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Piecewise spectral collocation method for system of Volterra integral equations

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Abstract The main purpose of this paper is to investigate the piecewise spectral collocation method for system of Volterra integral equations. The provided convergence analysis shows that the presented method performs better than global spectral collocation method and piecewise polynomial collocation method. Numerical experiments are carried out to confirm these theoretical results.

Keywords Spectral collocation method · System of Volterra integral equations · Convergence analysis

Mathematics Subject Classification (2010) 65M70 · 45D05

1 Introduction

Spectral methods are important numerical methods for differential equations. They are famous for their high accuracy. The monographs by Canuto et al. [4, 5], Shen et al. [25, 26] contain a lot of useful information on spectral methods. Guo and Wang [13, 14] established results on orthogonal projections and interpolations theory for the Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces. These

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results serves as an important tool in the analysis of the numerical error estimate for the spectral method.

Spectral collocation methods, spectral Galerkin methods, and the corresponding error analysis have been provided recently for Volterra type integral equations. Tang et al. [31] proposed a Legendre spectral-collocation method to solve VIEs (Volterra integral equations) of the second kind whose kernel and solutions are sufficiently smooth. Chen and her coworkers [7–12, 33–38] done a series of work on Volterra type integral equations. Xie et al. [39] provided spectral and pseudo-spectral Jacobi-Galerkin approaches for the second kind VIEs. Li et al. [17] proposed a parallel in time method to solve VIEs. Sheng et al. [27] proposed a multi-step spectral method to solve nonlinear VIEs.

The SVIEs (system of Volterra integral equations) appear in scientific applications in engineering, physics, chemistry and populations growth models. For examples, Volterra [32] refined "predator-prey" model as a system of nonlinear Volterra integro-differential equations which can be changed to be a SVIEs. Many high order Volterra integro-differential equations can be changed to be SVIEs. Studies of systems of integral equations have attracted much concern in applied sciences. The general ideas and the essential features of these systems are of wide applicability. So numerically solution of SVIEs is very meaningful. There exist many numerical methods for SVIEs. Capobianco et al. [6] proposed fast implicit and explicit Runge-Kutta methods for SVIEs of Hammerstein type. Sorkun and Yalcinbas [28] solved SVIEs by transforming the integral system into the matrix equation with the help of Taylor series. Using the Bessel polynomials and the collocation points, Sahin et al. [23] solved SVIEs by transforming the system of linear Volterra integral equations into a matrix equation. Taghvafard and Erjaee [29] solved SVIEs by using the fractional differential transform method. Samadi and Tohidi [24] extended the spectral collocation method in [31] to solve SVIEs. Mirzaee and Bimesl [20] investigated a new Euler matrix method for SVIEs. References [2, 3, 15, 16, 18, 19, 22, 30, 40] also provide numerical method for SVIEs.

In this paper, we provide a hp-version spectral collocation method to solove SVIEs of the following form

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_0^t \mathbf{K}(t,s)\mathbf{y}(s)ds, t \in [0, T], \tag{1}$$

where the unknown function is

$$\mathbf{y}(t) := [y_0(t), y_1(t), \cdots, y_Q(t)]'.$$

We assume that the functions describing the above equation all possess continuous derivatives of at least order $m \geq 1$ on their respective domains, i.e.,

$$\begin{aligned} &\mathbf{g}(t) := [g_0(t), g_1(t), \cdots, g_Q(t)]', g_q(t) \in C^m([0,T]), q = 0, 1, \cdots, Q, \\ &\mathbf{K}(t,s) := \Big[K_{ij}(t,s)\Big]_{(Q+1)\times(Q+1)}, K_{ij}(t,s) \in C^m(\Omega), \Omega := \{(t,s) : 0 \leq s \leq t \leq T\}. \end{aligned} \tag{2}$$

where $C^m([a,b])$ is the space of functions possessing continuous derivatives of at least order $m \geq 1$ on their domain $[a,b]$. From Theorem 2.1.7 in [1] we know that the unique solution $\mathbf{y}(t)$ of Eq. 1 with the data functions (2) lies in $C^m([0,T])$.

In the present method, we change (1) to be a new equation defined on $[-1, 1]$, then divide $[-1, 1]$ into $M + 1$ subintervals $[\eta_\mu, \eta_{\mu+1}]$, $\mu = 0, 1, \cdots, M$, $\eta_0 = -1$, $\eta_{M+1} = 1$. In each subinterval $[\eta_\mu, \eta_{\mu+1}]$ we set collocation points as $N + 1$ Legendre Gauss-Lobatto points. We provide convergence analysis to show that the numerical errors decay in the rate $h^{m-1/2}N^{1/2-m}$ and $h^m N^{-m}$ in spaces $L^\infty(-1, 1)$ and $L^2(-1, 1)$ respectively, where $h := \max\{(\eta_{\mu+1} - \eta_\mu)/2 : \mu = 0, 1, \cdots, M\}$. These theoretical results imply that the convergence behavior of errors relates to h, N and m. Therefore, refining mesh (h becomes smaller), adding more collocation points (N becomes bigger) and given functions possessing better regularity (m becomes bigger) will improve the accuracy of the numerical solution. We provide numerical examples to confirm these theoretical results.

In order to compare the piecewise spectral collocation method (PSC) with piecewise polynomial collocation method (PPC) (see [1]) and global spectral collocation (GSC) (see [24]), we introduce PPC method and GSC method briefly. For the PPC method, the interval $[0, T]$ is divided into subintervals $[\xi_\mu, \xi_{\mu+1}]$, $\mu = 0, 1, \cdots, M$. The convergence analysis result (see [1], page 95) showed that no matter how many collocation points is employed in the subinterval $[\xi_\mu, \xi_{\mu+1}]$, the decay of errors will not exceed the rate $\widetilde{h}^m$, where $\widetilde{h} := \max\{\widetilde{h}_\mu : \widetilde{h}_\mu = \xi_{\mu+1} - \xi_\mu, \mu = 0, 1, \cdots, M\}$. For the GSC method (see [31]), the interval $[0, T]$ is transformed to $[-1, 1]$. $N + 1$ Gusss type points are chosen as collocation points on the global interval $[-1, 1]$. The errors decay at the rate N^{-m}. Comparing $h^{m-1/2}N^{1/2-m}$ (PSC) with $\widetilde{h}^m$ (PPC) and N^{-m} (GSC), we conclude that PSC method is sharper than PPC method and GSC method.

This paper is organized as follows. In Section 2, we deduce the numerical scheme of hp-version spectral collocation method for SVIEs (1). The lemmas for convergence analysis are provided in Section 3. The convergence analysis for the proposed method is presented in Section 4. Numerical examples are given in Section 5. Finally, in Section 6, we end with the conclusion and remark.

2 Numerical scheme

For ease of analysis we change the interval $[0, T]$ to the standard interval $[-1, 1]$. Precisely we use the variable transformation

$$\begin{aligned} t(x) &= \tfrac{T}{2}(x + 1), x \in [-1, 1], \\ s(z) &= \tfrac{T}{2}(z + 1), z \in [-1, x]. \end{aligned} \tag{3}$$

Then Eq. 1 can be written as

$$\mathbf{u}(x) = \mathbf{f}(x) + \int_{-1}^{x} \mathbf{R}(x, z)\mathbf{u}(z)dz, x \in [-1, 1], \tag{4}$$

where

$$\begin{aligned} \mathbf{u}(x) &:= \mathbf{y}(t(x)), \mathbf{f}(x) = \mathbf{g}(t(x)), \\ \mathbf{R}(x, z) &:= \tfrac{T}{2}\mathbf{K}(t(x), s(z)). \end{aligned} \tag{5}$$

We divide the interval $[-1, 1]$ into $M+1$ subintervals $\sigma_\mu := (\eta_\mu, \eta_{\mu+1}]$, $h_\mu := \eta_{\mu+1} - \eta_\mu$, $\mu = 0, 1, \cdots, M$. Set the collocation points as the follows

$$X_N := \bigcup_{\mu=0}^{N-1} X^\mu,\ X^\mu := \{x_n^\mu : \eta_\mu = x_0^\mu < x_1^\mu < \cdots < x_N^\mu = \eta_{\mu+1}\},$$

where

$$x_i^\mu := h_\mu x_i + \frac{\eta_{\mu+1} + \eta_\mu}{2}, \tag{6}$$

here $h_\mu := \frac{\eta_{\mu+1} - \eta_\mu}{2}$; $x_i, i = 0, 1, \cdots, N$ are the Legendre Gauss-Lobatto points in the standard interval $[-1, 1]$. Then Eq. 4 holds at x_i^μ, $i = 0, 1, \cdots, N, \mu = 0, 1, \cdots, M$,

$$\mathbf{u}(x_i^\mu) = \mathbf{f}(x_i^\mu) + \int_{-1}^{x_i^\mu} \mathbf{R}(x_i^\mu, z)\mathbf{u}(z)dz, \tag{7}$$

which can be written as follows,

$$\mathbf{u}(x_i^\mu) = \mathbf{f}(x_i^\mu) + \sum_{r=0}^{\mu-1} \int_{\eta_r}^{\eta_{r+1}} \mathbf{R}(x_i^\mu, z)\mathbf{u}|_{\sigma_r}(z)dz + \int_{\eta_\mu}^{x_i^\mu} \mathbf{R}(x_i^\mu, z)\mathbf{u}|_{\sigma_\mu}(z)dz, \tag{8}$$

where

$$\mathbf{u}|_{\sigma_r}(x) := [u_0|_{\sigma_r}(x), u_1|_{\sigma_r}(x), \cdots, u_Q|_{\sigma_r}(x)]',$$

and $u_q|_{\sigma_r}(x)$ is the restriction of $u_q(x)$ to the subinterval $[\eta_r, \eta_{r+1}]$, i.e.,

$$u_q|_{\sigma_r}(x) := u_q(x), x \in \sigma_r, r = 0, 1, \cdots, M.$$

We use u_{qi}^μ to approximate $u_q(x_i^\mu)$, $i = 0, 1, \cdots, N$, $\mu = 0, 1, \cdots, M$, and use

$$U_q^\mu(x) := \sum_{j=0}^{N} u_{qj}^\mu F_j^\mu(x),\ x \in (\eta_\mu, \eta_{\mu+1}]$$

to approximate $u_q|_{\sigma_\mu}(x)$, where $F_j^\mu(x)$ is the j-th Lagrange interpolation basic function associated with the collocation points x_i^μ, $i = 0, 1, \cdots, N$ in the interval $[\eta_\mu, \eta_{\mu+1}]$. $U_q^\mu(x)$ is of the polynomial function whose definition domain is $(\eta_\mu, \eta_{\mu+1}]$. Eventually $u_q(x)$ can be approximated by

$$U_q(x) := U_q^\mu(x),\ \text{if } x \in (\eta_\mu, \eta_{\mu+1}],\ \mu = 0, 1, \cdots, M. \tag{9}$$

$U_q(x)$ is a continuous function defined on $[-1, 1]$, and its restriction to the subinterval $(\eta_\mu, \eta_{\mu+1}]$ is $U_q^\mu(x)$. Consequently $\mathbf{u}(x)$ can be approximated by

$$\mathbf{U}(x) := [U_0(x), U_1(x), \cdots, U_Q(x)]'. \tag{10}$$

Then Eq. 8 can be approximated by

$$\mathbf{U}_i^\mu \approx \mathbf{f}(x_i^\mu) + \sum_{r=0}^{\mu-1} \int_{\eta_r}^{\eta_{r+1}} \mathbf{R}(x_i^\mu, z)\mathbf{U}^r(z)dz + \int_{\eta_\mu}^{x_i^\mu} \mathbf{R}(x_i^\mu, z)\mathbf{U}^\mu(z)dz. \tag{11}$$

where

$$\mathbf{U}_i^\mu := [u_{0i}^\mu, u_{1i}^\mu, \cdots, u_{Qi}^\mu]', i = 0, 1, \cdots, N, \mu = 0, 1, \cdots, M,$$

$$\mathbf{U}^{\mu}(z) := [U_0^{\mu}(z), U_1^{\mu}(z), \cdots, U_Q^{\mu}(z)]', \mu = 0, 1, \cdots, M.$$

For simplicity, we denote

$$z_r(v) := \frac{\eta_{r+1} - \eta_r}{2} v + \frac{\eta_{r+1} + \eta_r}{2}, v \in [-1, 1],$$

$$\zeta_i(v) := \frac{x_i + 1}{2} v + \frac{x_i - 1}{2}, v \in [-1, 1].$$

Then Eq. 11 can be written as follows

$$\begin{aligned} \mathbf{U}_i^{\mu} \approx \mathbf{f}(x_i^{\mu}) + \sum_{r=0}^{\mu-1} h_r \int_{-1}^{1} \mathbf{R}\Big(x_i^{\mu}, z_r(v)\Big)\mathbf{U}^r\Big(z_r(v)\Big)dv \\ + h_{\mu}\tfrac{x_i+1}{2} \int_{-1}^{1} \mathbf{R}\Big(x_i^{\mu}, z_{\mu}(\zeta_i(v))\Big)\mathbf{U}^{\mu}\Big(z_{\mu}(\zeta_i(v))\Big)dv. \end{aligned} \tag{12}$$

Using the Gauss quadrature formula for the integral term we can approximate (12) as follows

$$\mathbf{U}_i^{\mu} = \mathbf{f}(x_i^{\mu}) + \mathbf{S}(x_i^{\mu}), \tag{13}$$

where

$$\mathbf{S}(x_i^{\mu}) := \sum_{r=0}^{\mu-1} h_r \mathbf{S}_r + h_{\mu} \frac{x_i + 1}{2} \mathbf{S}_{\mu}, \tag{14}$$

$$\mathbf{S}_r := [S_0^r, S_1^r, \cdots, S_Q^r]', \quad S_p^r := \sum_{q=0}^{Q} \sum_{j=0}^{N} u_{qj}^r R_{pq}(x_i^{\mu}, z_r(v_j))\omega_j,$$

$$\mathbf{S}_{\mu} := [S_0^{\mu}, S_1^{\mu}, \cdots, S_Q^{\mu}]', \quad S_p^{\mu} := \sum_{q=0}^{Q} \sum_{j=0}^{N} u_{qj}^{\mu} \sum_{k=0}^{N} R_{pq}(x_i^{\mu}, z_{\mu}(\zeta_i(v_k)))F_j(\zeta_i(v_k))\omega_k,$$

here $v_k, k = 0, 1, \cdots, N$ are the Legendre Gauss-Lobatto points in the standard interval [-1,1], corresponding to the weights $\omega_k, k = 0, 1, \cdots, N$. Our goal is to find $\mathbf{U}_i^{\mu}$ such that

$$\begin{aligned} \mathbf{U}_i^{\mu} = \mathbf{f}(x_i^{\mu}) + \sum_{r=0}^{\mu-1} h_r \mathbf{S}_r + h_{\mu} \frac{x_i + 1}{2} \mathbf{S}_{\mu}, \\ i = 0, 1, \cdots, N, \mu = 0, 1, \cdots, M. \end{aligned} \tag{15}$$

The approximation to $\mathbf{y}(t)$ is $\mathbf{U}(\frac{2}{T}t - 1)$. An efficient computation of $F_j(s)$ can be found in [4] or [31].

In order to solve the discrete system (13) easily by computer, we write it into matrix form. Let

$$\mathbb{U}_q^{\mu} := [u_{q0}^{\mu}, u_{q1}^{\mu}, \cdots, u_{qN}^{\mu}], \mu = 0, 1, \cdots, M,$$

$$\mathbb{U}^{\mu} := [\mathbb{U}_0^{\mu}, \mathbb{U}_1^{\mu}, \cdots, \mathbb{U}_Q^{\mu}]', \mu = 0, 1, \cdots, M,$$

$$\mathbb{F}_q^{\mu} := [f_q(x_0^{\mu}), f_q(x_1^{\mu}), \cdots, f_q(x_N^{\mu})], \mu = 0, 1, \cdots, M,$$

$$\mathbb{F}^{\mu} := [\mathbb{F}_0^{\mu}, \mathbb{F}_1^{\mu}, \cdots, \mathbb{F}_Q^{\mu}]', \mu = 0, 1, \cdots, M.$$

Then Eq. 13 can be written as

$$\mathbb{U}^{\mu} = \mathbb{F}^{\mu} + \sum_{r=0}^{\mu-1} h_r \mathbb{A}_r \mathbb{U}^r + h_{\mu} \mathbb{A}_{\mu} \mathbb{U}^{\mu}, \tag{16}$$

where

$$\mathbb{A}_r := [A_{qp}]_{(Q+1)\times(Q+1)}, r = 0, 1, \cdots, \mu.$$
$$A_{qp} := [a_{ij}]_{(N+1)\times(N+1)},$$
$$a_{ij} := R_{qp}(x_i^{\mu}, z_r(v_j))w_j, \text{ if } r = 0, 1, \cdots, \mu - 1,$$
$$a_{ij} := \frac{x_i + 1}{2} \sum_{k=0}^{N} R_{qp}(x_i^{\mu}, z_{\mu}(\zeta_i(v_k)))F_j(\zeta_i(v_k))w_k, \text{ if } r = \mu.$$

Since $\mathbf{f}(t)$, $\mathbf{R}(x, z)$, $F_j(z)$ are continuous on their definition domain, the elements of matrix $\mathbb{F}^{\mu}$, $\mathbb{A}_r$ and $\mathbb{A}_{\mu}$, $\mu = 0, 1, \cdots, M$ are all bounded. The Neumann Lemma (see [21], page 26, or [1], page 87) then shows that the inverse of the matrix

$$\mathcal{B}^{(\mu)} := I - h_{\mu}\mathbb{A}_{\mu}$$

exists whenever

$$h_{\mu}\|\mathbb{A}_{\mu}\| < 1$$

for some matrix norm. This clearly holds whenever h_{μ} is sufficiently small. This ensures (16) possesses an unique solution.

3 Some useful lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of error estimate in Section 4. In order to give the subsequent lemmas conveniently, we first introduce some spaces. For simplicity, we denote by $\partial_x^k u(x)$ the k-th derivative of u, i.e., $\partial_x^k u(x) := \frac{\mathrm{d}^k u}{\mathrm{d}x^k}(x)$.

Let (a, b) be a bounded interval of the real line. We denote by $L^2(a, b)$ the space of the measurable functions $u : (a, b) \to \mathbb{R}$ such that $\int_a^b |u(x)|^2 dx < +\infty$. It is a Hilbert space for the inner product

$$(u, v) := \int_a^b u(x)v(x)dx,$$

which induces the norm

$$\|v\|_{L^2(a,b)} := \Big(\int_a^b |v(x)|^2 dx\Big)^{1/2}.$$

Let $m \geq 1$ be an integer. We define $H^m(a, b)$ to be the vector space of the functions $v \in L^2(a, b)$ such that all the distribution of v of order up to m can be represented by functions in $L^2(a, b)$. In short,

$$H^m(a, b) := \{v \in L^2(a, b) : \text{ for } 0 \leq k \leq m, \partial_x^k v(x) \in L^2(a, b)\}.$$

$H^m(a, b)$ is endowed with the inner product

$$(u, v)_m = \sum_{k=0}^{m} \int_a^b \partial_x^k u(x) \partial_x^k v(x) dx$$

for which $H^m(a, b)$ is a Hilbert space. The associated norm is

$$\|v\|_{H^m(a,b)} := \Big((v, v)_m \Big)^{\frac{1}{2}}.$$

In bounding from the above approximation error, only some of the L^2-norms appearing on the right-hand side of the above norm enter into play. Thus, for a nonnegative integer N, it is convenient to introduce the semi-norm

$$|v|_{H^{m;N}(a,b)} := \Big(\sum_{k=\min(m,N+1)}^{m} \|\partial_x^k v(x)\|_{L^2(a,b)}^2 \Big)^{\frac{1}{2}},$$

which implies that if $N > m - 1$ then $|v|_{H^{m;N}(a,b)} = \|\partial_x^m v\|_{L^2(a,b)}$.

The space $L^\infty(a, b)$ is the Banach space of the measurable functions u that are bounded outside a set of measure zero, equipped the norm

$$\|u\|_{L^\infty(a,b)} := \operatorname*{ess\,sup}_{x \in (a,b)} |u(x)|.$$

We denote by $C([a, b])$ the space of continuous functions on the interval $[a, b]$.

We define an interpolation operator I_N associated with the collocation points X_N as follows: for any continuous functions $u \in C([-1, 1])$,

$$I_N u(x) := I_N^\mu (u|_{\sigma_\mu})(x), \text{ if } x \in (\eta_\mu, \eta_{\mu+1}], 0 \le \mu \le M, \tag{17}$$

where $u|_{\sigma_\mu}(x)$ is the restriction of $u(x)$ to the subinterval $[\eta_\mu, \eta_{\mu+1}]$, and I_N^μ is the interpolation operator associated with the collocation points X^μ in the subinterval $[\eta_\mu, \eta_{\mu+1}]$, i.e.,

$$I_N^\mu (u|_{\sigma_\mu})(x) := \sum_{j=0}^{N} u|_{\sigma_\mu}(x_j^\mu) F_j^\mu(x), x \in [\eta_\mu, \eta_{\mu+1}].$$

If $\mathbf{A}(t) = (a_{ij}(t))_{m \times n}$ is a matrix function of $t \in (a, b)$, we define

$$|\mathbf{A}(t)| := \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}(t)|,$$

it is a non-negative real function with respect to t. We define this function's norms in $L^\infty(a, b)$ and $L^2(a, b)$ as the follows respectively,

$$\|\mathbf{A}\|_{L^\infty(a,b)} := \operatorname*{ess\,sup}_{t \in (a,b)} |\mathbf{A}(t)|,$$

$$\|\mathbf{A}\|_{L^2(a,b)} := \int_a^b |\mathbf{A}(t)|^2 dt.$$

Hereafter, C denotes a generic positive constant that is independent of N.

Lemma 1 *Assume that* $\boldsymbol{u} \in H^m(-1,1), m \geq 1$, $\boldsymbol{v}(x)$ *is a bounded function. Then there exists a constant* C *independent of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *such that for* $N > m-1$,

$$\|\boldsymbol{u} - J_N\boldsymbol{u}\|_{L^2(-1,1)} \leq CN^{-m}\|\partial_x^m \boldsymbol{u}\|_{L^2(-1,1)}, \tag{18}$$

$$\| \boldsymbol{u} - J_N\boldsymbol{u} \|_{L^\infty(-1,1)} \leq CN^{1/2-m}\|\partial_x^m \boldsymbol{u}\|_{L^2(-1,1)}, \tag{19}$$

$$\sup_N \| J_N\boldsymbol{v} \|_{L^2(-1,1)} \leq C\|\boldsymbol{v}\|_{L^\infty(-1,1)}, \tag{20}$$

$$\|J_N\boldsymbol{v}\|_{L^\infty(-1,1)} \leq CN^{1/2}\|\boldsymbol{v}\|_{L^\infty(-1,1)}, \tag{21}$$

where J_N *is the interpolation operator associated with the* $N+1$*-point Legendre Gauss-Lobatto points in the interval* $[-1,1]$.

Proof The interpolation error estimate in L^2-norm (see [4] page315), i.e.,

$$\|u_q - J_N u_q\|_{L^2(-1,1)} \leq CN^{-m}\|\partial_x^m u_q\|_{L^2(-1,1)}$$

helps to deduce the inequality (*) in the following derivation.

$$\begin{aligned}
\|(I-J_N)\mathbf{u}\|_{L^2(-1,1)} &= \left\| \sum_{q=0}^{Q} |(I-J_N)u_q| \right\|_{L^2(-1,1)} \\
&= \left(\int_{-1}^{1} \left(\sum_{q=0}^{Q} |(I-J_N)u_q| \right)^2 dx \right)^{1/2} \\
&\leq \left(\int_{-1}^{1} \left(\left(\sum_{q=0}^{Q} |(I-J_N)u_q|^2 \right)^{1/2} \left(\sum_{q=0}^{Q} 1 \right)^{1/2} \right)^2 dx \right)^{1/2} \\
&\leq C \left(\sum_{q=0}^{Q} \|(I-J_N)u_q\|^2_{L^2(-1,1)} \right)^{1/2} \\
&\overset{(*)}{\leq} CN^{-m} \left(\sum_{q=0}^{Q} \|\partial_x^m u_q\|^2_{L^2(-1,1)} \right)^{1/2} \\
&= CN^{-m} \left(\sum_{q=0}^{Q} \int_{-1}^{1} (\partial_x^m u_q(x))^2 dx \right)^{1/2} \\
&= CN^{-m} \left(\int_{-1}^{1} \sum_{q=0}^{Q} (\partial_x^m u_q(x))^2 dx \right)^{1/2} \\
&\leq CN^{-m} \left(\int_{-1}^{1} (|\partial_x^m \mathbf{u}|(x))^2 dx \right)^{1/2} \\
&= CN^{-m}\|\partial_x^m \mathbf{u}\|_{L^2(-1,1)}.
\end{aligned} \tag{22}$$

In short

$$\|(I - J_N)\mathbf{u}\|_{L^2(-1,1)} \leq CN^{-m}\|\partial_x^m \mathbf{u}\|_{L^2(-1,1)}. \tag{23}$$

This is Eq. 18.

For u_q, using the Sobolev inequality ([4], page 490), we have

$$\|(I - J_N)u_q\|_{L^\infty(-1,1)} \leq C\|(I - J_N)u_q\|_{L^2(-1,1)}^{1/2}\|(I - J_N)u_q\|_{H^1(-1,1)}^{1/2}.$$

Applying the result (18) to $\|(I - J_N)u_q\|_{L^2(-1,1)}^{1/2}$ makes the above inequality become

$$\|(I - J_N)u_q\|_{L^\infty(-1,1)} \leq CN^{-m/2}\|\partial_x^m u_q\|_{L^2(-1,1)}^{1/2}\|(I - J_N)u_q\|_{H^1(-1,1)}^{1/2}, \tag{24}$$

which leads to Eq. 19 because $\|(I - J_N)u_q\|_{H^1(-1,1)}^{1/2}$ can be estimated as follows ([4], page 289),

$$\|(I - J_N)u_q\|_{H^1(-1,1)}^{1/2} \leq CN^{(1-m)/2}\|\partial_x^m u_q\|_{L^2(-1,1)}^{1/2}.$$

Then

$$\begin{aligned}
\|(I - J_N)\mathbf{u}\|_{L^\infty(-1,1)} &= \left\| \sum_{q=0}^{Q} |(I - J_N)u_q| \right\|_{L^\infty(-1,1)} \\
&\leq \sum_{q=0}^{Q} \|(I - J_N)u_q\|_{L^\infty(-1,1)} \\
&\leq CN^{1/2-m} \sum_{q=0}^{Q} \|\partial_x^m u_q\|_{L^\infty(-1,1)} \\
&\leq CN^{1/2-m}(Q + 1)\|\partial_x^m \mathbf{u}\|_{L^\infty(-1,1)} \\
&= CN^{1/2-m}\|\partial_x^m \mathbf{u}\|_{L^\infty(-1,1)}.
\end{aligned} \tag{25}$$

This gives (19).

The following result will give (20),

$$\begin{aligned}
\sup_N \|J_N \mathbf{v}\|_{L^2(-1,1)} &\leq \sup_N \|(J_N - I)\mathbf{v}\|_{L^2(-1,1)} + \|\mathbf{v}\|_{L^2(-1,1)} \\
&\overset{(**)}{\leq} C\|\mathbf{v}\|_{L^\infty(-1,1)} + \|\mathbf{v}\|_{L^\infty(-1,1)} \\
&\leq C\|\mathbf{v}\|_{L^\infty(-1,1)},
\end{aligned} \tag{26}$$

where the inequality (**) is derived by Eq. 18 with $m = 0$.

The following results give (21),

$$\begin{aligned}
\|J_N \mathbf{v}\|_{L^\infty(-1,1)} &= \|(J_N - I)\mathbf{v}\|_{L^\infty(-1,1)} + \|\mathbf{v}\|_{L^\infty(-1,1)} \\
&\overset{(***)}{\leq} CN^{1/2}\|\mathbf{v}\|_{L^\infty(-1,1)} + \|\mathbf{v}\|_{L^\infty(-1,1)} \\
&\leq CN^{1/2}\|\mathbf{v}\|_{L^\infty(-1,1)}
\end{aligned} \tag{27}$$

where the inequality (***) is derived by Eq. 19 with $m = 0$. □

The above lemma will help us to deduce the following lemma.

Lemma 2 *Let* $\boldsymbol{u}(x)$ *be the exact solution to* Eq. 4 *with the data functions possessing continuous derivatives of order m.* $I_N\boldsymbol{u}(x)$ *is the interpolation function defined in* Eq. 17 *where* $N+1$ *means the number of collocation points in the intervals* $[\eta_\mu, \eta_{\mu+1}]$, $\mu = 0, 1, \cdots, N-1$. *Then the following estimates hold for* $N > m-1$,

$$\|(I - I_N)\boldsymbol{u}\|_{L^2(-1,1)} \le Ch^m N^{-m} \|\partial_x^m \boldsymbol{u}\|_{L^2(-1,1)}, \tag{28}$$

$$\|(I - I_N)\boldsymbol{u}\|_{L^\infty(-1,1)} \le Ch^{m-\frac{1}{2}} N^{\frac{1}{2}-m} \|\partial_x^m \boldsymbol{u}\|_{L^2(-1,1)}, \tag{29}$$

$$\sup_N \| I_N\boldsymbol{u} \|_{L^2(-1,1)} \le C\|\boldsymbol{u}\|_{L^\infty(-1,1)}, \tag{30}$$

$$\|I_N\boldsymbol{u}\|_{L^\infty(-1,1)} \le CN^{1/2}\|\boldsymbol{u}\|_{L^\infty(-1,1)}. \tag{31}$$

Proof The estimate (18) helps to deduce the inequality (****) in the following derivation.

$$\begin{aligned}
\|(I - I_N)\mathbf{u}\|^2_{L^2(-1,1)} &= \sum_{\mu=0}^{M} \int_{\eta_\mu}^{\eta_{\mu+1}} |(I - I_N^\mu)(\mathbf{u}|_{\sigma_\mu})(z)|^2 dz \\
&= \sum_{\mu=0}^{M} h_\mu \int_{-1}^{1} |(I - I_N^\mu)(\mathbf{u}|_{\sigma_\mu})(z_\mu(v))|^2 dv \\
&= \sum_{\mu=0}^{M} h_\mu \|(I - I_N^\mu)(\mathbf{u}|_{\sigma_\mu})(z_\mu(\cdot))\|^2_{L^2(-1,1)} \\
&= \sum_{\mu=0}^{M} h_\mu \|(I - J_N)(\mathbf{u}|_{\sigma_\mu})(z_\mu(\cdot))\|^2_{L^2(-1,1)} \\
&\overset{(****)}{\le} CN^{-2m} \sum_{\mu=0}^{M} h_\mu \left\| \partial_v^m \Big((\mathbf{u}|_{\sigma_\mu})(z_\mu(\cdot)) \Big) \right\|^2_{L^2(-1,1)} \\
&\le CN^{-2m} \sum_{\mu=0}^{M} h_\mu^{2m+1} \left\| \Big(\partial_z^m (\mathbf{u}|_{\sigma_\mu}) \Big)(z_\mu(\cdot)) \right\|^2_{L^2(-1,1)} \\
&\le Ch^{2m} N^{-2m} \sum_{\mu=0}^{M} \left\| \partial_z^m (\mathbf{u}|_{\sigma_\mu}) \right\|^2_{L^2(\sigma_\mu)} \\
&= Ch^{2m} N^{-2m} \left\| \partial_z^m \mathbf{u} \right\|^2_{L^2(-1,1)}.
\end{aligned} \tag{32}$$

By the definition of I_N^μ we know that the $(I_N^\mu(u|_{\sigma_\mu}))(x)$ is a function defined on the subinterval $[\eta_\mu, \eta_{\mu+1}]$. The variable transformation $x = x_\mu(z)$ changes it to be a function valued on the standard interval $[-1, 1]$, i.e.,

$$(I_N^\mu(u|_{\sigma_\mu}))(x_\mu(z)) = \sum_{j=0}^{N} u|_{\sigma_\mu}(x_j^\mu) F_j^\mu(x_\mu(z)) = \sum_{j=0}^{N} u|_{\sigma_\mu}(x_j^\mu) F_j(z), z \in [-1, 1]. \tag{33}$$

The second equality above holds because we note that $F_j^\mu(x_\mu(z)) = F_j(z), z \in [-1, 1]$. In another hand, we note that $u|_{\sigma_\mu}(x_\mu(z))$ is a function defined on the interval $[-1, 1]$. Its interpolation polynomial associated with Legendre Gauss-Lobatto points z_j, $j = 0, 1, \cdots, N$ in the interval $[-1, 1]$ is

$$J_N\Big(u|_{\sigma_\mu}(x_\mu(z))\Big) = \sum_{j=0}^{N} u|_{\sigma_\mu}(x_\mu(z_j)) F_j(z), z \in [-1, 1]. \tag{34}$$

Note that $x_\mu(z_j) = x_j^\mu$, $j = 0, 1, \cdots, N$. Plugging this into the right hand side of Eq. 34 yields

$$J_N\Big(u|_{\sigma_\mu}(x_\mu(z))\Big) = \sum_{j=0}^{N} u|_{\sigma_\mu}(x_j^\mu) F_j(z), z \in [-1, 1]. \tag{35}$$

Combining Eq. 33 with Eq. 35 yields

$$\Big(I_N^\mu(u|_{\sigma_\mu})\Big)(x_\mu(z)) = J_N\Big(u|_{\sigma_\mu}(x_\mu(z))\Big), z \in [-1, 1]. \tag{36}$$

Now we begin to prove (29). Using Eqs. 19 and 36, we have

$$\begin{aligned}
\Big\|(I - I_N)\mathbf{u}\Big\|_{L^\infty(-1,1)} &\le \max_{0\le\mu\le M}\Big\{\Big\|(I - I_N^\mu)(\mathbf{u}|_{\sigma_\mu})\Big\|_{L^\infty(\sigma_\mu)}\Big\} \\
&= \max_{0\le\mu\le M}\Big\{\Big\|(I - J_N)(\mathbf{u}|_{\sigma_\mu}(z_\mu(\cdot)))\Big\|_{L^\infty(-1,1)}\Big\} \\
&\le CN^{\frac{1}{2}-m} \max_{0\le\mu\le M}\Big\{\Big\|\partial_v^m \mathbf{u}|_{\sigma_\mu}(z_\mu(\cdot)))\Big\|_{L^2(-1,1)}\Big\} \\
&\le CN^{\frac{1}{2}-m} \max_{0\le\mu\le M}\Big\{h_\mu^m\Big\|\partial_z^m(\mathbf{u}|_{\sigma_\mu})(z_\mu(\cdot))\Big\|_{L^2(-1,1)}\Big\} \\
&\le Ch^{m-1/2}N^{\frac{1}{2}-m} \max_{0\le\mu\le M}\Big\{\Big\|\partial_z^m(\mathbf{u}|_{\sigma_\mu})\Big\|_{L^2(\sigma_\mu)}\Big\} \\
&\le Ch^{m-1/2}N^{\frac{1}{2}-m}\Big\|\partial_z^m \mathbf{u}\Big\|_{L^2(-1,1)}.
\end{aligned} \tag{37}$$

Now we begin to prove (30). The result (20) is used in the following derivation,

$$\begin{aligned}\sup_N \|I_N\mathbf{u}\|^2_{L^2(-1,1)} &= \sup_N \int_{-1}^{1} |I_N\mathbf{u}(z)|^2 dz = \sup_N \sum_{\mu=0}^{M} \int_{\eta_\mu}^{\eta_{\mu+1}} \left|I_N^\mu \mathbf{u}|_{\sigma_\mu}(z)\right|^2 dz \\ &= \sup_N \sum_{\mu=0}^{M} h_\mu \int_{-1}^{1} \left|J_N(\mathbf{u}|_{\sigma_\mu}(z_\mu(v)))\right|^2 dv \\ &\le \sum_{\mu=0}^{M} h_\mu \sup_N \left\|J_N(\mathbf{u}|_{\sigma_\mu}(z_\mu(\cdot)))\right\|^2_{L^2(-1,1)} \\ &\le C\sum_{\mu=0}^{M} h_\mu \|\mathbf{u}|_{\sigma_\mu}(z_\mu(\cdot))\|^2_{L^\infty(-1,1)} \\ &= C\sum_{\mu=0}^{M} h_\mu \|\mathbf{u}|_{\sigma_\mu}\|^2_{L^\infty(\sigma_\mu)} \le C\sum_{\mu=0}^{M} h_\mu \|\mathbf{u}\|^2_{L^\infty(-1,1)} \\ &= C\|\mathbf{u}\|^2_{L^\infty(-1,1)}, \end{aligned} \tag{38}$$

which lead to the desired result (30).

Now we begin to prove (31). It is clear that

$$\left\|I_N\mathbf{u}(x)\right\|_{L^\infty(-1,1)} \le \max_{0\le\mu\le M}\left\{\left\|I_N^\mu(\mathbf{u}|_{\sigma_\mu})\right\|_{L^\infty(\sigma_\mu)}\right\}. \tag{39}$$

We use Eq. 21 to estimate $\|I_N^\mu(\mathbf{u}|_{\sigma_\mu})\|_{L^\infty(\sigma_\mu)}$ as follows,

$$\begin{aligned}\left\|I_N^\mu(\mathbf{u}|_{\sigma_\mu})\right\|_{L^\infty(\sigma_\mu)} &= \left\|(I_N^\mu(\mathbf{u}|_{\sigma_\mu}))(z_\mu(\cdot))\right\|_{L^\infty(-1,1)} = \left\|J_N(\mathbf{u}|_{\sigma_\mu}(z_\mu(\cdot)))\right\|_{L^\infty(-1,1)} \\ &\le CN^{1/2}\|\mathbf{u}|_{\sigma_\mu}(z_\mu(\cdot))\|_{L^\infty(-1,1)} = CN^{1/2}\|\mathbf{u}\|_{L^\infty(\sigma_\mu)} \\ &\le CN^{1/2}\|\mathbf{u}\|_{L^\infty(-1,1)}, \end{aligned} \tag{40}$$

which together with Eq. 39 leads to the desired result (31). □

Lemma 3 [4, 25] *Assume that $v \in H^m(-1,1)$ for some $m \ge 1$ and $\varphi \in \mathcal{P}_N$, which denotes the space of all polynomials of degree not exceeding $N > m-1$. Then there exists a constant C independent of N such that*

$$\begin{aligned}&\left|\int_{-1}^{1} v(x)\varphi(x)dx - \sum_{j=0}^{N} v(x_j)\varphi(x_j)\omega_j\right| \\ &\le CN^{-m}\|\partial_x^m v\|_{L^2(-1,1)} \parallel \varphi \parallel_{L^2(-1,1)},\end{aligned}$$

where x_j, $j = 0, 1, \cdots, N$ are the Legendre Gauss-Lobatto points in the interval $[-1, 1]$, *corresponding weights ω_j, $j = 0, 1, \cdots, N$.*

Lemma 4 *Assume that $\boldsymbol{R}(x,z)$,$\boldsymbol{U}(z)$ and $\boldsymbol{S}(x)$ are defined in* Eqs. 5, 10 *and* 13 *respectively. Then for sufficiently large $N > m-1$ there exists a constant C independent of N such that*

$$\Big|\int_{-1}^{x}\boldsymbol{R}(x,z)\boldsymbol{U}(z)dz-\boldsymbol{S}(x)\Big| \le h^m N^{-m}\|\partial_z^m \boldsymbol{R}(x,\cdot)\|_{L^2(-1,x)}\|\boldsymbol{U}(\cdot)\|_{L^2(-1,x)}. \quad (41)$$

Proof Note that

$$\Big|\int_{-1}^{x}\mathbf{R}(x,z)\mathbf{U}(z)dz-\mathbf{S}(x)\Big| \le \sum_{r=0}^{\mu-1}\Big|\Big(\int_{\eta_r}^{\eta_{r+1}}\mathbf{R}(x,z)\mathbf{U}^r(z)dz-h_r\mathbf{S}_r(x)\Big)\Big| + \Big|\int_{\eta_\mu}^{x}\mathbf{R}(x,z)\mathbf{U}^\mu(z)dz-h_\mu\tfrac{\widetilde{x}+1}{2}\mathbf{S}_\mu(x)\Big|,\ x\in\sigma_\mu, \quad (42)$$

where $\widetilde{x} := \frac{2}{\eta_{\mu+1}-\eta_\mu}(x-\frac{\eta_{\mu+1}+\eta_\mu}{2})$. By Lemma 3,

$$\begin{aligned}
&\Big|\int_{\eta_r}^{\eta_{r+1}}\mathbf{R}(x,z)\mathbf{U}^r(z)dz-h_r\mathbf{S}_r(x)\Big|\\
&=\sum_{p=0}^{Q}\left|\sum_{q=0}^{Q}h_r\left(\int_{-1}^{1}R_{pq}(x,z_r(v))U_q^r(z_r(v))dv-S_p^r(x)\right)\right|\\
&\le C\sum_{p=0}^{Q}\Big|\sum_{q=0}^{Q}h_rN^{-m}\|\partial_v^m(R_{pq}(x,z_r(\cdot)))\|_{L^2(-1,1)}\|U_q^r(z_r(v))\|_{L^2(-1,1)}\Big|\\
&\le C\sum_{p=0}^{Q}\Big|\sum_{q=0}^{Q}h_r^{m+1}N^{-m}\|(\partial_z^m R_{pq})(x,z_r(\cdot))\|_{L^2(-1,1)}\|U_q^r(z_r(v))\|_{L^2(-1,1)}\Big|\\
&\le C\sum_{p=0}^{Q}\Big|\sum_{q=0}^{Q}h_r^{m}N^{-m}\|(\partial_z^m R_{pq})(x,\cdot)\|_{L^2(\sigma_r)}\|U_q^r(\cdot)\|_{L^2(\sigma_r)}\Big|\\
&\le Ch_r^mN^{-m}\sum_{p=0}^{Q}\Big|\left(\sum_{q=0}^{Q}\|(\partial_z^m R_{pq})(x,\cdot)\|_{L^2(\sigma_r)}^2\right)^{1/2}\left(\sum_{q=0}^{Q}\|U_q^r(\cdot)\|_{L^2(\sigma_r)}^2\right)^{1/2}\Big|\\
&\le Ch_r^mN^{-m}\|\mathbf{U}^r(\cdot)\|_{L^2(\sigma_r)}\sum_{p=0}^{Q}\left(\sum_{q=0}^{Q}\|(\partial_z^m R_{pq})(x,\cdot)\|_{L^2(\sigma_r)}^2\right)^{1/2}\\
&\le Ch_r^mN^{-m}\|\mathbf{U}^r(\cdot)\|_{L^2(\sigma_r)}\|\partial_z^m\mathbf{R}(x,\cdot)\|_{L^2(\sigma_r)}.
\end{aligned} \quad (43)$$

Similarly

$$\Big|\int_{\eta_\mu}^{x}\mathbf{R}(x,z)\mathbf{U}^\mu(z)dz-h_\mu\frac{\widetilde{x}+1}{2}\mathbf{S}_\mu(x)\Big| \le Ch_\mu^mN^{-m}\|\mathbf{U}^\mu(\cdot)\|_{L^2(\eta_\mu,x)}\|\partial_z^m\mathbf{R}(x,\cdot)\|_{L^2(\eta_\mu,x)}. \quad (44)$$

Then by the Cauchy inequality

$$\sum_{r=0}^{\mu}a_rb_r \le \Big(\sum_{r=0}^{\mu}a_r^2\Big)^{1/2}\Big(\sum_{r=0}^{\mu}b_r^2\Big)^{1/2},$$

where we let

$$a_r = \|\partial_z^m \mathbf{R}(x,\cdot)\|_{L^2(\sigma_r)}, b_r = \|\mathbf{U}^r(\cdot)\|_{L^2(\sigma_r)}, r = 0, 1, \cdots, \mu - 1,$$

$$a_\mu = \|\partial_z^m \mathbf{R}(x,\cdot)\|_{L^2(\eta_\mu, x)}, b_\mu = \|\mathbf{U}^\mu(\cdot)\|_{L^2(\eta_\mu, x)},$$

we obtain that

$$\begin{aligned} \left|\int_{-1}^{x} \mathbf{R}(x,z)\mathbf{U}(z)dz - \mathbf{S}(x)\right| &\le Ch^m N^{-m} \left(\sum_{r=0}^{\mu} a_r^2\right)^{1/2} \left(\sum_{r=0}^{\mu} b_r^2\right)^{1/2} \\ &\le Ch^m N^{-m} \|\partial_z^m \mathbf{R}(x,\cdot)\|_{L^2(-1,x)} \|\mathbf{U}(\cdot)\|_{L^2(-1,x)}. \end{aligned} \tag{45}$$

This is Eq. 41. □

Lemma 5 *If an integrable matrix function* $\boldsymbol{e}(x)$ *satisfies*

$$\boldsymbol{e}(x) = \boldsymbol{v}(x) + \int_{-1}^{x} \boldsymbol{R}(x,z)\boldsymbol{e}(z)dz, x \in [-1, 1], \tag{46}$$

where $\boldsymbol{v}(x)$ *is also a nonnegative integrable matrix function and* $\boldsymbol{R}(x,z)$ *is a continuous matrix function. Then*

$$\|\boldsymbol{e}(x)\|_{L^p(-1,1)} \le C\|\boldsymbol{v}(x)\|_{L^p(-1,1)}, p = 2, \infty.$$

Proof From Eq. 46 we have

$$\begin{aligned} |\mathbf{e}(x)| &\le |\mathbf{v}(x)| + \left|\int_{-1}^{x} \mathbf{R}(x,z)\mathbf{e}(z)\mathrm{d}z\right| \\ &\le |\mathbf{v}(x)| + \int_{-1}^{x} |\mathbf{R}(x,z)||\mathbf{e}(z)|\mathrm{d}z \\ &\le |\mathbf{v}(x)| + C\int_{-1}^{x} |\mathbf{e}(z)|\mathrm{d}z, \end{aligned} \tag{47}$$

where C is a constant dependent on $\mathbf{R}(x,z)$. By Gronwall inequality (see [31]) we obtain the conclusion of this lemma. □

4 Convergence analysis

This section is devoted to provide a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed approximations. Firstly, we will carry out convergence analysis in $L^\infty(-1,1)$ space.

Theorem 1 *Let* $\boldsymbol{u}(x)$ *be the exact solution to* Eq. 4 *with the data functions possessing continuous derivatives of order* m. $\boldsymbol{U}(x)$ *is the approximate solution obtained by using the spectral collocation schemes* (13)*, where* $N+1$ *means the number of collo-*

cation points in the intervals $[\eta_\mu, \eta_{\mu+1}]$, $\mu = 0, 1, \cdots, N-1$. *Then for sufficiently large* $N \geq m-1$,

$$\|\boldsymbol{u}(x) - \boldsymbol{U}(x)\|_{L^\infty(-1,1)} \leq Ch^{m-1/2}N^{1/2-m}\Big(\widetilde{\boldsymbol{R}}\|\boldsymbol{u}\|_{L^\infty(-1,1)} + \|\partial_x^m \boldsymbol{u}\|_{L^2(-1,1)}\Big), \tag{48}$$

where

$$\widetilde{\boldsymbol{R}} := \max_{x\in[-1,1]} \left\|\partial_z^m \boldsymbol{R}(x,\cdot)\right\|_{L^2(-1,x)}.$$

From Eq. 48 we can see that the convergence rate of the numerical errors decay in the rate $h^{m-1/2}N^{1/2-m}$ which relates to N, h and m. This implies that if the data functions have better regularity, i.e., m is larger, the errors decay faster. If we employ more collocation points, i.e., N is larger, we can obtained higher accuracy. If we refining the mesh, i.e., h is smaller, the accuracy will become higher. It is worth mentioning that N, h and m is independent of each other.

Proof Subtracting Eq. 13 from Eq. 7 yields

$$\mathbf{u}(x_i^\mu) - \mathbf{U}_i^\mu = \int_{-1}^{x_i^\mu} \mathbf{R}(x_i^\mu, z)\mathbf{e}(z)dz + \mathbf{E}(x_i^\mu), \tag{49}$$

where

$$\mathbf{e}(x) := \mathbf{u}(x) - \mathbf{U}(x), x \in [-1,1],$$
$$\mathbf{E}(x) := \int_{-1}^{x} \mathbf{R}(x,z)\mathbf{U}(z)dz - \mathbf{S}(x), x \in [-1,1].$$

Multiplying $F_i^\mu(x)$ to both side of Eq. 49 and summing from $i=0$ to N

$$I_N^\mu(\mathbf{u}|_{\sigma_\mu})(x) - \mathbf{U}^\mu(x) = I_N^\mu(\textstyle\int_{-1}^x \mathbf{R}(x,z)\mathbf{e}(z)dz) + I_N^\mu(\mathbf{E}|_{\sigma_\mu})(x), x \in \sigma_\mu. \tag{50}$$

By the definition of I_N we obtain that

$$I_N\mathbf{u}(x) - \mathbf{U}(x) = I_N(\textstyle\int_{-1}^x \mathbf{R}(x,z)\mathbf{e}(z)dz) + I_N\mathbf{E}(x), x \in [-1,1]. \tag{51}$$

Then

$$\mathbf{e}(x) = (I - I_N)\mathbf{u}(x) + (I_N - I)\mathbf{b}(x) + I_N\mathbf{E}(x) + \mathbf{b}(x), x \in [-1,1], \tag{52}$$

where

$$\mathbf{b}(x) := \int_{-1}^{x} \mathbf{R}(x,z)\mathbf{e}(z)dz, x \in [-1,1].$$

By Lemma 5,

$$\|\mathbf{e}\|_{L^\infty(-1,1)} \leq C\Big(\|(I-I_N)\mathbf{u}\|_{L^\infty(-1,1)} + \|(I_N-I)\mathbf{b}\|_{L^\infty(-1,1)} + \|I_N\mathbf{E}\|_{L^\infty(-1,1)}\Big). \tag{53}$$

We estimate each term of the right hand side of the above inequality one by one. Applying (29) to $\mathbf{u}(x)$, we have

$$\|(I - I_N)\mathbf{u}\|_{L^\infty(-1,1)} < Ch^{m-1/2}N^{1/2-m}\|\partial_x^m \mathbf{u}\|_{L^2(-1,1)}. \tag{54}$$

Now we begin to estimate $\|(I_N - I)\mathbf{b}\|_{L^\infty(-1,1)}$. Applying (29) with $m=1$ to $\mathbf{b}$ yields

$$\|(I_N - I)\mathbf{b}\|_{L^\infty(-1,1)} \leq Ch^{1/2}N^{-1/2}\|\partial_x^1 \mathbf{b}\|_{L^2(-1,1)}. \tag{55}$$

Note that

$$
\begin{aligned}
|\partial_x^1 \mathbf{b}| &= \left| \mathbf{R}(x,x)\mathbf{e}(x) + \int_{-1}^{x} \frac{\partial \mathbf{R}}{\partial x}(x,z)\mathbf{e}(z)dz \right| \\
&\le |\mathbf{R}(x,x)||\mathbf{e}(x)| + \int_{-1}^{x} |\frac{\partial \mathbf{R}}{\partial x}(x,z)||\mathbf{e}(z)|dz \\
&\le C\|\mathbf{e}\|_{L^\infty(-1,1)}.
\end{aligned} \tag{56}
$$

Combining Eq. 55 with Eq. 56 yields

$$
\|(I_N - I)\mathbf{b}\|_{L^\infty(-1,1)} \le Ch^{1/2}N^{-1/2}\|\mathbf{e}\|_{L^\infty(-1,1)}. \tag{57}
$$

Now we begin to estimate $\|I_N\mathbf{E}\|_{L^\infty(-1,1)}$. By Eq. 31 we have

$$
\|I_N\mathbf{E}\|_{L^\infty(-1,1)} \le CN^{1/2}\|\mathbf{E}\|_{L^\infty(-1,1)}. \tag{58}
$$

By Eq. 41, we have

$$
\begin{aligned}
|\mathbf{E}(x)| &\le Ch^m N^{-m}\|\partial_z^m \mathbf{R}(x,\cdot)\|_{L^2(-1,x)}\|\mathbf{U}\|_{L^2(-1,x)} \\
&\le Ch^m N^{-m}\|\partial_z^m \mathbf{R}(x,\cdot)\|_{L^2(-1,x)}\Big(\|\mathbf{u}\|_{L^\infty(-1,x)} + \|\mathbf{e}\|_{L^\infty(-1,x)}\Big).
\end{aligned} \tag{59}
$$

Then

$$
\|\mathbf{E}\|_{L^\infty(-1,1)} \le Ch^m N^{-m}\widetilde{\mathbf{R}}\Big(\|\mathbf{u}\|_{L^\infty(-1,1)} + \|\mathbf{e}\|_{L^\infty(-1,1)}\Big), \tag{60}
$$

where

$$
\widetilde{\mathbf{R}} := \sup_{x\in[-1,1]} \|\partial_z^m \mathbf{R}(x,\cdot)\|_{L^2(-1,x)}.
$$

Combining Eq. 58 with Eq. 59 we have

$$
\|I_N\mathbf{E}\|_{L^\infty(-1,1)} \le Ch^m N^{1/2-m}\widetilde{\mathbf{R}}\Big(\|\mathbf{u}\|_{L^\infty(-1,1)} + \|\mathbf{e}\|_{L^\infty(-1,1)}\Big), \tag{61}
$$

which together with Eqs. 53, 54, 57 and 61 yield the desired result (48). □

Next, we will give the error estimate in $L^2(-1,1)$ space.

Theorem 2 *Let $\boldsymbol{u}(x)$ be the exact solution to* Eq. 4 *with the data functions possessing continuous derivatives of order m. $\boldsymbol{U}(x)$ is the approximate solution obtained by using the spectral collocation schemes* (13)*, where $N+1$ means the number of collocation points in the intervals $[\eta_\mu, \eta_{\mu+1}]$, $\mu = 0, 1, \cdots, N-1$. Then for sufficiently large $N \ge m-1$,*

$$
\|\boldsymbol{u} - \boldsymbol{U}\|_{L^2(-1,1)} \le Ch^m N^{-m}(\widetilde{\boldsymbol{R}} + 1)^2\Big(\|\boldsymbol{u}\|_{L^\infty(-1,1)} + \|\partial_x^m \boldsymbol{u}\|_{L^2(-1,1)}\Big).
$$

Proof By Lemma 5, it follows from Eq. 52 that

$$
\|\mathbf{e}\|_{L^2(-1,1)} \le C\Big(\|(I - I_N)\mathbf{u}\|_{L^2(-1,1)} + \|(I - I_N)\mathbf{b}\|_{L^2(-1,1)} + \|I_N\mathbf{E}\|_{L^2(-1,1)}\Big). \tag{62}
$$

Now we begin to estimate each term of the right hand side one by one. Applying Lemma 2 to $\mathbf{u}(x)$, we have

$$\|(I - I_N)\mathbf{u}\|_{L^2(-1,1)} \le Ch^m N^{-m} \|\partial_x^m \mathbf{u}\|_{L^2(-1,1)}. \tag{63}$$

As the same analysis in Eqs. 55, 56 and 57, using Eq. 28 in Lemma 2 with $m = 1$ for $\mathbf{b}(x)$, we obtain

$$\|(I - I_N)\mathbf{b}\|_{L^2(-1,1)} \le ChN^{-1} \|\mathbf{e}\|_{L^\infty(-1,1)}. \tag{64}$$

By Theorem 1, we get

$$\|(I - I_N)\mathbf{b}(x)\|_{L^2(-1,1)} \le Ch^{m+1/2} N^{-m-1/2} \Big(\widetilde{R} \|\mathbf{u}\|_{L^\infty(-1,1)} + \|\partial_x^m \mathbf{u}\|_{L^2(-1,1)} \Big). \tag{65}$$

Now we begin to estimate $\|I_N \mathbf{E}\|_{L^2(-1,1)}$. Applying (30) to $\mathbf{E}(x)$ yields

$$\|I_N \mathbf{E}(x)\|_{L^2(-1,1)} \le C \|\mathbf{E}(x)\|_{L^\infty(-1,1)}. \tag{66}$$

By Eq. 60 and Theorem 1,

$$\|I_N \mathbf{E}\|_{L^2(-1,1)} \le Ch^m N^{-m} (\widetilde{\mathbf{R}} + 1) \Big(\|\mathbf{u}\|_{L^\infty(-1,1)} + \|\partial_x^m \mathbf{u}\|_{L^2(-1,1)} \Big). \tag{67}$$

Combining Eq. 62 with Eqs. 65 and 67, we obtain the desired conclusion of this theorem. □

5 Numerical examples

In this section, we give numerical examples to confirm the theoretical results obtained in the previous section.

Example 1 Consider the following system of Volterra integral equations,

$$\begin{cases} y_1(t) = g_1(t) + \displaystyle\int_0^t e^{t-s} y_1(s)ds + \int_0^t \cos t \sin s y_2(s)ds + \int_0^t ts y_3(s)ds, \\ y_2(t) = g_2(t) + \displaystyle\int_0^t e^{\frac{t-s}{2}} y_1(s)ds + \int_0^t t y_2(s)ds + \int_0^t (t+s) y_3(s)ds, \\ y_3(t) = g_3(t) + \displaystyle\int_0^t t y_1(s)ds + \int_0^t e^t y_2(s)ds + \int_0^t \sin s y_3(s)ds, \end{cases} \tag{68}$$

where

$$g_1(t) = (1-t)e^t + \frac{1}{2}\cos t(\cos 2t - 1) - \frac{1}{3}t^4,$$

$$g_2(t) = \cos t - 2e^{t/2}(e^{t/2} - 1) - t\sin t - \frac{5}{6}t^3,$$

$$g_3(t) = t - t(e^t - 1) - e^t \sin t + t\cos t - \sin t.$$

The corresponding exact solution is given by $y_1(t) = e^t$, $y_2(t) = \cos t$, $y_3(t) = t$, $t \in [0, 2]$.

Errors versus N and $1/h$ are given in Table 1 from which we can see that the numerical results match well the theoretical results.

Table 1 Example 1: The errors versus N and $1/h$

N	2	4	6	8	10	12
L^∞-error$(1/h = 5)$	5.45e-02	1.04e-08	5.15e-12	1.75e-12	8.19e-12	1.47e-13
1/h	5	10	20	30	40	50
L^∞-error$(N = 3)$	3.08e-05	4.76e-07	7.41e-09	6.45e-10	1.10e-10	2.46e-11

The following example is provided to show that, in some case, multi interval formulation is more effective than single interval formulation.

Example 2 Consider (1) with given functions

$$K_{11} = \sin(30(t-s)),\ K_{12} = \cos(30(t-s)),\ K_{13} = \sin(30(t-s)),$$

$$K_{21} = \sin(20(t-s)),\ K_{22} = \cos(20(t-s)),\ K_{23} = \sin(20(t-s)),$$

$$K_{31} = \sin(10(t-s)),\ K_{32} = \cos(10(t-s)),\ K_{33} = \sin(10(t-s)),$$

$$g_1(t) = 1 - \frac{1}{30}(2(1-\cos(30t)) + \sin(30t)),$$

$$g_1(t) = 1 - \frac{1}{20}(2(1-\cos(20t)) + \sin(20t)),$$

$$g_1(t) = 1 - \frac{1}{10}(2(1-\cos(30t)) + \sin(10t)).$$

The corresponding exact solution is $y_1(t) = 1$, $y_2(t) = 1$, $y_3(t) = 1$, $t \in [0, 2]$.

In this example, given functions possess oscillation property. More accurately approximating given functions or integral term needs employing more collocation points. For the single interval formulation, employing more collocation points to approximate given functions or integral term may result that matrix order of Eq. 16 increase. Computation time cost of solving this matrix equation will increase significantly if the matrix order increase. For the multi interval formulation, Eq. 16 is a low-order matrix equation. Computation time cost for solving this matrix equation is much less than the one for single interval formulation. Table 2 records numerical errors versus computation time cost for both single interval formulation and multi interval formulation. Numerical results show that, obtaining the same precision, computation time cost for multi interval formulation is much less than the one for single interval formulation.

Now we give an example to underline the role of m in the behavior of the error convergence.

Table 2 Example 2: The errors versus time cost (seconds) for single interval formulation and multi interval formulation ($N = 5$)

N	8	12	16	20	24	28	32
L^∞-error	2.55	1.91	5.70e-01	2.60e-03	1.22e-06	1.10e-10	5.00e-15
time cost	1.1031	4.8458	14.837	36.624	78.174	151.68	265.17
1/h	5	10	15	20	25	30	40
L^∞-error	5.19e-05	5.62e-09	7.96e-11	4.24e-12	4.41e-13	7.07e-14	4.89e-15
time cost	2.6676	5.9327	9.7885	14.279	19.545	25.207	38.605

Example 3 Consider the following system of Volterra integral equations,

$$\begin{cases} y_1(t)=g_1(t) + \int_0^t t^{m+1/2} y_1(s)ds + \int_0^t t^{m+1/2} s^{m+1/2} y_2(s)ds + \int_0^t s^{m+1/2} y_3(s)ds, \\ y_2(t)=g_2(t) + \int_0^t t^{m+1/2} s^{m+1/2} y_1(s)ds + \int_0^t s^{m+1/2} y_2(s)ds + \int_0^t t^{m+1/2} y_3(s)ds, \\ y_3(t)=g_3(t) + \int_0^t s^{m+1/2} y_1(s)ds + \int_0^t t^{m+1/2} y_2(s)ds + \int_0^t t^{m+1/2} s^{m+1/2} y_3(s)ds, \end{cases} \tag{69}$$

where

$$g_1(t) = t^{m+1/2} - \frac{10m+11}{(2m+3)(2m+2)} t^{2m+2} - \frac{1}{2m+2} t^{3m+5/2} - \frac{4}{2m+3} t^{m+3/2},$$

$$g_2(t) = 1 + t^{m+1/2} - \frac{1}{2m+2} t^{3m+5/2} - \frac{4m+8}{2m+3} t^{m+3/2} - \frac{6m+7}{(2m+3)(2m+2)} t^{2m+2},$$

$$g_3(t) = 2 + t^{m+1/2} - \frac{14m+15}{(2m+3)(2m+2)} t^{2m+2} - \frac{1}{2m+2} t^{3m+5/2},$$

The corresponding exact solution is given by $y_1(t) = t^{m+1/2}$, $y_2(t) = 1 + t^{m+1/2}$, $y_3(t) = 2 + t^{m+1/2}$, $t \in [0, 2]$.

It is worth noting that each integral kernel possesses continuous derivatives of m order but the derivatives of $m + 1$ order is singular at the point $t = 0$. From Table 3 we can see that the bigger m may lead to higher accuracy for the numerical solution. In other words, better regularity of the given functions may help to obtain higher accuracy numerical solution. This confirms theoretical results.

Table 3 Example 3: The errors versus N ($1/h = 1$) and $1/h$ ($N = 5$)

N	10	12	14	16	18	20
L^∞-error with m=0	5.58e-01	3.26e-01	2.07e-01	1.39e-01	9.83e-02	7.19e-02
L^∞-error with m=1	8.81e-02	3.78e-02	1.80e-02	9.41e-03	5.30e-03	3.17e-03
1/h	5	10	20	30	40	50
L^∞-error with m=0	2.62e-01	9.62e-02	3.45e-02	1.89e-02	1.23e-02	8.81e-03
L^∞-error with m=1	2.06e-02	3.72e-03	6.64e-04	2.42e-04	1.18e-04	6.76e-05

Consider the high order Volterra integral differential equation

$$y^{(Q)}(t) = g(t) + \sum_{q=0}^{Q-1} a_q(t) y^{(q)}(t) + \int_0^t \sum_{q=0}^{Q} K_q(t,s) y^{(q)}(s) ds,$$
$$y^{(q)}(0) = c_q, q = 0, 1, \cdots, Q-1. \tag{70}$$

It can be transformed to a system of Volterra integral equations if we let $y_q(t) := y^{(q)}(t)$,

$$\begin{aligned} y_q(t) &= c_q + \int_0^t y_{q+1}(s)ds, q = 0, 1, \cdots, Q-1, \\ y_Q(t) &= g(t) + \sum_{q=0}^{Q-1} a_q(t) \left(c_q + \int_0^t y_{q+1}(s)ds \right) \\ &\quad + \int_0^t \sum_{q=0}^{Q} K_q(t,s) y_q(s) ds. \end{aligned} \tag{71}$$

Example 4 Consider (71) with $Q = 2$ and

$$a_0(t) = \cos t, a_1(t) = e^t, c_0 = 0, c_1 = 1,$$

$$K_0(t,s) = ts, K_1(t,s) = \sin t \sin s, K_2(t,s) = e^{t/2+s},$$

$$\begin{aligned} g(t) = & -\sin t - e^t \cos t - \cos t \sin t - \tfrac{1}{2} e^{t/2}[e^t(\cos t - \sin t) - 1] \\ & + \tfrac{1}{4} \sin t[\cos 2t - 1] + t^2 \cos t - t \sin t. \end{aligned} \tag{72}$$

The corresponding exact solution is given by $y(t) = \sin t$, $t \in [0, 2]$.

Errors versus N and $1/h$ are given in Table 4 from which we can see that our method is suitable for high order Volterra integro-differential equations.

Consider a nonlinear case of Eq. 1

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_0^t \mathbf{K}(t, s, y_0(s), \cdots, y_Q(s))ds, t \in [0, T], \tag{73}$$

where $\mathbf{K}(t, s, y_0(s), \cdots, y_Q(s))$ is of the form

$$\mathbf{K}(t, s, y_0(s), \cdots, y_Q(s)) := [K_0(t, s, y_0(s), \cdots, y_Q(s)), \cdots, K_Q(t, s, y_0(s), \cdots, y_Q(s))]'.$$

Table 4 Example 4: The errors versus N ($1/h = 5$) and $1/h$ ($N = 3$)

N	2	3	4	6	8	10
L^∞-error	81.4	1.17	5.98e-04	1.30e-07	1.59e-08	5.99e-11
1/h	5	10	20	40	60	80
L^∞-error	1.167	1.51e-05	1.28e-06	1.36e-07	6.72e-08	4.47e-09

Using a variable transformation it can be changed to be a new system

$$\mathbf{u}(x) = \mathbf{f}(x) + \int_{-1}^{x} \mathbf{R}(x, z, u_0(z), \cdots, u_Q(z))dz, x \in [-1, 1]. \tag{74}$$

Similarly to Eq. 13, the numerical scheme for Eq. 74 is

$$\begin{aligned} &\mathbf{U}_i^{\mu} = \mathbf{f}(x_i^{\mu}) + \sum_{r=0}^{\mu-1} h_r \mathbf{S}_r + h_{\mu} \frac{x_i + 1}{2} \mathbf{S}_{\mu}, \\ &i = 0, 1, \cdots, N, \mu = 0, 1, \cdots, M, \end{aligned} \tag{75}$$

$$\mathbf{S}_r := [S_0^r, S_1^r, \cdots, S_Q^r]', \quad S_p^r := \sum_{j=0}^{N} R_p(x_i^{\mu}, z_r(v_j), u_{0j}^r, \cdots, u_{Qj}^r)\omega_j,$$

$$\mathbf{S}_{\mu} := [S_0^{\mu}, S_1^{\mu}, \cdots, S_Q^{\mu}]',$$

$$S_p^{\mu} := \sum_{k=0}^{N} R_p \left(x_i^{\mu}, z_{\mu}(\zeta_i(v_k)), \sum_{j=0}^{N} u_{0j}^{\mu} F_j(\zeta_i(v_k)), \cdots, \sum_{j=0}^{N} u_{Qj}^{\mu} F_j(\zeta_i(v_k)) \right) \omega_k.$$

Discrete system (75) is a nonlinear system of equations with unknown elements $u_{qi}^{\mu}, 0 \le q \le Q, 0 \le i \le N+1, 0 \le \mu \le M$. In general, It can be solve by iterative method. It is worthy noting that the above approach can be used for nonlinear problems just based on a numerical perspective. The corresponding convergence analysis results need to be further studied and established.

Example 5 Consider (74) with $Q = 2$ and

$$K_0(t, s, y_0(s), y_1(s), y_2(s)) = y_0^2(s) + y_1^2(s) + y_2^2(s),$$

$$K_1(t, s, y_0(s), y_1(s), y_2(s)) = y_0(s)y_1(s) + y_0(s)y_2(s) + y_1(s)y_2(s),$$

$$K_2(t, s, y_0(s), y_1(s), y_2(s)) = y_0(s)y_1(s)y_2(s).$$

If we choose

$$g_0(t) = e^t - [\tfrac{1}{2}(e^{2t} - 1) + \tfrac{1}{3}t^3 + \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t], \tag{76}$$

$$g_1(t) = t - [e^t t - e^t + 1 + \frac{1}{2}(e^t \sin t - e^t \cos t + 1) - t\cos t + \sin t],$$

$$g_2(t) = \sin t - \frac{1}{2}\{-te^t \cos t + te^t \sin t - \frac{1}{2}(e^t \sin t - e^t \cos t + 1) + e^t \cos t - 1 + e^t \sin t\},$$

then the corresponding exact solution is $y_0(t) = e^t, y_1(t) = t, y_2(t) = \sin t, t \in [0, 2]$.

Table 5 presents errors versus N and $1/h$. From these data we can see that our method is available for nonlinear SVIEs based on numerical perspective.

A more general nonlinear SVIEs is

$$\begin{aligned} &y_q(t) = G_q\left(t, y_0(t), \cdots, y_Q(t), \int_0^t K_q(t, s, y_0(s), \cdots, y_Q(s))ds\right), t \in [0, T], \\ &q = 0, 1, \cdots, Q. \end{aligned} \tag{77}$$

Table 5 Example 5: The errors versus N $(1/h = 5)$ and $1/h$ $(N = 3)$

N	2	3	4	5	8	9
L^∞-error	8.81	3.39e-01	7.61e-05	8.37e-09	5.89e-09	3.79e-10
1/h	5	15	30	40	60	70
L^∞-error	3.39e-01	2.38e-04	3.66e-06	6.32e-07	2.86ee-08	1.96e-09

Using a variable transformation it can be changed to be

$$u_q(x)=H_q\left(x,u_0(x),\cdots,u_Q(x),\int_{-1}^x R_q(x,z,u_0(z),\cdots,u_Q(z))dz\right),\ x\in[-1,1],\quad q=0,1,\cdots,Q. \tag{78}$$

Its numerical scheme is

$$u_{qi}^\mu = H_q\left(x_i^\mu, u_{0i}^\mu,\cdots,u_{Qi}^\mu, \sum_{r=0}^{\mu-1} h_r S_q^r + h_\mu \frac{x_i+1}{2} S_q^\mu\right), \quad i=0,1,\cdots,N,\ \mu=0,1,\cdots,M,\ q=0,1,\cdots,Q, \tag{79}$$

where

$$S_q^r := \sum_{j=0}^N R_q(x_i^\mu, z_r(v_j), u_{0j}^r,\cdots,u_{Qj}^r)\omega_j,$$

$$S_q^\mu := \sum_{k=0}^N R_q\left(x_i^\mu, z_\mu(\zeta_i(v_k)), \sum_{j=0}^N u_{0j}^\mu F_j(\zeta_i(v_k)),\cdots,\sum_{j=0}^N u_{Qj}^\mu F_j(\zeta_i(v_k))\right)\omega_k.$$

Nonlinear Volterra integral and integro-differential equation have been used as mathematical models of population growth and related phenomena in biology. Volterra [32] refined "predator-prey" model as a system of nonlinear Volterra integro-differential equations

$$\begin{aligned} N_1'(t) &= N_1(t)\left(\varepsilon_1 - \gamma_1 N_2(t) - \int_0^t K_1(t-\tau)N_1(\tau)d\tau\right),\\ N_2'(t) &= N_2(t)\left(-\varepsilon_2 + \gamma_2 N_1(t) + \int_0^t K_2(t-\tau)N_2(\tau)d\tau\right), \end{aligned} \tag{80}$$

with $\varepsilon_i > 0, \gamma_i \geq 0$ and continuous $K_i(t) \geq 0$, where $N_1(t)$ and $N_2(t)$ represent the size of two population (prey and predator) at time $t \geq 0$. These equations can be extended naturally to describe the dynamics of multi-species ecological systems.

In general case, it is very difficult to obtain the expression of the solution of Eq. 80. In order to test the availability of our method to Eq. 80, we consider the following case

$$\begin{aligned} N_1'(t) &= a_1(t) + N_1(t)\left(\varepsilon_1 - \gamma_1 N_2(t) - \int_0^t K_1(t-s)N_1(s)ds\right),\\ N_2'(t) &= a_2(t) + N_2(t)\left(-\varepsilon_2 + \gamma_2 N_1(t) + \int_0^t K_2(t-s)N_2(s)ds\right), \end{aligned} \tag{81}$$

Table 6 Example 6: The errors versus N $(1/h = 5)$ and $1/h$ $(N = 3)$

N	2	3	4	5	7	9
L^∞-error	4.04e-06	3.06e-08	1.98e-10	1.41e-12	3.22e-15	6.66e-16
1/h	5	15	30	40	50	65
L^∞-error	3.06e-08	1.19e-10	3.66e-12	8.65e-13	2.83e-13	7.84e-14

which is equivalent to the SVIEs

$$\begin{aligned} N_1(t) &= N_1(0) + \int_0^t N_1'(s)ds, \\ N_1'(t) &= a_1(t) + N_1(t)\left(\varepsilon_1 - \gamma_1 N_2(t) - \int_0^t K_1(t-s)N_1(s)ds\right), \\ N_2(t) &= N_2(0) + \int_0^t N_2'(s)ds, \\ N_2'(t) &= a_2(t) + N_2(t)\left(-\varepsilon_2 + \gamma_2 N_1(t) + \int_0^t K_2(t-s)N_2(s)ds\right). \end{aligned} \tag{82}$$

This is a nonlinear SVIEs of the form (77) where unknown functions are $N_1(t), N_1'(t), N_2(t), N_2'(t), t \in [0, T]$.

Example 6 Consider (82) with

$$\begin{gathered} \varepsilon_1 = \varepsilon_2 = \gamma_1 = \gamma_2 = \frac{1}{2}, \\ N_1(0) = 0, N_2(0) = 1, K_1(t-s) = K_2(t-s) = t - s, \\ a_1(t) = \cos t - \sin t\left(\frac{1}{2} - \frac{1}{2}\cos t + \sin t - t\right), \\ a_2(t) = -\sin t - \cos t\left(\frac{1}{2} + \frac{1}{2}\sin t - \cos t\right). \end{gathered}$$

Then the exact solution is $N_1(t) = \sin t$, $N_2(t) = \cos t$, $t \in [0, 2]$.

Errors versus N and $1/h$ are presented in Table 6 from which we can see that our method can solve well Volterra's population models.

6 Conclusion and remark

In this paper, a hp-version spectral collocation method is proposed for SVIEs. The provided convergence analysis for the proposed method show that numerical errors decay exponentially. Numerical experiments are carried out to confirm theoretical results. Further more, we test the availability of the proposed method for high order Volterra integro-differential equations including the nonlinear case. Basic errors estimate theories of piecewise spectral collocation for SVIEs are established in this paper.

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