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Numerical analysis of a second order algorithm for simplified magnetohydrodynamic flows

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Abstract In this paper, we construct a second order algorithm based on the spectral deferred correction method for the time-dependent magnetohydrodynamics flows at a low magnetic Reynolds number. We present a complete theoretical analysis to prove that this algorithm is unconditionally stable, consistent and second order accuracy. Finally, two numerical examples are given to illustrate the convergence and effectiveness of our algorithm.

Keywords Spectral deferred correction · Backward-Euler · Finite element method · Magnetohydrodynamics

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1 Introduction

Magnetohydrodynamics (MHD) is the study of the interaction of electromagnetic fields and conducting fluids. The MHD systems describe the behavior of the electrically conducting fluids in the presence of an external magnetic field. The field of MHD is initiated by Alfvén [1] and widely develop to other science fields including astrophysics, geophysics and engineering, such as liquid metal cooling of nuclear reactors, controlled thermonuclear fusion and sea water propulsion, see [2–4]. Both in mathematical theory and practical applications, the study of MHD is very important and significant. In general, most terrestrial applications, such as liquid metals, involve small magnetic Reynolds number. In these cases, the magnetic field induced by the electrically conducting fluid motion can usually be negligible compared with the external magnetic field. Neglecting the induced magnetic field can reduce the general MHD flows to the simplified MHD flows, which are studied in this paper.

The MHD modeling consists of a coupling between the Navier-Stokes equations of fluid dynamics and the Maxwell equations of electromangetism. One can find the theoretical analysis and mathematical modeling of the MHD equations in [5]. Gunzburger, Meir, and Peterson [7] proved the existence and uniqueness of weak solutions of stationary incompressible MHD equations. An optimal convergence estimate of a finite element discretization of the equilibrium MHD equations is given by Meir and Schmidt in [9]. Other study of stationary MHD equations can be found in [10–12]. For unsteady MHD equations, a formulation for evolutionary MHD was presented by Schmidt in [13], where they established the existence of global-in-time weak solutions through the Galerkin method. Yuksel and Ingram [8] gave a comprehensive error analysis for both the semi-discrete and a fully-discrete approximate of time dependent MHD flow at small magnetic Reynolds number. Layton, Tran and Trenchea [14] introduced two partitioned methods to solve evolutionary MHD at low magnetic Reynold numbers and gave a complete stability and error analysis. Yuksel and Isik [15] provided a stability and convergence analysis of an finite element discretization for time-dependent MHD flows with Backward-Euler discretization at low magnetic Reynold numbers.

In this paper, we aim to construct a second order algorithm for the time-dependent MHD flows at a low magnetic Reynolds number. To that end, we employ the spectral deferred correction (SDC) method. The SDC method was proposed for stiff ordinary differential equations (ODEs) by Dutt, Greengrad, and Rokhlin in [16] and further developed by Minion et al., see [17, 18] and the references therein. The SDC methods allow one to automatically increase the accuracy of a stable low order time-stepping method through using spectral integration on Gaussian quadrature nodes and constructing the corrections. This can avoid instabilities and conditioning problems associated with repeated differentiations, such as the backward differentiation formulas (BDF) based high order methods. Wilson, Labovsky, and Trenchea proposed a second order scheme based on SDC method for the evolutionary full MHD equations at high magnetic Reynolds number in [19]. To construct a second order algorithm, the SDC method is employed in this paper and performs as follows. First, we have an unconditionally stable first order time-stepping method based on Backward-Euler time discretization, whose stability and error estimate were completely proved in [15].

Then we introduce the SDC method to this first order time-stepping method to improve its accuracy and obtain a second order time-stepping method without losing stability. Using this second order time-stepping method, we need to compute two first order accurate approximations instead of a single second order accurate approximation, which is thought to be less costly.

The simplified MHD flows at a low magnetic Reynolds number considered in this paper are the following, see, e.g. [8, 14, 15]: Given body force **f**, magnetic **B**, and time $T > 0$, find fluid velocity $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, pressure $p : \Omega \times [0, T] \to \mathbb{R}$ and electric potential $\phi : \Omega \times [0, T] \to \mathbb{R}$ such that:

$$\begin{aligned} &\frac{1}{N}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \frac{1}{M^2}\Delta\mathbf{u} + \nabla p = \mathbf{f} + \mathbf{B}\times\nabla\phi + (\mathbf{u}\times\mathbf{B})\times\mathbf{B}, \\ &\nabla\cdot\mathbf{u} = 0, \\ &\Delta\phi = \nabla\cdot(\mathbf{u}\times\mathbf{B}), \end{aligned} \tag{1}$$

subject to the homogeneous Dirichlet boundary conditions and the initial condition

$$\begin{aligned} &\mathbf{u} = 0 \qquad on\ \partial\Omega\times[0, T], \\ &\phi = 0 \qquad on\ \partial\Omega\times[0, T], \\ &\mathbf{u}(x, 0) = \mathbf{u}_0(x) \quad \forall\, x\in\Omega. \end{aligned} \tag{2}$$

Here, the domain $\Omega \subset \mathbb{R}^d$ $(d = 2\, or\, 3)$ is a convex polygon or polyhedra, N is interaction parameter and M is Hartmann number. Further, $\mathbf{u}_0(x) \in H_0^1(\Omega)^d$ and $\nabla\cdot\mathbf{u}_0 = 0$.

This paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we present the first order Backward-Euler method. A complete theoretical analysis of stability, consistency and error estimate is also shown in Section 3. In Section 4, the second order method based on the SDC method and its stability, consistency and error estimate are offered. Two numerical examples are given in Section 5. One is to compute the rate of convergence of our second order method to testify the correctness of our theoretical analysis. The other one compares the effectiveness of our second order method with Crank-Nicolson method and shows that our method performs better in this example.

2 Notations and preliminaries

Throughout this paper, we denote the $L^2(\Omega)$ inner products and corresponding norms by $(\cdot, \cdot)$ and$\|\cdot\|$, respectively. The $L^p(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$. The Sobolev spaces $W_p^k(\Omega)$, $k \geq 0$, see [6], are equipped the norms with $\|\cdot\|_{W_p^k}$ and the corresponding semi-norms $|\cdot|_{W_p^k}$. We write $H^k(\Omega) := W_2^k(\Omega)$, and denote $\|\cdot\|_k$ as the norms in $H^k(\Omega)$. The spaces $H^{-k}(\Omega)$ denote the dual spaces of $H_0^k(\Omega)$. C is a positive constant which is different in different places but independent of mesh size and time step. In addition, we define the following function spaces:

$$L^p(0,T;L^s(\Omega)) := \{v \in L^p(0,T;L^s(\Omega)) : (\int_0^T \|v(\cdot,t)\|_{L^s}^p dt)^{\frac{1}{p}} < \infty\},$$
$$L^\infty(0,T;L^s(\Omega)) := \{v \in L^\infty(0,T;L^s(\Omega)) : EssSup_{[0,T]}\|v(\cdot,t)\|_{L^s} < \infty\},$$
$$L^p(0,T;W_q^k(\Omega)) := \{v \in L^p(0,T;W_q^k(\Omega)) : (\int_0^T \|v(\cdot,t)\|_{W_q^k}^p dt)^{\frac{1}{p}} < \infty\},$$
$$L^\infty(0,T;W_q^k(\Omega)) := \{v \in L^\infty(0,T;W_q^k(\Omega)) : EssSup_{[0,T]}\|v(\cdot,t)\|_{W_q^k} < \infty\},$$

Here, $1 \le p < \infty$, $1 \le s \le \infty$, $1 \le q < \infty$. Furthermore, we have the following denotations.

$$\|v\|_{p,k} := (\int_0^T \|v(\cdot,t)\|_k^p dt)^{\frac{1}{p}} \qquad for\ v \in L^p(0,T;H^k(\Omega)),$$
$$\|v\|_{\infty,k} := EssSup_{[0,T]}\|v(\cdot,t)\|_k \qquad for\ v \in L^\infty(0,T;H^k(\Omega)).$$

The velocity, pressure, and electric potentials spaces are denoted as follows.

$$X := H_0^1(\Omega)^d = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_{\partial\Omega} = 0\},$$
$$Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\},$$
$$S := H_0^1(\Omega) = \{\psi \in H^1(\Omega) : \psi|_{\partial\Omega} = 0\}.$$

The divergence free space V is given by

$$V := \{\mathbf{v} \in X : (\nabla \cdot \mathbf{v}, q) = 0 \quad \forall q \in Q)\}.$$

Then, a weak formulation of Eq. 1 with Eq. 2 is: Find $\mathbf{u} : [0,T] \to X$, $p : [0,T] \to Q$ and $\phi : [0,T] \to S$ for $t \in (0,T]$ satisfying

$$\begin{aligned}
&\frac{1}{N}(\mathbf{u}_t, \mathbf{v}) + \frac{1}{N}(\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v}) + \frac{1}{M^2}(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) \\
&\qquad\qquad + (-\nabla\phi + (\mathbf{u}\times\mathbf{B}), \mathbf{v}\times\mathbf{B}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in X, \\
&(\nabla\cdot\mathbf{u}, q) = 0 \quad \forall q \in Q, \\
&(-\nabla\phi + (\mathbf{u}\times\mathbf{B}), -\nabla\psi) = 0 \quad \forall \psi \in S, \\
&\mathbf{u}(x,0) = \mathbf{u}_0(x) \quad a.e.\ x \in \Omega.
\end{aligned} \tag{3}$$

We define the trilinear form as usual.

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w}, \mathbf{v}).$$

The following lemmas are basic and widely used in the study of Navier-Stokes equations and MHD equations, i.e., Yuksel [15].

Lemma 1 *If* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$, *then*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le C(\Omega)\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|,$$
$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le C(\Omega)\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|.$$

Lemma 2 (The discrete Gronwall's lemma) *Suppose that n and N are nonnegative integers, $n \leq N$. The real numbers $a_n, b_n, c_n, \kappa_n, \triangle t, C$ are nonnegative and satisfy that*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \leq \Delta t \sum_{n=0}^{N} \kappa_n a_n + \Delta t \sum_{n=0}^{N} c_n + C.$$

Then,

$$a_N + \Delta t \sum_{n=0}^{N} b_n \leq exp\left(\Delta t \sum_{n=0}^{N} \frac{\kappa_n}{1 - \Delta t \kappa_n}\right)\left(\Delta t \sum_{n=0}^{N} c_n + C\right),$$

provided that $\Delta t \kappa_n \leq 1$ for each n.

Let Π_h be a set of triangulations of Ω with $\overline{\Omega} = \bigcup_{K \in \Pi_h} K$, which is assumed to be uniformly regular as $h \to 0$. Here $h = \sup_{K \in \Pi_h} diam(K)$. We choose the finite element spaces $X_h \subset X$, $Q_h \subset Q$, $S_h \subset S$, and assume that $X_h \times Q_h$ satisfies the usual discrete inf-sup condition, $\inf_{q \in Q_h} \sup_{\mathbf{v} \in X_h} \frac{(q, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\| \|q\|} \geq C > 0$. Besides, we also assume that X_h, Q_h, and S_h satisfy approximation properties of piecewise polynomials on quasi-uniform meshes of local degrees $k, k-1, k$ respectively. That is to say

$$\begin{aligned}
&\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\| \leq Ch^{k+1} \|\mathbf{u}\|_{k+1} && \mathbf{u} \in H^{k+1}(\Omega)^d,\\
&\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\|_1 \leq Ch^{k} \|\mathbf{u}\|_{k+1} && \mathbf{u} \in H^{k+1}(\Omega)^d,\\
&\inf_{\psi \in S_h} \|\phi - \psi\|_1 \leq Ch^{k} \|\phi\|_{k+1} && \phi \in H^{k+1}(\Omega),\\
&\inf_{q \in Q_h} \|p - q\| \leq Ch^{k} \|p\|_{k} && p \in H^{k}(\Omega).
\end{aligned}$$

Define the subspace V_h of X_h as follows.

$$V_h := \{\mathbf{v} \in X_h : (\nabla \cdot \mathbf{v}, q) = 0 \quad \forall q \in Q_h\}.$$

Let $t^n = n\Delta t$ for $n = 0, 1, 2, \cdots, m$. We denote $\mathbf{u}^n = \mathbf{u}(t^n)$ and similarly for other variables. Then, define the following discrete norms.

$$|\|\mathbf{v}\||_{\infty,k} := \max_{0 \leq n \leq m} \|\mathbf{v}^n\|_k, \quad |\|\mathbf{v}\||_{p,k} := (\Delta t \sum_{n=0}^{m} \|\mathbf{v}^n\|_k^p)^{\frac{1}{p}}.$$

Further, define the Galerkin projection operator $P_h : (V, Q) \to (V_h, Q_h)$, $P_h(\mathbf{u}, p) = (P_h\mathbf{u}, P_h p)$ satisfying

$$\frac{1}{M^2}(\nabla(\mathbf{u} - P_h\mathbf{u}), \nabla \mathbf{v}_h) - (p - P_h p, \nabla \cdot \mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in X_h, \tag{4}$$

or

$$\frac{1}{M^2}(\nabla(\mathbf{u} - P_h\mathbf{u}), \nabla \mathbf{v}_h) - (p - \lambda_h, \nabla \cdot \mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in V_h,\ \forall \lambda_h \in Q_h. \tag{5}$$

Then, the approximation property holds, see [20],

$$\frac{1}{M^2}\|\mathbf{u}-P_h\mathbf{u}\|+\frac{h}{M^2}\|\nabla(\mathbf{u}-P_h\mathbf{u})\|+h\|p-P_hp\|\leq Ch^{k+1}(\frac{1}{M^2}\|\mathbf{u}\|_{k+1}+\|p\|_k) \tag{6}$$

for $k=0,1$.

Lastly, in order to show the consistency of our method, we define the consistency error as follows. Given a weak formulation

$$L(\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v}),\quad \forall\mathbf{v}\in X, \tag{7}$$

where the operator $L:X\times X\to R$, and a full-discrete finite element approximation of Eq. 7

$$L_{\Delta t,h}(\mathbf{u}_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h),\quad \forall\mathbf{v}_h\in X_h, \tag{8}$$

where the operator $L_{\Delta t,h}:X_h\times X_h\to R$. We say that the finite approximation method (8) is consistent with Eq.7 if the consistency error

$$\sup_{\mathbf{v}\in X_h}\frac{|L_{\Delta t,h}(\mathbf{u},\mathbf{v})-(\mathbf{f},\mathbf{v})|}{\|\mathbf{v}\|}\to 0\quad as\quad \Delta t,h\to 0. \tag{9}$$

3 First order unconditionally stable Backward-Euler method

In this section, we introduce the first order Backward-Euler method ,see [15]. We firstly present the stability and then provide consistency, convergence and their proof. Besides, we estimate the time difference of the error, which is useful to the error analysis of the second order method based on the SDC method in the next section. Finally, a corollary about the estimate of the time difference of errors' time difference is given, which is useful to the consistency analysis in next section.

The full-discrete approximation via Backward-Euler time-stepping of Eq. 3 (see [15]) is : Given $\mathbf{u_0}\in V$, find $(\mathbf{u}_{1,h}^{n+1},p_{1,h}^{n+1},\phi_{1,h}^{n+1})\in X_h\times Q_h\times S_h$, for each $n=0,1,2,\cdots m-1$ $(m=\frac{T}{\Delta t})$, satisfying

$$\begin{aligned}
&\frac{1}{N}(\frac{\mathbf{u}_{1,h}^{n+1}-\mathbf{u}_{1,h}^{n}}{\Delta t},\mathbf{v}_h)+\frac{1}{N}b(\mathbf{u}_{1,h}^{n+1},\mathbf{u}_{1,h}^{n+1},\mathbf{v}_h)+\frac{1}{M^2}(\nabla\mathbf{u}_{1,h}^{n+1},\nabla\mathbf{v}_h)\\
&\qquad -(p_{1,h}^{n+1},\nabla\cdot\mathbf{v}_h)+(-\nabla\phi_{1,h}^{n+1}+\mathbf{u}_{1,h}^{n+1}\times\mathbf{B},\mathbf{v}_h\times\mathbf{B})=(\mathbf{f}^{n+1},\mathbf{v}_h)\quad\forall\mathbf{v}_h\in X_h,\\
&(\nabla\cdot\mathbf{u}_{1,h}^{n+1},q_h)=0\quad\forall q_h\in Q_h,\\
&(-\nabla\phi_{1,h}^{n+1}+\mathbf{u}_{1,h}^{n+1}\times\mathbf{B},-\nabla\psi_h)=0\quad\forall\psi_h\in S_h.
\end{aligned} \tag{10}$$

First, we present the unconditional stability of the method (10). Theorem 1 was introduced and proved in [15]. The proof of Theorem 1 is omitted here and the reader can refer to [15].

Theorem 1 *Let $\mathbf{u}_{1,h}^{n+1} \in X_h$, $p_{1,h}^{n+1} \in Q_h$, $\phi_{1,h}^{n+1} \in S_h$ satisfying (10) for each $n = 0, 1, 2, \cdots, m-1$. Then,*

$$\frac{1}{N}\|\mathbf{u}_{1,h}^{m}\|^2 + \frac{1}{N}\sum_{n=0}^{m-1}\|\mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{1,h}^{n}\|^2 + \frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla\mathbf{u}_{1,h}^{n+1}\|^2 + \Delta t\sum_{n=0}^{m-1}\| -\nabla\phi_{1,h}^{n+1} + \mathbf{u}_{1,h}^{n+1} \times \mathbf{B}\|^2$$
$$\leq \frac{1}{N}\|\mathbf{u}_{1,h}^{0}\|^2 + M^2\Delta t\sum_{n=0}^{m-1}\|\mathbf{f}^{n+1}\|_{-1}^2. \tag{11}$$

Remark 1 The electric current density $\mathbf{J}$ is an important electromagnetic quantity in MHD flows, see [21], which is defined by $\mathbf{J} = \sigma(-\nabla\phi + \mathbf{u} \times \mathbf{B})$. Here, the electrical conductivity σ is a constant. Obviously, the stability of $\mathbf{J}$ is directly related to the term $\Delta t \sum_{n=0}^{m-1} \| -\nabla\phi_{1,h}^{n+1} + \mathbf{u}_{1,h}^{n+1} \times \mathbf{B}\|^2$ in Eq. 11.

Next, we show the consistency of the method (10) through Theorem 2.

Theorem 2 *The first order Backward-Euler method (10) is consistent and the consistency error is $O(\Delta t)$.*

Proof At time t^{n+1}, the true solution $(\mathbf{u}, p, \phi)$ satisfies

$$\begin{aligned}
&\frac{1}{N}\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{\Delta t}, \mathbf{v}_h\right) + \frac{1}{N}b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h) + \frac{1}{M^2}(\nabla\mathbf{u}^{n+1}, \nabla\mathbf{v}_h) - (p^{n+1}, \nabla\cdot\mathbf{v}_h) \\
&\quad + (-\nabla\phi^{n+1} + \mathbf{u}^{n+1} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) = (\mathbf{f}^{n+1}, \mathbf{v}_h) + \frac{1}{N}\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{v}_h\right) \quad \forall \mathbf{v}_h \in X_h, \\
&(\nabla\cdot\mathbf{u}^{n+1}, q_h) = 0 \qquad \forall q_h \in Q_h, \\
&(-\nabla\phi^{n+1} + \mathbf{u}^{n+1} \times \mathbf{B}, -\nabla\psi_h) = 0 \qquad \forall \psi_h \in S_h.
\end{aligned} \tag{12}$$

Since the last two equations in Eq. 10 are obviously consistent with those in Eq. 12, we analyze the consistency of the first equation in Eq. 10. Based on our definition of consistency error, see Eq. 9, we have the following estimate of consistency error E_1.

$$E_1 = \sup_{\mathbf{v}_h \in X_h} \frac{\left|\frac{1}{N}\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{v}_h\right)\right|}{\|\mathbf{v}_h\|} \leq \frac{1}{N}\left\|\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{\Delta t} - \mathbf{u}_t(t^{n+1})\right\|. \tag{13}$$

Taking the Taylor expansion of $\mathbf{u}$ at time t^{n+1}, we obtain

$$\mathbf{u}^{n} = \mathbf{u}^{n+1} - \Delta t\,\mathbf{u}_t(t^{n+1}) + \frac{1}{2}\Delta t^2\,\mathbf{u}_{tt}(t^{n+1}) - O(\Delta t^3). \tag{14}$$

This gives us

$$E_1 \leq \frac{1}{N}\left\| -\frac{1}{2}\Delta t\,\mathbf{u}_{tt}(t^{n+1}) + O(\Delta t^2)\right\| \leq C\Delta t + O(\Delta t^2), \tag{15}$$

which completes the proof. □

Moreover, we analysis the convergence of method (10). To establish the optimal error estimate, we define $\mathbf{j} = -\nabla\phi + \mathbf{u}\times\mathbf{B}$ and $\mathbf{j}_{1,h}^{n+1} = -\nabla\phi_{1,h}^{n+1} + \mathbf{u}_{1,h}^{n+1}\times\mathbf{B}$. Denote the errors by $\mathbf{e}_{\mathbf{u}_1}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}_{1,h}^{n+1}$, $e_{\phi_1}^{n+1} = \phi^{n+1} - \phi_{1,h}^{n+1}$ and $\mathbf{e}_{\mathbf{j}_1}^{n+1} = \mathbf{j}^{n+1} - \mathbf{j}_{1,h}^{n+1}$. Obviously, $\mathbf{e}_{\mathbf{j}_1}^{n+1} = -\nabla e_{\phi_1}^{n+1} + \mathbf{e}_{\mathbf{u}_1}^{n+1}\times\mathbf{B}$.

Theorem 3 *Suppose that* $(\mathbf{u}_{1,h}^{n+1}, p_{1,h}^{n+1}, \phi_{1,h}^{n+1})$ *is given by method* (10). *Assume that the true solution* $(\mathbf{u}, p, \phi)$ *satisfies the following regularity*

$$\begin{aligned}&\mathbf{u} \in L^\infty(0,T;(H^{k+1}(\Omega))^d)\bigcap L^\infty(0,T;(W_4^1(\Omega))^d),\\ &\mathbf{u}_t \in L^2(0,T;(H^{k+1}(\Omega))^d), \quad \mathbf{u}_{tt} \in L^2(0,T;(L^2(\Omega))^d),\\ &\phi \in L^2(0,T;H^{k+1}(\Omega)).\end{aligned} \tag{16}$$

Then, we have

$$\|\mathbf{e}_{\mathbf{u}_1}^m\|^2 + \frac{N}{M^2}\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2 + N\Delta t\sum_{n=0}^{m-1}\|\mathbf{e}_{\mathbf{j}_1}^{n+1}\|^2 + N\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_{\phi_1}^{n+1}\|^2 \le C(\Delta t^2 + h^{2k}), \tag{17}$$

provided that Δt *is sufficiently small.*

Proof At time t^{n+1}, the true solution $(\mathbf{u}, p, \phi)$ satisfies (12). We decompose the errors as follows.

$$\begin{aligned}&\mathbf{e}_{\mathbf{u}_1}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}_{1,h}^{n+1} = \eta^{n+1} + \mathbf{U}_h^{n+1}, \quad \eta^{n+1} := \mathbf{u}^{n+1} - \widetilde{\mathbf{u}}_h^{n+1}, \quad \mathbf{U}_h^{n+1} := \widetilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_{1,h}^{n+1}.\\ &e_{\phi_1}^{n+1} = \phi^{n+1} - \phi_{1,h}^{n+1} = \varsigma^{n+1} + \Phi_h^{n+1}, \quad \varsigma^{n+1} := \phi^{n+1} - \widetilde{\phi}_h^{n+1}, \quad \Phi_h^{n+1} := \widetilde{\phi}_h^{n+1} - \phi_{1,h}^{n+1}.\\ &\mathbf{e}_{\mathbf{j}_1}^{n+1} = \mathbf{j}^{n+1} - \mathbf{j}_{1,h}^{n+1} = \chi^{n+1} + \mathbf{J}_h^{n+1}, \ \chi^{n+1} := -\nabla\varsigma^{n+1} + \eta^{n+1}\times\mathbf{B}, \ \mathbf{J}_h^{n+1} := -\nabla\Phi_h^{n+1} + \mathbf{U}_h^{n+1}\times\mathbf{B}.\end{aligned}$$

Here, $\widetilde{\mathbf{u}}_h^{n+1} = P_h\mathbf{u}^{n+1}$, $\widetilde{\phi}_h^{n+1}$ are the interpolation of ϕ^{n+1} in S_h.

Subtract Eq. 12 from Eq. 10 and set $\mathbf{v}_h = \mathbf{U}_h^{n+1}$, $\psi_h = \Phi_h^{n+1}$ to obtain

$$\begin{aligned}&\frac{1}{2N\Delta t}(\|\mathbf{U}_h^{n+1}\|^2 - \|\mathbf{U}_h^n\|^2 + \|\mathbf{U}_h^{n+1} - \mathbf{U}_h^n\|^2) + \frac{1}{M^2}\|\nabla\mathbf{U}_h^{n+1}\|^2 + (\mathbf{J}_h^{n+1}, \mathbf{U}_h^{n+1}\times\mathbf{B})\\ &= -\frac{1}{N}(\frac{\eta^{n+1}-\eta^n}{\Delta t}, \mathbf{U}_h^{n+1}) - \frac{1}{N}b(\eta^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1}) - \frac{1}{N}b(\mathbf{U}_h^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1})\\ &\quad - \frac{1}{N}b(\mathbf{u}_{1,h}^{n+1}, \eta^{n+1}, \mathbf{U}_h^{n+1}) - \frac{1}{M^2}(\nabla\eta^{n+1}, \nabla\mathbf{U}_h^{n+1}) + (p^{n+1} - \lambda_h^{n+1}, \nabla\cdot\mathbf{U}_h^{n+1})\\ &\quad - (\chi^{n+1}, \mathbf{U}_h^{n+1}\times\mathbf{B}) + \frac{1}{N}(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{U}_h^{n+1}),\end{aligned} \tag{18}$$

$$(\chi^{n+1} + \mathbf{J}_h^{n+1}, -\nabla\Phi_h^{n+1}) = 0, \tag{19}$$

for $\forall \lambda_h^{n+1} \in Q_h$. Using Eq. 5, we can obtain

$$\frac{1}{2N\Delta t}(\|\mathbf{U}_h^{n+1}\|^2 - \|\mathbf{U}_h^n\|^2 + \|\mathbf{U}_h^{n+1} - \mathbf{U}_h^n\|^2) + \frac{1}{M^2}\|\nabla \mathbf{U}_h^{n+1}\|^2 + (\mathbf{J}_h^{n+1}, \mathbf{U}_h^{n+1} \times \mathbf{B})$$
$$= -\frac{1}{N}(\frac{\eta^{n+1} - \eta^n}{\Delta t}, \mathbf{U}_h^{n+1}) - \frac{1}{N}b(\eta^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1}) - \frac{1}{N}b(\mathbf{U}_h^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1})$$
$$- \frac{1}{N}b(\mathbf{u}_{1,h}^{n+1}, \eta^{n+1}, \mathbf{U}_h^{n+1}) - (\chi^{n+1}, \mathbf{U}_h^{n+1} \times \mathbf{B}) + \frac{1}{N}(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{U}_h^{n+1}). \tag{20}$$

Adding Eq. 20 to Eq. 19, we have

$$\frac{1}{2N\Delta t}(\|\mathbf{U}_h^{n+1}\|^2 - \|\mathbf{U}_h^n\|^2 + \|\mathbf{U}_h^{n+1} - \mathbf{U}_h^n\|^2) + \frac{1}{M^2}\|\nabla \mathbf{U}_h^{n+1}\|^2 + \|\mathbf{J}_h^{n+1}\|^2$$
$$= -\frac{1}{N}(\frac{\eta^{n+1} - \eta^n}{\Delta t}, \mathbf{U}_h^{n+1}) - \frac{1}{N}b(\eta^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1}) - \frac{1}{N}b(\mathbf{U}_h^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1})$$
$$- \frac{1}{N}b(\mathbf{u}_{1,h}^{n+1}, \eta^{n+1}, \mathbf{U}_h^{n+1}) - (\chi^{n+1}, \mathbf{J}_h^{n+1}) + \frac{1}{N}(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{U}_h^{n+1}). \tag{21}$$

Note that Eq. 19 can be written

$$\|\nabla \Phi_h^{n+1}\|^2 = (\chi^{n+1} + \mathbf{U}_h^{n+1} \times \mathbf{B}, \nabla \Phi_h^{n+1}). \tag{22}$$

Add Eq. 22 to Eq. 21 to obtain

$$\frac{1}{2N\Delta t}(\|\mathbf{U}_h^{n+1}\|^2 - \|\mathbf{U}_h^n\|^2 + \|\mathbf{U}_h^{n+1} - \mathbf{U}_h^n\|^2) + \frac{1}{M^2}\|\nabla \mathbf{U}_h^{n+1}\|^2 + \|\mathbf{J}_h^{n+1}\|^2 + \|\nabla \Phi_h^{n+1}\|^2$$
$$= -\frac{1}{N}(\frac{\eta^{n+1} - \eta^n}{\Delta t}, \mathbf{U}_h^{n+1}) - \frac{1}{N}b(\eta^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1}) - \frac{1}{N}b(\mathbf{U}_h^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1})$$
$$- \frac{1}{N}b(\mathbf{u}_{1,h}^{n+1}, \eta^{n+1}, \mathbf{U}_h^{n+1}) - (\chi^{n+1}, \mathbf{J}_h^{n+1}) + (\chi^{n+1} + \mathbf{U}_h^{n+1} \times \mathbf{B}, \nabla \Phi_h^{n+1})$$
$$+ \frac{1}{N}(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{U}_h^{n+1}). \tag{23}$$

We now bound each term in the right hand side of Eq. 23. For arbitrary $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$,

$$-\frac{1}{N}(\frac{\eta^{n+1} - \eta^n}{\Delta t}, \mathbf{U}_h^{n+1}) \le C\|\frac{\eta^{n+1} - \eta^n}{\Delta t}\|_{-1}^2 + \varepsilon\|\nabla \mathbf{U}_h^{n+1}\|^2. \tag{24}$$

For the three nonlinear terms, we bound them as follows.

$$\begin{aligned} -\frac{1}{N}b(\eta^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1}) &\le C\|\nabla \eta^{n+1}\|\|\nabla \mathbf{u}^{n+1}\|\|\nabla \mathbf{U}_h^{n+1}\| \\ &\le C\|\mathbf{u}\|_{\infty,1}^2\|\nabla \eta^{n+1}\|^2 + \varepsilon\|\nabla \mathbf{U}_h^{n+1}\|^2. \end{aligned} \tag{25}$$

$$\begin{aligned} -\frac{1}{N}b(\mathbf{U}_h^{n+1}, \mathbf{u}^{n+1}, \mathbf{U}_h^{n+1}) &\le C\|\nabla \mathbf{U}_h^{n+1}\|^{\frac{1}{2}}\|\mathbf{U}_h^{n+1}\|^{\frac{1}{2}}\|\nabla \mathbf{u}^{n+1}\|\|\nabla \mathbf{U}_h^{n+1}\| \\ &\le C\|\nabla \mathbf{u}^{n+1}\|^4\|\mathbf{U}_h^{n+1}\|^2 + \varepsilon\|\nabla \mathbf{U}_h^{n+1}\|^2 \\ &\le C\|\mathbf{u}\|_{\infty,1}^4\|\mathbf{U}_h^{n+1}\|^2 + \varepsilon\|\nabla \mathbf{U}_h^{n+1}\|^2. \end{aligned} \tag{26}$$

Since

$$-\frac{1}{N}b(\mathbf{u}_{1,h}^{n+1},\eta^{n+1},\mathbf{U}_h^{n+1}) = -\frac{1}{N}b(\mathbf{u}^{n+1},\eta^{n+1},\mathbf{U}_h^{n+1}) + \frac{1}{N}b(\eta^{n+1},\eta^{n+1},\mathbf{U}_h^{n+1}) + \frac{1}{N}b(\mathbf{U}_h^{n+1},\eta^{n+1},\mathbf{U}_h^{n+1}), \tag{27}$$

and

$$-\frac{1}{N}b(\mathbf{u}^{n+1},\eta^{n+1},\mathbf{U}_h^{n+1}) \le C\|\mathbf{u}\|_{\infty,1}^2\|\nabla\eta^{n+1}\|^2 + \varepsilon\|\nabla\mathbf{U}_h^{n+1}\|^2, \tag{28}$$

$$\frac{1}{N}b(\eta^{n+1},\eta^{n+1},\mathbf{U}_h^{n+1}) \le C\|\nabla\eta^{n+1}\|^4 + \varepsilon\|\nabla\mathbf{U}_h^{n+1}\|^2, \tag{29}$$

$$\begin{aligned}\frac{1}{N}b(\mathbf{U}_h^{n+1},\eta^{n+1},\mathbf{U}_h^{n+1}) &\le C\|\nabla\mathbf{U}_h^{n+1}\|^{\frac{1}{2}}\|\mathbf{U}_h^{n+1}\|^{\frac{1}{2}}\|\nabla\eta^{n+1}\|\|\nabla\mathbf{U}_h^{n+1}\|\\ &\le C\|\nabla\eta^{n+1}\|^4\|\mathbf{U}_h^{n+1}\|^2 + \varepsilon\|\nabla\mathbf{U}_h^{n+1}\|^2\\ &\le C\|\mathbf{u}\|_{\infty,1}^4\|\mathbf{U}_h^{n+1}\|^2 + \varepsilon\|\nabla\mathbf{U}_h^{n+1}\|^2.\end{aligned} \tag{30}$$

From Eqs. 27–30, we can obtain

$$\begin{aligned}-\frac{1}{N}b(\mathbf{u}_{1,h}^{n+1},\eta^{n+1},\mathbf{U}_h^{n+1}) \le& C\|\mathbf{u}\|_{\infty,1}^2\|\nabla\eta^{n+1}\|^2 + C\|\nabla\eta^{n+1}\|^4\\ &+ C\|\mathbf{u}\|_{\infty,1}^4\|\mathbf{U}_h^{n+1}\|^2 + 3\varepsilon\|\nabla\mathbf{U}_h^{n+1}\|^2.\end{aligned} \tag{31}$$

The remaining terms are bounded as follows.

$$-(\chi^{n+1},\mathbf{J}_h^{n+1}) \le C\|\chi^{n+1}\|^2 + \varepsilon_1\|\mathbf{J}_h^{n+1}\|^2. \tag{32}$$

$$\begin{aligned}(\chi^{n+1}+\mathbf{U}_h^{n+1}\times\mathbf{B},\nabla\Phi_h^{n+1}) &\le C\|\chi^{n+1}+\mathbf{U}_h^{n+1}\times\mathbf{B}\|^2 + \varepsilon_2\|\nabla\Phi_h^{n+1}\|^2\\ &\le C\|\chi^{n+1}\|^2 + C\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{U}_h^{n+1}\|^2 + \varepsilon_2\|\nabla\Phi_h^{n+1}\|^2.\end{aligned} \tag{33}$$

$$\frac{1}{N}(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1}),\mathbf{U}_h^{n+1}) \le C\|\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1})\|^2 + \varepsilon\|\nabla\mathbf{U}_h^{n+1}\|^2. \tag{34}$$

Setting $\varepsilon = \frac{1}{14M^2}$, $\varepsilon_1 = \frac{1}{2}$ and $\varepsilon_2 = \frac{1}{2}$, and applying Eqs. 24–34 to Eq. 23, we have

$$\begin{aligned}&\frac{1}{2N\Delta t}(\|\mathbf{U}_h^{n+1}\|^2 - \|\mathbf{U}_h^n\|^2 + \|\mathbf{U}_h^{n+1}-\mathbf{U}_h^n\|^2) + \frac{1}{2M^2}\|\nabla\mathbf{U}_h^{n+1}\|^2 + \frac{1}{2}\|\mathbf{J}_h^{n+1}\|^2 + \frac{1}{2}\|\nabla\Phi_h^{n+1}\|^2\\ &\le C(\|\mathbf{B}\|_{L^\infty}^2 + \|\mathbf{u}\|_{\infty,1}^4)\|\mathbf{U}_h^{n+1}\|^2 + C\|\frac{\eta^{n+1}-\eta^n}{\Delta t}\|_{-1}^2 + C\|\mathbf{u}\|_{\infty,1}^2\|\nabla\eta^{n+1}\|^2\\ &\quad + C\|\nabla\eta^{n+1}\|^4 + C\|\chi^{n+1}\|^2 + C\|\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1})\|^2.\end{aligned} \tag{35}$$

Let $\kappa = C(\|\mathbf{B}\|^2_{L^\infty} + \|\mathbf{u}\|^4_{\infty,1})$. Summing from $n = 0$ to $n = m - 1$ and applying Lemma 2 give

$$\|\mathbf{U}_h^m\|^2+\sum_{n=0}^{m-1}\|\mathbf{U}_h^{n+1}-\mathbf{U}_h^n\|^2+\frac{N}{M^2}\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{U}_h^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|\mathbf{J}_h^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|\nabla\Phi_h^{n+1}\|^2$$
$$\le \exp((m+1)\frac{\Delta t\kappa}{1-\Delta t\kappa})[\|\mathbf{U}_h^0\|^2+C\Delta t\sum_{n=0}^{m-1}\|\frac{\eta^{n+1}-\eta^n}{\Delta t}\|^2_{-1}+C\|\mathbf{u}\|^2_{\infty,1}\Delta t\sum_{n=0}^{m-1}\|\nabla\eta^{n+1}\|^2$$
$$+C\Delta t\sum_{n=0}^{m-1}\|\nabla\eta^{n+1}\|^4+C\Delta t\sum_{n=0}^{m-1}\|\chi^{n+1}\|^2+C\Delta t\sum_{n=0}^{m-1}\|\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}-\mathbf{u}_t(t^{n+1})\|^2], \tag{36}$$

provided that $\Delta t\kappa < 1$.

The right hand side of Eq. 36 is bounded as follows.

$$\|\mathbf{U}_h^0\|^2 \le 2\|\mathbf{u}^0-\mathbf{u}^0_{1,h}\|^2+2\|\eta^0\|^2 \le 2\|\mathbf{u}^0-\mathbf{u}^0_{1,h}\|^2+Ch^{2k+2}|\|\mathbf{u}\||^2_{\infty,k+1}. \tag{37}$$

$$\Delta t\sum_{n=0}^{m-1}\|\frac{\eta^{n+1}-\eta^n}{\Delta t}\|^2_{-1} \le \sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\eta_t\|^2dt \le Ch^{2k+2}\|\mathbf{u}_t\|^2_{2,k+1}. \tag{38}$$

$$\|\mathbf{u}\|^2_{\infty,1}\Delta t\sum_{n=0}^{m-1}\|\nabla\eta^{n+1}\|^2 \le C\Delta t\sum_{n=0}^{m-1}h^{2k}\|\mathbf{u}^{n+1}\|^2_{k+1} = Ch^{2k}|\|\mathbf{u}\||^2_{2,k+1}. \tag{39}$$

$$\Delta t\sum_{n=0}^{m-1}\|\nabla\eta^{n+1}\|^4 \le Ch^{4k}|\|\mathbf{u}\||^4_{4,k+1}. \tag{40}$$

$$\begin{aligned}\Delta t\sum_{n=0}^{m-1}\|\chi^{n+1}\|^2 &\le \Delta t\sum_{n=0}^{m-1}\|-\nabla\varsigma^{n+1}\|^2+\Delta t\sum_{n=0}^{m-1}\|\eta^{n+1}\times\mathbf{B}\|^2\\ &\le C\Delta t\sum_{n=0}^{m-1}h^{2k}\|\phi^{n+1}\|^2_{k+1}+C\Delta t\sum_{n=0}^{m-1}h^{2k+2}\|\mathbf{u}^{n+1}\|^2_{k+1}\\ &\le Ch^{2k}|\|\phi\||^2_{2,k+1}+Ch^{2k+2}|\|\mathbf{u}\||^2_{2,k+1}.\end{aligned} \tag{41}$$

$$\Delta t\sum_{n=0}^{m-1}\|\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}-\mathbf{u}_t(t^{n+1})\|^2 \le \Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\mathbf{u}_{tt}\|^2dt = \Delta t^2\|\mathbf{u}_{tt}\|^2_{2,0}. \tag{42}$$

Then, combining Eqs. 37–42 with Eq. 36 to obtain

$$\begin{aligned}&\|\mathbf{U}_h^m\|^2+\sum_{n=0}^{m-1}\|\mathbf{U}_h^{n+1}-\mathbf{U}_h^n\|^2+\frac{N}{M^2}\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{U}_h^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|\mathbf{J}_h^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|\nabla\Phi_h^{n+1}\|^2\\ &\le C(\|\mathbf{u}^0-\mathbf{u}^0_{1,h}\|^2+h^{2k+2}|\|\mathbf{u}\||^2_{\infty,k+1}+h^{2k+2}\|\mathbf{u}_t\|^2_{2,k+1}+h^{2k}|\|\mathbf{u}\||^2_{2,k+1}+h^{4k}|\|\mathbf{u}\||^4_{4,k+1}\\ &\quad+h^{2k}|\|\phi\||^2_{2,k+1}+h^{2k+2}|\|\mathbf{u}\||^2_{2,k+1}+\Delta t^2\|\mathbf{u}_{tt}\|^2_{2,0})\\ &\le C(\Delta t^2+h^{2k}).\end{aligned} \tag{43}$$

Finally, using the triangle inequality completes the proof. □

Remark 2 In the proof of Theorem 3, we mainly use Lemmas 1, 2, the approximation properties assumption of (X_h, Q_h, S_h), the *Hölder* inequality and the Young's inequality to bound the terms, which are similarly used in the later proof.

Next, we estimate the time difference $\frac{\mathbf{e}_{\mathbf{u}_1}^{n+1}-\mathbf{e}_{\mathbf{u}_1}^{n}}{\Delta t}$, which will be used in the error analysis of the second order method. Denote $S_{\mathbf{u}_1}^{n+1}=\frac{\mathbf{e}_{\mathbf{u}_1}^{n+1}-\mathbf{e}_{\mathbf{u}_1}^{n}}{\Delta t}$, $S_{\phi_1}^{n+1}=\frac{e_{\phi_1}^{n+1}-e_{\phi_1}^{n}}{\Delta t}$ and $S_{\mathbf{j}_1}^{n+1}=\frac{\mathbf{e}_{\mathbf{j}_1}^{n+1}-\mathbf{e}_{\mathbf{j}_1}^{n}}{\Delta t}$. The main result is shown in Theorem 4.

Theorem 4 *Suppose that* $(\mathbf{u}_{1,h}^{n+1}, p_{1,h}^{n+1}, \phi_{1,h}^{n+1})$ *is given by method* (10). *Let the assumption of Theorem 3 be satisfied. Assume also that the true solution* $\mathbf{u}$ *satisfies the following regularity*

$$\begin{aligned} &\nabla\mathbf{u}_t \in L^2(0,T;(L^\infty(\Omega))^d), \quad \mathbf{u}_{ttt} \in L^2(0,T;(L^2(\Omega))^d),\\ &p_t \in L^2(0,T;H^k(\Omega)). \end{aligned} \tag{44}$$

Then, we have

$$\|S_{\mathbf{u}_1}^m\|^2+\frac{N}{M^2}\Delta t\sum_{n=0}^{m-1}\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|S_{\mathbf{j}_1}^{n+1}\|^2 \le C(\Delta t^2+h^{2k}), \tag{45}$$

provided that Δt *is sufficiently small.*

Proof At time t^{n+1}, we have

$$\begin{aligned} &\frac{1}{N}(\frac{\mathbf{e}_{\mathbf{u}_1}^{n+1}-\mathbf{e}_{\mathbf{u}_1}^{n}}{\Delta t},\mathbf{v}_h)+\frac{1}{N}b(\mathbf{e}_{\mathbf{u}_1}^{n+1},\mathbf{u}^{n+1},\mathbf{v}_h)+\frac{1}{N}b(\mathbf{u}_{1,h}^{n+1},\mathbf{e}_{\mathbf{u}_1}^{n+1},\mathbf{v}_h)+\frac{1}{M^2}(\nabla\mathbf{e}_{\mathbf{u}_1}^{n+1},\nabla\mathbf{v}_h)\\ &\quad-(p^{n+1}-p_{1,h}^{n+1},\nabla\cdot\mathbf{v}_h)+(\mathbf{e}_{\mathbf{j}_1}^{n+1},\mathbf{v}_h\times\mathbf{B})=\frac{1}{N}(\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}-\mathbf{u}_t(t^{n+1}),\mathbf{v}_h)\quad\forall\mathbf{v}_h\in X_h,\\ &(\nabla\cdot\mathbf{e}_{\mathbf{u}_1}^{n+1},q_h)=0\qquad\forall q_h\in Q_h,\\ &(\mathbf{e}_{\mathbf{j}_1}^{n+1},-\nabla\psi_h)=0\qquad\forall\psi_h\in S_h. \end{aligned} \tag{46}$$

Consider Eq. 46 at the previous time t^n to get new equations. Setting $\mathbf{v}_h = S_{\mathbf{u}_1}^{n+1}$, $\psi_h=S_{\phi_1}^{n+1}$ and subtracting Eq. 46 from the new equations, we can obtain

$$\begin{aligned} &\frac{1}{2N}(\|S_{\mathbf{u}_1}^{n+1}\|^2-\|S_{\mathbf{u}_1}^{n}\|^2+\|S_{\mathbf{u}_1}^{n+1}-S_{\mathbf{u}_1}^{n}\|^2)+\frac{\Delta t}{M^2}\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2+\Delta t\|S_{\mathbf{j}_1}^{n+1}\|^2\\ &=-\frac{\Delta t}{N}b(S_{\mathbf{u}_1}^{n+1},\mathbf{u}^{n+1},S_{\mathbf{u}_1}^{n+1})-\frac{\Delta t}{N}b(\frac{\mathbf{u}_{1,h}^{n+1}-\mathbf{u}_{1,h}^{n}}{\Delta t},\mathbf{e}_{\mathbf{u}_1}^{n},S_{\mathbf{u}_1}^{n+1})\\ &\quad-\frac{\Delta t}{N}b(\mathbf{e}_{\mathbf{u}_1}^{n},\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t},S_{\mathbf{u}_1}^{n+1})+\Delta t(\frac{(p^{n+1}-\lambda_h^{n+1})-(p^n-\lambda_h^n)}{\Delta t},\nabla\cdot S_{\mathbf{u}_1}^{n+1})\\ &\quad+\frac{1}{N}((\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}-\mathbf{u}_t(t^{n+1}))-(\frac{\mathbf{u}^{n}-\mathbf{u}^{n-1}}{\Delta t}-\mathbf{u}_t(t^{n})),S_{\mathbf{u}_1}^{n+1}), \end{aligned} \tag{47}$$

for $\forall\ \lambda_h\in Q_h$.

We now bound each term in the right hand side of Eq. 47 as follows. For $\forall\ \varepsilon > 0$,

$$-\frac{\Delta t}{N}b(S_{\mathbf{u}_1}^{n+1}, \mathbf{u}^{n+1}, S_{\mathbf{u}_1}^{n+1}) \leq C\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^{\frac{1}{2}}\|S_{\mathbf{u}_1}^{n+1}\|^{\frac{1}{2}}\|\nabla \mathbf{u}^{n+1}\|\|\nabla S_{\mathbf{u}_1}^{n+1}\| \\ \leq C\Delta t\|\mathbf{u}\|_{\infty,1}^4\|S_{\mathbf{u}_1}^{n+1}\|^2 + \varepsilon\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2. \tag{48}$$

$$-\frac{\Delta t}{N}b(\mathbf{e}_{\mathbf{u}_1}^n, \frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}, S_{\mathbf{u}_1}^{n+1}) \leq C\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|\|\nabla \mathbf{u}_t(\xi^{n+1})\|\|\nabla S_{\mathbf{u}_1}^{n+1}\| \\ \leq C\Delta t\|\nabla \mathbf{u}_t(\xi^{n+1})\|^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2 + \varepsilon\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2, \tag{49}$$

where $\xi^{n+1} \in (t^n, t^{n+1})$.

$$\begin{aligned} &-\frac{\Delta t}{N}b(\frac{\mathbf{u}_{1,h}^{n+1}-\mathbf{u}_{1,h}^n}{\Delta t}, \mathbf{e}_{\mathbf{u}_1}^n, S_{\mathbf{u}_1}^{n+1}) \\ &= -\frac{\Delta t}{N}b(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}, \mathbf{e}_{\mathbf{u}_1}^n, S_{\mathbf{u}_1}^{n+1}) + \frac{\Delta t}{N}b(S_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_1}^n, S_{\mathbf{u}_1}^{n+1}) \\ &\leq C\Delta t\|\nabla \mathbf{u}_t(\xi^{n+1})\|\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|\|\nabla S_{\mathbf{u}_1}^{n+1}\| + C\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^{\frac{1}{2}}\|S_{\mathbf{u}_1}^{n+1}\|^{\frac{1}{2}}\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|\|\nabla S_{\mathbf{u}_1}^{n+1}\| \\ &\leq C\Delta t\|\nabla \mathbf{u}_t(\xi^{n+1})\|^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2 + C\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^4\|S_{\mathbf{u}_1}^{n+1}\|^2 + 2\varepsilon\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2. \end{aligned} \tag{50}$$

For the term $\Delta t(\frac{(p^{n+1}-\lambda_h^{n+1})-(p^n-\lambda_h^n)}{\Delta t}, \nabla\cdot S_{\mathbf{u}_1}^{n+1})$,

$$\Delta t(\frac{(p^{n+1}-\lambda_h^{n+1})-(p^n-\lambda_h^n)}{\Delta t}, \nabla\cdot S_{\mathbf{u}_1}^{n+1}) \\ \leq C\Delta t\|\frac{(p^{n+1}-\lambda_h^{n+1})-(p^n-\lambda_h^n)}{\Delta t}\|^2 + \varepsilon\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2. \tag{51}$$

For the last term,

$$\frac{1}{N}((\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1})) - (\frac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\Delta t} - \mathbf{u}_t(t^n)), S_{\mathbf{u}_1}^{n+1}) \\ = \frac{1}{N}\Delta t^2(\mathbf{u}_{ttt}(\theta^{n+1}), S_{\mathbf{u}_1}^{n+1}) \leq C\Delta t^3\|\mathbf{u}_{ttt}(\theta^{n+1})\|^2 + \varepsilon\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2, \tag{52}$$

where $\theta^{n+1} \in (t^{n-1}, t^{n+1})$. Combining Eqs. 48–52 with Eq. 47, and setting $\varepsilon = \frac{1}{12M^2}$, we have

$$\|S_{\mathbf{u}_1}^{n+1}\|^2 - \|S_{\mathbf{u}_1}^n\|^2 + \frac{N}{M^2}\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + N\Delta t\|S_{\mathbf{j}_1}^{n+1}\|^2 \\ \leq C\Delta t(\|\mathbf{u}\|_{\infty,1}^4 + \|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^4)\|S_{\mathbf{u}_1}^{n+1}\|^2 + C\Delta t\|\nabla \mathbf{u}_t(\xi^{n+1})\|^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2 \\ + C\Delta t\|\frac{(p^{n+1}-\lambda_h^{n+1})-(p^n-\lambda_h^n)}{\Delta t}\|^2 + C\Delta t^3\|\mathbf{u}_{ttt}(\theta^{n+1})\|^2. \tag{53}$$

In order to use Lemma 2, we need $\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^4 \leq C$ uniformly for all n. It can be easily deduced from Theorem 3 that

$$\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^4 = \frac{(\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2)^2}{\Delta t} \leq C(\Delta t^3 + \Delta t h^{4k} + \frac{h^{4k}}{\Delta t}). \tag{54}$$

On the other hand, using the inverse inequality and Theorem 3, we have

$$\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^4 \leq C\Delta t h^{-4}\|\mathbf{e}_{\mathbf{u}_1}^n\|^4 \leq C(\frac{\Delta t^5}{h^4} + \Delta t^3 h^{2k-4} + \Delta t h^{4k-4}). \tag{55}$$

Thus, if $h^{4k} \leq \Delta t$, we use Eq. 54 to bound $\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^4$, otherwise use Eq. 55. In any case, we have $\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^4 \leq C$. Furthermore, since $S_{\mathbf{u}_1}^n = \frac{\mathbf{e}_{\mathbf{u}_1}^n - \mathbf{e}_{\mathbf{u}_1}^{n-1}}{\Delta t}$ is not defined for $n = 0$, we can only sum Eq. 53 from $n = 1$ to $n = m - 1$, which impels us to bound the terms $\|S_{\mathbf{u}_1}^1\|^2$, $\Delta t\|\nabla S_{\mathbf{u}_1}^1\|^2$ and $\Delta t\|S_{\mathbf{j}_1}^1\|^2$. We bound these terms as follows.

At time t^1, substracting Eq. 12 from Eq. 10 to obtain

$$\begin{aligned}
\frac{1}{N}(\frac{\mathbf{e}_{\mathbf{u}_1}^1 - \mathbf{e}_{\mathbf{u}_1}^0}{\Delta t}, \mathbf{v}_h) + \frac{1}{N}b(\mathbf{e}_{\mathbf{u}_1}^1, \mathbf{u}^1, \mathbf{v}_h) + \frac{1}{N}b(\mathbf{u}_{1,h}^1, \mathbf{e}_{\mathbf{u}_1}^1, \mathbf{v}_h) + \frac{1}{M^2}(\nabla \mathbf{e}_{\mathbf{u}_1}^1, \nabla \mathbf{v}_h) \\
- (p^1 - p_{1,h}^1, \nabla \cdot \mathbf{v}_h) + (\mathbf{e}_{\mathbf{j}_1}^1, -\nabla\psi_h + \mathbf{v}_h \times \mathbf{B}) \\
= \frac{1}{N}\left(\frac{\mathbf{u}^1 - \mathbf{u}^0}{\Delta t} - \mathbf{u}_t(t^1, \mathbf{v}_h)\right) \quad \forall \mathbf{v}_h \in X_h,\ \forall\psi_h \in S_h.
\end{aligned} \tag{56}$$

Set $\mathbf{u}_{1,h}^0 = \widetilde{\mathbf{u}}_h^0 = \mathbf{u}^0$, we have $\mathbf{e}_{\mathbf{u}_1}^0 = 0$, $\eta^0 = 0$, $\mathbf{U}_h^0 = 0$. Setting $\mathbf{v}_h = \mathbf{e}_{\mathbf{u}_1}^1$, $\psi_h = e_{\phi_1}^1$ in Eq. 56, it follows

$$\frac{1}{N}(\frac{\mathbf{e}_{\mathbf{u}_1}^1}{\Delta t}, \mathbf{e}_{\mathbf{u}_1}^1) + \frac{1}{N}b(\mathbf{e}_{\mathbf{u}_1}^1, \mathbf{u}^1, \mathbf{e}_{\mathbf{u}_1}^1) + \frac{1}{M^2}(\nabla \mathbf{U}_h^1, \nabla \mathbf{e}_{\mathbf{u}_1}^1) + (\mathbf{e}_{\mathbf{j}_1}^1, \mathbf{e}_{\mathbf{j}_1}^1) = \frac{1}{N}(\frac{\mathbf{u}^1 - \mathbf{u}^0}{\Delta t} - \mathbf{u}_t(t^1), \mathbf{e}_{\mathbf{u}_1}^1). \tag{57}$$

Multiplying Eq. 57 by $N\Delta t$, we can obtain

$$\begin{aligned}
\|\mathbf{e}_{\mathbf{u}_1}^1\|^2 + \frac{N}{M^2}\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^2 + N\Delta t\|\mathbf{e}_{\mathbf{j}_1}^1\|^2 = -\Delta t\, b(\mathbf{e}_{\mathbf{u}_1}^1, \mathbf{u}^1, \mathbf{e}_{\mathbf{u}_1}^1) + \frac{N}{M^2}\Delta t(\nabla\eta^1, \nabla \mathbf{e}_{\mathbf{u}_1}^1) \\
+ \Delta t(\frac{\mathbf{u}^1 - \mathbf{u}^0}{\Delta t} - \mathbf{u}_t(t^1), \mathbf{e}_{\mathbf{u}_1}^1).
\end{aligned} \tag{58}$$

We bound the three terms of the right hand side in Eq. 58 as follows. For $\forall\, \varepsilon > 0$,

$$\begin{aligned}
-\Delta t\, b(\mathbf{e}_{\mathbf{u}_1}^1, \mathbf{u}^1, \mathbf{e}_{\mathbf{u}_1}^1) &\leq C\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^{\frac{1}{2}}\|\mathbf{e}_{\mathbf{u}_1}^1\|^{\frac{1}{2}}\|\nabla \mathbf{u}^1\|\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\| \\
&\leq C\Delta t\|\mathbf{e}_{\mathbf{u}_1}^1\|^2 + \varepsilon\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^2.
\end{aligned} \tag{59}$$

$$\begin{aligned}
\frac{N}{M^2}\Delta t(\nabla\eta^1, \nabla \mathbf{e}_{\mathbf{u}_1}^1) &\leq C\Delta t\|\nabla\eta^1\|^2 + \varepsilon\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^2 \\
&= C\Delta t^3\|\frac{\nabla\eta^1 - \nabla\eta^0}{\Delta t}\|^2 + \varepsilon\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^2 \\
&\leq C\Delta t^2 h^{2k} + \varepsilon\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^2.
\end{aligned} \tag{60}$$

$$\Delta t(\frac{\mathbf{u}^1 - \mathbf{u}^0}{\Delta t} - \mathbf{u}_t(t^1), \mathbf{e}_{\mathbf{u}_1}^1) = \Delta t^2(\frac{\frac{\mathbf{u}^1 - \mathbf{u}^0}{\Delta t} - \mathbf{u}_t(t^1)}{\Delta t}, \mathbf{e}_{\mathbf{u}_1}^1) \quad \leq C\Delta t^4 + \frac{1}{2}\|\mathbf{e}_{\mathbf{u}_1}^1\|^2. \tag{61}$$

Setting $\varepsilon = \frac{N}{4M^2}$ and combining Eqs. 59–61 with Eq. 58, we have

$$(\frac{1}{2} - C\Delta t)\|\mathbf{e}_{\mathbf{u}_1}^1\|^2 + \frac{N}{2M^2}\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^2 + N\Delta t\|\mathbf{e}_{\mathbf{j}_1}^1\|^2 \leq C\Delta t^2(\Delta t^2 + h^{2k}). \tag{62}$$

Since Δt is sufficiently small, we can easily obtain

$$\|\mathbf{e}_{\mathbf{u}_1}^1\|^2 + \frac{N}{M^2}\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^2 + N\Delta t\|\mathbf{e}_{\mathbf{j}_1}^1\|^2 \le C\Delta t^2(\Delta t^2 + h^{2k}). \tag{63}$$

Hence

$$\begin{aligned}&\|S_{\mathbf{u}_1}^1\|^2 + \frac{N}{M^2}\Delta t\|\nabla S_{\mathbf{u}_1}^1\|^2 + N\Delta t\|S_{\mathbf{j}_1}^1\|^2\\ &= \Delta t^{-2}(\|\mathbf{e}_{\mathbf{u}_1}^1\|^2 + \frac{N}{M^2}\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^1\|^2 + N\Delta t\|\mathbf{e}_{\mathbf{j}_1}^1\|^2)\\ &\le C(\Delta t^2 + h^{2k}).\end{aligned} \tag{64}$$

We sum Eq. 53 from $n = 1$ to $n = m - 1$ and use Lemma 2 to obtain

$$\begin{aligned}&\|S_{\mathbf{u}_1}^m\|^2 + \frac{N}{M^2}\sum_{n=1}^{m-1}\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + N\Delta t\sum_{n=1}^{m-1}\|S_{\mathbf{j}_1}^{n+1}\|^2\\ &\le C(\Delta t\sum_{n=1}^{m-1}\|\nabla \mathbf{u}_t(\xi^{n+1})\|^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2 + \Delta t\sum_{n=1}^{m-1}\|\frac{(p^{n+1}-\lambda_h^{n+1}) - (p^n - \lambda_h^n)}{\Delta t}\|^2\\ &\quad + \Delta t^3\sum_{n=1}^{m-1}\|\mathbf{u}_{ttt}(\theta^{n+1})\|^2 + \|S_{\mathbf{u}_1}^1\|^2 + \frac{N}{M^2}\Delta t\|\nabla S_{\mathbf{u}_1}^1\|^2 + N\Delta t\|S_{\mathbf{j}_1}^1\|^2).\end{aligned} \tag{65}$$

Using Theorem 3, the terms in the right side hand of Eq. 65 are bounded as follows.

$$\Delta t\sum_{n=1}^{m-1}\|\nabla \mathbf{u}_t(\xi^{n+1})\|^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2 \le C\Delta t\sum_{n=1}^{m-1}\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2 \le C(\Delta t^2 + h^{2k}). \tag{66}$$

$$\begin{aligned}\Delta t\sum_{n=1}^{m-1}\|\frac{(p^{n+1}-\lambda_h^{n+1}) - (p^n - \lambda_h^n)}{\Delta t}\|^2 &\le \Delta t\sum_{n=0}^{m-1}\|(p-\lambda_h)_t(\delta^{n+1})\|^2\\ &\le Ch^{2k}\sum_{n=0}^{m-1}\Delta t\|p_t(\delta^{n+1})\|_k^2\\ &\le Ch^{2k},\end{aligned} \tag{67}$$

where $\delta^{n+1} \in (t^n, t^{n+1})$.

$$\Delta t^3\sum_{n=1}^{m-1}\|\mathbf{u}_{ttt}(\theta^{n+1})\|^2 \le C\Delta t^2. \tag{68}$$

Combining Eqs. 64 and 66–68 with Eq. 65 completes the proof. □

Corollary 1 *Suppose that Δt is sufficiently small, we have the following estimate.*

$$\|K_{\mathbf{u}_1}^m\|^2 + \frac{N}{M^2}\Delta t\sum_{n=1}^{m-1}\|\nabla K_{\mathbf{u}_1}^{n+1}\|^2 + N\Delta t\sum_{n=1}^{m-1}\|K_{\mathbf{j}_1}^{n+1}\|^2 \le C(\Delta t^2 + h^{2k}), \tag{69}$$

where $K_{\mathbf{u}_1}^{n+1} := \frac{S_{\mathbf{u}_1}^{n+1} - S_{\mathbf{u}_1}^n}{\Delta t}$.

Remark 3 The proof of Corollary 1 is similar to that of Theorem 4, thus we omit the proof.

4 Second order unconditionally stable method based on SDC method

In this section, we present the second order method based on the SDC method. We first prove the unconditionally stability of the second order method. Then we give a complete theoretical analysis of its consistency and error estimate, which shows that this method is second accurate.

Based on the method (10), we now present the second order method, using the SDC method. Given $(\mathbf{u}_{1,h}^{n}, p_{1,h}^{n}, \phi_{1,h}^{n})$ and $(\mathbf{u}_{1,h}^{n+1}, p_{1,h}^{n+1}, \phi_{1,h}^{n+1})$ of the method (10), find $(\mathbf{u}_{2,h}^{n+1}, p_{2,h}^{n+1}, \phi_{2,h}^{n+1}) \in (X_h, Q_h, S_h)$ satisfying

$$
\begin{aligned}
&\frac{1}{N}(\frac{\mathbf{u}_{2,h}^{n+1}-\mathbf{u}_{2,h}^{n}}{\Delta t}, \mathbf{v}_h) + \frac{1}{N} b(\mathbf{u}_{2,h}^{n+1}, \mathbf{u}_{2,h}^{n+1}, \mathbf{v}_h) + \frac{1}{M^2}(\nabla \mathbf{u}_{2,h}^{n+1}, \nabla \mathbf{v}_h) - (p_{2,h}^{n+1}, \nabla \cdot \mathbf{v}_h) \\
&+ (-\nabla \phi_{2,h}^{n+1} + \mathbf{u}_{2,h}^{n+1} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) = \frac{1}{2N} b(\mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{1,h}^{n+1}, \mathbf{v}_h) - \frac{1}{2N} b(\mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n}, \mathbf{v}_h) \\
&+ \frac{1}{2M^2}(\nabla \mathbf{u}_{1,h}^{n+1}, \nabla \mathbf{v}_h) - \frac{1}{2M^2}(\nabla \mathbf{u}_{1,h}^{n}, \nabla \mathbf{v}_h) - \frac{1}{2}(p_{1,h}^{n+1} - p_{1,h}^{n}, \nabla \cdot \mathbf{v}_h) \\
&+ \frac{1}{2}(-\nabla \phi_{1,h}^{n+1} + \mathbf{u}_{1,h}^{n+1} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) - \frac{1}{2}(-\nabla \phi_{1,h}^{n} + \mathbf{u}_{1,h}^{n} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) \\
&+ (\frac{\mathbf{f}^{n} + \mathbf{f}^{n+1}}{2}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in X_h, \\
&(\nabla \cdot \mathbf{u}_{2,h}^{n+1}, q_h) = 0 \quad \forall q_h \in Q_h, \\
&(-\nabla \phi_{2,h}^{n+1} + \mathbf{u}_{2,h}^{n+1} \times \mathbf{B}, -\nabla \psi_h) = 0 \quad \forall \psi_h \in S_h.
\end{aligned} \tag{70}
$$

Remark 4 The time step Δt and mesh size h in Eq. 70 are the same as in Eq. 10.

We now present the unconditional stability of the method (70).

Theorem 5 *Let the assumptions of Theorems 3 and 4 be satisfied. Let* $\mathbf{u}_{2,h}^{n+1} \in X_h$, $p_{2,h}^{n+1} \in Q_h$, $\phi_{2,h}^{n+1} \in S_h$ *satisfying* (70) *for each* $n = 0, 1, 2, \cdots, m-1$. *Then,*

$$
\begin{aligned}
&\frac{1}{N}\|\mathbf{u}_{2,h}^{m}\|^2 + \frac{1}{N}\sum_{n=0}^{m-1}\|\mathbf{u}_{2,h}^{n+1} - \mathbf{u}_{2,h}^{n}\|^2 + \frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2 + \Delta t \sum_{n=0}^{m-1}\| - \nabla \phi_{2,h}^{n+1} + \mathbf{u}_{2,h}^{n+1} \times \mathbf{B}\|^2 \\
&\leq C(\|\mathbf{u}_{1,h}^{0}\|^2 + \|\mathbf{u}_{2,h}^{0}\|^2 + \|\nabla \mathbf{u}_{1,h}^{0}\|^2 + \|\mathbf{j}_{1,h}^{0}\|^2 + \Delta t \sum_{n=0}^{m}\|\mathbf{f}^{n}\|_{-1}^2).
\end{aligned} \tag{71}
$$

Proof Setting $\mathbf{v}_h = \mathbf{u}_{2,h}^{n+1}$, $q_h = \mathbf{u}_{2,h}^{n+1}$ and $\psi_h = \phi_{2,h}^{n+1}$ in Eq. 70 and multiplying it by $2\Delta t$, we have

$$\frac{1}{N}(\|\mathbf{u}_{2,h}^{n+1}\|^2 - \|\mathbf{u}_{2,h}^{n}\|^2 + \|\mathbf{u}_{2,h}^{n+1} - \mathbf{u}_{2,h}^{n}\|^2) + \frac{2\Delta t}{M^2}\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2 + 2\Delta t\|-\nabla\phi_{2,h}^{n+1} + \mathbf{u}_{2,h}^{n+1} \times \mathbf{B}\|^2$$
$$= \frac{\Delta t}{N}(b(\mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{2,h}^{n+1}) - b(\mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n}, \mathbf{u}_{2,h}^{n+1})) + \frac{\Delta t}{M^2}(\nabla \mathbf{u}_{1,h}^{n+1} - \nabla \mathbf{u}_{1,h}^{n}, \nabla \mathbf{u}_{2,h}^{n+1}) \quad (72)$$
$$+ \Delta t(\mathbf{j}_{1,h}^{n+1} - \mathbf{j}_{1,h}^{n}, \mathbf{u}_{2,h}^{n+1} \times \mathbf{B}) + 2\Delta t(\frac{\mathbf{f}^n + \mathbf{f}^{n+1}}{2}, \mathbf{u}_{2,h}^{n+1}).$$

We bound each terms in the right hand side of Eq. 72 as follows. For an arbitrary $\varepsilon > 0$,

$$\frac{\Delta t}{N}(b(\mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{2,h}^{n+1}) - b(\mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n}, \mathbf{u}_{2,h}^{n+1}))$$
$$= \frac{\Delta t}{N}(b(\mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{2,h}^{n+1}) + b(\mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{1,h}^{n}, \mathbf{u}_{2,h}^{n+1})). \quad (73)$$

For the term $b(\mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{2,h}^{n+1})$, using the result of Theorem 4, we have

$$\begin{aligned} b(\mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{2,h}^{n+1}) &= b(\mathbf{u}^{n+1} - \mathbf{u}^{n}, \mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{2,h}^{n+1}) - \Delta t\, b(S_{\mathbf{u}_1}^{n+1}, \mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{2,h}^{n+1}) \\ &\le C\|\nabla \mathbf{u}^{n+1} - \nabla \mathbf{u}^{n}\|\|\nabla \mathbf{u}_{1,h}^{n+1}\|\|\nabla \mathbf{u}_{2,h}^{n+1}\| \\ &\quad + C\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|\|\nabla \mathbf{u}_{1,h}^{n+1}\|\|\nabla \mathbf{u}_{2,h}^{n+1}\| \\ &\le C(\|\nabla \mathbf{u}^{n+1}\|^2 + \|\nabla \mathbf{u}^{n}\|^2)\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 \\ &\quad + C\Delta t^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 + \tfrac{\varepsilon}{2}\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2 \\ &\le C\|\mathbf{u}\|_{\infty,1}^2\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 + C\Delta t^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 \\ &\quad + \tfrac{\varepsilon}{2}\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2 \\ &\le C\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 + C\Delta t^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 \\ &\quad + \tfrac{\varepsilon}{2}\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2 \\ &\le C\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 + \tfrac{\varepsilon}{2}\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2. \end{aligned} \quad (74)$$

Similar to bound $b(\mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{1,h}^{n}, \mathbf{u}_{2,h}^{n+1})$, we can obtain

$$\frac{\Delta t}{N}(b(\mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{2,h}^{n+1}) - b(\mathbf{u}_{1,h}^{n}, \mathbf{u}_{1,h}^{n}, \mathbf{u}_{2,h}^{n+1}))$$
$$\le C\Delta t(\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 + \|\nabla \mathbf{u}_{1,h}^{n}\|^2) + \varepsilon\Delta t\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2. \quad (75)$$

The remaining terms are bound as follows.

$$\frac{\Delta t}{M^2}(\nabla \mathbf{u}_{1,h}^{n+1} - \nabla \mathbf{u}_{1,h}^{n}, \nabla \mathbf{u}_{2,h}^{n+1}) \le C\Delta t(\|\nabla \mathbf{u}_{1,h}^{n+1}\|^2 + \|\nabla \mathbf{u}_{1,h}^{n}\|^2) + \varepsilon\Delta t\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2. \quad (76)$$

$$\begin{aligned} \Delta t(\mathbf{j}_{1,h}^{n+1} - \mathbf{j}_{1,h}^{n}, \mathbf{u}_{2,h}^{n+1} \times \mathbf{B}) &\le C\Delta t\|\mathbf{j}_{1,h}^{n+1} - \mathbf{j}_{1,h}^{n}\|\|\nabla \mathbf{u}_{2,h}^{n+1}\|\|\mathbf{B}\|_{L^\infty} \\ &\le C\Delta t(\|\mathbf{j}_{1,h}^{n+1}\|^2 + \|\mathbf{j}_{1,h}^{n}\|^2) + \varepsilon\Delta t\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2. \end{aligned} \quad (77)$$

$$2\Delta t(\frac{\mathbf{f}^n + \mathbf{f}^{n+1}}{2}, \mathbf{u}_{2,h}^{n+1}) \le C\Delta t\|\frac{\mathbf{f}^n + \mathbf{f}^{n+1}}{2}\|_{-1}^2 + \varepsilon\Delta t\|\nabla \mathbf{u}_{2,h}^{n+1}\|^2. \quad (78)$$

Set $\varepsilon = \frac{1}{4M^2}$. Combining Eqs. 75 to 78 with Eq. 72 and summing Eq. 72 from $n = 0$ to $n = m - 1$, we have

$$\frac{1}{N}\|\mathbf{u}_{2,h}^{m}\|^2 + \frac{1}{N}\sum_{n=0}^{m-1}\|\mathbf{u}_{2,h}^{n+1} - \mathbf{u}_{2,h}^{n}\|^2 + \frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla\mathbf{u}_{2,h}^{n+1}\|^2 + \Delta t\sum_{n=0}^{m-1}\| - \nabla\phi_{2,h}^{n+1} + \mathbf{u}_{2,h}^{n+1} \times \mathbf{B}\|^2 \le \frac{1}{N}\|\mathbf{u}_{2,h}^{0}\|^2 + C\Delta t\sum_{n=0}^{m-1}(\|\nabla\mathbf{u}_{1,h}^{n+1}\|^2 + \|\nabla\mathbf{u}_{1,h}^{n}\|^2 + \|\mathbf{j}_{1,h}^{n+1}\|^2 + \|\mathbf{j}_{1,h}^{n}\|^2 + \|\frac{\mathbf{f}^n + \mathbf{f}^{n+1}}{2}\|_{-1}^2). \tag{79}$$

Finally, using the result of Theorem 1 completes the proof. □

Next, we analyze the consistency and convergence of the method (70) and show that the method (70) is second accurate. Similar to the previous denotations, we denote $\mathbf{j}_{2,h}^{n+1} = -\nabla\phi_{2,h}^{n+1} + \mathbf{u}_{2,h}^{n+1} \times \mathbf{B}$, $\mathbf{e}_{\mathbf{u}_2}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}_{2,h}^{n+1}$, $e_{\phi_2}^{n+1} = \phi^{n+1} - \phi_{2,h}^{n+1}$ and $\mathbf{e}_{\mathbf{j}_2}^{n+1} = \mathbf{j}^{n+1} - \mathbf{j}_{2,h}^{n+1}$. It is also obvious that $\mathbf{e}_{\mathbf{j}_2}^{n+1} = -\nabla e_{\phi_2}^{n+1} + \mathbf{e}_{\mathbf{u}_2}^{n+1} \times \mathbf{B}$.

In addition, we define the function $F := \frac{1}{N}\mathbf{u}_t = -\frac{1}{N}\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{M^2}\Delta\mathbf{u} - \nabla p + \mathbf{f} + \mathbf{B}\times\nabla\phi + (\mathbf{u}\times\mathbf{B})\times\mathbf{B}$. Then the first continuous momentum equation can be written as

$$\frac{1}{N}\frac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^{n}) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F\, d\tau. \tag{80}$$

Using the trapezoid rule that

$$\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F\, d\tau = \frac{F^n + F^{n+1}}{2} + C\Delta t^2 F_{tt}(\alpha^{n+1}),$$

where $\alpha^{n+1} \in (t^n, t^{n+1})$, we can obtain

$$\begin{aligned}
\frac{1}{N}\frac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^{n}) = &-\frac{1}{2N}\mathbf{u}^n\cdot\nabla\mathbf{u}^n - \frac{1}{2N}\mathbf{u}^{n+1}\cdot\nabla\mathbf{u}^{n+1} + \frac{1}{2M^2}\Delta\mathbf{u}^n + \frac{1}{2M^2}\Delta\mathbf{u}^{n+1}\\
&- \frac{1}{2}\nabla p^n - \frac{1}{2}\nabla p^{n+1} + \frac{1}{2}\mathbf{f}^n + \frac{1}{2}\mathbf{f}^{n+1}\\
&+ \frac{1}{2}\mathbf{B}\times\nabla\phi^n + (\mathbf{u}^n\times\mathbf{B})\times\mathbf{B} + \frac{1}{2}\mathbf{B}\times\nabla\phi^{n+1} + (\mathbf{u}^{n+1}\times\mathbf{B})\times\mathbf{B}\\
&+ C\Delta t^2 F_{tt}(\alpha^{n+1}),\\
\nabla\cdot\mathbf{u} = 0, \qquad \Delta\phi &= \nabla\cdot(\mathbf{u}\times\mathbf{B}).
\end{aligned} \tag{81}$$

Then, at time t^{n+1}, we have the following weak formulation.

$$
\begin{aligned}
&\frac{1}{N}(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}, \mathbf{v}_h) + \frac{1}{N}b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h) + \frac{1}{M^2}(\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}_h) - (p^{n+1}, \nabla \cdot \mathbf{v}_h) \\
&+ (-\nabla \phi^{n+1} + \mathbf{u}^{n+1} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) = \frac{1}{2N}b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h) - \frac{1}{2N}b(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h) \\
&+ \frac{1}{2M^2}(\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}_h) - \frac{1}{2M^2}(\nabla \mathbf{u}^n, \nabla \mathbf{v}_h) - \frac{1}{2}(p^{n+1} - p^n, \nabla \cdot \mathbf{v}_h) \\
&+ \frac{1}{2}(-\nabla \phi^{n+1} + \mathbf{u}^{n+1} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) - \frac{1}{2}(-\nabla \phi^n + \mathbf{u}^n \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) \\
&+ (\frac{\mathbf{f}^n + \mathbf{f}^{n+1}}{2}, \mathbf{v}_h) + C\Delta t^2 (F_{tt}(\alpha^{n+1}), \mathbf{v}_h) \quad \forall \mathbf{v}_h \in X_h, \\
&(\nabla \cdot \mathbf{u}^{n+1}, q_h) = 0 \qquad \forall q_h \in Q_h, \\
&(-\nabla \phi^{n+1} + \mathbf{u}^{n+1} \times \mathbf{B}, -\nabla \psi_h) = 0 \qquad \forall \psi_h \in S_h.
\end{aligned} \tag{82}
$$

The followed theorem shows the consistency of method (70).

Theorem 6 *The method* (70) *is consistent and the consistency error is* $O(\Delta t^2 + \Delta t\, h^k)$.

Proof It is obvious that the last two equations in Eq. 70 are consistent with those in Eq. 82. We only need to analyze the consistency of the first equation in Eq. 70. Based on our definition of consistency error, see Eq. 9, then from Eqs. 70 and 82, we have the following estimate of the consistency error E_2.

$$
E_2 = \sup_{\mathbf{v}_h \in X_h} \frac{1}{\|\mathbf{v}_h\|} |E_{21} + E_{22} - E_{23}|, \tag{83}
$$

where E_{21}, E_{22} and E_{23} are given as follows.

$$
E_{21} = C\Delta t^2 (F_{tt}(\alpha^{n+1}), \mathbf{v}_h). \tag{84}
$$

$$
\begin{aligned}
E_{22} =& \frac{1}{2N}b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h) - \frac{1}{2N}b(\mathbf{u}_{1,h}^{n+1}, \mathbf{u}_{1,h}^{n+1}, \mathbf{v}_h) \\
&+ \frac{1}{2M^2}(\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}_h) - \frac{1}{2M^2}(\nabla \mathbf{u}_{1,h}^{n+1}, \nabla \mathbf{v}_h) \\
&- \frac{1}{2}(p^{n+1}, \nabla \cdot \mathbf{v}_h) + \frac{1}{2}(p_{1,h}^{n+1}, \nabla \cdot \mathbf{v}_h) \\
&+ \frac{1}{2}(-\nabla \phi^{n+1} + \mathbf{u}^{n+1} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) - \frac{1}{2}(-\nabla \phi_{1,h}^{n+1} + \mathbf{u}_{1,h}^{n+1} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}).
\end{aligned} \tag{85}
$$

$$
\begin{aligned}
E_{23} =&\frac{1}{2N}b(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h) - \frac{1}{2N}b(\mathbf{u}^n_{1,h}, \mathbf{u}^n_{1,h}, \mathbf{v}_h)\\
&+ \frac{1}{2M^2}(\nabla \mathbf{u}^n, \nabla \mathbf{v}_h) - \frac{1}{2M^2}(\nabla \mathbf{u}^n_{1,h}, \nabla \mathbf{v}_h)\\
&- \frac{1}{2}(p^n, \nabla \cdot \mathbf{v}_h) + \frac{1}{2}(p^n_{1,h}, \nabla \cdot \mathbf{v}_h)\\
&+ \frac{1}{2}(-\nabla \phi^n + \mathbf{u}^n \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) - \frac{1}{2}(-\nabla \phi^n_{1,h} + \mathbf{u}^n_{1,h} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}).
\end{aligned} \tag{86}
$$

We need to bound the above three terms. For the term E_{21},

$$
|E_{21}| \leq C\Delta t^2 \|\mathbf{v}_h\|. \tag{87}
$$

In order to bound E_{22} and E_{23}, we subtract the first equation of Eq. 12 from that of Eq. 10, then rearrange the terms on both sides of the equation to obtain

$$
E_{22} = \frac{1}{2N}[(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{v}_h) - (S^{n+1}_{\mathbf{u}_1}, \mathbf{v}_h)]. \tag{88}
$$

Similarly, we also have

$$
E_{23} = \frac{1}{2N}[(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} - \mathbf{u}_t(t^n), \mathbf{v}_h) - (S^n_{\mathbf{u}_1}, \mathbf{v}_h)]. \tag{89}
$$

From Eqs. 88 and 89, we obtain

$$
\begin{aligned}
|E_{22} - E_{23}| =&|\frac{1}{2N}[(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{v}_h) - (S^{n+1}_{\mathbf{u}_1}, \mathbf{v}_h)]\\
&- \frac{1}{2N}[(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} - \mathbf{u}_t(t^n), \mathbf{v}_h) - (S^n_{\mathbf{u}_1}, \mathbf{v}_h)]|\\
\leq&\frac{1}{2N}\|(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t^{n+1})) - (\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} - \mathbf{u}_t(t^n))\|\|\mathbf{v}_h\|\\
&+ \frac{1}{2N}\|S^{n+1}_{\mathbf{u}_1} - S^n_{\mathbf{u}_1}\|\|\mathbf{v}_h\|\\
\leq&\frac{1}{2N}\Delta t^2\|\mathbf{u}_{ttt}\|_{\infty,0}\|\mathbf{v}_h\| + \frac{1}{2N}\Delta t\|K^{n+1}_{\mathbf{u}_1}\|\|\mathbf{v}_h\|\\
\leq&C\Delta t^2\|\mathbf{v}_h\| + C\Delta t(\Delta t + h^k)\|\mathbf{v}_h\|,
\end{aligned} \tag{90}
$$

where we use the result of Corollary 1.

Combining Eqs. 90, 87 and 83, we have

$$
E_2 \leq \sup_{\mathbf{v}_h \in X_h} \frac{|E_{21}| + |E_{22} - E_{23}|}{\|\mathbf{v}_h\|} \leq C\Delta t^2 + C\Delta t\, h^k, \tag{91}
$$

which completes the proof. □

Finally, we give the convergence result of method (70).

Theorem 7 *Let the assumptions of Theorems 3 and 4 be satisfied.* $(\mathbf{u}_{2,h}^{n+1}, p_{2,h}^{n+1}, \phi_{2,h}^{n+1})$ *is given by method* (70)*. Then, we have*

$$\|\mathbf{e}_{\mathbf{u}_2}^m\|^2+\frac{N}{M^2}\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|\mathbf{e}_{\mathbf{j}_2}^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_{\phi_2}^{n+1}\|^2\le C(\Delta t^4+h^{2k}), \quad (92)$$

provided that Δt *is sufficiently small.*

Proof At time t^{n+1}, the true solution $(\mathbf{u}, p, \phi)$ satisfies (82). Subtracting Eq. 82 from Eq. 70 and then setting $\mathbf{v}_h = \mathbf{e}_{\mathbf{u}_2}^{n+1}$, $\psi_h = e_{\phi_2}^{n+1}$, we can obtain

$$\begin{aligned}
&\frac{1}{2N\Delta t}(\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2-\|\mathbf{e}_{\mathbf{u}_2}^{n}\|^2+\|\mathbf{e}_{\mathbf{u}_2}^{n+1}-\mathbf{e}_{\mathbf{u}_2}^{n}\|^2)+\frac{1}{M^2}\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2+\|\mathbf{e}_{\mathbf{j}_2}^{n+1}\|^2+\|\nabla\mathbf{e}_{\phi_2}^{n+1}\|^2\\
&=-\frac{1}{N}b(\mathbf{e}_{\mathbf{u}_2}^{n+1},\mathbf{u}^{n+1},\mathbf{e}_{\mathbf{u}_2}^{n+1})+(\mathbf{e}_{\mathbf{u}_2}^{n+1}\times\mathbf{B},\nabla e_{\phi_2}^{n+1})+\frac{1}{2}(\mathbf{e}_{\mathbf{j}_1}^{n+1}-\mathbf{e}_{\mathbf{j}_1}^{n},\mathbf{e}_{\mathbf{u}_2}^{n+1}\times\mathbf{B})\\
&\quad+\frac{1}{2M^2}(\nabla\mathbf{e}_{\mathbf{u}_1}^{n+1}-\nabla\mathbf{e}_{\mathbf{u}_1}^{n},\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1})+\frac{1}{2N}b(\mathbf{e}_{\mathbf{u}_1}^{n+1}-\mathbf{e}_{\mathbf{u}_1}^{n},\mathbf{u}^{n+1},\mathbf{e}_{\mathbf{u}_2}^{n+1})\\
&\quad+\frac{1}{2N}b(\mathbf{e}_{\mathbf{u}_1}^{n},\mathbf{u}^{n+1}-\mathbf{u}^{n},\mathbf{e}_{\mathbf{u}_2}^{n+1})+\frac{1}{2N}b(\mathbf{u}_{1,h}^{n+1}-\mathbf{u}_{1,h}^{n},\mathbf{e}_{\mathbf{u}_1}^{n+1},\mathbf{e}_{\mathbf{u}_2}^{n+1})\\
&\quad+\frac{1}{2N}b(\mathbf{u}_{1,h}^{n},\mathbf{e}_{\mathbf{u}_1}^{n+1}-\mathbf{e}_{\mathbf{u}_1}^{n},\mathbf{e}_{\mathbf{u}_2}^{n+1})+\left(\frac{p^{n+1}+p^n}{2}-\frac{\lambda_h^{n+1}+\lambda_h^n}{2},\nabla\cdot\mathbf{e}_{\mathbf{u}_2}^{n+1}\right)\\
&\quad+C\Delta t^2(F_{tt}(\alpha^{n+1}),\mathbf{e}_{\mathbf{u}_2}^{n+1}),
\end{aligned} \quad (93)$$

for $\forall\,\lambda_h \in Q_h$.

Next, we bound each term in the right hand side of Eq. 93. For $\forall\,\varepsilon, \varepsilon_1 > 0$,

$$\begin{aligned}
-\frac{1}{N}b(\mathbf{e}_{\mathbf{u}_2}^{n+1},\mathbf{u}^{n+1},\mathbf{e}_{\mathbf{u}_2}^{n+1}) &\le C\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^{\frac{1}{2}}\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^{\frac{1}{2}}\|\nabla\mathbf{u}^{n+1}\|\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|\\
&\le C\|\mathbf{u}\|_{\infty,1}^4\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2+\varepsilon\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2\\
&\le C\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2+\varepsilon\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2.
\end{aligned} \quad (94)$$

For the term $(\mathbf{e}_{\mathbf{u}_2}^{n+1}\times\mathbf{B}, \nabla e_{\phi_2}^{n+1})$,

$$\begin{aligned}
(\mathbf{e}_{\mathbf{u}_2}^{n+1}\times\mathbf{B},\nabla e_{\phi_2}^{n+1}) &\le C\|\mathbf{B}\|_{L^\infty}\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|\|\nabla e_{\phi_2}^{n+1}\|\\
&\le C\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2+\varepsilon_1\|\nabla e_{\phi_2}^{n+1}\|\\
&\le C\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2+\varepsilon_1\|\nabla e_{\phi_2}^{n+1}\|.
\end{aligned} \quad (95)$$

For the term $\frac{1}{2}(\mathbf{e}_{\mathbf{j}_1}^{n+1}-\mathbf{e}_{\mathbf{j}_1}^{n}, \mathbf{e}_{\mathbf{u}_2}^{n+1}\times\mathbf{B})$,

$$\begin{aligned}
\frac{1}{2}(\mathbf{e}_{\mathbf{j}_1}^{n+1}-\mathbf{e}_{\mathbf{j}_1}^{n},\mathbf{e}_{\mathbf{u}_2}^{n+1}\times\mathbf{B}) &\le C\Delta t\|\mathbf{B}\|_{L^\infty}\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|\|S_{\mathbf{j}_1}^{n+1}\|\\
&\le C\Delta t^2\|\mathbf{B}\|_{L^\infty}^2\|S_{\mathbf{j}_1}^{n+1}\|^2+\varepsilon\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2\\
&\le C\Delta t^2\|S_{\mathbf{j}_1}^{n+1}\|^2+\varepsilon\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2.
\end{aligned} \quad (96)$$

For the term $\frac{1}{2M^2}(\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1} - \nabla \mathbf{e}_{\mathbf{u}_1}^{n}, \nabla \mathbf{e}_{\mathbf{u}_2}^{n+1})$,

$$\frac{1}{2M^2}(\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1} - \nabla \mathbf{e}_{\mathbf{u}_1}^{n}, \nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}) \leq C\Delta t^2 \|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \varepsilon \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2. \tag{97}$$

For the nonlinear terms,

$$\begin{aligned}\frac{1}{2N} b(\mathbf{e}_{\mathbf{u}_1}^{n+1} - \mathbf{e}_{\mathbf{u}_1}^{n}, \mathbf{u}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) &\leq C\Delta t \|\nabla S_{\mathbf{u}_1}^{n+1}\| \|\nabla \mathbf{u}^{n+1}\| \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\| \\ &\leq C\Delta t^2 \|\mathbf{u}\|_{\infty,1}^2 \|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \varepsilon \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2 \\ &\leq C\Delta t^2 \|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \varepsilon \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2.\end{aligned} \tag{98}$$

$$\begin{aligned}\frac{1}{2N} b(\mathbf{e}_{\mathbf{u}_1}^{n}, \mathbf{u}^{n+1} - \mathbf{u}^{n}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) &\leq C\Delta t \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n}\| \|\nabla \mathbf{u}_t(\beta^{n+1})\| \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\| \\ &\leq C\Delta t^2 \|\mathbf{u}_t\|_{\infty,1}^2 \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n}\|^2 + \varepsilon \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2 \\ &\leq C\Delta t^2 \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n}\|^2 + \varepsilon \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2,\end{aligned} \tag{99}$$

where $\beta^{n+1} \in (t^n, t^{n+1})$. Since

$$\frac{1}{2N} b(\mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{1,h}^{n}, \mathbf{e}_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) = \frac{\Delta t}{2N} b(\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{\Delta t}, \mathbf{e}_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) - \frac{\Delta t}{2N} b(S_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}),$$

and

$$\begin{aligned}\frac{\Delta t}{2N} b(\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{\Delta t}, \mathbf{e}_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) &\leq C\Delta t \|\nabla \mathbf{u}_t(\beta^{n+1})\| \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\| \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\| \\ &\leq C\Delta t^2 \|\mathbf{u}_t\|_{\infty,1}^2 \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2 + \varepsilon \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2 \\ &\leq C\Delta t^2 \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2 + \frac{\varepsilon}{2} \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2,\end{aligned}$$

$$\begin{aligned}-\frac{\Delta t}{2N} b(S_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) &\leq C\Delta t \|\nabla S_{\mathbf{u}_1}^{n+1}\| \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\| \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\| \\ &\leq C\Delta t^2 \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2 \|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \frac{\varepsilon}{2} \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2,\end{aligned}$$

we can obtain

$$\frac{1}{2N} b(\mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{1,h}^{n}, \mathbf{e}_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) \leq C\Delta t^2 \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2 + C\Delta t^2 \|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2 \|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \varepsilon \|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2. \tag{100}$$

Since

$$\frac{1}{2N}b(\mathbf{u}_{1,h}^n, \mathbf{e}_{\mathbf{u}_1}^{n+1} - \mathbf{e}_{\mathbf{u}_1}^n, \mathbf{e}_{\mathbf{u}_2}^{n+1}) = \frac{\Delta t}{2N}b(\mathbf{u}^n, S_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) - \frac{\Delta t}{2N}b(\mathbf{e}_{\mathbf{u}_1}^n, S_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}),$$

and

$$\begin{aligned}\frac{\Delta t}{2N}b(\mathbf{u}^n, S_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) &\leq C\Delta t\|\nabla \mathbf{u}^n\|\|\nabla S_{\mathbf{u}_1}^{n+1}\|\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|\\ &\leq C\Delta t^2\|\mathbf{u}\|_{\infty,1}^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \frac{\varepsilon}{2}\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2\\ &\leq C\Delta t^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \frac{\varepsilon}{2}\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2,\end{aligned}$$

$$\begin{aligned}-\frac{\Delta t}{2N}b(\mathbf{e}_{\mathbf{u}_1}^n, S_{\mathbf{u}_1}^{n+1}, \mathbf{e}_{\mathbf{u}_2}^{n+1}) &\leq C\Delta t\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|\|\nabla S_{\mathbf{u}_1}^{n+1}\|\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|\\ &\leq C\Delta t^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \frac{\varepsilon}{2}\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2,\end{aligned}$$

we have

$$\frac{1}{2N}b(\mathbf{u}_{1,h}^n, \mathbf{e}_{\mathbf{u}_1}^{n+1} - \mathbf{e}_{\mathbf{u}_1}^n, \mathbf{e}_{\mathbf{u}_2}^{n+1}) \leq C\Delta t^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + C\Delta t^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + \varepsilon\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2. \tag{101}$$

For the remaining terms,

$$\left(\frac{p^{n+1}+p^n}{2} - \frac{\lambda_h^{n+1}+\lambda_h^n}{2}, \nabla\cdot\mathbf{e}_{\mathbf{u}_2}^{n+1}\right) \leq C\left\|\frac{p^{n+1}+p^n}{2} - \frac{\lambda_h^{n+1}+\lambda_h^n}{2}\right\|^2 + \varepsilon\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2. \tag{102}$$

$$C\Delta t^2(F_{tt}(\alpha^{n+1}), \mathbf{e}_{\mathbf{u}_2}^{n+1}) \leq C\Delta t^4\|F_{tt}(\alpha^{n+1})\|^2 + \varepsilon\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2. \tag{103}$$

Combining Eqs. 94–103 with Eq. 93 and setting $\varepsilon = \frac{1}{18M^2}$, $\varepsilon_1 = \frac{1}{2}$, we have

$$\begin{aligned}&\frac{1}{2N\Delta t}(\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2 - \|\mathbf{e}_{\mathbf{u}_2}^n\|^2 + \|\mathbf{e}_{\mathbf{u}_2}^{n+1} - \mathbf{e}_{\mathbf{u}_2}^n\|^2) + \frac{1}{2M^2}\|\nabla \mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2 + \frac{1}{2}\|\mathbf{e}_{\mathbf{j}_2}^{n+1}\|^2 + \frac{1}{2}\|\nabla \mathbf{e}_{\phi_2}^{n+1}\|^2\\ &\leq C\|\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2 + C\Delta t^2\|S_{\mathbf{j}_1}^{n+1}\|^2 + C\Delta t^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + C\Delta t^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2 + C\Delta t^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2\\ &\quad + C\Delta t^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2 + C\Delta t^2\|\nabla \mathbf{e}_{\mathbf{u}_1}^n\|^2\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2\\ &\quad + C\left\|\frac{p^{n+1}+p^n}{2} - \frac{\lambda_h^{n+1}+\lambda_h^n}{2}\right\|^2 + C\Delta t^4\|F_{tt}(\alpha^{n+1})\|^2.\end{aligned} \tag{104}$$

Multiplying Eq. 104 by $2N\Delta t$, summing from $n = 0$ to $n = m-1$ and using Lemma 2, we can deduce

$$\begin{aligned}
&\|\mathbf{e}_{\mathbf{u}_2}^m\|^2+\sum_{n=0}^{m-1}(\|\mathbf{e}_{\mathbf{u}_2}^{n+1}-\mathbf{e}_{\mathbf{u}_2}^n\|^2)+\frac{N}{M^2}\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_{\mathbf{u}_2}^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|\mathbf{e}_{\mathbf{j}_2}^{n+1}\|^2+N\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_{\phi_2}^{n+1}\|^2\\
&\le C(\|\mathbf{e}_{\mathbf{u}_2}^0\|^2+\Delta t^2\sum_{n=0}^{m-1}\Delta t\|S_{\mathbf{j}_1}^{n+1}\|^2+\Delta t^2\sum_{n=0}^{m-1}\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2+\Delta t^2\sum_{n=0}^{m-1}\Delta t\|\nabla\mathbf{e}_{\mathbf{u}_1}^n\|^2\\
&\quad+\Delta t^2\sum_{n=0}^{m-1}\Delta t\|\nabla\mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2+\Delta t\sum_{n=0}^{m-1}\Delta t\|\nabla\mathbf{e}_{\mathbf{u}_1}^{n+1}\|^2\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2\\
&\quad+\Delta t\sum_{n=0}^{m-1}\Delta t\|\nabla\mathbf{e}_{\mathbf{u}_1}^{n}\|^2\Delta t\|\nabla S_{\mathbf{u}_1}^{n+1}\|^2+\sum_{n=0}^{m-1}\Delta t\|\frac{p^{n+1}+p^n}{2}-\frac{\lambda_h^{n+1}+\lambda_h^n}{2}\|^2\\
&\quad+\Delta t^4\sum_{n=0}^{m-1}\Delta t\|F_{tt}(\alpha^{n+1})\|^2).
\end{aligned} \tag{105}$$

Finally, using the results of Theorems 3 and 4 completes the proof. □

5 Numerical experiment

In this section, we provide two numerical experiments to testify the convergence and effectiveness of the second order method based on the SDC method (BE-SDC) presented in Section 4. In the first example, we computed the rate of convergence of BE-SDC method to testify the correctness of our theoretical analysis of convergence. Then, in the second example, we compared BE-SDC method with Crank-Nicolson (CN) method, see [8], and showed that BE-SDC converged but CN did not converge in this example, which illustrates that BE-SDC was more effective in this example. The experiment results were obtained by using the software package $FreeFEM{+}{+}$, see [22].

Table 1 Errors and convergence rate of BE-SDC

Δt	$\|\|\mathbf{u}_{2,h}-\mathbf{u}\|\|_{\infty,0}$	Rate	$\|\|\nabla(\mathbf{u}_{2,h}-\mathbf{u})\|\|_{2,0}$	Rate	$\|\|\nabla(\phi_{2,h}-\phi)\|\|_{2,0}$	Rate
1/16	1.269e-4		1.607e-1		1.808e-2	
1/24	4.397e-5	2.61	7.599e-2	1.85	8.565e-3	1.84
1/32	2.191e-5	2.42	4.406e-2	1.90	4.967e-3	1.89
1/48	8.663e-6	2.29	2.017e-2	1.93	2.275e-3	1.93
1/64	4.607e-6	2.20	1.151e-2	1.95	1.298e-3	1.95
1/96	1.939e-6	2.13	5.193e-3	1.99	5.856e-4	1.96

Table 2 Effectiveness of BE-SDC method for Experiment 2

	$T=1$	$T=2$	$T=5$
$\|(\mathbf{u}_h-\mathbf{u})(T)\|$	2.5343e-1	2.1529e-1	1.3461e-1
$\|(p_h-p)(T)\|$	4.8600e-1	5.6450e-1	2.6013e-1
$\|(\phi_h-\phi)(T)\|$	1.2549e-2	7.6146e-3	1.1702e-2

5.1 Experiment 1

Let the domain $\Omega=[0,1]\times[0,1]$, $T=1$, $Re=1$, $N=1$, $M=1$ and $\mathbf{B}=(0,0,1)$. Consider the true solution $(\mathbf{u},p,\phi)$ given as follows.

$$\begin{aligned}
&\mathbf{u}(x,y,t)=(2\pi\cos(2\pi x)\sin(2\pi y),-2\pi\sin(2\pi x)\cos(2\pi y),0)e^{-5t},\\
&p(x,y,t)=0,\\
&\phi(x,y,t)=(\cos(2\pi x)\cos(2\pi y)+x^2-y^2)e^{-5t}.
\end{aligned}$$

The $\mathbf{f}$, boundary condition and initial condition are determined by the true solution.

We utilize the P2-P1 Taylor-Hood mixed finite elements for fluid velocity and pressure and P2 finite element for electric potential. Besides, we set the time step equal to the mesh size, $\Delta t=h$, to testify the theoretical convergence result of BE-SDC. We choose the step sizes $\Delta t=\frac{1}{16},\frac{1}{24},\frac{1}{32},\frac{1}{48},\frac{1}{64},\frac{1}{96}$ to compute $|||\mathbf{u}_{2,h}-\mathbf{u}|||_{\infty,0}$, $|||\nabla(\mathbf{u}_{2,h}-\mathbf{u})|||_{2,0}$ and $|||\nabla(\phi_{2,h}-\phi)|||_{2,0}$. Table 1 shows the convergence accuracy of BE-SDC. It is clear that BE-SDC is second accurate. Hence, the numerical experiment result is consistent with the theoretical analysis.

5.2 Experiment 2

Let the domain $\Omega=[0,1]\times[0,1]$ and $\mathbf{B}=(0,0,1)$. Similar to the example studied in [15], select $Re=6766$, $M=20$ and then $N=\frac{M^2}{Re}=0.059$. Set $\Delta t=\frac{1}{100}$, $h=\frac{1}{10}$ and choose $T=1,2,5$. Consider the same true solution $(\mathbf{u},p,\phi)$ and $\mathbf{f}$ as given in the above experiment. We use BE-SDC method and CN method to compute $\|(\mathbf{u}_h-\mathbf{u})(T)\|$, $\|(p_h-p)(T)\|$ and $\|(\phi_h-\phi)(T)\|$. Tables 2 and 3 present the computing results of BE-SDC method and CN method, respectively. We can see that BE-SDC method converges but CN method does not converge, which shows that our second method performs better in this experiment.

Table 3 Effectiveness of CN method for Experiment 2

	$T=1$	$T=2$	$T=5$
$\|(\mathbf{u}_h-\mathbf{u})(T)\|$	5.6775e-1	3.8484e+3	1.3134e+3
$\|(p_h-p)(T)\|$	4.1328	1.0715e+8	5.2221e+9
$\|(\phi_h-\phi)(T)\|$	2.4490e-2	8.6007e+1	3.7110e+1

6 Conclusion

In this paper, we introduce a second order algorithm based on the SDC method to solve the simplified MHD flows at a low magnetic Reynolds number. We give a complete theoretical analysis about the stability, consistency and error estimate of our algorithm. We prove that our algorithm is unconditionally stable, consistent and second accurate. The numerical experiments testify the rightness of our theoretical analysis. For further research, we plan to study high accuracy stable decoupled algorithm for MHD flows.

References

1. Alfvén, H.: Existence of electromagnetic-hydrodynamic waves. Nature **150**, 405–406 (1942)
2. Barleon, L., Casal, V., Lenhart, L.: MHD Flow in liquid-metal-cooled blankets. Fusion Eng. Des. **14**, 401–412 (1991)
3. Davidson, P.A.: Magnetohydrodynamics in material processing. Annu. Rev. Fluid Mech. **31**, 273–300 (1999)
4. Lin, T.F., Gilbert, J.B., Kossowsky, R.: Sea-water magnetohydrodynamic propulsion for next-generation undersea vehicles. Pennsylvania State University State College Applied Research Laboratory (1990)
5. Gerbeau, J.F., Bris, C.L., Lelièvre, T.: Mathmatical Methods for the Magnetohydrodynamics of Liquid Metals. Oxford University Press (2006)
6. Adams, R.A.: Sobolev Spaces. Academic Press (2003)
7. Gunzburger, M.D., Meir, A.J., Peterson, J.: On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible Magnetohydrodynamics. Math. Comput. **56**(194), 523–563 (1991)
8. Yuksel, G., Ingram, R.: Numerical analysis of a finite element Crank-Nicolson discretization for MHD flows at small magnetic Reynolds numbers. Int. J. Numer. Anal. Model. **10**(1), 74–98 (2013)
9. Meir, A.J., Schmidt, P.G.: Analysis and numerical approximation of stationary MHD flow problem with nonideal boundary. SIAM J. Numer. Anal. **36**(4), 1304–1332 (1999)
10. Meir, A.J., Schmidt, P.G.: Variational methods for stationary MHD flow under natural interface conditions. Nonlinear Anal. **26**(4), 659–689 (1996)
11. Meir, A.J.: The equations of stationary, incompressible magnetohydynamics with mixed boundary conditions. Comput. Math. Appl. **25**(12), 13–29 (1993)
12. Layton, W.J., Meir, A.J., Schmidt, P.G.: A two-level discretization method for the stationary MHD equations. Electron. Trans. Numer. Anal. **6**, 198–210 (1997)
13. Schmidt, P.G.: A Galerkin method for time-dependent MHD flow with nonideal boundaries. Commun. Appl. Anal. **3**(3), 383–398 (1999)
14. Layton, W.J., Tran, H., Trenchea, C.: Numerical analysis of two partitioned methods for uncoupling evolutionary MHD flows. Numer. Meth. Part D.E. **30**(4), 1083–1102 (2014)
15. Yuksel, G., Isik, O.R.: Numerical analysis of Backward-Euler discretization for simplified magnetohydynamic flows. Applied Mathematical Modelling. **39**, 1889–1898 (2015)
16. Dutt, A., Greengard, L., Rokhlin, V.: Spectral deferred correction methods for ordinary differential equations. BIT. **40**(2), 241–266 (2000)
17. Minion, M.L.: Semi-implicit projection methods for incompressible flow based on spectral deferred corrections. Appl. Numer. Math. **48**(3-4), 369–387 (2004)
18. Minion, M.L.: Semi-Implicit Projection methods for ordinary differential equations. Comm. Math. Sci. **1**(3), 471–500 (2003)
19. Wilson, N., Labovsky, A., Trenchea, C.: High accuracy method for magnetohydynamic system in Elsässer variables. Computational Methods in Applied Mathematics **15**(1), 97–110 (2014)
20. He, Y.: Unconditional convergence of the Euler semi-implicit scheme for the three-dimensional incompressible MHD equations. IMA Journal of Numerical Analysis. dru015 (2014)
21. Roberts, P.H.: An Introduction to Magnetohydynamics. Elsevier, USA (1967)
22. Hecht, F., Pironneau, O.: Freefem++. Webpage: http://www.freefem.org