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Hierarchical spline spaces: quasi-interpolants and local approximation estimates

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Abstract A local approximation study is presented for hierarchical spline spaces. Such spaces are composed of a hierarchy of nested spaces and provide a flexible framework for local refinement in any dimensionality. We provide approximation estimates for general hierarchical quasi-interpolants expressed in terms of the truncated hierarchical basis. Under some mild assumptions, we prove that such hierarchical quasi-interpolants and their derivatives possess optimal local approximation power in the general q-norm with $1 \leq q \leq \infty$. In addition, we detail a specific family of hierarchical quasi-interpolants defined on uniform hierarchical meshes in any dimensionality. The construction is based on cardinal B-splines of degree p and central factorial numbers of the first kind. It guarantees polynomial reproduction of degree p and it requires only function evaluations at grid points (odd p) or half-grid points (even p). This results in good approximation properties at a very low cost, and is illustrated with some numerical experiments.

Keywords Local approximation · Quasi-interpolation · Hierarchical bases · Local refinement · Tensor-product B-splines

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1 Introduction

Quasi-interpolation is a general term denoting the construction, with a low computational cost, of accurate approximants to a given set of data or a given function. A quasi-interpolant is usually obtained as a linear combination of a set of blending functions that form a convex partition of unity and have small local support. The computation of the corresponding coefficients only involves a local portion of the given data/function. These properties ensure both numerical stability and local control of the constructed approximant. Quasi-interpolants in polynomial spline spaces are a common and powerful approximation tool, see e.g. [1, 3, 19, 21].

In this paper we focus on quasi-interpolants in hierarchical spline spaces. Such spline spaces provide a flexible framework for local refinement coupled with a remarkable intrinsic simplicity. Hierarchical B-splines are defined in terms of a hierarchy of locally refined meshes, reflecting different levels of refinement. They were introduced in [10] as an accumulation of tensor-product B-splines with nested knot vectors. The hierarchical approach has been successfully applied in various areas, ranging from approximation theory to numerical simulation, see e.g. [6, 10, 13, 20, 26]. A similar local refinement approach can also be found in [8].

The original set of hierarchical B-splines in [10] lacks some important properties such as linear independence and partition of unity. The issue of linear independence has been solved in [14, 15]. In order to obtain a partition of unity, an alternative basis for the same hierarchical spline space has been proposed in [11] and is called truncated hierarchical B-spline (THB-spline) basis. Besides forming a convex partition of unity, the THB-spline basis possesses some other interesting properties, like strong stability with respect to the supremum norm and the property of preservation of coefficients, see [12].

More recently, in [25], the above properties of THB-splines have been exploited to develop a general and very simple procedure for the construction of quasi-interpolants in hierarchical spline spaces. Thanks to the preservation of coefficients, the construction is basically effortless. It suffices to consider a quasi-interpolant in each space associated with a particular level in the hierarchy, which will be referred to as a one-level quasi-interpolant. Then, the coefficients of the proposed hierarchical quasi-interpolant are nothing else than a proper subset of the coefficients of the one-level quasi-interpolants. Important properties – like polynomial reproduction – of the one-level quasi-interpolants are preserved in the hierarchical construction. A basic local approximation study (using the infinity norm) was also provided in [25]. Here, we complete this study and we prove that such hierarchical quasi-interpolants and their derivatives possess optimal local approximation power in the general q-norm with $1 \leq q \leq \infty$, under some mild assumptions on the underlying hierarchical meshes and the one-level quasi-interpolants.

The concept of truncated hierarchical basis is not confined to tensor-product polynomial spline spaces. It has also been investigated in [23] for hierarchical spline spaces over Powell–Sabin triangulations, and in [12] for a broad class of hierarchical spaces. The truncated hierarchical bases in this general setting maintain properties like convex partition of unity, stability, and preservation of coefficients. The above framework for the construction of hierarchical quasi-interpolants can also be applied in this general setting, see [25].

The remainder of the paper is organized as follows. Section 2 provides some preliminary results on polynomial approximation. In Section 3 we recall the definition of THB-splines and the general construction of hierarchical quasi-interpolants based on the THB-spline representation as developed in [25]. Section 4 is devoted to the local approximation study of such hierarchical quasi-interpolants and contains our main results. Under some mild assumptions, we prove that these hierarchical quasi-interpolants and their derivatives possess optimal local approximation power in the general q-norm with $1 \leq q \leq \infty$. In Section 5 we detail a simple but effective construction of a specific family of hierarchical quasi-interpolants on uniform hierarchical meshes. The construction is based on cardinal B-splines of degree p and central factorial numbers of the first kind. It guarantees polynomial reproduction of degree p and it requires only function evaluations at grid points (odd p) or half-grid points (even p). This results in good approximation properties at a very low cost. Their effectiveness is illustrated with numerical experiments. We end in Section 6 with some concluding remarks.

2 Preliminaries on polynomials

We denote the usual q-norm over a given set $\Upsilon \subseteq \mathbb{R}^d$ by

$$\|f\|_{\infty,\Upsilon} := \sup_{\boldsymbol{x}\in\Upsilon} |f(\boldsymbol{x})|, \qquad \|f\|_{q,\Upsilon} := \left(\int_\Upsilon |f(\boldsymbol{x})|^q \mathrm{d}\boldsymbol{x}\right)^{1/q}, \quad 1 \leq q < \infty,$$

and the usual semi-norm in the Sobolev space $W_q^k(\Upsilon)$ by

$$|f|_{k,\infty,\Upsilon} := \max_{|\boldsymbol{\alpha}|=k} \|D^{\boldsymbol{\alpha}} f\|_{\infty,\Upsilon}, \quad |f|_{k,q,\Upsilon} := \left(\sum_{|\boldsymbol{\alpha}|=k} \|D^{\boldsymbol{\alpha}} f\|_{q,\Upsilon}^q\right)^{1/q}, \quad 1 \leq q < \infty,$$

where $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $|\boldsymbol{\alpha}| := \sum_{i=1}^d \alpha_i$. Let $\mathrm{diam}(\Upsilon)$ be the diameter of Υ, and let $\mathrm{conv}(\Upsilon)$ be the convex hull of Υ. Furthermore, $\mathrm{chunk}(\Upsilon)$ stands for the usual chunkiness parameter of Υ defined by

$$\mathrm{chunk}(\Upsilon) := \frac{\mathrm{diam}(\Upsilon)}{r_\Upsilon},$$

with r_Υ the radius of the largest ball contained in Υ, and $\mathrm{chunk}(\Upsilon, \Psi)$ stands for the combined chunkiness parameter of $\Upsilon \subseteq \Psi$ defined by

$$\mathrm{chunk}(\Upsilon, \Psi) := \frac{\mathrm{diam}(\Psi)}{r_\Upsilon}.$$

It is clear that for any $\Upsilon \subseteq \Psi$,

$$\mathrm{chunk}(\Upsilon) \leq \mathrm{chunk}(\Upsilon, \Psi), \quad \mathrm{chunk}(\Psi) \leq \mathrm{chunk}(\Upsilon, \Psi).$$

Let Π_p be the space of d-variate polynomials of total degree at most p. We start by providing some preliminary properties of multivariate polynomials, of interest in the later sections. Note that C stands for a generic constant which may have a different value throughout the paper.

Lemma 1 *Let Υ be a convex body in $\mathbb{R}^d$, and let r_Υ be the radius of the largest ball contained in Υ. For any $g \in \Pi_p$ and $1 \le q \le \infty$, we have*

$$\|D^{\boldsymbol{\alpha}} g\|_{q,\Upsilon} \le \frac{C}{(r_\Upsilon)^{|\boldsymbol{\alpha}|}} \|g\|_{q,\Upsilon}, \quad 0 \le |\boldsymbol{\alpha}| \le p, \tag{1}$$

where C is a constant independent of g and Υ.

Proof The result is trivial for $|\boldsymbol{\alpha}| = 0$. The case $|\boldsymbol{\alpha}| = 1$ is a classical Markov-type estimate for multivariate polynomials. Indeed, we know (see e.g. [16, Theorem 3]) that for any $g \in \Pi_p$,

$$\|\,|\nabla g|\,\|_{\infty,\Upsilon} \le \frac{C_\infty}{\omega_\Upsilon} \|g\|_{\infty,\Upsilon},$$

where $|\nabla g| := \sup_{|\boldsymbol{\epsilon}|=1} |D^{\boldsymbol{\epsilon}} g|$ is the magnitude of the gradient and ω_Υ is the minimal distance between two parallel supporting hyper-planes for Υ. The constant C_∞ only depends on p. It is clear that $2r_\Upsilon \le \omega_\Upsilon$, and so we obtain the inequality (1) for any $|\boldsymbol{\alpha}| = 1$ and $q = \infty$. Moreover, from [17, Theorem 1] we know that for any $g \in \Pi_p$ and $1 \le q < \infty$,

$$\|\,|\nabla g|\,\|_{q,\Upsilon} \le \frac{C_q}{r_\Upsilon} \|g\|_{q,\Upsilon},$$

where C_q is a constant independent of g and Υ. This implies the inequality (1) for any $|\boldsymbol{\alpha}| = 1$ and $1 \le q < \infty$. The inequality (1) for general $\boldsymbol{\alpha}$ follows by applying the above inequalities repeatedly for the higher order derivatives. □

We now introduce the so-called *averaged Taylor polynomials*. More details can be found in [5, Chapter 4]. Let $B := \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x} - \boldsymbol{x}_0| < r\}$ be the ball centered around the point $\boldsymbol{x}_0$ with radius r, and consider the cut-off function

$$\varphi_B(\boldsymbol{x}) := \begin{cases} c\, e^{-\left(1-|\boldsymbol{x}-\boldsymbol{x}_0|^2/r^2\right)^{-1}}, & \text{if } \boldsymbol{x} \in B, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where the constant c is chosen such that $\int_{\mathbb{R}^d} \varphi_B(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 1$. For any smooth function $f \in C^p(\mathbb{R}^d)$, the Taylor polynomial of degree p of f at $\boldsymbol{y}$ is given by

$$\mathfrak{T}_{p,\boldsymbol{y}} f(\boldsymbol{x}) := \sum_{|\boldsymbol{\alpha}| \le p} \frac{(\boldsymbol{x} - \boldsymbol{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} f(\boldsymbol{y}), \tag{3}$$

where $\boldsymbol{\alpha}! := \prod_{i=1}^d \alpha_i!$ and $\boldsymbol{z}^{\boldsymbol{\alpha}} := \prod_{i=1}^d (z_i)^{\alpha_i}$ for $\boldsymbol{z} := (z_1, \ldots, z_d)$.

Definition 1 The averaged Taylor polynomial of degree p over the ball B is defined as

$$\mathfrak{F}_{p,B} f(\boldsymbol{x}) := \int_B \mathfrak{T}_{p,\boldsymbol{y}} f(\boldsymbol{x})\, \varphi_B(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y},$$

where $\mathfrak{T}_{p,\boldsymbol{y}} f$ is the Taylor polynomial (3) and φ_B is the cut-off function (2).

An averaged Taylor polynomial is well-defined for any integrable function $f \in L_1(B)$. It can be verified that $\mathfrak{F}_{p,B} f$ is a polynomial of total degree at most p. It holds that

$$\mathfrak{F}_{p,B}\, g = g, \quad \forall g \in \Pi_p.$$

The next lemma describes the approximation power of polynomials, using an averaged Taylor polynomial. This result is a slightly simplified version of the Bramble–Hilbert Lemma (see e.g. [5, Lemma 4.3.8]).

Lemma 2 *Let Υ be a convex body in $\mathbb{R}^d$, and let B_Υ be a ball contained in Υ with largest radius r_Υ. For any $f \in W_q^{p+1}(\Upsilon)$ and $1 \le q \le \infty$, we have*

$$\|D^{\boldsymbol{\alpha}}(f - \mathfrak{F}_{p,B_\Upsilon} f)\|_{q,\Upsilon} \le C\,(\operatorname{diam}(\Upsilon))^{p+1-|\boldsymbol{\alpha}|}\,|f|_{p+1,q,\Upsilon}, \quad 0 \le |\boldsymbol{\alpha}| \le p,$$

where C is a constant independent of f and $\operatorname{diam}(\Upsilon)$*, but dependent on* $\operatorname{chunk}(\Upsilon)$.

Another error bound for averaged Taylor polynomials is given in the next lemma, detailing the influence of the chunkiness parameter in the bound. It is a particular modification of [5, Proposition 4.3.2] and extends the result of [24, Lemma 7].

Lemma 3 *Let Υ be a convex body in $\mathbb{R}^d$, and let B be a ball contained in Υ with center $\boldsymbol{x}_0$ and radius r. For any $f \in W_q^{p+1}(\Upsilon)$, we have*

$$\|f - \mathfrak{F}_{p,B} f\|_{\infty,\Upsilon} \le C\,(1 + \operatorname{chunk}(B,\Upsilon))^d\,(\operatorname{diam}(\Upsilon))^{p+1-d/q}|f|_{p+1,q,\Upsilon},$$

provided that $1 < q < \infty$ and $p > d/q - 1$, or $q = 1$ and $p \ge d - 1$, or $q = \infty$ and $p \ge 0$. The constant C is independent of f, $\operatorname{diam}(\Upsilon)$ *and* $\operatorname{chunk}(\Upsilon)$.

Proof Suppose $1 < q < \infty$. From [5, Proposition 4.2.8] we know that the remainder $f - \mathfrak{F}_{p,B} f$ has the following pointwise representation for any point $\boldsymbol{x} \in \Upsilon$:

$$(f - \mathfrak{F}_{p,B} f)(\boldsymbol{x}) = (p+1) \sum_{|\boldsymbol{\alpha}|=p+1} \int_{B_{\boldsymbol{x}}} \frac{(\boldsymbol{x}-\boldsymbol{z})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\, k(\boldsymbol{x},\boldsymbol{z})\, D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\, \mathrm{d}\boldsymbol{z},$$

where $B_{\boldsymbol{x}} := \operatorname{conv}(\{\boldsymbol{x}\} \cup B)$. The function $k(\boldsymbol{x},\boldsymbol{z})$ can be bounded as

$$|k(\boldsymbol{x},\boldsymbol{z})| \le \frac{C_1}{|\boldsymbol{z}-\boldsymbol{x}|^d}\left(1 + \frac{|\boldsymbol{x}-\boldsymbol{x}_0|}{r}\right)^d.$$

Because $B_{\boldsymbol{x}} \subseteq \Upsilon$, we find that

$$|k(\boldsymbol{x},\boldsymbol{z})| \le \frac{C_1}{|\boldsymbol{z}-\boldsymbol{x}|^d}\left(1 + \frac{\operatorname{diam}(\Upsilon)}{r}\right)^d = \frac{C_1}{|\boldsymbol{z}-\boldsymbol{x}|^d}\,(1 + \operatorname{chunk}(B,\Upsilon))^d.$$

Hence, the remainder can be estimated by

$$\begin{aligned}|(f - \mathfrak{F}_{p,B} f)(\boldsymbol{x})| &\le K_1 \sum_{|\boldsymbol{\alpha}|=p+1} \int_{B_{\boldsymbol{x}}} \frac{|(\boldsymbol{x}-\boldsymbol{z})^{\boldsymbol{\alpha}}|}{|\boldsymbol{z}-\boldsymbol{x}|^d}\,|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z}\\ &\le K_1 \sum_{|\boldsymbol{\alpha}|=p+1} \int_{\Upsilon} |\boldsymbol{z}-\boldsymbol{x}|^{p+1-d}|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z},\end{aligned}$$

with $K_1 := C_1(p+1)\,(1+\text{chunk}(B,\Upsilon))^d$. Let $1/q + 1/q' = 1$, then Hölder's inequality for integrals and sums implies that

$$
\begin{aligned}
|(f - \mathfrak{F}_{p,B} f)(\boldsymbol{x})| &\le K_1 \sum_{|\boldsymbol{\alpha}|=p+1} \left(\int_\Upsilon |\boldsymbol{z}-\boldsymbol{x}|^{(p+1-d)q'} \,\mathrm{d}\boldsymbol{z} \right)^{1/q'} \left(\int_\Upsilon |D^{\boldsymbol{\alpha}} f(\boldsymbol{z})|^q \,\mathrm{d}\boldsymbol{z} \right)^{1/q} \\
&\le K_1 \left(\int_\Upsilon |\boldsymbol{z}-\boldsymbol{x}|^{(p+1-d)q'} \,\mathrm{d}\boldsymbol{z} \right)^{1/q'} \left(\sum_{|\boldsymbol{\alpha}|=p+1} 1 \right)^{1/q'} \left(\sum_{|\boldsymbol{\alpha}|=p+1} \int_\Upsilon |D^{\boldsymbol{\alpha}} f(\boldsymbol{z})|^q \,\mathrm{d}\boldsymbol{z} \right)^{1/q} \\
&= K_1 \left(C_2 \int_\Upsilon |\boldsymbol{z}-\boldsymbol{x}|^{(p+1-d)q'} \,\mathrm{d}\boldsymbol{z} \right)^{1/q'} |f|_{p+1,q,\Upsilon}.
\end{aligned}
$$

Now suppose $(p+1-d)q' + d > 0$, and consider the ball D with center $\boldsymbol{x}$ and radius $\text{diam}(\Upsilon)$. It holds that $\Upsilon \subset D$. We apply a transformation to polar coordinates to obtain that

$$
\begin{aligned}
|(f - \mathfrak{F}_{p,B} f)(\boldsymbol{x})| &\le K_1 \left(C_2 \int_D |\boldsymbol{z}-\boldsymbol{x}|^{(p+1-d)q'} \,\mathrm{d}\boldsymbol{z} \right)^{1/q'} |f|_{p+1,q,\Upsilon} \\
&= K_1 \left(C_3 \int_0^{\text{diam}(\Upsilon)} y^{(p+1-d)q'+d-1} \,\mathrm{d}y \right)^{1/q'} |f|_{p+1,q,\Upsilon} \\
&= K_1\, C_4\, (\text{diam}(\Upsilon))^{p+1-d+d/q'} |f|_{p+1,q,\Upsilon} \\
&= C\,(1+\text{chunk}(B,\Upsilon))^d\, (\text{diam}(\Upsilon))^{p+1-d/q} |f|_{p+1,q,\Upsilon}.
\end{aligned}
$$

It remains to check when $(p+1-d)q' + d > 0$. Since $1 < q < \infty$, this means that $p+1-d > -d/q' = d/q - d$ or $p > d/q - 1$.

The cases $q = 1$ and $q = \infty$ can be proved with a similar line of arguments. If $q = 1$, then $p+1-d \ge 0$ or $p \ge d-1$. If $q = \infty$, then $p+1-d+d > 0$ or $p \ge 0$. $\square$

3 Quasi-interpolation framework based on THB-splines

This section is devoted to the general framework of quasi-interpolants in hierarchical spline spaces developed in [25]. We summarize the main results and follow the same notations as in [25]. One of the key ingredients is the representation of the quasi-interpolants in terms of the truncated hierarchical basis.

Let D be a hyper-rectangle in $\mathbb{R}^d$. We consider a (finite) nested sequence of tensor-product d-variate spline function spaces defined on D,

$$\mathbb{V}^0 \subset \mathbb{V}^1 \subset \mathbb{V}^2 \subset \cdots \subset \mathbb{V}^{n-1}. \tag{4}$$

Any element of $\mathbb{V}^\ell$ is a piecewise polynomial defined over a partition of D consisting of hyper-rectangles, which will be called *cells* of level ℓ. Let N_ℓ be the dimension of $\mathbb{V}^\ell$; we denote by

$$\mathscr{B}^\ell := \{B_{i,\ell},\ i = 1, \ldots, N_\ell\} \tag{5}$$

the normalized tensor-product B-spline basis of $\mathbb{V}^\ell$. It is well known (see e.g. [2, 22]) that these basis functions are locally linearly independent, they have local support, they are nonnegative, and they form a partition of unity.

In addition to the spaces $\mathbb{V}^\ell$ and the corresponding bases $\mathscr{B}^\ell$, we consider a (finite) nested sequence of closed subsets of D,

$$\Omega^0 \supseteq \Omega^1 \supseteq \Omega^2 \supseteq \cdots \supseteq \Omega^{n-1}, \tag{6}$$

where each Ω^ℓ is the union of a selection of cells of level ℓ. We will refer to $\mathbf{\Omega}_n := \{\Omega^0, \Omega^1, \ldots, \Omega^{n-1}\}$ as the *hierarchy of subsets of D of depth n*. Moreover, the collection of the corresponding cells in the hierarchy will be referred to as the *hierarchical mesh of depth n*. Note that a hierarchical mesh is not a partition of the domain because it consists of overlapping cells, in contrast to the conventional concept of (one-level) mesh. Finally, we denote by $\text{supp}(f)$ the intersection of the support of the function f with Ω^0.

Given a sequence of spaces and bases as in Eqs. 4–5 and a hierarchy of subsets as in Eq. 6, we define the hierarchical basis functions as elements of $\bigcup_{\ell=0}^{n-1} \mathscr{B}^\ell$ following a specific selection mechanism, see [11, 12, 26].

Definition 2 The hierarchical basis $\mathscr{H}_{\mathbf{\Omega}_n}$ associated with the hierarchy of subsets $\mathbf{\Omega}_n$ is recursively constructed as follows:

i) $\mathscr{H}^0 := \{B_{i,0} \in \mathscr{B}^0 : \text{supp}(B_{i,0}) \neq \emptyset\}$;

ii) for $\ell = 0, \ldots, n-2$:

$$\mathscr{H}^{\ell+1} := \mathscr{H}_C^{\ell+1} \cup \mathscr{H}_F^{\ell+1},$$

where

$$\begin{aligned} \mathscr{H}_C^{\ell+1} &:= \{B_{i,j} \in \mathscr{H}^\ell : \text{supp}(B_{i,j}) \not\subseteq \Omega^{\ell+1}\}, \\ \mathscr{H}_F^{\ell+1} &:= \{B_{i,\ell+1} \in \mathscr{B}^{\ell+1} : \text{supp}(B_{i,\ell+1}) \subseteq \Omega^{\ell+1}\}; \end{aligned}$$

iii) $\mathscr{H}_{\mathbf{\Omega}_n} := \mathscr{H}^{n-1}$.

The elements in $\mathscr{H}_{\mathbf{\Omega}_n}$ are linearly independent functions, see [26]. Moreover, it is clear that they are nonnegative, but they do not form a partition of unity. The space

$$\mathbb{S}_{\mathbf{\Omega}_n} := \left\langle B_{i,j} : B_{i,j} \in \mathscr{H}_{\mathbf{\Omega}_n} \right\rangle$$

will be referred to as the *hierarchical spline space associated with* $\mathbf{\Omega}_n$. Let $I_{\ell,\mathbf{\Omega}_n}$ be the set of indices of the elements in $\mathscr{B}^\ell$ belonging to $\mathscr{H}_{\mathbf{\Omega}_n}$, i.e.,

$$I_{\ell,\mathbf{\Omega}_n} := \{i \ : \ B_{i,\ell} \in \mathscr{B}^\ell \cap \mathscr{H}_{\mathbf{\Omega}_n}\}, \quad \ell = 0, \ldots, n-1.$$

Then, we can uniquely represent any element $s \in \mathbb{S}_{\mathbf{\Omega}_n}$ as

$$s = \sum_{\ell=0}^{n-1} \sum_{i \in I_{\ell,\mathbf{\Omega}_n}} d_{i,\ell} B_{i,\ell}.$$

An alternative basis with enhanced properties can be obtained by truncating the elements in $\mathscr{H}_{\mathbf{\Omega}_n}$, see [11, 12]. To this end, we first need to define the concept of truncation. Let $s \in \mathbb{V}^\ell \subset \mathbb{V}^{\ell+1}$ be represented with respect to the tensor-product B-spline basis $\mathscr{B}^{\ell+1}$, i.e.,

$$s = \sum_{i=1}^{N_{\ell+1}} c_{i,\ell+1}\, B_{i,\ell+1}, \tag{7}$$

where $c_{i,k}$ denotes the coefficient of $s \in \mathbb{V}^k$ with respect to the basis element $B_{i,k} \in \mathscr{B}^k$. We define the *truncation* of s at level $\ell+1$ as the sum of the terms appearing in Eq. 7 related to the basis functions whose support is not a subset of $\Omega^{\ell+1}$, i.e.,

$$\operatorname{trunc}^{\ell+1}(s) := \sum_{i\,:\,\operatorname{supp}(B_{i,\ell+1}) \not\subseteq \Omega^{\ell+1}} c_{i,\ell+1}\, B_{i,\ell+1}.$$

By using successive truncations of the functions constructed in Definition 2, we can define a new set of basis functions as follows.

Definition 3 The truncated hierarchical basis $\mathscr{T}_{\boldsymbol{\Omega}_n}$ associated with the hierarchy of subsets $\boldsymbol{\Omega}_n$ is recursively constructed as follows:

i) $\mathscr{T}^0 := \{B_{i,0} \in \mathscr{B}^0 : \operatorname{supp}(B_{i,0}) \neq \emptyset\}$;
ii) for $\ell = 0, \ldots, n-2$:

$$\mathscr{T}^{\ell+1} := \mathscr{T}_C^{\ell+1} \cup \mathscr{T}_F^{\ell+1},$$

where

$$\mathscr{T}_C^{\ell+1} := \{\operatorname{trunc}^{\ell+1}(B_{i,j}^{\mathscr{T}}) : B_{i,j}^{\mathscr{T}} \in \mathscr{T}^\ell,\ \operatorname{supp}(B_{i,j}^{\mathscr{T}}) \not\subseteq \Omega^{\ell+1}\},$$
$$\mathscr{T}_F^{\ell+1} := \{B_{i,\ell+1} \in \mathscr{B}^{\ell+1} : \operatorname{supp}(B_{i,\ell+1}) \subseteq \Omega^{\ell+1}\};$$

iii) $\mathscr{T}_{\boldsymbol{\Omega}_n} := \mathscr{T}^{n-1}$.

In similarity to the hierarchical basis $\mathscr{H}_{\boldsymbol{\Omega}_n}$, we denote the elements of $\mathscr{T}_{\boldsymbol{\Omega}_n}$ by

$$\{B_{i,\ell,\boldsymbol{\Omega}_n}^{\mathscr{T}},\ i \in I_{\ell,\boldsymbol{\Omega}_n},\ \ell = 0, \ldots, n-1\},$$

and we will refer to them as the *truncated hierarchical B-splines (THB-splines)*. The THB-spline $B_{i,\ell,\boldsymbol{\Omega}_n}^{\mathscr{T}}$ is said to be of level ℓ. According to [11], the THB-splines form an alternative basis of $\mathbb{S}_{\boldsymbol{\Omega}_n}$. They are nonnegative and sum up to one, so they form a convex partition of unity. Moreover, the truncation process ensures that the THB-splines have the same or smaller support than in the case of the classical hierarchical basis.

Following the approach in [25], good quasi-interpolants (QIs) in $\mathbb{S}_{\boldsymbol{\Omega}_n}$ can be easily obtained by considering the truncated hierarchical basis. We now detail the construction and the properties of such QIs in $\mathbb{S}_{\boldsymbol{\Omega}_n}$.

Given a function f on Ω^0, we first consider a (one-level) QI to f in each of the spaces $\mathbb{V}^\ell$,

$$\mathfrak{Q}^\ell f := \sum_{i=1}^{N_\ell} \lambda_{i,\ell}(f) B_{i,\ell}, \quad \ell = 0, \ldots, n-1, \tag{8}$$

where $\lambda_{i,\ell}$ are suitable linear functionals. We say that $\lambda_{i,\ell}$ is *supported on* $\Lambda_{i,\ell}$ if

$$f|_{\Lambda_{i,\ell}} = 0 \;\Rightarrow\; \lambda_{i,\ell}(f) = 0. \tag{9}$$

The linear functionals $\lambda_{i,\ell}$ can have various expressions. Popular choices involve function values, derivative values and/or integrals of f taken in the neighborhood of $\text{supp}(B_{i,\ell})$, possibly with some restrictions on the smoothness of f. Examples of interesting QIs of the form (8) can be found in [1, 3, 18, 19, 21].

Then, in order to construct a suitable QI to f in the hierarchical space $\mathbb{S}_{\boldsymbol{\Omega}_n}$, we select as coefficient for each basis element $B^{\mathscr{T}}_{i,\ell,\boldsymbol{\Omega}_n}$ the coefficient of the corresponding basis element $B_{i,\ell}$ in Eq. 8. More precisely, we set

$$\mathfrak{Q} f := \sum_{\ell=0}^{n-1} \sum_{i \in I_{\ell,\boldsymbol{\Omega}_n}} \lambda_{i,\ell}(f) B^{\mathscr{T}}_{i,\ell,\boldsymbol{\Omega}_n}. \tag{10}$$

We will refer to QIs of the form (10) as *hierarchical QIs*.

Given a set of degrees $\boldsymbol{p} := (p_1, \ldots, p_d) \in \mathbb{N}^d$, let $\mathbb{P}_{\boldsymbol{p}}$ be the space of tensor-product polynomials of degree p_i in the i-th coordinate. Note that Π_p is a subset of any $\mathbb{P}_{\boldsymbol{p}}$ where $p_i \geq p$, $i = 1, \ldots, d$. From [25, Theorem 3] we know that it is easy to construct hierarchical QIs reproducing polynomials in $\mathbb{P}_{\boldsymbol{p}}$.

Theorem 1 *Let $\mathfrak{Q}^\ell$ be a given sequence of QIs as in* Eq. 8*, and let $\mathfrak{Q}$ be the corresponding hierarchical QI as in* Eq. 10*. If*

$$\mathfrak{Q}^\ell g = g, \quad \forall g \in \mathbb{P}_{\boldsymbol{p}}, \quad \ell = 0, \ldots, n-1,$$

then

$$\mathfrak{Q} g = g, \quad \forall g \in \mathbb{P}_{\boldsymbol{p}}.$$

Furthermore, we can construct hierarchical QIs which are projectors onto $\mathbb{S}_{\boldsymbol{\Omega}_n}$ according to [25, Theorem 4].

Theorem 2 *Let $\mathfrak{Q}^\ell$ be a given sequence of QIs as in* Eq. 8*, and let $\mathfrak{Q}$ be the corresponding hierarchical QI as in* Eq. 10*. Assume*

$$\mathfrak{Q}^\ell s = s, \quad \forall s \in \mathbb{V}^\ell, \quad \ell = 0, \ldots, n-1,$$

and each $\lambda_{i,\ell}$ used in Eq. 10 *is supported on $\Omega^\ell \setminus \Omega^{\ell+1}$. Then,*

$$\mathfrak{Q} s = s, \quad \forall s \in \mathbb{S}_{\boldsymbol{\Omega}_n}.$$

Finally, it is noteworthy to recall from [25, Corollary 3] that the hierarchical QI given in Eq. 10 can also be expressed in terms of the classical hierarchical basis (Definition 2) as follows.

Theorem 3 *Let $\mathfrak{Q}^\ell$ be a given sequence of QIs as in* Eq. 8*, and let $\mathfrak{Q}$ be the corresponding hierarchical QI as in* Eq. 10*. Assume*

$$\mathfrak{Q}^\ell s = s, \quad \forall s \in \mathbb{V}^\ell, \quad \ell = 0, \ldots, n-1,$$

then

$$\mathfrak{Q} f = \sum_{\ell=0}^{n-1} f^{(\ell)},$$

where

$$f^{(0)} := \sum_{i \in I_{0,\mathbf{\Omega}_n}} \lambda_{i,0}(f)\, B_{i,0},$$

$$f^{(\ell)} := \sum_{i \in I_{\ell,\mathbf{\Omega}_n}} \lambda_{i,\ell}\left(f - f^{(0)} - f^{(1)} - \ldots - f^{(\ell-1)}\right) B_{i,\ell}, \quad \ell = 1, \ldots, n-1.$$

4 Local approximation estimates for hierarchical quasi-interpolants

In this section we focus on the local approximation power of the hierarchical QIs of the form (10) and their derivatives in the general q-norm with $1 \le q \le \infty$. This extends and completes the basic approximation results in [25, Section 5].

For the sake of simplicity, we introduce the notation $\|f\|_\infty := \|f\|_{\infty,\Omega^0}$. For a given quasi-interpolant $\mathfrak{Q}$, we denote by $\|\mathfrak{Q}\|_\infty$ the usual induced norm, namely

$$\|\mathfrak{Q}\|_\infty := \sup_{\|f\|_\infty = 1} \|\mathfrak{Q} f\|_\infty.$$

Let us fix for each linear functional $\lambda_{i,\ell}$ a set $\Lambda_{i,\ell}$ satisfying (9). This set is assumed to be taken as small as possible, and we will refer to it as the support of $\lambda_{i,\ell}$. Since the basis $\mathscr{T}_{\mathbf{\Omega}_n}$ forms a convex partition of unity, for any QI of the form (10) it is easy to see that

$$\|\mathfrak{Q}\|_\infty \le \sup_{\ell=0,\ldots,n-1;\ i \in I_{\ell,\mathbf{\Omega}_n}} \|\lambda_{i,\ell}\|_{\infty,\Lambda_{i,\ell}} =: C_{\mathfrak{Q}}, \tag{11}$$

where $\|\lambda_{i,\ell}\|_{\infty,\Lambda_{i,\ell}}$ stands for the induced norm of $\lambda_{i,\ell}$ on $\Lambda_{i,\ell}$. Moreover, for each cell Υ in the hierarchical mesh of $\mathbf{\Omega}_n$, we define the extended set

$$\Lambda_\Upsilon := \operatorname{conv}\left(\bigcup_{(i,\ell)\,:\,\operatorname{supp}(B^{\mathscr{T}}_{i,\ell,\mathbf{\Omega}_n}) \bigcap \Upsilon \neq \emptyset} \Lambda_{i,\ell} \cup \Upsilon \right). \tag{12}$$

It is reasonable to assume that the values $\operatorname{chunk}(\Upsilon)$ and $\operatorname{chunk}(\Lambda_\Upsilon)$ are both bounded by a constant. The set Λ_Υ gives rise to the definition of the hierarchical mesh parameter $\delta_{\mathbf{\Omega}_n}$, called *mesh level disparity*. This definition is inspired by a similar concept used in [24].

Definition 4 For each cell Υ in the hierarchical mesh of $\mathbf{\Omega}_n$, let δ_Υ be the largest difference between the levels of the THB-splines supported on Υ (i.e., all THB-splines involved in the definition of Λ_Υ, see Eq. 12). The mesh level disparity $\delta_{\mathbf{\Omega}_n}$ is defined as the maximum of the values δ_Υ related to all cells Υ which are not further refined in the hierarchical mesh of $\mathbf{\Omega}_n$.

We are now ready to formulate our main approximation result for hierarchical QIs.

Theorem 4 *Let $\mathbb{S}_{\mathbf{\Omega}_n}$ be a hierarchical spline space associated with the hierarchy of subsets $\mathbf{\Omega}_n$. Let Υ be a cell which is not further refined in the hierarchical mesh of*

$\mathbf{\Omega}_n$, *and let* Λ_Υ *be the corresponding set as defined in* Eq. 12. *Let* $\mathfrak{Q}$ *be a hierarchical quasi-interpolant of the form* (10) *in* $\mathbb{S}_{\mathbf{\Omega}_n}$, *such that* $C_{\mathfrak{Q}}$ *in* Eq. 11 *is bounded and*

$$\mathfrak{Q}g|_\Upsilon = g|_\Upsilon, \quad \forall g \in \mathbb{P}_{\boldsymbol{p}}, \quad \boldsymbol{p} := (p, \ldots, p). \tag{13}$$

If $f \in W_q^{p+1}(\Lambda_\Upsilon)$ *such that* $\mathfrak{Q}f$ *is well-defined, then for any* $0 \leq |\boldsymbol{\alpha}| \leq p$, $1 \leq q \leq \infty$ *we have*

$$\|D^{\boldsymbol{\alpha}}(f - \mathfrak{Q}f)\|_{q,\Upsilon} \leq K\,(\mathrm{diam}(\Lambda_\Upsilon))^{p+1-|\boldsymbol{\alpha}|}\,|f|_{p+1,q,\Lambda_\Upsilon}, \tag{14}$$

and

$$K := C\left(1 + C_{\mathfrak{Q}}\,(\mathrm{chunk}(\Upsilon, \Lambda_\Upsilon))^{|\boldsymbol{\alpha}|}\right),$$

provided that $1 < q < \infty$ *and* $p > d/q - 1$, *or* $q = 1$ *and* $p \geq d - 1$, *or* $q = \infty$ *and* $p \geq 0$. *The constant C is independent of f and* $\mathrm{diam}(\Lambda_\Upsilon)$.

Proof Thanks to Eq. 13, for any polynomial $g \in \mathbb{P}_{\boldsymbol{p}}$, we have

$$\begin{aligned}\|D^{\boldsymbol{\alpha}}(f - \mathfrak{Q}f)\|_{q,\Upsilon} &= \|D^{\boldsymbol{\alpha}}(f - g + g - \mathfrak{Q}f)\|_{q,\Upsilon} \\ &\leq \|D^{\boldsymbol{\alpha}}(f - g)\|_{q,\Upsilon} + \|D^{\boldsymbol{\alpha}}\mathfrak{Q}(f - g)\|_{q,\Upsilon}.\end{aligned} \tag{15}$$

Since $\Pi_p \subset \mathbb{P}_{\boldsymbol{p}}$ we can take $g = \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f$, i.e., the averaged Taylor polynomial of degree p as defined in Lemma 2 using the ball B_{Λ_Υ} contained in Λ_Υ with largest radius. The first term in the right-hand side of Eq. 15 can then be bounded by Lemma 2,

$$\begin{aligned}\|D^{\boldsymbol{\alpha}}(f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f)\|_{q,\Upsilon} &\leq \|D^{\boldsymbol{\alpha}}(f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f)\|_{q,\Lambda_\Upsilon} \\ &\leq C_1\,(\mathrm{diam}(\Lambda_\Upsilon))^{p+1-|\boldsymbol{\alpha}|}\,|f|_{p+1,q,\Lambda_\Upsilon}.\end{aligned} \tag{16}$$

We now examine in more detail the second term in the right-hand side of Eq. 15. Any hierarchical spline in $\mathbb{S}_{\mathbf{\Omega}_n}$ over the cell Υ is a polynomial, so we can apply the inequality in Lemma 1. This means that

$$\|D^{\boldsymbol{\alpha}}\mathfrak{Q}(f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f)\|_{q,\Upsilon} \leq \frac{C_2}{(r_\Upsilon)^{|\boldsymbol{\alpha}|}}\|\mathfrak{Q}(f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f)\|_{q,\Upsilon}. \tag{17}$$

Furthermore, by the convex partition of unity property of the basis used in Eq. 10 and by Eqs. 11 and 12, we have

$$\begin{aligned}\|\mathfrak{Q}(f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f)\|_{\infty,\Upsilon} &\leq \sup_{(i,\ell)\,:\,\mathrm{supp}(B^{\mathscr{T}}_{i,\ell,\mathbf{\Omega}_n}) \bigcap \Upsilon \neq \emptyset} \|\lambda_{i,\ell}\|_{\infty,\Lambda_{i,\ell}}\|f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f\|_{\infty,\Lambda_{i,\ell}} \\ &\leq C_{\mathfrak{Q}}\|f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f\|_{\infty,\Lambda_\Upsilon}.\end{aligned}$$

When applying Lemma 3 we obtain

$$\|f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f\|_{\infty,\Lambda_\Upsilon} \leq C_3\left(1 + \mathrm{chunk}(B_{\Lambda_\Upsilon}, \Lambda_\Upsilon)\right)^d (\mathrm{diam}(\Lambda_\Upsilon))^{p+1-d/q}|f|_{p+1,q,\Lambda_\Upsilon}.$$

From its definition it follows that $\mathrm{chunk}(B_{\Lambda_\Upsilon}, \Lambda_\Upsilon) = \mathrm{chunk}(\Lambda_\Upsilon, \Lambda_\Upsilon) = \mathrm{chunk}(\Lambda_\Upsilon)$, which is bounded by assumption. Hence, we arrive at

$$\begin{aligned}\|\mathfrak{Q}(f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f)\|_{q,\Upsilon} &\leq (\mathrm{diam}(\Upsilon))^{d/q}\|\mathfrak{Q}(f - \mathfrak{F}_{p,B_{\Lambda_\Upsilon}} f)\|_{\infty,\Upsilon} \\ &\leq C_4\,C_{\mathfrak{Q}}(\mathrm{diam}(\Lambda_\Upsilon))^{p+1}|f|_{p+1,q,\Lambda_\Upsilon}.\end{aligned} \tag{18}$$

By combining inequalities (15)–(18), we obtain our approximation estimate (14). □

A special instance of the result in Theorem 4 can be found in [25, Theorem 6], which only considers the supremum norm and without estimates for derivatives.

Remark 1 The bound $C_{\mathfrak{Q}}$ for the supremum norm of the quasi-interpolation operator (see Eq. 11) is used in the general q-norm approximation result in Theorem 4. Actually, the proof could be simplified by involving a bound for the q-norm of the quasi-interpolation operator. However, in practice, a bound for the supremum norm is more useful as it is much simpler to compute for many quasi-interpolants.

Remark 2 In order to get a useful (local) approximation result, we need to have a grip on all QI parameters in Eq. 14. The value $C_{\mathfrak{Q}}$ can be easily controlled as explained in Remark 3. Moreover, it is preferred that $\mathrm{diam}(\Lambda_\Upsilon)$ is of the same order as $\mathrm{diam}(\Upsilon)$, which will also have a positive effect on the parameter $\mathrm{chunk}(\Upsilon, \Lambda_\Upsilon)$. To this purpose, we have to control both the supports of the linear functionals $\lambda_{i,\ell}$ and those of the truncated basis elements $B^{\mathcal{T}}_{i,\ell,\mathbf{\Omega}_n}$. The former supports are discussed in Remark 4. The latter supports depend on the hierarchies of the subsets (6) and the spaces (4); they can be controlled by bounding the mesh level disparity $\delta_{\mathbf{\Omega}_n}$, see Remark 5.

Remark 3 As already observed in [25, Remark 3], the value $C_{\mathfrak{Q}}$ does not depend on $\mathrm{diam}(\Lambda_\Upsilon)$ when the main ingredients in the recipe of $\mathfrak{Q}$, namely $\mathfrak{Q}^\ell$ in Eq. 8, have a similar property, i.e., when

$$C_{\mathfrak{Q}^\ell} := \sup_{i=1,\ldots,N_\ell} \|\lambda_{i,\ell}\|_{\infty,\Lambda_{i,\ell}}$$

does not depend on $\mathrm{diam}(\Lambda_\Upsilon)$. For examples of one-level QIs with such a property, see e.g. [18] or Section 5.

Remark 4 The supports of the linear functionals $\lambda_{i,\ell}$ can be fixed by a proper selection of the QIs in Eq. 8. There are several examples of good spline QIs where each $\lambda_{i,\ell}$ is supported on the support of $B_{i,\ell}$, see e.g. [18, 19] or Section 5.

Remark 5 It is possible to provide selection strategies for Ω^ℓ, $\ell = 0, \ldots, n-1$, such that the mesh level disparity $\delta_{\mathbf{\Omega}_n}$ is bounded independently of the number of levels in the hierarchy. This means that only basis elements $B^{\mathcal{T}}_{i,\ell,\mathbf{\Omega}_n}$ corresponding to a bounded number of consecutive levels are nonzero on a given cell Υ (which is not further refined in the hierarchical mesh of $\mathbf{\Omega}_n$). Such meshes are called *admissible meshes* in [6]. As a consequence, this limits the total number of overlapping basis elements on the cell[1], and also the span of their supports with respect to the cell. For example, the strategy in [12, Appendix A] ensures that only basis elements corresponding to at most two consecutive levels $\ell-1,\ \ell$ are nonzero on a cell of level ℓ in $\Omega^\ell \setminus \Omega^{\ell+1}$.

[1] An upper bound for the total number of overlapping basis functions $B^{\mathcal{T}}_{i,\ell,\mathbf{\Omega}_n}$ on the cell Υ is given by $(\delta_{\mathbf{\Omega}_n}+1)\prod_{i=1}^d (p_i+1)$ when all tensor-product spaces $\mathbb{V}^\ell$ are of degrees $\boldsymbol{p} := (p_1, \ldots, p_d)$.

Remark 6 Both the definition of the truncated hierarchical bases and the construction of the related hierarchical QIs have been presented here in terms of tensor-product B-splines due to their relevant interest. However, for both of them it is not necessary to use tensor-product B-splines as building blocks, see [12] (and also [25, Section 6]) for a broad framework of sets of basis functions that can be used as well. The proof of Theorem 4 can be rephrased towards this broad framework. For example, in the hierarchical context of Powell–Sabin splines, a similar result can be found in [24].

Finally, we apply Theorem 4 to a specific case of interest where we consider a hierarchy of nested uniform spline spaces. In this case we can quantify better the cell parameters. We consider a sequence of spaces as in Eq. 4 where $\mathbb{V}^k$ is a tensor-product d-variate spline space of degree $\boldsymbol{p} := (p, \dots, p)$ on a uniform mesh consisting of hyper-cubes with edge size h_k, and $\mathbb{V}^k$ is obtained from $\mathbb{V}^{k-1}$ by dyadic refinement. Hence, we have $h_k = h_{k-1}/2$. Let us take the set Υ in Theorem 4 as a cell of a given level ℓ in $\Omega^\ell \setminus \Omega^{\ell+1}$, and it will be denoted by Υ_ℓ. It is easy to check that

$$\operatorname{diam}(\Upsilon_\ell) = h_\ell\sqrt{d}, \quad \operatorname{chunk}(\Upsilon_\ell) = \frac{h_\ell\sqrt{d}}{h_\ell/2} = 2\sqrt{d}.$$

Moreover, we assume that the linear functionals $\lambda_{i,k}$ are locally supported, namely

$$\operatorname{diam}(\Lambda_{i,k}) \leq C_\lambda h_k, \quad \forall i \in I_{k,\boldsymbol{\Omega}_n}, \quad k = 0, \dots, n-1, \tag{19}$$

where C_λ is a constant independent of h_k. Then, taking into account the mesh level disparity $\delta_{\boldsymbol{\Omega}_n}$, we can bound $\operatorname{diam}(\Lambda_{\Upsilon_\ell})$ as

$$\operatorname{diam}(\Lambda_{\Upsilon_\ell}) \leq C_\Lambda 2^{\delta_{\boldsymbol{\Omega}_n}} h_\ell,$$

where C_Λ is a constant independent of h_ℓ. As a consequence,

$$\operatorname{chunk}(\Upsilon_\ell, \Lambda_{\Upsilon_\ell}) \leq C_\Lambda 2^{\delta_{\boldsymbol{\Omega}_n}+1}.$$

Hence, we obtain the following result in the case of a hierarchical spline space $\mathbb{S}_{\boldsymbol{\Omega}_n}$ defined on a uniform hierarchical mesh of $\boldsymbol{\Omega}_n$ as described above, with a mesh level disparity $\delta_{\boldsymbol{\Omega}_n}$ bounded independently of the number of levels in the hierarchy (see Remark 5).

Corollary 1 *Consider a sequence of spaces as in* Eq. 4 *where $\mathbb{V}^k$ is a tensor-product d-variate spline space of degree $\boldsymbol{p} := (p, \dots, p)$ on a uniform mesh consisting of hyper-cubes with edge size h_k, and $\mathbb{V}^k$ is obtained from $\mathbb{V}^{k-1}$ by dyadic refinement. Let $\mathbb{S}_{\boldsymbol{\Omega}_n}$ be a hierarchical spline space based on the above spaces $\mathbb{V}^k$ and defined on a hierarchical mesh of $\boldsymbol{\Omega}_n$ with a mesh level disparity $\delta_{\boldsymbol{\Omega}_n}$ bounded independently of the number of levels in the hierarchy. Let Υ_ℓ be a cell of level ℓ in $\Omega^\ell \setminus \Omega^{\ell+1}$, and let Λ_{Υ_ℓ} be the corresponding set as defined in* Eq. 12. *Let $\mathfrak{Q}$ be a hierarchical quasi-interpolant of the form* (10) *in $\mathbb{S}_{\boldsymbol{\Omega}_n}$, such that $C_{\mathfrak{Q}}$ in* Eq. 11 *is bounded and each $\lambda_{i,\ell}$ is locally supported as in* Eq. 19. *Moreover, suppose*

$$\mathfrak{Q}g|_{\Upsilon_\ell} = g|_{\Upsilon_\ell}, \quad \forall g \in \mathbb{P}_{\boldsymbol{p}}, \quad \boldsymbol{p} := (p, \dots, p).$$

If $f \in W_q^{p+1}(\Lambda_{\Upsilon_\ell})$ *such that* $\mathfrak{Q} f$ *is well-defined, then for any* $0 \le |\boldsymbol{\alpha}| \le p$, $1 \le q \le \infty$ *we have*

$$\|D^{\boldsymbol{\alpha}}(f - \mathfrak{Q} f)\|_{q,\Upsilon_\ell} \le C\,(h_\ell)^{p+1-|\boldsymbol{\alpha}|}\,|f|_{p+1,q,\Lambda_{\Upsilon_\ell}},$$

provided that $1 < q < \infty$ *and* $p > d/q - 1$, *or* $q = 1$ *and* $p \ge d - 1$, *or* $q = \infty$ *and* $p \ge 0$. *The constant* C *is independent of* f *and* h_ℓ.

Note that the spline spaces $\mathbb{V}^k$ can have arbitrary smoothness as long as they form a nested sequence. A special instance of the result in Corollary 1 can be found in [25, Example 2] considering the supremum norm and a specific mesh refinement strategy.

5 Construction of a family of hierarchical quasi-interpolants

In this section we elaborate a family of hierarchical QIs that fit in the framework described in Section 3, where the underlying bases $\mathscr{B}^\ell$ consist of tensor-product B-splines over uniform nested sequences of knots. The presented construction of hierarchical QIs requires a small number of function evaluations for a good approximation performance.

We start by describing the one-level QI in the univariate case. Let ϕ_p be the *cardinal B-spline* of degree $p \in \mathbb{N}$ defined over the uniform knot sequence $\{0, 1, \ldots, p+1\}$. It is a piecewise polynomial of class $C^{p-1}(\mathbb{R})$, and can be computed through the following recurrence formula [1]:

$$\phi_p(x) := \frac{x}{p}\phi_{p-1}(x) + \frac{p+1-x}{p}\phi_{p-1}(x-1), \quad p \ge 1,$$

starting with

$$\phi_0(x) := \begin{cases} 1, & \text{if } x \in [0,1), \\ 0, & \text{elsewhere.} \end{cases}$$

For the construction of a good quasi-interpolant based on translates of cardinal B-splines, we use the *central factorial numbers of the first kind*. They are denoted by $t(m,k)$ and can be computed recursively as follows [7, Proposition 2.1]:

$$t(m,k) := \begin{cases} 0, & \text{if } k > m, \\ 1, & \text{if } k = m, \\ t(m-2,k-2) - \left(\dfrac{m-2}{2}\right)^2 t(m-2,k), & \text{if } 2 \le k < m, \end{cases}$$

starting with

$$t(m,0) := 0, \quad t(m,1) := \prod_{i=1}^{m-1}\left(\frac{m}{2} - i\right), \quad m \ge 2,$$

and $t(0,0) = t(1,1) = 1$, $t(0,1) = t(1,0) = 0$. We also recall the definition of the *central differences*:

$$\delta f(x) := f\left(x + \frac{1}{2}\right) - f\left(x - \frac{1}{2}\right),$$
$$\delta^k f(x) := \delta^{k-1} f\left(x + \frac{1}{2}\right) - \delta^{k-1} f\left(x - \frac{1}{2}\right).$$

Note that the central differences of even order can be explicitly expressed by

$$\delta^{2k} f(x) = \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} f(x + k - i).$$

Then, inspired by [7, Proposition 6.2.2], we define the following spline approximant to a given univariate function f:

$$\mathfrak{Q}_p f(x) := \sum_{j \in \mathbb{Z}} f\left(j + \frac{p+1}{2}\right) \psi_p(x - j), \tag{20}$$

where

$$\psi_p(x) := \sum_{k=0}^{r} \frac{t(2k+p+1, p+1)}{(2k)!\binom{2k+p+1}{p+1}} \delta^{2k} \phi_p(x)$$
$$= \sum_{k=0}^{r} \frac{t(2k+p+1, p+1)}{\binom{2k+p+1}{p+1}} \sum_{i=0}^{2k} \frac{(-1)^i}{i!\,(2k-i)!} \phi_p(x + k - i),$$

and $r := \lceil (p+1)/2 \rceil - 1$. Note that $\psi_p(x)$ is supported on $[-r, r+p+1]$. This QI can be rewritten as

$$\mathfrak{Q}_p f(x) = \sum_{j \in \mathbb{Z}} \mu_{j,p}(f)\, \phi_p(x - j), \tag{21}$$

where

$$\mu_{j,p}(f) := \sum_{k=0}^{r} \frac{t(2k+p+1, p+1)}{\binom{2k+p+1}{p+1}} \sum_{i=0}^{2k} \frac{(-1)^i}{i!\,(2k-i)!} f\left(k + j - i + \frac{p+1}{2}\right). \tag{22}$$

For example, formula (22) gives for $p = 1, \ldots, 5$:

$$\begin{aligned}
\mu_{j,1}(f) &= f(j+1), \\
\mu_{j,2}(f) &= -\frac{1}{8} f\left(j+\frac{1}{2}\right) + \frac{5}{4} f\left(j+\frac{3}{2}\right) - \frac{1}{8} f\left(j+\frac{5}{2}\right), \\
\mu_{j,3}(f) &= -\frac{1}{6} f(j+1) + \frac{4}{3} f(j+2) - \frac{1}{6} f(j+3), \\
\mu_{j,4}(f) &= \frac{47}{1152} f\left(j+\frac{1}{2}\right) - \frac{107}{288} f\left(j+\frac{3}{2}\right) + \frac{319}{192} f\left(j+\frac{5}{2}\right) \\
&\quad -\frac{107}{288} f\left(j+\frac{7}{2}\right) + \frac{47}{1152} f\left(j+\frac{9}{2}\right), \\
\mu_{j,5}(f) &= \frac{13}{240} f(j+1) - \frac{7}{15} f(j+2) + \frac{73}{40} f(j+3) - \frac{7}{15} f(j+4) \\
&\quad +\frac{13}{240} f(j+5).
\end{aligned}$$

The QI in Eqs. 20 and 21 reproduces polynomials of degree up to p. It requires only function evaluations at either integer points (odd p) or half-integer points (even p). These points are nothing else than the Greville points of the cardinal B-spline basis $\{\phi_p(x-j),\ j \in \mathbb{Z}\}$. Moreover, it is easy to verify that $\mu_{j,p}$ is supported on $M_{j,p} := [j, j+p+1]$, see Eq. 9. When writing (22) in the form $\mu_{j,p}(f) = \sum_m w_{m,p} f(j+m+\frac{p+1}{2})$, it is clear that

$$\|\mu_{j,p}(f)\|_{\infty,M_{j,p}} \leq C_p \|f\|_{\infty,M_{j,p}}, \tag{23}$$

with $C_p := \sum_m |w_{m,p}|$ which only depends on p. This constant is plotted for $p = 1, \ldots, 10$ in Fig. 1 and seems to grow like $(4/3)^{p-1}$. Hence, the QI in Eqs. 20 and 21 satisfies the requirement discussed in Remark 3.

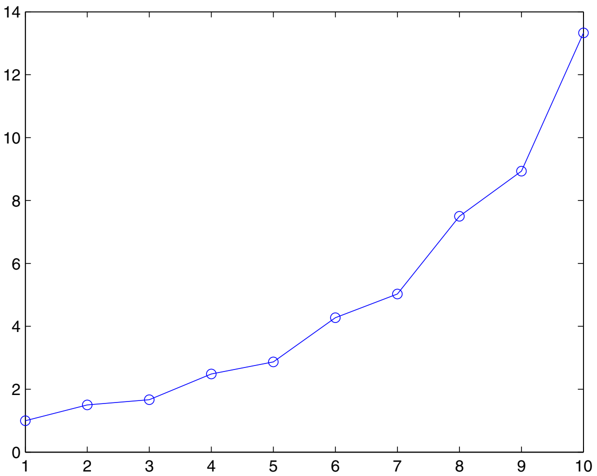

Fig. 1 The graph of C_p in Eq. 23 with respect to $p = 1, \ldots, 10$

The one-level QI in the multivariate setting can be constructed by taking the tensor product of univariate schemes of the form (21). More precisely, we define the following spline approximant to a given d-variate function f:

$$\mathfrak{Q}_{\boldsymbol{p}} f(\boldsymbol{x}) := \sum_{\boldsymbol{j} \in \mathbb{Z}^d} \mu_{\boldsymbol{j},\boldsymbol{p}}(f)\, \phi_{\boldsymbol{p}}(\boldsymbol{x} - \boldsymbol{j}), \tag{24}$$

where $\boldsymbol{p} := (p_1, \ldots, p_d) \in \mathbb{N}^d$, $\boldsymbol{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\boldsymbol{j} := (j_1, \ldots, j_d) \in \mathbb{Z}^d$, and

$$\phi_{\boldsymbol{p}}(\boldsymbol{x}) := \phi_{p_1}(x_1) \cdots \phi_{p_d}(x_d), \quad \mu_{\boldsymbol{j},\boldsymbol{p}}(f) := (\mu_{j_1,p_1} \cdots \mu_{j_d,p_d})(f),$$

assuming that μ_{j_i,p_i} is the linear functional (22) operating on functions of the variable x_i. The QI in Eq. 24 reproduces tensor-product polynomials of degrees up to $\boldsymbol{p}$. From its construction it is clear that $\mu_{\boldsymbol{j},\boldsymbol{p}}$ is supported on $[j_1, j_1 + p_1 + 1] \times \cdots \times [j_d, j_d + p_d + 1]$.

We now describe a possible hierarchical QI to a given d-variate function f. We consider a sequence of spaces as in Eq. 4 where $\mathbb{V}^\ell$ is a tensor-product d-variate spline space of degree $\boldsymbol{p} := (p, \ldots, p)$ and of class $C^{p-1}(\mathbb{R}^d)$ defined on a uniform mesh consisting of hyper-cubes with edge size h_ℓ, and $\mathbb{V}^\ell$ is obtained from $\mathbb{V}^{\ell-1}$ by dyadic refinement. Since each space $\mathbb{V}^\ell$ is uniform, the corresponding tensor-product B-splines $B_{i,\ell} \in \mathscr{B}^\ell$ can be seen as translated and dilated versions of the tensor-product cardinal B-spline $\phi_{\boldsymbol{p}}$. Hence, formula (24) can be used to construct a QI in the space $\mathbb{V}^\ell$, denoted by

$$\widetilde{\mathfrak{Q}}^\ell(f) := \sum_{i=1}^{N_\ell} \widetilde{\lambda}_{i,\ell}(f) B_{i,\ell}.$$

According to Eq. 10, the sequence of quasi-interpolants $\widetilde{\mathfrak{Q}}^\ell$, $\ell = 0, \ldots, n-1$, leads to the hierarchical quasi-interpolant

$$\widetilde{\mathfrak{Q}}(f) := \sum_{\ell=0}^{n-1} \sum_{i \in I_{\ell,\boldsymbol{\Omega}_n}} \widetilde{\lambda}_{i,\ell}(f) B^{\mathscr{T}}_{i,\ell,\boldsymbol{\Omega}_n}, \tag{25}$$

which ensures polynomial reproduction as explained in Theorem 1.

Example 1 To illustrate the performance of the hierarchical QI in Eq. 25, we approximate the bivariate function

$$f(x, y) = \frac{\tanh(9y - 9x) + 1}{9}, \tag{26}$$

on the square $\Omega^0 = [-1, 1] \times [-1, 1]$. This function is taken from [9, 25] and simulates a sharp diagonal rise, see Fig. 2 (left). We now show that an approximation with a given accuracy can be obtained with a small(er) number of degrees of freedom by using a proper locally refined hierarchical mesh. To this end, we consider a sequence of hierarchical meshes of depths $n = 1, \ldots, 5$, which are locally refined in the neighborhood of the diagonal rise. The final mesh of depth $n = 5$ is depicted in Fig. 2 (right). The remaining meshes of depth $n \leq 4$ can be obtained by taking the first n levels of the final mesh. The coarsest mesh ($n = 1$) consists of 8×8 cells,

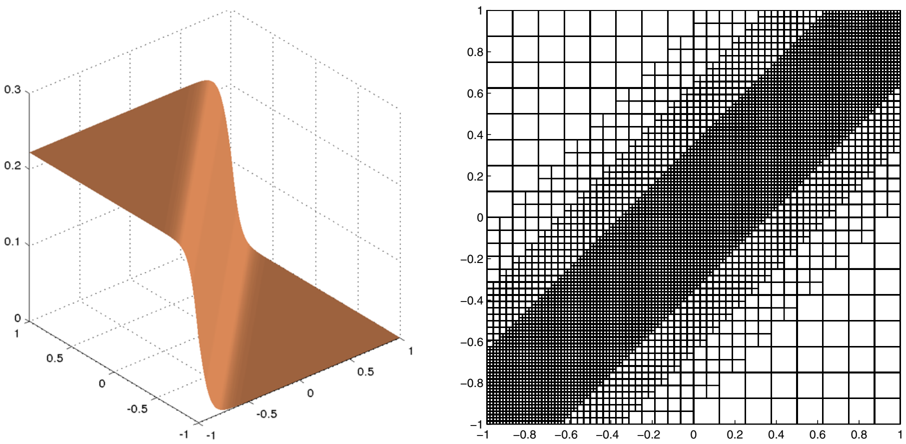

Fig. 2 Plot of the function f in Eq. 26, and a related locally refined hierarchical mesh of depth 5

and any cell of level ℓ is of size $h_\ell := 2^{-(\ell+2)}$, $\ell = 0, \ldots, 4$. For comparison, we also consider a sequence of five uniform (globally refined) meshes. We then compute the hierarchical quasi-interpolants $\widetilde{\mathfrak{Q}}(f)$ of bidegree (p, p) with $p = 2, 4$, on both sequences of meshes. Table 1 reports the maximum errors

$$e_{\max} := \max |f - \widetilde{\mathfrak{Q}}(f)|, \quad e^{(1)}_{\max} := \max |D_x(f - \widetilde{\mathfrak{Q}}(f))| = \max |D_y(f - \widetilde{\mathfrak{Q}}(f))|,$$

for the different refined meshes. All maximum errors are computed on a uniform 150×150 grid over Ω^0. Recall that $D_x = D^{(1,0)}$ and $D_y = D^{(0,1)}$. The dimension (number of degrees of freedom) of the corresponding hierarchical spline spaces is also shown in Table 1. Since f is a smooth function, from Corollary 1 we expect for all the sequences of QIs that the function error $e_{\max}$ converges with order $p + 1$, and that the derivative error $e^{(1)}_{\max}$ converges with order p. These orders are roughly observed in the table for both degrees $p = 2, 4$ (when the meshes are fine enough). Note that the locally refined QIs obtain the same accuracy as the corresponding globally refined ones of the same depth, but of course with a lower dimension.

Other examples of hierarchical QIs that fit in the framework described in Section 3 can be found in [25, Section 7] and [4]. In particular, we refer to the results presented in [25, Example 6] where a quasi-interpolant $\widetilde{\mathfrak{Q}}^{\mathrm{s}}(f)$ was applied to the same function f as in Eq. 26. This QI is a projector onto the hierarchical spline space, and it requires only function evaluations (just like $\widetilde{\mathfrak{Q}}(f)$). Let us now make a global comparison between the two hierarchical quasi-interpolants $\widetilde{\mathfrak{Q}}^{\mathrm{s}}(f)$ and $\widetilde{\mathfrak{Q}}(f)$, both of bidegree (p, p).

– The linear functionals $\widetilde{\lambda}^{\mathrm{s}}_{i,\ell}(f)$ used in $\widetilde{\mathfrak{Q}}^{\mathrm{s}}(f)$ are only supported on a single cell (of level ℓ), and guarantee hierarchical spline reproduction. However, this locality comes at a price: since $\widetilde{\lambda}^{\mathrm{s}}_{i,\ell}(f)$ requires to solve a local interpolation problem on a uniform $(p + 1) \times (p + 1)$ tensor-product grid over the cell, it means that

Table 1 Dimensions and maximum errors of spline quasi-interpolants $\widetilde{\mathfrak{Q}}(f)$ (and their derivatives) applied to the function f in Eq. 26, using bidegrees (p, p) with $p = 2, 4$, and defined on globally and locally refined hierarchical meshes of different depths $n = 1, \ldots, 5$ (with smallest mesh size $h_{n-1} = 2^{-n-1}$), see Example 1

		Global refinement			Local refinement		
p	n	dim	$e_{\max}$	$e^{(1)}_{\max}$	dim	$e_{\max}$	$e^{(1)}_{\max}$
2	1	100	3.413e-02	5.294e-01	100	3.413e-02	5.294e-01
	2	324	1.042e-02	2.297e-01	324	1.042e-02	2.297e-01
	3	1156	1.595e-03	5.521e-02	876	1.595e-03	5.521e-02
	4	4356	1.337e-04	9.791e-03	2522	1.337e-04	9.791e-03
	5	16900	1.310e-05	3.057e-03	6382	1.310e-05	3.057e-03
4	1	144	3.136e-02	4.980e-01	144	3.136e-02	4.980e-01
	2	400	8.206e-03	1.869e-01	400	8.206e-03	1.869e-01
	3	1296	8.425e-04	2.757e-02	952	8.425e-04	2.757e-02
	4	4624	3.174e-05	1.413e-03	2558	3.174e-05	1.413e-03
	5	17424	6.547e-07	3.753e-05	6258	7.330e-07	3.753e-05

(roughly speaking) the approximation needs on average p^2 function evaluations per degree of freedom.

– The linear functionals $\widetilde{\lambda}_{i,\ell}(f)$ used in $\widetilde{\mathfrak{Q}}(f)$ are supported on $(p+1)^2$ cells (of level ℓ), and guarantee polynomial reproduction. The cost is quite small because it requires only function evaluations at grid points (odd p) or half-grid points (even p) which are shared between different $\widetilde{\lambda}_{i,\ell}(f)$ of the same level. Hence, the approximation needs on average only 1 function evaluation per degree of freedom.

The above properties can be interpreted as follows. The theory predicts a (local) approximation order of $p+1$ for both QIs. On the one hand, for a fixed spline dimension, $\widetilde{\mathfrak{Q}}^{\mathrm{s}}(f)$ is expected to have a better approximation quality than $\widetilde{\mathfrak{Q}}(f)$ thanks to the locality of the linear functionals and the hierarchical spline reproduction, and so the same accuracy can be obtained by $\widetilde{\mathfrak{Q}}^{\mathrm{s}}(f)$ on a coarser (or more locally refined) hierarchical mesh. On the other hand, for a fixed cost (in terms of function evaluations), $\widetilde{\mathfrak{Q}}(f)$ is expected to perform better than $\widetilde{\mathfrak{Q}}^{\mathrm{s}}(f)$. This behavior is confirmed numerically by the results in Example 1 and [25, Example 6].

6 Conclusions

In this paper we have presented a full approximation study for the general procedure of QIs in hierarchical spline spaces developed in [25]. Such spaces are composed of a hierarchy of nested spaces and provide a flexible framework for local refinement in any dimensionality. The considered hierarchical QIs are described in terms of the truncated hierarchical basis and their construction is basically effortless. Indeed, it suffices to consider a sequence of one-level QIs, i.e., a QI in each space associated with a particular level in the hierarchy. Then, the coefficients of the hierarchical QI are nothing else than a proper subset of the coefficients of the one-level QIs. Under some mild assumptions on the underlying hierarchical meshes and the one-level QIs, we have shown that the corresponding hierarchical QIs and their derivatives possess optimal local approximation power in the general q-norm with $1 \leq q \leq \infty$.

We have detailed the construction and the approximation theory of hierarchical QIs in terms of hierarchies of truncated tensor-product B-splines, the so-called THB-splines. Nevertheless, the approach is completely general and can be rephrased towards the broad framework of sets of basis functions defined in [12] (see also [25, Section 6]).

Finally, we have described a specific family of hierarchical QIs defined on uniform hierarchical meshes. The construction is based on cardinal B-splines of degree p and central factorial numbers of the first kind. It guarantees polynomial reproduction of degree p and it only requires function evaluations at grid points (odd p) or half-grid points (even p). This results in good approximation properties at a very low cost, and has been confirmed by some numerical experiments.

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References

1. de Boor, C.: Quasi interpolants and approximation power of multivariate splines. In: Dahmen, W., Gasca, M., Micchelli, C.A. (eds.) Computation of Curves and Surfaces, pp. 313–345. Kluwer (1990)
2. de Boor, C.: A Practical Guide to Splines. Revised Edition. Springer (2001)
3. de Boor, C., Fix, G.J.: Spline approximation by quasi-interpolants. J. Approx. Theory **8**, 19–45 (1973)
4. Bracco, C., Giannelli, C., Mazzia, F., Sestini, A.: Bivariate hierarchical Hermite spline quasi-interpolation. Preprint (2015)
5. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, 2nd Edition. Springer-Verlag (2002)
6. Buffa, A., Giannelli, C.: Adaptive isogeometric methods with hierarchical splines: Error estimator and convergence. Math. Models Methods Appl. Sci. **26**, 1–25 (2016)
7. Butzer, P.L., Schmidt, K., Stark, E.L., Vogt, L.: Central factorial numbers; their main properties and some applications. Numer. Funct. Anal. Optim. **10**, 419–488 (1989)
8. Davydov, O.: On local refinement of smooth finite elements and splines. Matem. Mod. **14**, 109–116 (2002)
9. Dyn, N., Levin, D., Rippa, S.: Data dependent triangulations for piecewise linear interpolation. IMA J. Numer. Anal. **10**, 137–154 (1990)
10. Forsey, D.R., Bartels, R.H.: Hierarchical B-spline refinement. Comput. Graph. **22**, 205–212 (1988)
11. Giannelli, C., Jüttler, B., Speleers, H.: THB-splines: The truncated basis for hierarchical splines. Comput. Aided Geom. Des. **29**, 485–498 (2012)
12. Giannelli, C., Jüttler, B., Speleers, H.: Strongly stable bases for adaptively refined multilevel spline spaces. Adv. Comp. Math. **40**, 459–490 (2014)
13. Greiner, G., Hormann, K.: Interpolating and approximating scattered 3D-data with hierarchical tensor product B-splines. In: Le Méhauté, A., Rabut, C., Schumaker, L.L. (eds.) Surface Fitting and Multiresolution Methods, pp. 163–172. Vanderbilt University Press (1997)

14. Kraft, R.: Adaptive and linearly independent multilevel B-splines. In: Le Méhauté, A., Rabut, C., Schumaker, L.L. (eds.) Surface Fitting and Multiresolution Methods, pp. 209–218. Vanderbilt University Press, Nashville (1997)
15. Kraft, R.: Adaptive und linear unabhängige Multilevel B-Splines und ihre Anwendungen. Ph.D. thesis, Universität Stuttgart (1998)
16. Kroó, A., Révész, S.: On Bernstein and Markov-type inequalities for multivariate polynomials on convex bodies. J. Approx. Theory **99**, 134–152 (1999)
17. Kroó, A.: On Bernstein–Markov-type inequalities for multivariate polynomials in L_q-norm. J. Approx. Theory **159**, 85–96 (2009)
18. Lee, B.-G., Lyche, T., Mørken, K.: Some examples of quasi-interpolants constructed from local spline projectors. In: Lyche, T., Schumaker, L.L. (eds.) Mathematical Methods for Curves and Surfaces, Oslo 2000, pp. 243–252. Vanderbilt University Press (2001)
19. Lyche, T., Schumaker, L.L.: Local spline approximation methods. J. Approx. Theory **15**, 294–325 (1975)
20. Rabut, C.: Locally tensor product functions. Numer. Algor. **39**, 329–348 (2005)
21. Sablonnière, P.: Recent progress on univariate and multivariate polynomial or spline quasi-interpolants. In: de Brujn, M.G., Mache, D.H., Szabadoz, J. (eds.) Trends and Applications in Constructive Approximation, ISNM, vol. 151, pp. 229–245. Birhäuser Verlag Basel (2005)
22. Schumaker, L.L.: Spline Functions: Basic Theory, 3rd Edition. Cambridge University Press (2007)
23. Speleers, H., Dierckx, P., Vandewalle, S.: Quasi-hierarchical Powell–Sabin B-splines. Comput. Aided Geom. Des. **26**, 174–191 (2009)
24. Speleers, H., Dierckx, P., Vandewalle, S.: On the local approximation power of quasi-hierarchical Powell–Sabin splines. In: Dæhlen, M., Floater, M.S., Lyche, T., Merrien, J.L., Mørken, K., Schumaker, L.L. (eds.) Mathematical Methods for Curves and Surfaces, Lecture Notes in Computer Science, vol. 5862, pp. 419–433 (2010)
25. Speleers, H., Manni, C.: Effortless quasi-interpolation in hierarchical spaces. Numer. Math. **132**, 155–184 (2016)
26. Vuong, A.-V., Giannelli, C., Jüttler, B., Simeon, B.: A hierarchical approach to adaptive local refinement in isogeometric analysis. Comput. Methods Appl. Mech. Eng. **200**, 3554–3567 (2011)