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Homotopy analysis Sumudu transform method for time—fractional third order dispersive partial differential equation

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Abstract In this article, we apply the newly introduced numerical method which is a combination of Sumudu transforms and Homotopy analysis method for the solution of time fractional third order dispersive type PDE equations. It is also discussed generalized algorithm, absolute convergence and analytic result of the finite number of independent variables including time variable.

Keywords Dispersive partial differential equation · Homotopy analysis method · Homotopy analysis Sumudu transform method · Linear and nonlinear partial differential equation

Mathematics Subject Classification (2010) 26A33 · 34A08 · 60G22 · 65Gxx

1 Introduction

The Fractional calculus is as old as classical calculus because importance of this theory was marked as soon as the ideas of the classical calculus were born from the discussion of half derivative in epistle of Leibniz and L'Hpital in the year 1695. Further, many mathematicians contributed on this theory and strengthened the notion

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of generalized order differential and integrals viz. Liouville, Euler, Fourier, Abel, Riemann, Weyl. Liouville took initial steps for the fractional order integration and published the series of papers (1832–1837). The Riemann–Liouville operator was the most popular among mathematicians who solved fractional order integration problems [1]. Evidently, up to 300 years mentioned theory was the asset of only pure mathematicians due to unavailability of geometrical and physical interpretation of fractional order differential and integral operators. Caputo [2] described useful formula for generalized order derivatives. Oldham and Spanier [1] discussed the initial framework of application in diffusion problem, classical calculus with proper explanation. Ross [3] presented the chronological development of this theory after completing his PhD in fractional calculus and also published a monograph [4]. In the consequence, Podlubny [5], Kilbas et al. [6], Anatoly et al. [7], Diethelm [8], Caponetto et al. [9], Samko et al. [10] introduced the generalized differential and integral operators in more precise form with existence and uniqueness of results in application. Now a days, enormous model and physical phenomena like anomalous diffusion equation theory [11], mechanics of non-Hamiltonian systems [12], theory of long range interaction [13], astrophysics [14], optics [15], mechanics of fractal media [16], plasma physics [17, 18], physical kinetics [19], quantum mechanics [20], chaotic dynamics [21], which cannot meaningfully describe without means of fractional operators. Because in dynamical systems, integer order derivatives only evaluate a fixed number of derivatives wherein fractional derivatives can evaluate the value for any arbitrary order of derivative correspond to real numbers. Payable to its incredible scope and relevance in many branches of science and engineering, an extensive attention has been shown to find the solution of differential and integral equations involving the fractional derivatives. Except the modelling approach of mentioned differential equations and its solution procedure, including efficiency of convergence, divergence or junctions solutions of the model are uniformly important in numerical evaluation analysis. In order to achieve more convenient and highly adorable results, numerous numerical methods have been proposed to solve the differential equations of fractional order. Some of semi-analytic/analytic methods or numerical methods are differential transform method [22–24], Variational iteration method (VIM) [25, 26], fractional variational iteration method (FVIM)[27], Wavelet Operational matrix method [28], generalized differential transform method [29], Fractional sub equation method [30], Homotopy perturbation method [31–34], Homotopy analysis method [35–38], Homotopy analysis transform method [39–43], Fractional differential transform and Modified Fractional differential transform method [44, 45], Homotopy analysis fractional Sumudu transform method (HAFSTM) [46].

In order to convert the complex linear and nonlinear form of fractional order partial differential equations into simpler algebraic form many type of fractional integral and differential transforms have been applied to gain the exact and approximate solutions of FPDE‘s [47, 48]. Kumar and his co–workers successfully applied homotopy analysis transform method which is cumulation of Laplace transform and homotopy analysis method for the solution of fractional Fornberg–Whitham equation arising in wave breaking [39], volterra integral equation [40], fractional wave equations [41], coupled Boussinesq–Burger‘s equations arise in propagation of shallow water waves [42], unidirectional propagation of long waves in dispersive media [43]. Watugala [49] introduced the Sumudu transform and some properties discussed by Weerakoon [50, 51]. Further, Belgacem [52–59] provide precise definition of Sumudu transform and also discussed better implementations for the solution of FDE‘s, FPDE‘s using many results, properties and relations, which enhances the literature of this transform. It can easily convert many fractional order linear and nonlinear partial differential equations in time domain without loss of generality for different type of included fractional operator viz. Caputo, Riemann–Liouville, Ritzs space, etc. Multistage HAM is introduce in [60] for Solving non-linear Riccati Differential Equations. Since the homotopy analysis method applied to solve in wide variety of linear and nonlinear partial differential equations such as some fractional order smoking model [61], Lorenz system [62], a class of partial differential equations [63], space– and time-fractional kdv equation [64], Foam Drainage Equation with Space– and Time–Fractional Derivatives [65] and so on.The disadvantage of perturbation method is to solve each iteration and convergence region is very less. ADM, VIM, provide week convergent and not necessarily accurate always to exact solutions. DTM and FDTM, MFDTM require additional information and basic formula to evaluate the results. While HASTM is easily evaluate the nonlinear term with high accuracy due to independence of physical parameters and absolute convergence of series towards the exact solutions.

In this article we have applied Homotopy analysis Sumudu transform method to solve third- order fractional dispersive partial differential equations [66–71] included fractional derivative in caputo sense. The HASTM obtains semi analytic solutions in the form of series solutions. It is different from other transforms and semi analytic method, which does not require additional information except some initial and boundary conditions. It easily changes the original problem to lucid manner and then one can evaluate the result with high convergence and accuracy.

The article taxonomy is arranged as follows: In Section 2 rudimental definitions of fractional calculus and properties are discussed. The rudimental concept of HASTM is explained in Section 3. To demonstrate the method and advantages, three examples of fractional order dispersive partial differential equations are solved with discussion of convergence in Section 4. At the end concluding remark is presented in Section 5.

2 Basic definitions

Definition 2.1 A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p\,(> \mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu,\ m \in N$.

Definition 2.2 The Riemann–Liouville Fractional integral operator of order $\alpha \geq 0$, of a function $f(t) \in C_\mu$, and $\mu \geq -1$ is defined as [72, 73]
$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau,\ \alpha > 0,\ x > 0$ and $J^0 f(t) = f(t)$.

For the Riemann–Liouville fractional integral, we have

$$J^{\alpha} t^{y} = \frac{\Gamma(y+1)}{\Gamma(y+\alpha+1)} t^{\alpha+y}.$$

Definition 2.3 The fractional derivative of $f(t)$ in the Caputo sense is defined as [1]

$$D_t^{\alpha} f(t) = \begin{cases} J^{m-\alpha} D^{n} f(t), \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, \end{cases}$$

where $m-1 < \alpha \leq m,\ m \in N,\ t > 0$.

Definition 2.4 The Sumudu transform is defined over the set of functions [52–54]

$$A = \left\{ f(t) \middle| \exists M,\ \tau_1, \tau_2 > 0,\ |f(t)| < M e^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0, \infty) \right\},$$

by the following formula

$$\bar{f}(u) = \mathbb{S}[f(t)] = \int_0^{\infty} f(ut)\, e^{-t}\, dt,\ u \in (-\tau_1, \tau_2).$$

Definition 2.5 The Sumudu transform of $f(t) = t^{\alpha}$ is defined as [53]

$$\mathbb{S}\left[t^{\alpha}\right] = \int_0^{\infty} e^{-t} t^{\alpha}\, dt = \Gamma(\alpha+1)\, u^{\alpha},\ R(\alpha) > 0.$$

Definition 2.6 The Sumudu transform $\mathbb{S}[f(t)]$ of the Riemann - -Liouville fractional integral is defined as [53]

$$\mathbb{S}\left[J^{\alpha} f(t)\right] = u^{-\alpha} F(u).$$

Definition 2.7 The Sumudu transform $\mathbb{S}[f(t)]$ of the Caputo fractional derivative is defined as [53]

$$\mathbb{S}\left[D_t^{\alpha} f(t)\right] = u^{-\alpha} \mathbb{S}[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}\left(0^{+}\right), \text{ where } m-1 < \alpha \leq m.$$

3 Solution by Homotopy analysis Sumudu transform method

To illustrate the rudimental conception of the HASTM for the fractional partial differential equation, we consider the linear third order dispersive partial differential equations in following manner:

$$D_t^{n\alpha} \xi(x_1, x_2, ..., x_n, t) + \sum_{i=1}^{n} l_i \frac{\partial^3 \xi(x_1, x_2, ..., x_n, t)}{\partial x_i^3} = G(x_1, x_2, ..., x_n, t); \quad (1)$$

$\forall l_i, t > 0,\ \forall x_i \in R,\ n-1 < \alpha \leq n$, and the $G(x_1, x_2, ..., x_n, t)$ is the source function.

For simplicity, we ignore all initial and boundary conditions, which can be treated in a homogeneous way. Now the methodology consists of applying the Sumudu transform first on both sides of the Eq. 1, we get

$$\mathbb{S}\left[D_t^{n\alpha}\xi\left(x_1, x_2, ..., x_n, t\right)\right]+\mathbb{S}\left[\sum_{i=1}^{n} l_i \frac{\partial^3 \xi\left(x_1, x_2, ..., x_n, t\right)}{\partial x_i^3}\right]=\mathbb{S}[G\left(x_1, x_2, ..., x_n, t\right)]; \tag{2}$$

Using the definition (2.7) differentiation property of the Sumudu transform

$$u^{-\alpha}\mathbb{S}\left[\xi\left(x_1, x_2, ..., x_n, t\right)\right]-\sum_{k=0}^{n-1}\frac{\xi^{(k)}(0)}{u^{(\alpha-k)}}+\mathbb{S}\left[\sum_{i=1}^{n} l_i \frac{\partial^3 \xi\left(x_1, x_2, ..., x_n, t\right)}{\partial x_i^3}\right] = \mathbb{S}\left[G\left(x_1, x_2, ..., x_n, t\right)\right];$$

which gives

$$\mathbb{S}\left[\xi\left(x_1, x_2, ..., x_n, t\right)\right]-\sum_{k=0}^{n-1}\frac{\xi^{(k)}(0)}{u^{-k}}+u^{\alpha}\mathbb{S}\left[\sum_{i=1}^{n} l_i \frac{\partial^3 \xi\left(x_1, x_2, ..., x_n, t\right)}{\partial x_i^3} - G\left(x_1, x_2, ..., x_n, t\right)\right]=0; \tag{3}$$

we define nonlinear operator as

$$\mathbb{N}\left[\phi\left(x_1, x_2, ..., x_n, t; p\right)\right] = \mathbb{S}\left[\phi\left(x_1, x_2, ..., x_n, t; p\right)\right]-\sum_{k=0}^{n-1}\frac{\phi^{(k)}(0)}{u^{-k}} + u^{\alpha}\mathbb{S}\left[\sum_{i=1}^{n} l_i \frac{\partial^3 \phi(x_1, x_2, ..., x_n, t; p)}{\partial x_i^3} - G\left(x_1, x_2, ..., x_n, t; p\right)\right], \tag{4}$$

where $p \in [0, 1]$ be an embedding parameter and $\phi\left(x_1, x_2, ..., x_n, t; p\right)$ is a real function of $x_1, x_2, ..., x_n, t$ and p.

We construct a homotopy as follow:

$$\begin{aligned}&(1-p)\,\mathbb{S}\left[\phi\left(x_1, x_2, ..., x_n, t; p\right)-\xi_0\left(x_1, x_2, ..., x_n, t\right)\right]\\&= p\hbar\mathrm{H}\left(x_1, x_2, ..., x_n, t\right)\mathbb{N}\left[\phi\left(x_1, x_2, ..., x_n, t; p\right)\right];\end{aligned} \tag{5}$$

where $\hbar$ is a nonzero auxiliary parameter and $\mathrm{H}\left(x_1, x_2, ..., x_n, t\right) \neq 0$. An auxiliary function $\xi_0\left(x_1, x_2, ..., x_n, t\right)$ is an initial guess of $\xi\left(x_1, x_2, ..., x_n, t\right)$ and $\phi\left(x_1, x_2, ..., x_n, t; p\right)$ is an unknown function. It is important that one has great freedom to choose auxiliary parameter in HASTM. Obviously, when $p = 0$ and $p = 1$ it holds

$$\phi(x_1, x_2, ..., x_n, t; 0) = \xi_0\left(x_1, x_2, ..., x_n, t\right), \quad \phi(x_1, x_2, ..., x_n, t; 1) = \xi\left(x_1, x_2, ..., x_n, t\right). \tag{6}$$

Thus, as p increases from 0 to 1, the solution varies from initial guess $\xi_0\left(x_1, x_2, ..., x_n, t\right)$ to the solution$\xi\left(x_1, x_2, ..., x_n, t\right)$. Now, expanding $\phi\left(x_1, x_2, ..., x_n, t; p\right)$ on Taylor‘s series with respect to q, we get

$$\phi\left(x_1, x_2, ..., x_n, t; p\right) = \xi_0\left(x_1, x_2, ..., x_n, t\right) + \sum_{m=1}^{\infty} p^m \xi_m\left(x_1, x_2, ..., x_n, t\right), \tag{7}$$

where

$$\xi_m(x_1, x_2, ..., x_n, t) = \frac{1}{\Gamma(m+1)} \left.\frac{\partial^m \phi(x_1, x_2, ..., x_n, t;\, p)}{\partial p^m}\right|_{p=0}. \tag{8}$$

The convergence of the series solution (7) is controlled by $\hbar$. If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are properly chosen, the series (7) converges at $p = 1$. Hence we obtain

$$\xi(x_1, x_2, ..., x_n, t) = \xi_0(x_1, x_2, ..., x_n, t) + \sum_{m=1}^{\infty} \xi_m(x_1, x_2, ..., x_n, t), \tag{9}$$

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess $\xi_0(x_1, x_2, ..., x_n, t)$ and the exact solution $\xi(x_1, x_2, ..., x_n, t)$ by means of the terms $\xi_m(x_1, x_2, ..., x_n, t)$ $(m = 1, 2, 3, ...)$, which are still to be determined.

Define the vectors

$$\overrightarrow{\xi} = \{\xi_0(x_1,x_2,...,x_n,t), \xi_1(x_1,x_2,...,x_n,t), \xi_2(x_1,x_2,...,x_n,t), ..., \xi_m(x_1,x_2,...,x_n,t)\}. \tag{10}$$

Differentiating the zero order deformation Eq. 5 m times with respect to embedding parameter p and then setting $p = 0$, and finally dividing them by $\Gamma(m+1)$ we obtain the m^{th} order deformation equation as follows:

$$\begin{aligned} &\mathbb{S}\left[\xi_m(x_1, x_2, ..., x_n, t) - \chi_m \xi_{m-1}(x_1, x_2, ..., x_n, t)\right] \\ &= \hbar H(x_1, x_2, ..., x_n, t)\, R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right). \end{aligned} \tag{11}$$

Operating the inverse Sumudu transform of both sides, we get

$$\begin{aligned} &\xi_m(x_1, x_2, ..., x_n, t) = \chi_m \xi_{m-1}(x_1, x_2, ..., x_n, t) \\ &+\mathbb{S}^{-1}\left[\hbar H(x_1, x_2, ..., x_n, t)\, R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right)\right], \end{aligned} \tag{12}$$

where

$$R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right) = \frac{1}{\Gamma(m)} \left.\frac{\partial^{m-1} \varphi(x_1, x_2, ..., x_n, t;\, p)}{\partial p^{m-1}}\right|_{p=0}, \tag{13}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1 & m > 1. \end{cases} \tag{14}$$

In our case

$$\begin{aligned} R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right) = D_t^{n\alpha} \xi_{m-1}(x_1, x_2, ..., x_n, t) + \textstyle\sum_{i=1}^{n} l_i \frac{\partial^3 \xi_{m-1}(x_1, x_2, ..., x_n, t)}{\partial x_i^3} \\ - (1-\chi_m)\, G(x_1, x_2, ..., x_n, t). \end{aligned} \tag{15}$$

In this way, it is easy to obtain $\xi_m(x_1, x_2, ..., x_n, t)$ for $m \ge 1$, at M^{th} order, we have

$$\xi(x_1, x_2, ..., x_n, t) = \sum_{m=0}^{M} \xi_m(x_1, x_2, ..., x_n, t), \tag{16}$$

where $M \to \infty$, we obtain an accurate approximation of the original equation (1).

Theorem 3.1 (Convergence Theorem) *If the series* (16) *is converging for* $M \to \infty$, *where* $\xi_m(x_1, x_2, ..., x_n, t)$ *is obtained by* Eq. 12 *and using the conditions* (14) *and* (15). *Then, it must be the exact solution of original discussed partial differential equation* (1).

Proof Let the series (16) be the convergent series then

$$\sum_{m=0}^{\infty}\xi_m(x_1,x_2,...,x_n,t)=\xi_0(x_1,x_2,...,x_n,t)+\sum_{m=1}^{\infty}\xi_m(x_1,x_2,...,x_n,t)=K(x_1,x_2,...,x_n,t). \tag{17}$$

Now we have $\lim_{M\to\infty}\xi_m(x_1, x_2, ..., x_n, t) = 0$. Using definition of Eq. 11 we obtained

$$\lim_{M\to\infty}\left[\hbar H(x_1, x_2, ..., x_n, t)\sum_{m=1}^{M} R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right)\right]$$
$$= \lim_{M\to\infty}\left(\sum_{m=1}^{M}\mathbb{S}\left[\xi_m(x_1, x_2, ..., x_n, t) - \chi_m\xi_{m-1}(x_1, x_2, ..., x_n, t)\right]\right)$$
$$= \lim_{M\to\infty}\left(\sum_{m=1}^{M}\mathbb{S}\left[\xi_m(x_1, x_2, ..., x_n, t) - \chi_m\xi_{m-1}(x_1, x_2, ..., x_n, t)\right]\right)$$
$$= \mathbb{S}\left(\lim_{M\to\infty}\sum_{m=1}^{M}\left[\xi_m(x_1, x_2, ..., x_n, t) - \chi_m\xi_{m-1}(x_1, x_2, ..., x_n, t)\right]\right)$$
$$= \mathbb{S}\left(\lim_{M\to\infty}\xi_M(x_1, x_2, ..., x_n, t)\right)$$
$$= 0.$$

Since $\hbar \neq 0$, $H(x_1, x_2, ..., x_n, t) \neq 0$, therefore $\sum_{m=1}^{\infty} R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right) = 0$.

From (15)

$$\sum_{m=1}^{\infty} R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right) = \sum_{m=1}^{\infty}\left(D_t^{n\alpha}\xi_{m-1}(x_1, x_2, ..., x_n, t)\right.$$
$$\left.+\sum_{i=1}^{n} l_i\,\frac{\partial^3\xi_{m-1}(x_1, x_2, ..., x_n, t)}{\partial x_i^3} - (1-\chi_m)\,G(x_1, x_2, ..., x_n, t)\right)$$

$$\sum_{m=1}^{\infty} R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right) = \sum_{m=1}^{\infty} D_t^{n\alpha}\xi_{m-1}(x_1, x_2, ..., x_n, t)$$
$$+\sum_{m=1}^{\infty}\sum_{i=1}^{n} l_i\,\frac{\partial^3\xi_{m-1}(x_1, x_2, ..., x_n, t)}{\partial x_i^3} - \sum_{m=1}^{\infty}(1-\chi_m)\,G(x_1, x_2, ..., x_n, t)$$

$$\sum_{m=1}^{\infty} R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, ..., x_n, t\right) = D_t^{n\alpha} \sum_{m=0}^{\infty} \xi_m\left(x_1, x_2, ..., x_n, t\right)$$

$$+\sum_{i=1}^{n} l_i \frac{\partial^3 \sum\limits_{m=0}^{\infty} \xi_m\left(x_1, x_2, ..., x_n, t\right)}{\partial x_i^3} - G\left(x_1, x_2, ..., x_n, t\right)$$

$$D_t^{n\alpha} K\left(x_1, x_2, ..., x_n, t\right) + \sum_{i=1}^{n} l_i \frac{\partial^3 K\left(x_1, x_2, ..., x_n, t\right)}{\partial x_i^3} - G\left(x_1, x_2, ..., x_n, t\right) = 0. \tag{18}$$

Above equation (18) shows that, $K\left(x_1, x_2, ..., x_n, t\right)$ satisfies the original problem (1). □

4 Numerical illustrations

In this section we consider the time fractional dispersive partial differential equations to authenticate the method discussed in the previous section.

Example 4.1 We consider the linear time fractional KDV [71]

$$\xi_t^{\alpha}\left(x, t\right) + 2\frac{\partial \xi\left(x, t\right)}{\partial x} + \frac{\partial^3 \xi\left(x, t\right)}{\partial x^3} = 0,\ \ t > 0,\ \ 0 < \alpha \leq 1, \tag{19}$$

subject to the initial condition

$$\xi\left(x, 0\right) = \ Sin\, x. \tag{20}$$

The exact solution at $\alpha = 1$ is given by

$$\xi\left(x, t\right) = Sin\ \left(x - t\right). \tag{21}$$

Applying the Sumudu transform of both sides in Eq. (19) and after using the definition (2.7) of Sumudu transform for fractional derivative, we get

$$\mathbb{S}\left[\xi\left(x, t\right)\right] + u^{\alpha}\mathbb{S}\left[2\frac{\partial \xi\left(x, t\right)}{\partial x} + \frac{\partial^3 \xi\left(x, t\right)}{\partial x^3}\right] = 0,\ t > 0. \tag{22}$$

The nonlinear operator is

$$N[\phi\left(x, t; p\right)] = \mathbb{S}[\phi\left(x, t; p\right)] + u^{\alpha}\mathbb{S}\left[2\frac{\partial \phi(x, t; p)}{\partial x} + \frac{\partial^3 \phi(x, t; p)}{\partial x^3}\right] = 0, t > 0, 0 \leq p \leq 1, \tag{23}$$

and thus

$$R_m\left(\overrightarrow{\xi}_{m-1}, x, t\right) = \mathbb{S}\left[\xi_{m-1}(x, t)\right] + u^{\alpha}\mathbb{S}\left[2\frac{\partial \xi_{m-1}\left(x, t\right)}{\partial x} + \frac{\partial^3 \xi_{m-1}\left(x, t\right)}{\partial x^3}\right] = 0,\ t > 0. \tag{24}$$

The $m^{th}-$ order deformation equation is given by

$$\mathbb{S}\left[\xi_m\left(x, t\right) - \chi_m \xi_{m-1}\left(x, t\right)\right] = \hbar H\left(x, t\right) R_m\left(\overrightarrow{\xi}_{m-1}, x, t\right).$$

Applying the inverse Sumudu transform, we have

$$\xi_m(x,t) = \chi_m \xi_{m-1}(x,t) + \mathbb{S}^{-1}\left[\hbar H(x,t) R_m\left(\overrightarrow{\xi}_{m-1}, x, t\right)\right]. \tag{25}$$

On solving above equation for $m = 1, 2, \ldots,$. For simplicity, we choose $H(x,t) = 1$,

$$\xi_1(x,t) = \frac{t^{\alpha}\hbar\, Cos\, x}{\Gamma(1+\alpha)},$$

$$\xi_2(x,t) = \frac{t^{\alpha}\hbar\, Cos\, x}{\Gamma(1+\alpha)} + \frac{t^{2\alpha-1}\alpha\hbar^2\, Cos\, x\,\Gamma(\alpha)}{\Gamma(1+\alpha)\Gamma(2\alpha)} - \frac{t^{2\alpha}\hbar^2\, Sin\, x}{\Gamma(1+2\alpha)},$$

$$\begin{aligned}\xi_3(x,t) = {} & \frac{t^{\alpha}\hbar\, Cos\, x}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha-1}\alpha\hbar^2\, Cos\, x\,\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(1+\alpha)} - \frac{t^{3\alpha-2}\alpha\hbar^3\, Cos\, x\;\Gamma(\alpha)\;\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(1+\alpha)\Gamma(3\alpha-1)} \\ & + \frac{2t^{3\alpha-2}\alpha^2\hbar^3\, Cos\, x\;\Gamma(\alpha)\;\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(1+\alpha)\Gamma(3\alpha-1)} - \frac{3t^{\alpha}\hbar^3\, Cos\, x}{\Gamma(1+\alpha)} - \frac{t^{3\alpha-1}\alpha\,\hbar^3\,\Gamma(\alpha)\, Sin\, x}{\Gamma(1+\alpha)\Gamma(3\alpha)} \\ & - \frac{2t^{2\alpha}\hbar^2\, Sin\, x}{\Gamma(1+2\alpha)} - \frac{2t^{3\alpha-1}\alpha\,\hbar^3\Gamma(2\alpha)\, Sin\, x}{\Gamma(1+2\alpha)\Gamma(3\alpha)},\end{aligned}$$

$$\vdots$$

and so on.

Here, we consider the results upto $m = 10$ and rest of the components can be evaluated by iteration formula (24).

Therefore the solution of Eq. 19 is given by

$$\xi(x,t) = \xi_0(x,t) + \sum_{m=1}^{\infty}\xi_m(x,t)\,. \tag{26}$$

At $\hbar = -1$ we obtained the following approximation:

$$\begin{aligned}\xi(x,t) = {} & \frac{-3\,t^{\alpha}\, Cos\, x}{\Gamma(1+\alpha)} + \frac{3\,t^{2\alpha-1}\alpha\, Cos\, x\,\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(1+\alpha)} + \frac{t^{3\alpha-2}\alpha\, Cos\, x\,\Gamma(\alpha)\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(1+\alpha)\Gamma(3\alpha-1)} \\ & - \frac{2t^{3\alpha-2}\alpha^2\, Cos\, x\,\Gamma(\alpha)\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(1+\alpha)\Gamma(3\alpha-1)} + \frac{t^{3\alpha}\, Cos\, x}{\Gamma(1+3\alpha)} + Sin\, x + \frac{t^{3\alpha-1}\alpha\Gamma(\alpha)\; Sin\, x}{\Gamma(3\alpha)\Gamma(1+\alpha)} \\ & - \frac{3\,t^{2\alpha}\, Sin\, x}{\Gamma(1+2\alpha)} + \frac{2t^{3\alpha-1}\alpha\,\Gamma(2\alpha)\; Sin\, x}{\Gamma(3\alpha)\Gamma(1+2\alpha)} + \ldots,\end{aligned} \tag{27}$$

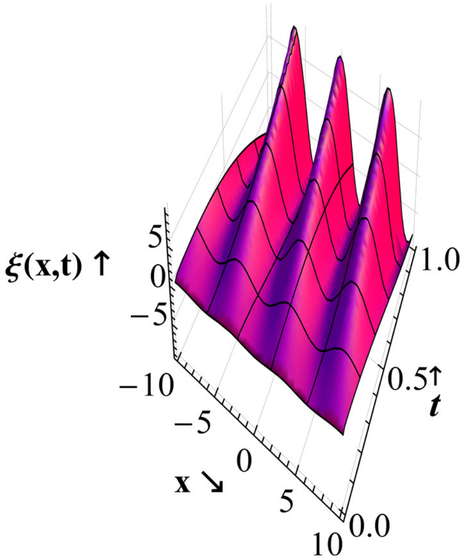
when $\alpha = 1$ Eq. 27 shows the similar results as [70] which is the exact solution of Eq. 19

$$\xi(x,t) = Sin\, x - \frac{1}{2}t^2\, Sin\, x - t\, Cos\, x + \frac{1}{6}t^3 Cos\, x + \ldots.$$

After simplification we get Eq. 21.

Figures 1, 2, 3 and 4 show that the nature of fractional derivative and fluctuation changes from $\alpha = 0.9,\ 95,\ 1$ and exact solution at $\alpha = 1$.

Fig. 1 Plot of $\xi\,(x,t)$ w.r.t x and t at $\alpha = 0.9$



Example 4.2 Consider the linear time fractional KDV equation in one dimensional space

$$\xi_t^{\alpha}\,(x,t) + 3\frac{\partial^3 \xi\,(x,t)}{\partial x^3} = 0,\ \ t > 0,\ 0 < x < 1,\ \ 0 < \alpha \leq 1, \tag{28}$$

subject to the initial condition

$$\xi\,(x,0) = \ Cos\,x,\ \ 0 \leq x \leq 1. \tag{29}$$

Fig. 2 Plot of $\xi\,(x,t)$ w.r.t x and t at $\alpha = 0.95$

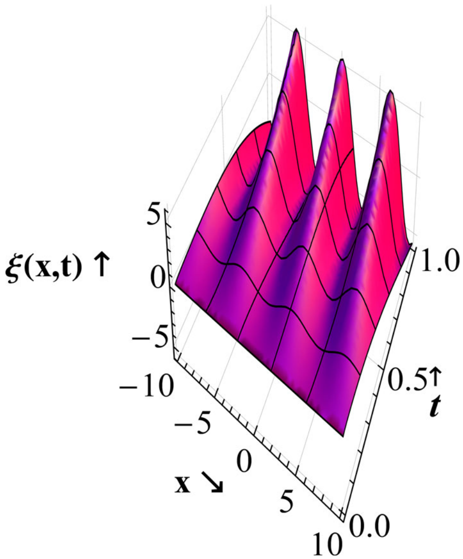
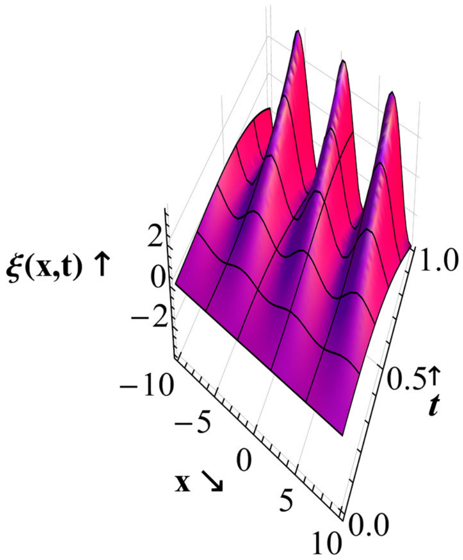


Fig. 3 Plot of $\xi\ (x, t)$ w.r.t x and t at $\alpha = 1$



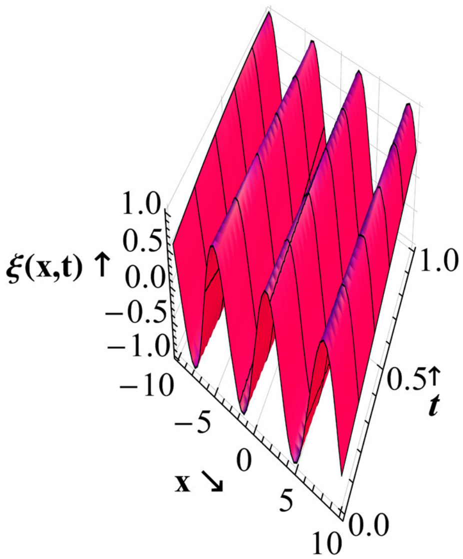
The exact solution at $\alpha = 1$ is given by

$$\xi\ (x, t) = Cos\ (x + 3t)\,. \tag{30}$$

Applying the Sumudu transform of both sides in Eq. 28 and after using the definition (2.7) of Sumudu transform for fractional derivative, we get

$$\mathbb{S}\left[\xi\ (x, t)\right] + u^{\alpha}\mathbb{S}\left[3\frac{\partial^3 \xi\ (x, t)}{\partial x^3}\right] = 0,\ t > 0. \tag{31}$$

Fig. 4 Plot of Exact Solution of $\xi\ (x, t)$ w.r.t x and t



The nonlinear operator is

$$N\left[\phi\left(x,t;\,p\right)\right]=\mathbb{S}\left[\phi\left(x,t;\,p\right)\right]+3u^{\alpha}\mathbb{S}\left[\frac{\partial^{3}\phi\left(x,t;\,p\right)}{\partial x^{3}}\right]=0,\ t>0,\ 0\leq p\leq 1, \tag{32}$$

and thus

$$R_{m}\left(\overrightarrow{\xi}_{m-1},x,t\right)=\mathbb{S}\left[\xi_{m-1}\left(x,t\right)\right]+3u^{\alpha}\mathbb{S}\left[\frac{\partial^{3}\xi_{m-1}\left(x,t\right)}{\partial x^{3}}\right]=0,\quad t>0. \tag{33}$$

The $m^{th}-$ order deformation equation is given by

$$\mathbb{S}\left[\xi_{m}\left(x,t\right)-\chi_{m}\xi_{m-1}\left(x,t\right)\right]=\hbar H\left(x,t\right)R_{m}\left(\overrightarrow{\xi}_{m-1}\left(x,t\right)\right).$$

Applying the inverse Sumudu transform, we have

$$\xi_{m}\left(x,t\right)=\chi_{m}\xi_{m-1}\left(x,t\right)+\mathbb{S}^{-1}\left[\hbar H\left(x,t\right)R_{m}\left(\overrightarrow{\xi}_{m-1}\left(x,t\right)\right)\right]. \tag{34}$$

On solving above equation for $m=1,2,\ \ldots,$. For simplicity, we choose $H\left(x,t\right)=1$

$$\xi_{1}\left(x,t\right)=\frac{3t^{\alpha}\hbar\,Sin\,x}{\Gamma\left(1+\alpha\right)},$$

$$\xi_{2}\left(x,t\right)=-\frac{9t^{2\alpha}\hbar^{2}\,Cos\,x}{\Gamma\left(1+2\alpha\right)}+\frac{3t^{\alpha}\hbar\,Sin\,x}{\Gamma\left(1+\alpha\right)}+\frac{3t^{2\alpha-1}\alpha\hbar^{2}\,Sin\,x\,\Gamma\left(\alpha\right)}{\Gamma\left(1+\alpha\right)\Gamma\left(2\alpha\right)},$$

$$\begin{aligned}\xi_{3}\left(x,t\right)=&-\frac{9t^{3\alpha-1}\alpha\,\hbar^{3}\,Cos\,x\Gamma\left(\alpha\right)}{\Gamma\left(1+\alpha\right)\Gamma\left(3\alpha\right)}-\frac{18t^{2\alpha}\,\hbar^{2}\,Cos\,x}{\Gamma\left(1+2\alpha\right)}\\&-\frac{18\,t^{3\alpha-1}\alpha\,\hbar^{3}\,Cos\,x\Gamma\left(2\alpha\right)}{\Gamma\left(1+2\alpha\right)\Gamma\left(3\alpha\right)}+\frac{3t^{\alpha}\hbar\,Sin\,x}{\Gamma\left(1+\alpha\right)}\\&+\frac{6t^{2\alpha-1}\alpha\hbar^{2}\,Sin\,x\,\Gamma\left(\alpha\right)}{\Gamma\left(1+\alpha\right)\Gamma\left(2\alpha\right)}-\frac{3t^{3\alpha-2}\alpha\hbar^{3}\,Sin\,x\,\Gamma\left(\alpha\right)\Gamma\left(2\alpha-1\right)}{\Gamma\left(2\alpha\right)\Gamma\left(1+\alpha\right)\Gamma\left(3\alpha-1\right)}\\&+\frac{6t^{3\alpha-2}\alpha^{2}\hbar^{3}\,Sin\,x\,\Gamma\left(\alpha\right)\Gamma\left(2\alpha-1\right)}{\Gamma\left(2\alpha\right)\Gamma\left(1+\alpha\right)\Gamma\left(3\alpha-1\right)}-\frac{27t^{3\alpha}\hbar^{3}\,Sin\,x}{\Gamma\left(3\alpha+1\right)},\end{aligned}$$

$$\vdots$$

and so on.

Here, we consider the results upto $m=10$ and rest of the components can evaluate by iteration formula (34).

Therefore the solution of Eq. 28 is given by

$$\xi\left(x,t\right)=\xi_{0}\left(x,t\right)+\sum_{m=1}^{\infty}\xi_{m}\left(x,t\right). \tag{35}$$

At $\hbar = -1$ we obtained the following approximation:

$$\begin{aligned}\xi(x,t) = & \frac{-3\,t^{\alpha}\,Cos\,x}{\Gamma(1+\alpha)} + \frac{3\,t^{2\alpha-1}\alpha\,Cos\,x\,\Gamma(\alpha)}{\Gamma(2\alpha)\,\Gamma(1+\alpha)} + \frac{t^{3\alpha-2}\alpha\,Cos\,x\,\Gamma(\alpha)\,\Gamma(2\alpha-1)}{\Gamma(2\alpha)\,\Gamma(1+\alpha)\,\Gamma(3\alpha-1)} \\ & -\frac{2t^{3\alpha-2}\alpha^{2}\,Cos\,x\,\Gamma(\alpha)\,\Gamma(2\alpha-1)}{\Gamma(2\alpha)\,\Gamma(1+\alpha)\,\Gamma(3\alpha-1)} + \frac{t^{3\alpha}\,Cos\,x}{\Gamma(1+3\alpha)} + Sin\,x + \frac{t^{3\alpha-1}\alpha\Gamma(\alpha)\;Sin\,x}{\Gamma(3\alpha)\,\Gamma(1+\alpha)} \\ & -\frac{3\,t^{2\alpha}\,Sin\,x}{\Gamma(1+2\alpha)} + \frac{2t^{3\alpha-1}\alpha\,\Gamma(2\alpha)\;Sin\,x}{\Gamma(3\alpha)\,\Gamma(1+2\alpha)} + \ldots.\end{aligned} \tag{36}$$

when $\alpha = 1$ Eq. 36 shows the similar results as [70] which is the exact solution of Eq. 28.

$$\xi(x,t) = \;Sin\,x\; - \frac{1}{2}\,t^{2}\;Sin\,x\; - \;t\;Cos\,x\; + \;\frac{1}{6}\,t^{3}Cos\,x + \ldots$$

After simplification we can get Eq. 30.

As dicussed in previous example Figs. 5, 6, 7 and 8 show that the nature of fractional derivative and fluctuation changes for $\alpha = 0.9$, 95, 1 and exact solution at $\alpha = 1$.

Example 4.3 Now, we consider the linear time fractional KDV equation in two dimensional space [71]

$$\xi_t^{\alpha}(x,y,t) + 2\,\frac{\partial^{3}\xi(x,y,t)}{\partial x^{3}} + \frac{\partial^{3}\xi(x,y,t)}{\partial y^{3}} = 0,\;\; t > 0,\;\; 0 < \alpha \leq 1, \tag{37}$$

subject to the initial condition

$$\xi(x,y,0) = Cos\;(x+y)\,. \tag{38}$$

Fig. 5 Plot of $\xi(x,t)$ w.r.t x and t at $\alpha = 0.9$

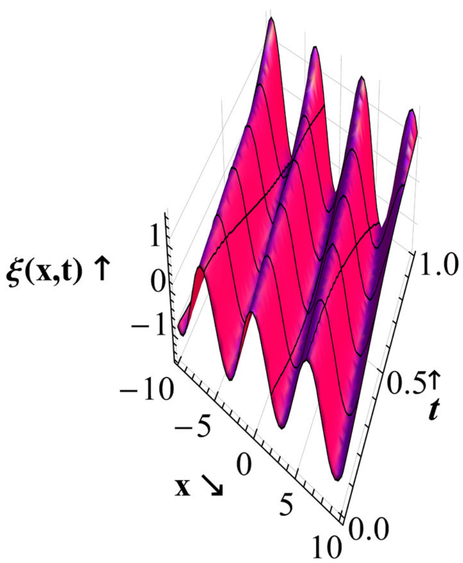
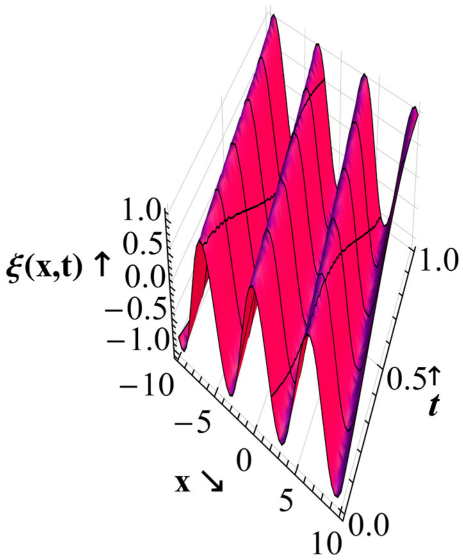


Fig. 6 Plot of $\xi\ (x, t)$ w.r.t x and t at $\alpha = 0.95$



The exact solution at $\alpha = 1$ is given by

$$\xi\ (x, y, t) = Sin\ (x + y + 2t)\,. \tag{39}$$

Applying the Sumudu transform of both sides in Eq. 37 and after using the definition (2.7) of Sumudu transform for fractional derivative, we get

$$\mathbb{S}\left[\xi\ (x, y, t)\right] + u^{\alpha}\mathbb{S}\left[2\,\frac{\partial^3 \xi\ (x, y, t)}{\partial x^3} + \frac{\partial^3 \xi\ (x, y, t)}{\partial y^3}\right] = 0,\ t > 0. \tag{40}$$

Fig. 7 Plot of $\xi\ (x, t)$ w.r.t x and t at $\alpha = 1$

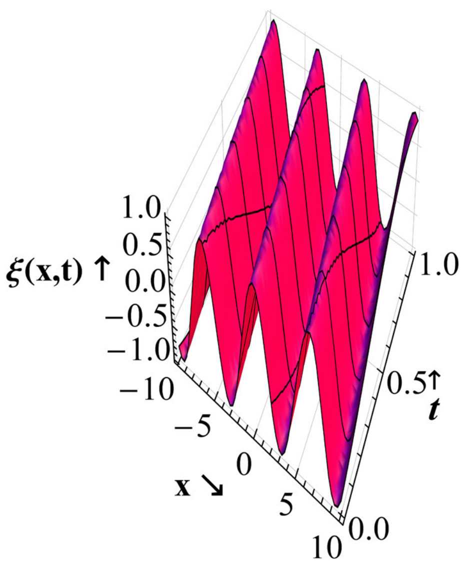
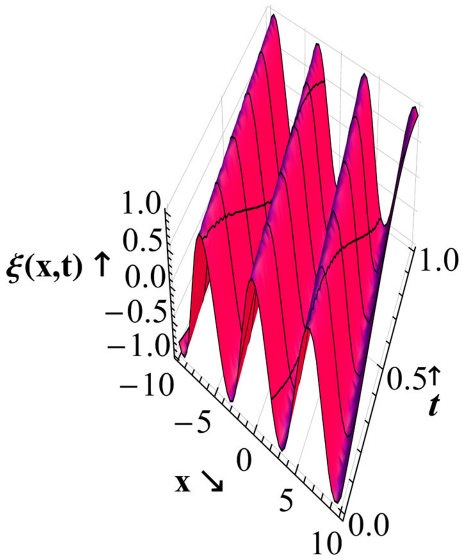


Fig. 8 Plot of Exact Solution of $\xi(x,t)$ w.r.t x and t



The nonlinear operator is

$$N[\phi(x,y,t;p)] = \mathbb{S}[\phi(x,y,t;p)] + u^{\alpha}\mathbb{S}\left[2\frac{\partial^3\phi(x,y,t;p)}{\partial x^3} + \frac{\partial^3\phi(x,y,t;p)}{\partial y^3}\right] = 0, \quad t > 0,\ 0 \le p \le 1, \tag{41}$$

and thus

$$R_m\left(\overrightarrow{\xi}_{m-1}, x, y, t\right) = \mathbb{S}\left[\xi_{m-1}(x,y,t)\right] + u^{\alpha}\mathbb{S}\left[2\frac{\partial^3\xi_{m-1}(x,y,t)}{\partial^3 x} + \frac{\partial^3\xi_{m-1}(x,y,t)}{\partial y^3}\right] = 0, \quad t > 0. \tag{42}$$

The $m^{th}-$ order deformation equation is given by

$$\mathbb{S}\left[\xi_m(x,y,t) - \chi_m\xi_{m-1}(x,y,t)\right] = \hbar H(x,y,t)\, R_m\left(\overrightarrow{\xi}_{m-1}(x,y,t)\right).$$

Applying the inverse Sumudu transform, we have

$$\xi_m(x,y,t) = \chi_m\xi_{m-1}(x,y,t) + \mathbb{S}^{-1}\left[\hbar H(x,y,t)\, R_m\left(\overrightarrow{\xi}_{m-1}(x,y,t)\right)\right]. \tag{43}$$

On solving above equation for $m = 1, 2, \ldots,$. For simplicity, we choose $H(x,t) = 1$,

$$\xi_1(x,y,t) = \frac{2t^{\alpha}\hbar\, Sin\ (x+y)}{\Gamma(1+\alpha)},$$

$$\xi_2(x,y,t) = -\frac{4t^{2\alpha}\hbar^2\, Cos\ (x+y)}{\Gamma(1+2\alpha)} + \frac{2t^{\alpha}\hbar\, Sin\ (x+y)}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha-1}\alpha\hbar^2\Gamma(\alpha)\, Sin\ (x+y)}{\Gamma(1+2\alpha)\,\Gamma(1+\alpha)},$$

$$\xi_3(x,t) = -\frac{4t^{3\alpha-1}\alpha\,\hbar^3 Cos\ (x+y)\,\Gamma(\alpha)}{\Gamma(1+\alpha)} - \frac{8t^{2\alpha}\hbar^2\, Cos\ (x+y)}{\Gamma(1+2\alpha)}$$
$$-\frac{8t^{3\alpha-1}\alpha\hbar^3\, Cos\ (x+y)\ \ \Gamma(2\alpha)}{\Gamma(3\alpha)\,\Gamma(1+2\alpha)} + \frac{2t^{\alpha}\hbar\, Sin\ (x+y)}{\Gamma(1+\alpha)}$$
$$+\frac{4t^{2\alpha-1}\alpha\hbar^2\Gamma(\alpha)\ Sin\ (x+y)}{\Gamma(1+\alpha)\,\Gamma(2\alpha)} - \frac{2t^{3\alpha-2}\alpha\,\hbar^3\,\Gamma(\alpha)\ \Gamma(2\alpha-1)\, Sin\ (x+y)}{\Gamma(1+\alpha)\,\Gamma(3\alpha)\,\Gamma(3\alpha-1)}$$
$$+\frac{4t^{3\alpha-2}\alpha\,\hbar^3\,\Gamma(\alpha)\ \Gamma(2\alpha-1)\, Sin\ (x+y)}{\Gamma(1+\alpha)\,\Gamma(2\alpha)\,\Gamma(3\alpha-1)} - \frac{8t^{3\alpha}\hbar^3\, Sin\ (x+y)}{\Gamma(1+3\alpha)}$$
$$\vdots$$

and so on.

Here, we consider the results upto $m = 10$ and rest of the components can be evaluated by iteration formula (43).

Therefore the solution of Eq. 19 is given by

$$\xi(x,y,t) = \xi_0(x,y,t) + \sum_{m=1}^{\infty}\xi_m(x,y,t)\ . \tag{44}$$

At $\hbar = -1$ we obtain the following approximation:

$$\xi(x,y,t) = Cos\ (x+y) + \frac{4\,t^{3\alpha-1}\alpha\, Cos\ (x+y)\,\Gamma(\alpha)}{\Gamma(3\alpha)\,\Gamma(1+\alpha)} - \frac{12\,t^{2\alpha}\, Cos\ (x+y)}{\Gamma(1+2\alpha)}$$
$$+\frac{8\,t^{3\alpha-1}\alpha\, Cos\ (x+y)\ \Gamma(2\alpha)}{\Gamma(3\alpha)\,\Gamma(1+2\alpha)} - \frac{6\,t^{\alpha}\ Sin\ (x+y)}{\Gamma(1+\alpha)} + \frac{6t^{2\alpha-1}\alpha\ \Gamma(\alpha)\ Sin\ (x+y)}{\Gamma(2\alpha)\,\Gamma(1+\alpha)}$$
$$+\frac{2\,t^{3\alpha-2}\alpha\,\Gamma(\alpha)\,\Gamma(2\alpha-1)\ Sin\ (x+y)}{\Gamma(2\alpha)\,\Gamma(1+\alpha)\,\Gamma(3\alpha-1)} + \frac{4\,t^{3\alpha-2}\alpha^2\,\Gamma(\alpha)\,\Gamma(2\alpha-1)\ Sin\ (x+y)}{\Gamma(2\alpha)\,\Gamma(1+\alpha)\,\Gamma(3\alpha-1)}$$
$$+\frac{8\,t^{3\alpha}\, Sin\ (x+y)}{\Gamma(3\alpha+\ 1)} \tag{45}$$

when $\alpha = 1$ Eq. 45 shows the similar results as [70] which is the exact solution of Eq. 37

$$\xi(x,y,t) = Cos\,(x+y) - 2\,t^2 Cos\ (x+y)\ - 2\,t\ Sin\ (x+y)\ + \frac{4}{3}\,t^3 Sin\ (x+y) + \ldots$$

After simplification we can get Eq. 39.

5 Conclusion

Here, we have applied HASTM for solving fractional third order dispersive partial differential equations. It is shown that HASTM is an effective alternate tool for the evaluation of linear and nonlinear partial differential equations, which is not require any physical perturbation quantity, lucid to understand. Thus, we can conclude that the transform method with HAM is very effective and highly accurate for any lower and higher order fractional partial differential equations.

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