$
$$\forall \beta < -1; \quad 2^{-\beta-1} \le x^{\beta+1} \le 1, \forall \beta \in (-1, 0),$$

we obtain for $\alpha < 0$, $\beta < 0$,

$$\int_{1/2}^{1} v^2(x)(1-x)^{\alpha-1}x^{\beta-1}dx \le 2^{1-\beta}\int_{1/2}^{1} v^2(x)(1-x)^{\alpha-1}dx$$
$$\le \frac{2^{3-\beta}}{-\alpha}\int_{1/2}^{1}(\partial_x v(x))^2(1-x)^{\alpha+1}dx \le c\int_{1/2}^{1}(\partial_x v(x))^2(1-x)^{\alpha+1}x^{\beta+1}dx. \tag{4.23}$$

Similarly, for $\alpha < 0$, $\beta < 0$ it follows from (4.21)

$$\int_{0}^{1/2} v^2(x)(1-x)^{\alpha-1}x^{\beta-1}dx \le c\int_{0}^{1/2}(\partial_x v(x))^2(1-x)^{\alpha+1}x^{\beta+1}dx. \tag{4.24}$$

Putting (4.23) and (4.24) together completes the proof. □

We will also need a Poincaré inequality in the weighted space $B^1_{\omega^{-1,-1,\lambda}}(I)$, which is given in the following lemma.

Lemma 4.7 *For all* $\lambda \in (0, 1]$*, it holds*

$$\|v\|_{0,\omega^{-1,-1,\lambda}} \le c\|\partial_x v\|_{0,\omega^{0,2/\lambda-2,\lambda}}, \quad \forall v \in B^1_{\omega^{-1,-1,\lambda}}(I).$$

Proof For all $v \in B^1_{\omega^{-1,-1,\lambda}}(I)$, we have

$$\begin{aligned}\|v\|^2_{0,\omega^{-1,-1,\lambda}} &= \int_0^1 \lambda(1-x^\lambda)^{-1}x^{-1}v^2(x)dx = \int_0^1 (1-s)^{-1}s^{-1}v^2(s^{1/\lambda})ds \\ &= \|v(x^{1/\lambda})\|^2_{0,\omega^{-1,-1,1}}, \; \|\partial_x v\|^2_{0,\omega^{0,2/\lambda-2,\lambda}} = \int_0^1 \lambda x^{1-\lambda}(\partial_x v(x))^2 dx \\ &= \int_0^1 \lambda^2(\partial_s v(s^{1/\lambda}))^2 ds = \lambda\|\partial_x v(x^{1/\lambda})\|^2_{0,\omega^{0,0,1}}.\end{aligned}$$

This means $v(x) \in B^1_{\omega^{-1,-1,\lambda}}(I)$ if and only if $v(x^{1/\lambda}) \in B^1_{\omega^{-1,-1,1}}(I)$. Moreover, using Lemma 4.6 with $\alpha = \beta = -1$ gives

$$\|v(x^{1/\lambda})\|_{0,\omega^{-1,-1,1}} \le c\|\partial_x v(x^{1/\lambda})\|_{0,\omega^{0,0,1}}.$$

Thus

$$\|v\|_{0,\omega^{-1,-1,\lambda}} \le c\|\partial_x v\|_{0,\omega^{0,2/\lambda-2,\lambda}}.$$

This proves the desired result. □

We are now in a position to establish the well-posedness of the weak problem and its fractional spectral approximation, and derive error estimates for the numerical solution.

Theorem 4.2 *For all* $f \in L^2_{\omega^{1,4/\lambda-3,\lambda}}(I)$*, the problem* (4.19) *is well-posed. Furthermore, if u is the solution of* (4.19)*, it holds*

$$\|u\|_{1,\omega^{-1,-1,\lambda}} \le c\|f\|_{0,\omega^{1,4/\lambda-3,\lambda}}. \tag{4.25}$$

Proof We employ the Lax-Milgram lemma to prove the well-posedness of problem (4.19). First, we prove the continuity and coercivity of the bilinear form $\mathcal{A}(\cdot,\cdot)$ in the space $B^1_{\omega^{-1,-1,\lambda}}(I)$. Applying the Hardy inequality (4.21) to $\partial_x u$ with $a=0,\ b=1,\ \gamma=1-\lambda$, we get

$$\int_0^1 x^{-1-\lambda}u^2(x)dx \le \frac{4}{\lambda}\int_0^1 x^{1-\lambda}(\partial_x u)^2 dx. \tag{4.26}$$

Then using the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mathcal{A}(u,v) &= (\partial_x u(x), \partial_x\{\omega^{0,2/\lambda-2,\lambda}(x)v(x)\})\\
&= (\partial_x u(x), \lambda(1-\lambda)x^{-\lambda}v(x)) + (\partial_x u(x),\ \lambda x^{1-\lambda}\partial_x v(x))\\
&= \left(x^{\frac{1-\lambda}{2}}\partial_x u(x), \lambda(1-\lambda)x^{\frac{-1-\lambda}{2}}v(x)\right) + (\partial_x u(x), \lambda x^{1-\lambda}\partial_x v(x))\\
&\le (1-\lambda)|u|_{1,\omega^{-1,-1,\lambda}}\left(\int_0^1 \lambda x^{-1-\lambda}v^2(x)dx\right)^{1/2} + |u|_{1,\omega^{-1,-1,\lambda}}|v|_{1,\omega^{-1,-1,\lambda}}\\
&\le (2(1-\lambda)+1)|u|_{1,\omega^{-1,-1,\lambda}}|v|_{1,\omega^{-1,-1,\lambda}}\\
&\le (3-\lambda)\|u\|_{1,\omega^{-1,-1,\lambda}}\|v\|_{1,\omega^{-1,-1,\lambda}}.
\end{aligned}$$

On the other side, in virtue of Lemma 4.7, we have

$$\begin{aligned}
\mathcal{A}(u,u) &= (\partial_x u(x),\ \lambda(1-\lambda)x^{-\lambda}u(x)) + (\partial_x u(x),\ \lambda x^{1-\lambda}\partial_x u(x))\\
&= \frac{\lambda(1-\lambda)}{2}\int_0^1 x^{-\lambda}(u^2(x))'dx + \|\partial_x u\|^2_{0,\omega^{0,2/\lambda-2,\lambda}}\\
&= \frac{\lambda^2(1-\lambda)}{2}\int_0^1 x^{-\lambda-1}u^2(x)dx + \|\partial_x u\|^2_{0,\omega^{0,2/\lambda-2,\lambda}}\\
&\ge \|\partial_x u\|^2_{0,\omega^{0,2/\lambda-2,\lambda}} \ge c\|u\|^2_{1,\omega^{-1,-1,\lambda}}.
\end{aligned} \tag{4.27}$$

Then, the inequality

$$|\mathcal{F}(v)| = |(f,v)_{\omega^{0,2/\lambda-2,\lambda}}| \le c\|f\|_{0,\omega^{1,4/\lambda-3,\lambda}}\|v\|_{1,\omega^{-1,-1,\lambda}} \tag{4.28}$$

means $\mathcal{F}(\cdot)$ is a continuous functional in the space $B^1_{\omega^{-1,-1,\lambda}}(I)$. Thus the well-posedness of problem (4.19) is guaranteed by the Lax-Milgram lemma. The stability inequality (4.25) is a direct consequence of (4.27) and (4.28). □

Theorem 4.3 *For all $f \in L^2_{\omega^{1,4/\lambda-3,\lambda}}(I)$, the fractional spectral approximation problem* (4.20) *admits a unique solution u_N, which satisfies*

$$\|u_N\|_{1,\omega^{-1,-1,\lambda}} \le c\|f\|_{0,\omega^{1,4/\lambda-3,\lambda}}. \tag{4.29}$$

Furthermore, if the solution of (4.19) *satisfies $u(x^{1/\lambda}) \in B^m_{\omega^{-1,-1,1}}(I)$, then we have*

$$\|u-u_N\|_{1,\omega^{-1,-1,\lambda}} \le cN^{1-m}\|\partial_x^m u(x^{1/\lambda})\|_{0,\omega^{m-1,m-1,1}}. \tag{4.30}$$

Proof The well-posedness of the approximation problem (4.20), and the stability inequality of its solution follow exactly the same lines as in Theorem 4.2. We now derive the error estimate. Using the standard error estimate for the Galerkin method of elliptic problems gives immediately

$$\|u-u_N\|_{1,\omega^{0,2/\lambda-2,\lambda}} \leq \inf_{v_N\in S_{N,\lambda}^{-1,-1}(I)} \|u-u_N\|_{1,\omega^{0,2/\lambda-2,\lambda}} \leq c\|u-\pi_{N,\omega^{-1,-1,\lambda}}u\|_{1,\omega^{0,2/\lambda-2,\lambda}}.$$

Then the estimate (4.30) follows from the approximation result established in Theorem 3.3 for the orthogonal projector $\pi_{N,\omega^{-1,-1,\lambda}}$. □

Remark 4.1 We have chosen this model elliptic problem for simplifying the analysis. However the main result remains valid for a more general equation with the additional term αu. In fact, in this case the corresponding bilinear form would be:

$$\mathcal{A}(u, v) := \left(\partial_x u(x), \partial_x\{\omega^{0,2/\lambda-2,\lambda}(x)v(x)\}\right) + \alpha\,(u(x), v(x))_{\omega^{0,2/\lambda-2,\lambda}(x)}.$$

It can be verified that using Lemma 4.7 the following control holds:

$$\|u\|_{0,\omega^{0,2/\lambda-2,\lambda}(x)} \leq \|u\|_{0,\omega^{-1,-1,\lambda}} \leq c\|\partial_x u\|_{0,\omega^{0,2/\lambda-2,\lambda}}.$$

Thus the well-posedness of the associated weak problem could be established by accordingly modifying the proof given in Section 4.3.

5 Implementation of the fractional spectral methods

We give in this section some implementation details of the proposed method. We will introduce suitable basis functions and numerical quadratures which allow efficient evaluations of the integrals involved in the discrete problems.

5.1 Integro-differential equation

For the fractional spectral method of the integro-differential (4.2), we propose to use the basis functions $\{J_{j+1}^{0,-1,\lambda}\}$, and express the numerical solution by $u_N(t) = \sum_{j=0}^{N} u_i J_{j+1}^{0,-1,\lambda}(t)$. Denote the unknown vector $U = (u_0, u_1, \ldots, u_N)^T$ and the matrices associated to the different terms by

$$\begin{aligned} A_{i,j} &= \left(\partial_t J_{j+1}^{0,-1,\lambda}, J_{i+1}^{-1,0,\lambda}\right), \\ B_{i,j} &= \left(J_{j+1}^{0,-1,\lambda}, J_{i+1}^{-1,0,\lambda}\right), \\ I_{i,j} &= \left({}_0I_t^{\mu} J_{j+1}^{0,-1,\lambda}, J_{i+1}^{-1,0,\lambda}\right) \end{aligned}$$

The entries of these matrices will be calculated in the following way:

$$
\begin{aligned}
A_{i,j} &= \int_0^1 \lambda(j+1)(1-t^\lambda)t^{\lambda-1}J_i^{1,0,\lambda}(t)J_j^{1,0,\lambda}(t)dt = \frac{1}{2}\delta_{i,j},\\
B_{i,j} &= \int_0^1 (1-t^\lambda)t^\lambda J_i^{1,0}(2t^\lambda-1)J_j^{0,1}(2t^\lambda-1)dt = \frac{1}{\lambda}\int_0^1 (1-s)s^{1/\lambda}J_i^{1,0}(2s-1)J_j^{0,1}(2s-1)ds\\
&= \frac{1}{\lambda}\sum_{k=0}^{N}\rho_k J_i^{1,0}(2\xi_k-1)J_j^{0,1}(2\xi_k-1),\\
I_{i,j} &= \lambda(j+1)\left({}_0I_t^{1+\mu}\left(t^{\lambda-1}J_j^{1,0,\lambda}(t)\right),\ (1-t^\lambda)J_i^{1,0,\lambda}(t)\right)\\
&= \frac{\lambda(j+1)}{\Gamma(1+\mu)}\int_0^1\int_0^t (t-s)^\mu s^{\lambda-1}J_j^{1,0}(2s^\lambda-1)ds(1-t^\lambda)J_i^{1,0}(2t^\lambda-1)dt\\
&= \frac{j+1}{\Gamma(1+\mu)}\int_0^1\int_0^1 t^{\lambda+\mu}(1-\sigma^{\frac{1}{\lambda}})^\mu J_j^{1,0}(2t^\lambda\sigma-1)d\sigma(1-t^\lambda)J_i^{1,0}(2t^\lambda-1)dt\\
&= \frac{j+1}{\lambda\Gamma(1+\mu)}\int_0^1\int_0^1 (1-\sigma^{\frac{1}{\lambda}})^\mu J_j^{1,0}(2s\sigma-1)d\sigma s^{\frac{\mu+1}{\lambda}}(1-s)J_i^{1,0}(2s-1)ds\\
&= \frac{j+1}{\lambda\Gamma(1+\mu)}\sum_{m=0}^{N}\sum_{n=0}^{M} J_j^{1,0}(2\zeta_m\hat{\zeta}_n-1)\hat{\omega}_n J_i^{1,0}(2\zeta_m-1)\omega_m,\quad \text{with } M=\left\lceil\frac{N-1}{2}\right\rceil,
\end{aligned}
$$

where the points sets $\{\xi_k\}_{k=0}^N$ and $\{\zeta_m\}_{m=0}^N$ are zeros of the shifted Jacobi polynomials $J_{N+1}^{1,1/\lambda}(2x-1)$ and $J_{N+1}^{1,\frac{\mu+1}{\lambda}}(2x-1)$, respectively. $\{\rho_k\}_{k=0}^N$ and $\{\omega_m\}_{m=0}^N$ are respectively the associated Gauss weights. $\{\hat{\zeta}_n\}_{n=0}^M$ and $\{\hat{\omega}_n\}_{n=0}^M$ are the zeros of the orthogonal polynomial of degree $M+1$ and the Gauss weights associated to the nonclassical weight function $(1-x^{\frac{1}{\lambda}})^\mu$. We present below a procedure to compute these nonclassical Gauss quadrature nodes and weights. We first note that the moments M_r defined in (2.8) corresponding to the weight function $\omega(x,\lambda)=(1-x^{\frac{1}{\lambda}})^\mu$ satisfies

$$M_r = \lambda B(\lambda(r+1),\ \mu+1),$$

where $B(\cdot,\cdot)$ is the Euler Beta function. In fact, by definition (2.8) we have

$$M_r = \int_0^1 x^r(1-x^{\frac{1}{\lambda}})^\mu dx.$$

Making the variable change $x=t^\lambda$ gives

$$M_r = \lambda\int_0^1 t^{\lambda r+\lambda-1}(1-t)^\mu dt = \lambda B(\lambda(r+1),\ \mu+1).$$

Then we can follow the three-step algorithm proposed in [19] to calculate $\{\hat{\zeta}_n\}_{n=0}^M$ and $\{\hat{\omega}_n\}_{n=0}^M$, and arrive at the linear system

$$(A+B+I)U = F,$$

satisfied by the solution of (4.20), where F is the right hand side vector defined by

$$F_i = \left(f, J_{i+1}^{-1,0,\lambda}\right), \ i = 0, 1, 2, \ldots, N.$$

The evaluation of F_i can be either exact or approximative, depending on the f used in the actual calculation.

Remark 5.1 As pointed in [19] the calculation of the moments M_r can be numerically problematic when the number of points is large: in order to obtain the double precision entries of the matrices, one would have to perform with about 40 digits operations. A way to deal with this problem is to avoid computing with the non-conventional weight function. This can be done by the following reformulation:

$$\begin{aligned}
I_{i,j} &= \lambda(j+1)\left({}_0I_t^{1+\mu}\left(t^{\lambda-1}J_j^{1,0,\lambda}(t)\right),\ (1-t^{\lambda})J_i^{1,0,\lambda}(t)\right)\\
&= \frac{j+1}{\lambda\Gamma(1+\mu)}\int_0^1\int_0^1 (1-\sigma^{\frac{1}{\lambda}})^{\mu} J_j^{1,0}(2s\sigma-1)d\sigma s^{\frac{\mu+1}{\lambda}}(1-s)J_i^{1,0}(2s-1)ds\\
&= \frac{j+1}{\lambda\Gamma(1+\mu)}\int_0^1\int_0^1 (1-\sigma)^{\mu}\left(\frac{1-\sigma^{\frac{1}{\lambda}}}{1-\sigma}\right)^{\mu} J_j^{1,0}(2s\sigma-1)d\sigma s^{\frac{\mu+1}{\lambda}}(1-s)J_i^{1,0}(2s-1)ds\\
&= \frac{j+1}{\lambda\Gamma(1+\mu)}\sum_{m=0}^{N}\sum_{n=0}^{M}\left(\frac{1-\hat{\zeta}_n^{\frac{1}{\lambda}}}{1-\hat{\zeta}_n}\right)^{\mu} J_j^{1,0}(2\zeta_m\hat{\zeta}_n-1)\hat{\omega}_n J_i^{1,0}(2\zeta_m-1)\omega_m,
\end{aligned}$$

where the points sets $\{\hat{\zeta}_n\}_{n=0}^{M}$ and $\{\zeta_m\}_{m=0}^{N}$ are zeros of the shifted Jacobi polynomials $J_{M+1}^{\mu,0}(2x-1)$ and $J_{N+1}^{1,\frac{\mu+1}{\lambda}}(2x-1)$, respectively. $\{\hat{\omega}_n\}_{n=0}^{M}$ and $\{\omega_m\}_{m=0}^{N}$ are respectively the associated Gauss weights. For the special cases $\lambda = 1/p$, $p = 1, 2, 3, \cdots$, we have $\frac{1-\sigma^{\frac{1}{\lambda}}}{1-\sigma} = \sum_{k=0}^{p-1}\sigma^k$. This treatment is particularly efficient in these cases.

5.2 Elliptic equation

Next we describe implementation details for problem (4.20). we use the basis functions $\{J_{j+2}^{-1,-1,\lambda}\}_{j=0}^{N}$ to present the numerical solution, i.e., $u_N(x) = \sum_{j=0}^{N} u_j J_{j+2}^{-1,-1,\lambda}(x)$. Bringing this expression into (4.20) results in a stiffness matrix $\hat{A}$ with matrix entries

$$\hat{A}_{i,j} = \left(\partial_x J_{j+2}^{-1,-1,\lambda}(x),\ \partial_x\{\lambda x^{1-\lambda} J_{i+2}^{-1,-1,\lambda}(x)\}\right),\quad i, j = 0, 1, 2, \ldots, N.$$

A direct calculation gives

$$
\begin{aligned}
\hat{A}_{i,j} &= \Big(\partial_x J_{j+2}^{-1,-1,\lambda}(x), \lambda(1-\lambda)x^{-\lambda} J_{i+2}^{-1,-1,\lambda}(x)\Big) + \Big(\partial_x J_{j+2}^{-1,-1,\lambda}(x), \lambda x^{1-\lambda}\partial_x J_{i+2}^{-1,-1,\lambda}(x)\Big)\\
&= -\lambda^2(1-\lambda)(j+1)\Big(x^{\lambda-1}L_{j+1}(2x^\lambda-1), (1-x^\lambda)J_i^{1,1}(2x^\lambda-1)\Big)\\
&\quad + \lambda^2(i+1)(j+1)\Big(x^{\lambda-1}L_{j+1}(2x^\lambda-1), \lambda L_{i+1}(2x^\lambda-1)\Big)\\
&= -\lambda^2(1-\lambda)(j+1)\int_0^1 (1-x^\lambda)x^{\lambda-1}L_{j+1}(2x^\lambda-1)J_i^{1,1}(2x^\lambda-1)dx + \frac{\lambda^2(i+1)^2}{2i+3}\delta_{ij}\\
&= -\lambda(1-\lambda)(j+1)\int_0^1 (1-s)L_{j+1}(2s-1)J_i^{1,1}(2s-1)ds + \frac{\lambda^2(i+1)^2}{2i+3}\delta_{ij}\\
&= -\lambda(1-\lambda)(j+1)\int_0^1 (1-s)\left[\frac{j+2}{2j+3}J_{j+1}^{1,0}(2s-1) - \frac{j+1}{2j+3}J_j^{1,0}(2s-1)\right]J_i^{1,1}(2s-1)ds\\
&\qquad + \frac{\lambda^2(i+1)^2}{2i+3}\delta_{ij}.
\end{aligned}
$$

Clearly, $\hat{A}_{i,j} = 0$ if $i < j$. For $i \geq j$, we have

$$
\hat{A}_{i,j} = -\lambda(1-\lambda)(j+1)\sum_{k=0}^{N} L_{j+1}(2x_k-1)J_i^{1,1}(2x_k-1)\bar{\omega}_k + \frac{\lambda^2(i+1)^2}{2i+3}\delta_{ij},
$$

where $\{x_k\}_{k=0}^N$ are the zeros of classical shifted Jacobi polynomials $J_{N+1}^{1,0}(2x-1)$, and $\{\bar{\omega}_k\}_{k=0}^N$ are the associated Gauss-Lobatto weights. Thus we obtain a lower triangular system to solve:

$$
\hat{A}\hat{U} = \hat{F},
$$

where $\hat{U} = (u_0, u_1, \cdots, u_N)^T$ and $\hat{F}_i = (f, J_{i+2}^{-1,-1,\lambda})_{\omega^{0,2/\lambda-2,\lambda}}$.

6 Numerical results

In this section, we present some numerical examples to demonstrate the accuracy of the proposed fractional spectral methods and to verify the error estimates derived in the previous section. When solving problem (4.1) in the following numerical tests, we fix the coefficients a_1 and a_2 to be 1.

Example 6.1 We start by considering the integro-differential problem (4.1) with $\mu = 1/2$ and the source term $f(t) = 1/2t^{-1/2} - \Gamma(3/2)t - t^{1/2}$.

It can be verified that the exact solution is $u(t) = t^{1/2}$, which has limited regularity at the left end point. The results obtained by using the scheme (4.2), with $\lambda = 1/8, 1/10$, and $1/12$ respectively, are plotted in Fig. 1, showing exponential decay of the errors with respect to N for all employed values of λ. This result is in a good agreement with the theoretical prediction given in Theorem 4.1, stating that the convergence of numerical solution is exponential if $u(x^{1/\lambda})$ is smooth.

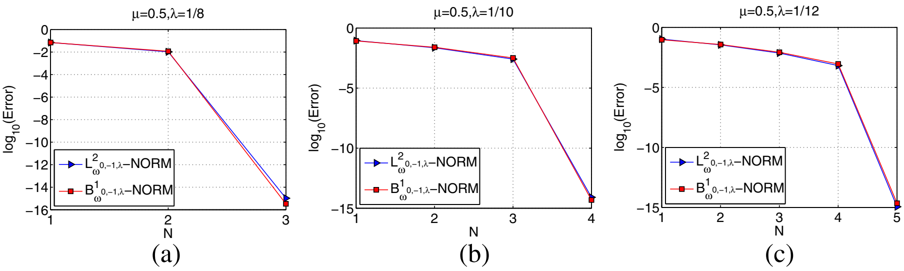

Fig. 1 Error decays with respect to N for problem (4.1) with exact solution $u(t) = t^{1/2}$ using fractional spectral approximation spaces with **a** $\lambda = \frac{1}{8}$; **b** $\lambda = \frac{1}{10}$; **c** $\lambda = \frac{1}{12}$

Example 6.2 Consider the (4.1) with $f(t) = \sqrt{3}t^{\sqrt{3}-1} - \frac{\Gamma(\sqrt{3}+1)}{\Gamma(\sqrt{3}+1+\mu)}t^{\sqrt{3}+\mu} - t^{\sqrt{3}}$ and $\mu = 0.1$ or 0.9. Its exact solution is $u(t) = t^{\sqrt{3}}$.

Notice that $\sqrt{3}$ is an irrational number, it is impossible to make $u(t^{1/\lambda})$ smooth with a rational λ. One can of course take $\lambda = 1/\sqrt{3}$ such that $u(t^{1/\lambda})$ being smooth, but what we want to demonstrate here is that by using reasonably small λ, one can make $u(t^{1/\lambda})$ smooth enough so that high accurate numerical solution can still be achieved using corresponding fractional spectral approximation. The numerical results shown in Fig. 2a–c for a number of parameters μ and λ confirm the above claim.

Example 6.3 Still consider (4.1), now with artificially constructed source term $f(t) = 20.5t^{19.5} + \frac{61}{3}t^{58/3} + \frac{1}{2}t^{-1/2} + \frac{1}{3}t^{-2/3} - \frac{\Gamma(21.5)}{\Gamma(21.5+\mu)}t^{20.5+\mu} - \frac{\Gamma(\frac{64}{3})}{\Gamma(\frac{64}{3}+\mu)}t^{61/3+\mu} - \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+\mu)}t^{1/2+\mu} - \frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{4}{3}+\mu)}t^{1/3+\mu} - (t^{1/2} + t^{1/3})(t^{20} + 1)$ and $\mu = 0.9$.

The corresponding exact solution is $u(t) = (t^{1/2} + t^{1/3})(t^{20} + 1)$, which has composite singularity in its first order derivative. The error decay history as a function of the fractional polynomial degree N is plotted in Fig. 3 for several values λ.

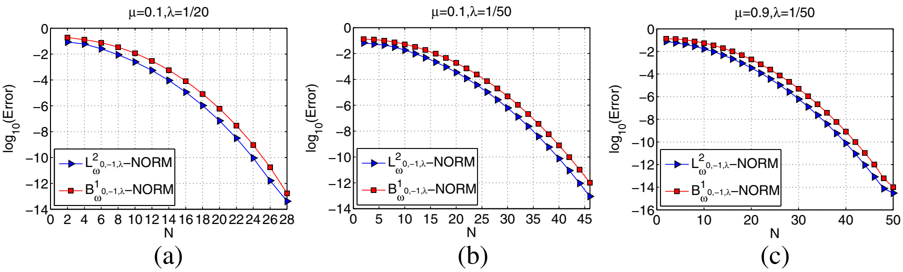

Fig. 2 Error decays for problem (4.1) with exact solution $u(t) = t^{\sqrt{3}}$ with several values of μ and fractional spaces parameter λ: **a** $\mu = 0.1, \lambda = \frac{1}{20}$; **b** $\mu = 0.1, \lambda = \frac{1}{50}$; **c** $\mu = 0.9, \lambda = \frac{1}{50}$;

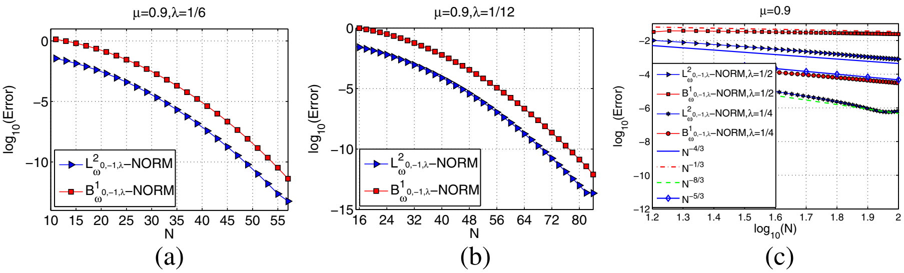

Fig. 3 Error behavior for problem (4.1) with $u(t) = (t^{1/2} + t^{1/3})(t^{20} + 1)$ for: **a** $\lambda = \frac{1}{6}$; **b** $\lambda = \frac{1}{12}$; **c** $\lambda = \frac{1}{2}$ and $\frac{1}{4}$

The subfigures Fig. 3a and b show that the exponential convergence is achieved as expected for $u(x^{1/\lambda})$ is smooth for $\lambda = 1/6$ and $1/12$. The values of λ used in the subfigure Fig. 3c are 1/2 and 1/4. In this case $u(x^{1/\lambda})$ is no longer smooth, and the method based on the corresponding fractional polynomial space leads to only algebraic convergence, as shown in Fig. 3c. For ease of observation, the lines of slopes $N^{-4/3}$, $N^{-1/3}$, $N^{-8/3}$, and $N^{-5/3}$ are also plotted in Fig. 3c, which clearly indicates that the convergence rate is close to $N^{-4/3}$ for $\lambda = \frac{1}{2}$ and $N^{-8/3}$ for $\lambda = \frac{1}{4}$ in the $L^2_{\omega^{0,-1,\lambda}}(I)$−norm, and the convergence rate in the $B^1_{\omega^{0,-1,\lambda}}(I)$−norm is approximately $N^{-1/3}$ for $\lambda = \frac{1}{2}$ and $N^{-5/3}$ for $\lambda = \frac{1}{4}$. This is in a quite good agreement with the theoretical estimate in (4.7).

Example 6.4 In this example, we take an arbitrary smooth force function $f(t) = \sin(4\pi t)$, for which the exact solution is unknown. In order to investigate the error behavior of the numerical solution, we use a numerical solution obtained with a big enough N, i.e., $N = 100$, as the "exact" solution. Fix $\mu = 0.1$.

The error history as a function of N for a number of λ is presented in Fig. 4. It is observed from Fig. 4a for $\lambda = 1$ that the convergence rate is close to $N^{-2(3+\mu)}$ in the $L^2_{\omega^{0,-1,\lambda}}(I)$−norm and $N^{-2(3+\mu)+1}$ in the $B^1_{\omega^{0,-1,\lambda}}(I)$−norm. The result for $\lambda = 1/4$ in Fig. 4b implies a convergence rate close to $N^{-8(3+\mu)}$ in the $L^2_{\omega^{0,-1,\lambda}}(I)$−norm and $N^{-8(3+\mu)+1}$ in the $B^1_{\omega^{0,-1,\lambda}}(I)$−norm. The error behavior for $\lambda = 1/10$ is plotted in Fig. 4c in semi-log scale. It can be seen that the error variations are almost linear versus the fractional polynomial degrees, which means that the convergence rate is exponential. According to Theorem 4.1, the obtained result in these three figures conjectures that the transformation $u(t^{1/\lambda})$ of the exact solution belongs to $B^{2(3+\mu)/\lambda}_{\omega^{0,-1,1}}(I)$, and it becomes smooth for suitable λ. *In fact, this conjecture can be proved by using some existing results. For instance, using a result in* [11]*, for smooth forcing function, the solution of* (4.1) *can be expressed as*

$$u(t) = \sum_{j,k=0,1,\ldots} \gamma_{j,k} t^{j+k\mu} + u_s(t), \quad t \in I, \tag{6.1}$$

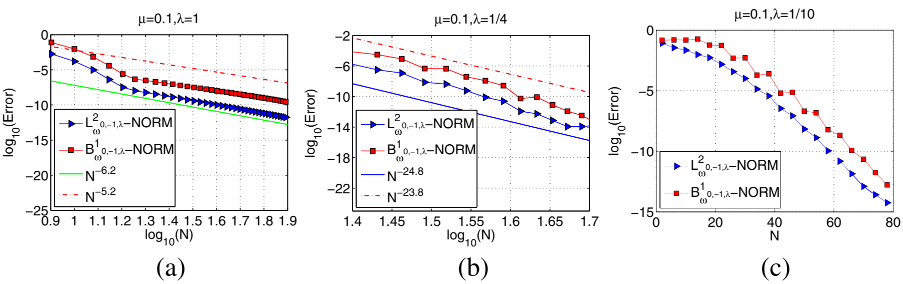

Fig. 4 Errors versus N for $\mu = 0.1$. **a** $\lambda = 1$; **b** $\lambda = \frac{1}{4}$; **c** $\lambda = \frac{1}{10}$

where $\gamma_{j,k}$ are constants, and $u_s(\cdot) \in C^\infty(I)$.
Furthermore, the (4.1) with $f(t) = \sin(4\pi t)$ implies $u(0) = 0, \partial_t u(0) = 0$. Differentiating the (4.1) one time yields $\partial_t^2 u(0) = 4\pi$. Differentiating (4.1) twice gives $\partial_t^3 u(0) = 0$. Repeating this operation one more time gives $\partial_t^4 u(0) \sim t^{\mu-1}$. Thus it follows from (6.1):

$$u(t) = 2\pi t^2 + \gamma_{3,1} t^{3+\mu} + \sum_{j+k\mu>3+\mu} \gamma_{j,k} t^{j+k\mu} + u_s(t). \tag{6.2}$$

That is, $u(t^{1/\lambda}) \in B^{2(3+\mu)/\lambda-\varepsilon}_{\omega^{0,-1,1}}(I)$ for any $\varepsilon > 0$.

Example 6.5 In this last example, we consider the elliptic problem (4.18) with two source terms: (i) $f(x) = \pi^2 \sin(\pi x)$; (ii) $f(x) = \frac{12}{169} x^{-14/13}$. In the case (i) the problem (4.18) admit a unique smooth solution $u(x) = sin(\pi x)$. In the case (ii) the exact solution is $u(x) = x^{12/13} - x$, which has a limited regularity at the left end point.

Figure 5a and b show the errors versus N for the smooth solution (i) for $\lambda = 1$ and 1/2. As expected, the errors exhibit exponential convergence decay, since in this case $u(x^{1/\lambda})$ is smooth for both values of λ. In the case the exact solution $u(x)$ has limited regularity, using the classical spectral method based on classical polynomial approximations, i.e., $\lambda = 1$, will result in poor convergence, as shown in Fig. 5c where the errors versus the polynomial degree N for the nonsmooth solution (ii) for $\lambda = 1$. It is observed from Fig. 5c that all the error curves are straight lines in the log-log representation, which indicates that only algebraic accuracy is obtained. This poor accuracy for the limited regular solution can be significantly improved by using the fractional spectral method introduced in this paper. The errors of the numerical solution using the fractional spectral method with $\lambda = 1/13$ are plotted in Fig. 5d. Clearly, the errors decay exponentially, and the accurate solution is obtained with $N = 11$.

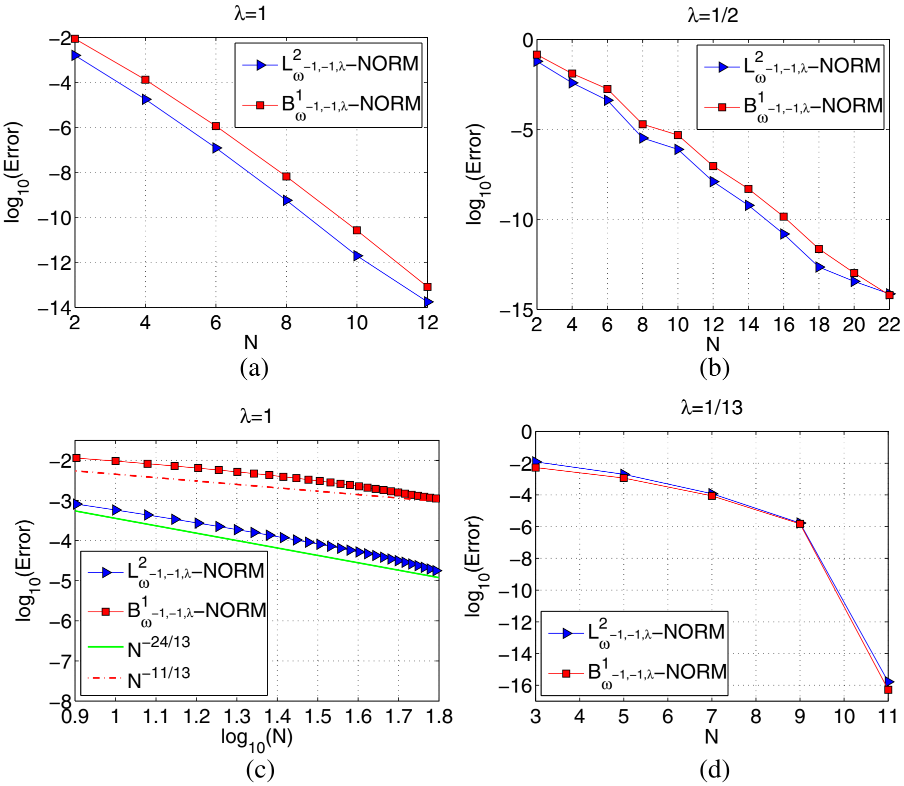

Fig. 5 (i) Errors versus N for the fractional spectral approximation to the exact solution $u(x) = \sin(\pi x)$ of (4.18) for **a** $\lambda = 1$; **b** $\lambda = \frac{1}{2}$. (ii) Errors versus N in the case the exact solution $u(x) = x^{12/13} - x$ for **c** $\lambda = 1$; **d** $\lambda = \frac{1}{13}$

7 Concluding remarks

We have developed and analyzed a fractional spectral method for a kind of integro-differential equations. The proposed method makes use of the fractional polynomials, also known as Müntz polynomials, constructed through a transformation of the traditional Jacobi polynomials. The most remarkable feature of the method is its capability to achieve spectral convergence for the solution with limited regularity. We derived useful error estimates for some weighted projection operators in the Müntz polynomial spaces. Based on these approximation results, several efficient fractional spectral methods for a class of integro-differential equations with singular kernels, fractional differential equations, and the classical elliptic equation with singular forcing function were constructed, together with some error estimates for the proposed approaches. A series of numerical experiments were carried out to verify the theoretical claims.

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